CHAPTER

# 4

# PRINCIPLE OF MATHEMATICAL INDUCTION

## CONCEPT TYPE QUESTIONS

**Directions** : This section contains multiple choice questions. Each question has four choices (a), (b), (c) and (d), out of which only one is correct.

**1.** Let P($n$) be statement $2^n < n!$. Where $n$ is a natural number, then P($n$) is true for:
(a) all $n$ (b) all $n > 2$
(c) all $n > 3$ (d) None of these

**2.** If $P(n) = 2 + 4 + 6 + ..... + 2n, n \in \mathbf{N}$, then $P(k) = k(k+1) + 2$
$\Rightarrow P(k+1) = (k+1)(k+2) + 2$ for all $k \in \mathbf{N}$. So we can conclude that $P(n) = n(n+1) + 2$ for
(a) all $n \in \mathbf{N}$ (b) $n > 1$
(c) $n > 2$ (d) nothing can be said

**3.** Let $T(k)$ be the statement $1 + 3 + 5 + .... + (2k - 1) = k^2 + 10$
Which of the following is correct?
(a) $T(1)$ is true
(b) $T(k)$ is true $\Rightarrow$ $T(k + 1)$ is true
(c) $T(n)$ is true for all $n \in \mathbf{N}$
(d) All above are correct

**4.** Let $S(K) = 1 + 3 + 5 ... + (2K - 1) = 3 + K^2$, then which of the following is true?
(a) Principle of mathematical induction can be used to prove the formula
(b) $S(K) \Rightarrow S(K+1)$
(c) $S(K) \not\Rightarrow S(K+1)$
(d) $S(1)$ is correct

**5.** Let P($n$) : "$2^n < (1 \times 2 \times 3 \times ... \times n)$". Then the smallest positive integer for which P($n$) is true is
(a) 1 (b) 2
(c) 3 (d) 4

**6.** A student was asked to prove a statement $P(n)$ by induction. He proved that $P(k + 1)$ is true whenever $P(k)$ is true for all $k > 5 \in$ N and also that $P(5)$ is true. On the basis of this he could conclude that P($n$) is true
(a) for all $n \in$ N
(b) for all $n > 5$
(c) for all $n \geq 5$
(d) for all $n < 5$

**7.** If $P(n) : 2 + 4 + 6 + ... + (2n), n \in N$, then $P(k) = k(k + 1) + 2$ implies $P(k + 1) = (k + 1)(k + 2) + 2$ is true for all $k \in N$. So statement $P(n) = n(n + 1) + 2$ is true for:
(a) $n \geq 1$
(b) $n \geq 2$
(c) $n \geq 3$
(d) None of these

**8.** If $P(n)$ : "$46^n + 16^n + k$ is divisible by 64 for $n \in$ N" is true, then the least negative integral value of $k$ is.
(a) $-1$ (b) 1
(c) 2 (d) $-2$

**9.** Use principle of mathematical induction to find the value of k, where $(10^{2n-1} + 1)$ is divisible by k.
(a) 11 (b) 12
(c) 13 (d) 9

**10.** For all $n \geq 1$, $1^2 + 2^2 + 3^2 + 4^2 + ..... + n^2 =$
(a) $\frac{n(n+1)}{6}$
(b) $n(n + 1)(2n - 1)$
(c) $\frac{n(n-1)(2n+1)}{2}$
(d) $\frac{n(n+1)(2n+1)}{6}$

**11.** P(n) : $2.7^n + 3.5^n - 5$ is divisible by
(a) 24, $\forall$ n $\in$ N
(b) 21, $\forall$ n $\in$ N
(c) 35, $\forall$ n $\in$ N
(d) 50, $\forall$ n $\in$ N

**12.** For all $n \geq 1$, $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ..... + \frac{1}{n(n+1)} =$
(a) n (b) $\frac{n}{n+1}$
(c) $\frac{(n+1)}{n}$ (d) $\frac{4n+3}{2n}$

**13.** For every positive integer n, $7^n - 3^n$ is divisible by
(a) 7 (b) 3
(c) 4 (d) 5

14. By mathematical induction,

$$\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+.....+\frac{1}{n(n+1)(n+2)}$$ is equal to

(a) $\frac{n(n+1)}{4(n+2)(n+3)}$

(b) $\frac{n(n+3)}{4(n+1)(n+2)}$

(c) $\frac{n(n+2)}{4(n+1)(n+3)}$

(d) None of these

15. By using principle of mathematical induction for every natural number, $(ab)^n$ =
(a) $a^n b^n$ (b) $a^n b$
(c) $ab^n$ (d) 1

16. If $n \in N$, then $11^{n+2} + 12^{2n+1}$ is divisible by
(a) 113 (b) 123
(c) 133 (d) None of these

17. For all $n \in N$, $41^n - 14^n$ is a multiple of
(a) 26 (b) 27
(c) 25 (d) None of these

18. The remainder when $5^{4n}$ is divided by 13, is
(a) 1 (b) 8
(c) 9 (d) 10

19. If m, n are any two odd positive integers with n < m, then the largest positive integer which divides all the numbers of the type $m^2 - n^2$ is
(a) 4 (b) 6
(c) 8 (d) 9

20. For natural number n, $2^n (n-1)! < n^n$, if
(a) $n < 2$ (b) $n > 2$ (c) $n \ge 2$ (d) $n > 3$

21. For all $n \in N$, $3.5^{2n+1} + 2^{3n+1}$ is divisible by
(a) 19 (b) 17
(c) 23 (d) 25

22. Principle of mathematical induction is used
(a) to prove any statement
(b) to prove results which are true for all real numbers
(c) to prove that statements which are formulated in terms of n, where n is positive integer
(d) in deductive reasoning

23. For all $n \in N$, $1.3 + 2.3^2 + 3.3^3 + ..... + n.3^n$ is equal to

(a) $\frac{(2n+1)3^{n+1}+3}{4}$

(b) $\frac{(2n-1)3^{n+1}+3}{4}$

(c) $\frac{(2n+1)3^{n}+3}{4}$

(d) $\frac{(2n-1)3^{n+1}+1}{4}$

24. For all $n \in N$,

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+.....+\frac{1}{1+2+3+.....+n}$$

is equal to

(a) $\frac{3n}{n+1}$ (b) $\frac{n}{n+1}$

(c) $\frac{2n}{n-1}$ (d) $\frac{2n}{n+1}$

25. $10^n + 3(4^{n+2}) + 5$ is divisible by ($n \in N$)
(a) 7 (b) 5
(c) 9 (d) 17

26. The statement P(n)
"$1 \times 1! + 2 \times 2! + 3 \times 3! + ..... + n \times n! = (n+1)! - 1$" is
(a) True for all $n > 1$ (b) Not true for any n
(c) True for all $n \in N$ (d) None of these

27. If n is a natural number, then $\left(\frac{n+1}{2}\right)^n \ge n!$ is true when
(a) $n > 1$ (b) $n \ge 1$
(c) $n > 2$ (d) $n \ge 2$

28. For natural number n, $(n!)^2 > n^n$, if
(a) $n > 3$ (b) $n > 4$
(c) $n \ge 4$ (d) $n \ge 3$

## STATEMENT TYPE QUESTIONS

**Directions** : Read the following statement and choose the correct option from the given below four options.

29. **Statement-I :** $1 + 2 + 3 + ..... + n < \frac{1}{8}(2n+1)^2$, $n \in N$.

**Statement-II :** $n(n+1)(n+5)$ is a multiple of 3, $n \in N$.
(a) Only Statement I is true
(b) Only Statement II is true
(c) Both Statements are true
(d) Both Statements are false

## ASSERTION - REASON TYPE QUESTIONS

**Directions :** Each of these questions contains two statements, Assertion and Reason. Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.
(a) Assertion is correct, Reason is correct; reason is a correct explanation for assertion.
(b) Assertion is correct, Reason is correct; reason is not a correct explanation for assertion
(c) Assertion is correct, Reason is incorrect
(d) Assertion is incorrect, Reason is correct.

30. **Assertion :** For every natural number $n \ge 2$,

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+.....+\frac{1}{\sqrt{n}} > \sqrt{n}$$

**Reason :** For every natural number $n \ge 2$,

$\sqrt{n(n+1)} < n + 1$.

**31.** **Assertion :** $11^{m+2} + 12^{2m+1}$ is divisible by 133 for all $m \in N$.

**Reason :** $x^n - y^n$ is divisible by $x + y$, $\forall\ n \in N, x \neq y$.

## CRITICALTHINKING TYPE QUESTIONS

**Directions** : This section contains multiple choice questions. Each question has four choices (a), (b), (c) and (d), out of which only one is correct.

**32.** The greatest positive integer, which divides $n(n+1)(n+2)(n+3)$ for all $n \in \mathbf{N}$, is
(a) 2 (b) 6
(c) 24 (d) 120

**33.** Let P($n$) : $n^2 + n + 1$ is an even integer. If $P(k)$ is assumed true then $P(k+1)$ is true. Therefore $P(n)$ is true.
(a) for $n > 1$ (b) for all $n \in N$
(c) for $n > 2$ (d) None of these

**34.** By the principle of induction $\forall\ n \in N$, $3^{2n}$ when divided by 8, leaves remainder
(a) 2 (b) 3
(c) 7 (d) 1

**35.** If n is a positive integer, then $5^{2n+2} - 24n - 25$ is divisible by
(a) 574 (b) 575
(c) 674 (d) 576

**36.** The greatest positive integer, which divides $(n + 1)(n + 2)(n + 3)\ .....\ (n + r)$ for all $n \in W$, is
(a) r (b) r!
(c) n + r (d) (r + 1)!

**37.** If $\frac{4^n}{n+1} < \frac{(2n)!}{(n!)^2}$, then P(n) is true for
(a) $n \geq 1$ (b) $n > 0$
(c) $n < 0$ (d) $n \geq 2$

**38.** For all $n \in N$,
$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right).....\left(1+\frac{(2n+1)}{n^2}\right)$$
is equal to
(a) $\frac{(n+1)^2}{2}$ (b) $\frac{(n+1)^3}{3}$
(c) $(n+1)^2$ (d) None of these

**39.** For all $n \in N$, the sum of $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is
(a) a negative integer (b) a whole number
(c) a real number (d) a natural number

**40.** For given series:
$1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 + .....,$
if $S_n$ is the sum of n terms, then
(a) $S_n = \frac{n(n+1)^2}{2}$, if n is even
(b) $S_n = \frac{n^2(n+1)}{2}$, if n is odd
(c) Both (a) and (b) are true
(d) Both (a) and (b) are false

**41.** When $2^{301}$ is divided by 5, the least positive remainder is
(a) 4 (b) 8
(c) 2 (d) 6

# HINTS AND SOLUTIONS

## CONCEPT TYPE QUESTIONS

**1.** **(c)** Let $P(n) : 2^n < n!$
Then $P(1) : 2^1 < 1!$, which is true
Now $P(2) : 2^2 < 2!$, which is not true
Also $P(3) : 2^3 < 3!$, which is not true
$P(4) : 2^4 < 4!$, which is true
Let $P(k)$ is true if $k \geq 4$
That is $2^k < k!$, $k \geq 4$
$\Rightarrow 2.2^k < 2(k!) \Rightarrow 2^{k+1} < k(k!)$ $[\because k \geq 4 > 2]$
$\Rightarrow 2^{k+1} < (k+1)! \Rightarrow P(k+1)$ is true.
Hence, we conclude that $P(n)$ is not true for $n = 2, 3$ but holds true for $n \geq 4$.

**2.** **(d)** We note that $P(1) = 2$ and hence,
$P(n) = n\,(n+1) + 2$ is not true for $n = 1$.
So the principle of mathematical induction is not applicable and nothing can be said about the validity of the statement $P(n) = n\,(n+1) + 2$.

**3.** **(b)** When $k = 1$, LHS = 1 but RHS = 1 + 10 = 11
$\therefore$ $T(1)$ is not true
Let $T(k)$ is true.
*i.e.,* $1 + 3 + 5 + ..... + (2k-1) = k^2 + 10$
Now, $1 + 3 + 5 + ..... + (2k-1) + (2k+1)$
$= k^2 + 10 + 2k + 1 = (k+1)^2 + 10$
$\therefore T(k+1)$ is true.
*i.e.,* $T(k)$ is true $\Rightarrow T(k+1)$ is true.
But $T(n)$ is not true for all $n \in \mathbf{N}$, as $T(1)$ is not true.

**4.** **(b)** $S(K) = 1+3+5+...+(2K-1) = 3 + K^2$
$S(1) = 1 = 3 + 1$, which is not true
$\because$ $S(1)$ is not true.
$\therefore$ P.M.I cannot be applied
Let $S(K)$ is true, i.e. $1 + 3 + 5 + .... + (2K-1) = 3 + K^2$
$\Rightarrow 1 + 3 + 5 + .... + (2K-1) + 2K + 1$
$= 3 + K^2 + 2K + 1 = 3 + (K+1)^2$
$\therefore S(K) \Rightarrow S(K+1)$

**5.** **(d)** Since $P(1) : 2 < 1$ is false
$P(2) : 2^2 < 1 \times 2$ is false
$P(3) : 2^3 < 1 \times 2 \times 3$ is false
$P(4) : 2^4 < 1 \times 2 \times 3 \times 4$ is true

**6.** **(c)** Since $P(5)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

**7.** **(d)** $P(1) = 2$ and $k(k+1) + 2 = 4$, So $P(1)$ is not true. Mathematical Induction is not applicable.

**8.** **(a)** For $n = 1$, $P(1)$: $65 + k$ is divisible by 64.
Thus $k$, should be –1
Since 65 – 1 = 64 is divisible by 64.

**9.** **(a)** Let P(n) be the statement given by
$P(n) : 10^{2n-1} + 1$ is divisible by 11
For n = 1, $P(1) : 10^{(2 \times 1)-1} + 1 = 11$,
which is divisible by 11.
So, P(1) is true.
Let P(k) be true, i.e. $10^{2k-1} + 1$ is divisible by 11
$\Rightarrow$ $10^{2k-1} + 1 = 11\lambda$, for some $\lambda \in N$ ... (i)
We shall now show that P(k + 1) is true. For this, we have to show that $10^{2(k+1)-1} + 1$ is divisible by 11.
Now, $10^{2(k+1)-1} + 1 = 10^{2k-1}\,10^2 + 1$
$= (11\lambda - 1)100 + 1$ [Using (i)]
$= 1100\lambda - 99 = 11(100\lambda - 9) = 11\mu$,
where $\mu = 100\lambda - 9 \in N$
$\Rightarrow$ $10^{2(k+1)-1} + 1$ is divisible by 11
$\Rightarrow$ P(k + 1) is true.
Thus, P(k + 1) is true, whenever P(k) is true.
Hence, by the principle of mathematical induction, P(k) is true for all n $\in$ N, i.e. $10^{2n-1} + 1$ is divisible by 11 for all n $\in$ N.

**10.** **(d)** Let the given statement be P(n), i.e.
$$P(n) : 1^2 + 2^2 + 3^2 + 4^2 + ..... + n^2 = \frac{n(n+1)(2n+1)}{6}$$
For n = 1,
$$P(1) : 1 = \frac{1(1+1)((2 \times 1)+1)}{6} = \frac{1 \times 2 \times 3}{6} = 1,$$
which is true.
Assume that P(k) is true for some positive integer k,
$$\text{i.e. } 1^2 + 2^2 + 3^2 + 4^2 + ..... + k^2 = \frac{k(k+1)(2k+1)}{6} \quad ... \text{(i)}$$
We shall now prove that P(k + 1) is also true,
i.e. $1^2 + 2^2 + 3^2 + 4^2 + ..... + k^2 + (k+1)^2$
$$= \frac{(k+1)(k+2)(2k+3)}{6}$$
Now, L.H.S. $= (1^2 + 2^2 + 3^2 + 4^2 + ..... + k^2) + (k+1)^2$
$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{[Using (i)]}$$

$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$

$= \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$ = R.H.S.

Thus, P(k + 1) is true, whenever P(k) is true.
Hence, from the principle of mathematical induction, the statement P(n) is true for all natural numbers n.

**11. (a)** P(n) : $2.7^n + 3.5^n - 5$ is divisible by 24.
For n = 1,
P(1) : 2.7 + 3.5 – 5 = 24, which is divisible by 24.
Assume that P(k) is true,
i.e. $2.7^k + 3.5^k - 5 = 24q$, where $q \in N$ ... (i)
Now, we wish to prove that P(k + 1) is true whenever P(k) is true, i.e. $2.7^{k+1} + 3.5^{k+1} - 5$ is divisible by 24.
We have,
$2.7^{k+1} + 3.5^{k+1} - 5 = 2.7^k \cdot 7^1 + 3.5^k \cdot 5^1 - 5$
$= 7[2.7^k + 3.5^k - 5 - 3.5^k + 5] + 3.5^k \cdot 5 - 5$
$= 7[24q - 3.5^k + 5] + 15.5^k - 5$
$= (7 \times 24q) - 21.5^k + 35 + 15.5^k - 5$
$= (7 \times 24q) - 6.5^k + 30 = (7\times 24q) - 6(5^k - 5)$
$= (7 \times 24q) - 6(4p)$ [$\because$ $(5^k - 5)$ is a multiple of 4]
$= (7 \times 24q) - 24p = 24(7q - p)$
$= 24 \times r$; $r = 7q - p$, is some natural number ... (ii)
Thus, P(k + 1) is true whenever P(k) is true.
Hence, by the principle of mathematical induction, P(n) is true for all $n \in N$.

**12. (b)** Let P(n) : $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ..... + \frac{1}{n(n+1)} = \frac{n}{n+1}$.
For n = 1,
P(1) : $\frac{1}{1.2} = \frac{1}{2} = \frac{1}{1+1}$, which is true.
Assume that P(k) is true for some natural number k,
i.e. $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ..... + \frac{1}{k(k+1)} = \frac{k}{k+1}$ ... (i)
We shall now prove that P(k + 1) is true, i.e.

$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ..... + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$

L.H.S. $= \left[\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ..... + \frac{1}{k(k+1)}\right] + \frac{1}{(k+1)(k+2)}$

$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$ [Using (i)]

$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{(k^2+2k+1)}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)}$

$= \frac{k+1}{k+2}$ = R.H.S.

Thus, P(k + 1) is true whenever P(k) is true. Hence, by the principle of mathematical induction, P(n) is true for all natural numbers.

**13. (c)** Let P(n) : $7^n - 3^n$ is divisible by 4.
For n = 1,
P(1) : $7^1 - 3^1 = 4$, which is divisible by 4. Thus, P(n) is true for n = 1.
Let P(k) be true for some natural number k,
i.e. P(k) : $7^k - 3^k$ is divisible by 4.
We can write $7^k - 3^k = 4d$, where $d \in N$ ... (i)
Now, we wish to prove that P(k + 1) is true whenever P(k) is true, i.e. $7^{k+1} - 3^{k+1}$ is divisible by 4.
Now, $7^{(k+1)} - 3^{(k+1)} = 7^{(k+1)} - 7.3^k + 7.3^k - 3^{(k+1)}$
$= 7(7^k - 3^k) + (7 - 3)3^k = 7(4d) + 4.3^k$ [using (i)]
$= 4(7d + 3^k)$, which is divisible by 4.
Thus, P(k + 1) is true whenever P(k) is true. Therefore, by the principle of mathematical induction the statement is true for every positive integer n.

**14. (b)** Let P(n) : $\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + ..... + \frac{1}{n(n+1)(n+2)}$

$= \frac{n(n+3)}{4(n+1)(n+2)}$

For n = 1,

L.H.S. $= \frac{1}{1\cdot2\cdot3} = \frac{1}{6}$

and R.H.S. $= \frac{1(1+3)}{4(1+1)(1+2)} = \frac{1}{6}$

$\therefore$ P(1) is true.
Let P(k) is true, then

P(k) : $\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + ..... + \frac{1}{k(k+1)(k+2)}$

$= \frac{k(k+3)}{4(k+1)(k+2)}$ ... (i)

For n = k + 1,

P(k + 1) : $\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + .....$

$+ \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$

$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$

L.H.S. $= \frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + .....$

$+ \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$

$= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$ [from (i)]

$= \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)} = \frac{(k+1)(k+4)}{4(k+2)(k+3)} =$ R.H.S.

Hence, P(k + 1) is true.

Hence, by principle of mathematical induction for all $n \in N$, P(n) is true.

**15. (a)** Let P(n) be the given statement,

i.e. $P(n) : (ab)^n = a^n b^n$

We note that P(n) is true for n = 1, since $(ab)^1 = a^1 b^1$

Let P(k) be true,

i.e. $(ab)^k = a^k b^k$ ... (i)

We shall now prove that P(k + 1) is true whenever P(k) is true.

Now, we have $(ab)^{k+1} = (ab)^k (ab)$

$= (a^k b^k)(ab)$ [by using (i)]

$= (a^k \cdot a^1)(b^k \cdot b^1) = a^{k+1} \cdot b^{k+1}$

Therefore, P(k + 1) is also true whenever P(k) is true. Hence, by principle of mathematical induction, P(n) is true for all $n \in N$.

**16. (c)** On putting n = 1 in $11^{n+2} + 12^{2n+1}$, we get $11^{1+2} + 12^{(2\times1)+1} = 11^3 + 12^3 = 3059$, which is divisible by 133 only.

**17. (b)** Let P(n) be the statement given by

$P(n) : 41^n - 14^n$ is a multiple of 27

For n = 1,

i.e. $P(1) = 41^1 - 14^1 = 27 = 1 \times 27$,

which is a multiple of 27.

$\therefore$ P(1) is true.

Let P(k) be true, i.e. $41^k - 14^k = 27\lambda$ ... (i)

For n = k + 1,

$41^{k+1} - 14^{k+1} = 41^k\, 41 - 14^k\, 14$

$= (27\lambda + 14^k)41 - 14^k\, 14$ [using (i)]

$= (27\lambda \times 41) + (14^k \times 41) - (14^k \times 14)$

$= (27\lambda \times 41) + 14^k(41 - 14)$

$= (27\lambda \times 41) + (14^k \times 27)$

$= 27(41\lambda + 14^k)$,

which is a multiple of 27.

Therefore, P(k + 1) is true when P(k) is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

**18. (a)** For n = 1,

$5^4 = 625 = (624 + 1) = (48 \times 13) + 1$,

i.e $5^4$ leaves 1 as remainder when divided by 13.

**19. (c)** Let m = 2k + 1, n = 2k – 1 ($k \in N$)

$\therefore m^2 - n^2 = 4k^2 + 1 + 4k - 4k^2 + 4k - 1 = 8k$

Hence, all the numbers of the form $m^2 - n^2$ are always divisible by 8.

**20. (b)** The condition $2^n (n-1)! < n^n$ is satisfied for n > 2.

**21. (b)** $3.5^{2n+1} + 2^{3n+1}$

Put n = 1, we get

$(3 \times 5^3) + 2^4 = 391$, which is divisible by 17.

Put n = 2, we get

$(3 \times 5^5) + 2^7 = 9503$, which is divisible by 17 only.

**22. (c)** In algebra or in other discipline of Mathematics, there are certain results or statements that are formulated in terms of n, where n is a positive integer. To prove such statement, the well-suited principle, i.e. used-based on the specific technique is known as the principle of mathematical induction.

**23. (b)** Let the statement P(n) be defined as

$P(n) = 1.3 + 2.3^2 + 3.3^3 + ..... + n.3^n$

$= \frac{(2n-1)3^{n+1} + 3}{4}$

**Step I :** For n = 1,

$P(1) : 1.3 = \frac{(2.1-1)3^{1+1} + 3}{4} = \frac{3^2 + 3}{4}$

$= \frac{9+3}{4} = \frac{12}{4} = 3 = 1.3$, which is true.

**Step II :** Let it is true for n = k,

i.e. $1.3 + 2.3^2 + 3.3^3 + ..... + k.3^k$

$= \frac{(2k-1)3^{k+1} + 3}{4}$ ... (i)

**Step III :** For n = k + 1,

$(1.3 + 2.3^2 + 3.3^3 + ..... + k.3^k) + (k+1)3^{k+1}$

$= \frac{(2k-1)3^{k+1} + 3}{4} + (k+1)3^{k+1}$

[Using equation (i)]

$= \frac{(2k-1)3^{k+1} + 3 + 4(k+1)3^{k+1}}{4}$

$= \frac{3^{k+1}(2k - 1 + 4k + 4) + 3}{4}$

[taking $3^{k+1}$ common in first and last term of numerator part]

$= \frac{3^{k+1}(6k+3) + 3}{4} = \frac{3^{k+1} \cdot 3(2k+1) + 3}{4}$

[taking 3 common in first term of numerator part]

$= \frac{3^{(k+1)+1}[2k + 2 - 1] + 3}{4}$

$= \frac{[2(k+1) - 1]3^{(k+1)+1} + 3}{4}$

Therefore, P(k + 1) is true when P(k) is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

**24. (d)** Let the statement P(n) be defined as

$$P(n) : 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + ..... + \frac{1}{1+2+3+.....+n} = \frac{2n}{n+1}$$

i.e. $P(n) : 1+\frac{1}{1+2}+\frac{1}{1+2+3}+.....+\frac{2}{n(n+1)}=\frac{2n}{n+1}$

$\left[\because \text{ sum of natural numbers} = \frac{n(n+1)}{2}\right]$

**Step I :** For n = 1,

$P(1) : 1 = \frac{2\times 1}{1+1} = \frac{2}{2} = 1$, which is true.

**Step II :** Let it is true for n = k,

i.e. $1 + \frac{1}{1+2}+\frac{1}{1+2+3}+.....+\frac{2}{k(k+1)}=\frac{2k}{k+1}$ ... (i)

**Step III :** For n = k + 1,

$\left(1+\frac{1}{1+2}+\frac{1}{1+2+3}+.....+\frac{2}{k(k+1)}\right)+\frac{2}{(k+1)(k+2)}$

$= \frac{2k}{k+1} + \frac{2}{(k+1)(k+2)}$ [using equation (i)]

$= \frac{2k(k+2)+2}{(k+1)(k+2)} = \frac{2[k^2+2k+1]}{(k+1)(k+2)}$

[taking 2 common in numerator part]

$= \frac{2(k+1)^2}{(k+1)(k+2)}$ [$\because (a+b)^2 = a^2 + 2ab + b^2$]

$= \frac{2(k+1)}{k+2} = \frac{2(k+1)}{(k+1)+1}$

Therefore, P(k + 1) is true, when P(k) is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

**25. (c)** $10^n + 3(4^{n+2}) + 5$
Taking n = 2;
$10^2 + 3 \times 4^4 + 5 = 100 + 768 + 5 = 873$
Therefore, this is divisible by 9.

**26. (c)** Check for n = 1, 2, 3, ....., it is true for all $n \in N$.

**27. (b)** Check through option, the condition $\left(\frac{n+1}{2}\right)^n \geq n!$ is true for $n \geq 1$.

**28. (d)** Check through option, condition $(n!)^2 > n^n$ is true when $n \geq 3$.

## STATEMENT TYPE QUESTIONS

**29. (c)** I. Let the statement P(n) be defined as

$P(n) : 1 + 2 + 3 + ..... + n < \frac{1}{8}(2n+1)^2$

**Step I :** For n = 1,

$P(1) : 1 < \frac{1}{8}(2.1+1)^2 \Rightarrow 1 < \frac{1}{8} \times 3^2$

$\Rightarrow 1 < \frac{9}{8}$, which is true.

**Step II :** Let it is true for n = k.

$1 + 2 + 3 + ..... + k < \frac{1}{8}(2k+1)^2$ ... (i)

**Step III :** For n = k + 1,

$(1+2+3+.....+k)+(k+1) < \frac{1}{8}(2k+1)^2 + (k+1)$

[using equation (i)]

$= \frac{(2k+1)^2}{8} + \frac{k+1}{1} = \frac{(2k+1)^2+8k+8}{8}$

$= \frac{4k^2+1+4k+8k+8}{8}$

$= \frac{4k^2+12k+9}{8} = \frac{(2k+3)^2}{8}$

$= \frac{(2k+2+1)^2}{8} = \frac{[2(k+1)+1]^2}{8}$

$\Rightarrow 1+2+3+.....+k+(k+1) < \frac{[2(k+1)+1]^2}{8}$

Therefore, P(k + 1) is true when P(k) is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

II. Let the statement P(n) be defined as
P(n) : n(n + 1) (n + 5) is a multiple of 3.
**Step I :** For n = 1,
$P(1) : 1(1 + 1)(1 + 5) = 1 \times 2 \times 6 = 12 = 3 \times 4$,
which is a multiple of 3, that is true.
**Step II :** Let it is true for n = k,
i.e. $k(k+1)(k+5) = 3\lambda$
$\Rightarrow k(k^2 + 5k + k + 5) = 3\lambda$
$\Rightarrow k^3 + 6k^2 + 5k = 3\lambda$ ... (i)
**Step III :** For n = k + 1, (k + 1) (k + 1 + 1) (k + 1 + 5)
$= (k+1)(k+2)(k+6) = (k^2+2k+k+2)(k+6)$
$= (k^2 + 3k + 2)(k + 6)$
$= k^3 + 6k^2 + 3k^2 + 18k + 2k + 12$

$= k^3 + 9k^2 + 20k + 12$
$= (3\lambda - 6k^2 - 5k) + 9k^2 + 20k + 12$ [using equation (i)]
$= 3\lambda + 3k^2 + 15k + 12$
$= 3(\lambda + k^2 + 5k + 4)$, which is a multiple of 3.
Therefore, P(k + 1) is true when P(k) is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.
Hence, both the statements are true.

## ASSERTION - REASON TYPE QUESTIONS

**30.** **(a)** **Assertion :** Let P(n) : $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ..... + \frac{1}{\sqrt{n}} > \sqrt{n}$

For n = 2,

P(2) : $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$, which is true.

Assume P(k) is true,

i.e. $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ..... + \frac{1}{\sqrt{k}} > \sqrt{k}$ ... (i)

For n = k + 1, we have to show that

$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ..... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$ ... (ii)

L.H.S. $= \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ..... + \frac{1}{\sqrt{k}}\right) + \frac{1}{\sqrt{k+1}}$ ... (iii)

**Reason :** For n = k,

$\sqrt{k(k+1)} < k + 1$

$\Rightarrow \sqrt{k}\sqrt{k+1} < \sqrt{k+1}\sqrt{k+1}$

$\Rightarrow \sqrt{k} < \sqrt{k+1}$

$\because \sqrt{k+1} > \sqrt{k}$ for $k \ge 2$

$\Rightarrow 1 > \frac{\sqrt{k}}{\sqrt{k+1}}$

$\Rightarrow \sqrt{k} > \frac{k}{\sqrt{k+1}}$, (Multiplying by $\sqrt{k}$)

$\Rightarrow \sqrt{k} > \frac{(k+1)-1}{\sqrt{k+1}} \Rightarrow \sqrt{k} > \sqrt{k+1} - \frac{1}{\sqrt{k+1}}$

$\Rightarrow \sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$ ... (iv)

From (iii) and (iv),

$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ..... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k}$

$+ \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$ [Using (i)]

Hence, (ii) is true for n = k + 1
Hence, P(n) is true for $n \ge 2$
So, Assertion and Reason are correct and Reason is the correct explanation of Assertion.

**31.** **(c)** If $11^{m+2} + 12^{2m+1}$ is divisible by 133, then
$11^{m+2} + 12^{2m+1} = 133\lambda, \lambda \in N$ ... (i)
Hence, $11^{(m+1)+2} + 12^{2(m+1)+1}$
$= (11^{m+2} \times 11) + (12^{2m+1} \times 12^2)$
$= (133\lambda - 12^{2m+1}) \times 11 + (144 \times 12^{2m+1})$ [using (i)]
$= (11 \times 133\lambda) - (11 \times 12^{2m+1}) + (144 \times 12^{2m+1})$
$= (11 \times 133\lambda) + (133 \times 12^{2m+1})$

## CRITICALTHINKING TYPE QUESTIONS

**32.** **(c)** The product of $r$ consecutive integers is divisible by $r!$. Thus $n(n+1)(n+2)(n+3)$ is divisible by $4! = 24$.

**33.** **(d)** $P(1)$ is not true (Principle of induction is not applicable). Also $n(n+1)+1$ is always an odd integer.

**34.** **(d)** Let P(n) be the statement given by
P(n) : $3^{2n}$ when divided by 8, the remainder is 1.
or P(n) : $3^{2n} = 8\lambda + 1$ for some $\lambda \in N$
For n = 1,
P(1) : $3^2 = (8 \times 1) + 1 = 8\lambda + 1$, where $\lambda = 1$
$\therefore$ P(1) is true.
Let P(k) be true.
Then, $3^{2k} = 8\lambda + 1$ for some $\lambda \in N$ ... (i)
We shall now show that P(k + 1) is true, for which we have to show that $3^{2(k+1)}$ when divided by 8, the remainder is 1.
Now, $3^{2(k+1)} = 3^{2k} \cdot 3^2 = (8\lambda + 1) \times 9$ [Using (i)]
$= 72\lambda + 9 = 72\lambda + 8 + 1 = 8(9\lambda + 1) + 1$
$= 8\mu + 1$, where $\mu = 9\lambda + 1 \in N$
$\Rightarrow$ P(k + 1) is true.
Thus, P(k + 1) is true, whenever P(k) is true.
Hence, by the principle of mathematical induction P(n) is true for all $n \in N$.

**35.** **(d)** Let P(n) be the statement given by
P(n) : $5^{2n+2} - 24n - 25$ is divisible by 576.
For n = 1,
P(1) : $5^{2+2} - 24 - 25 = 625 - 49 = 576$,
which is divisible by 576.
$\therefore$ P(1) is true.
Let P(k) be true,
i.e. P(k) : $5^{2k+2} - 24k - 25$ is divisible by 576.
$\Rightarrow 5^{2k+2} - 24k - 25 = 576\lambda$ ... (i)
We have to show that P(k + 1) is true,
i.e. $5^{2k+4} - 24k - 49$ is divisible by 576

Now, $5^{2k+4} - 24k - 49$

$= 5^{2k+2+2} - 24k - 49 = 5^{2k+2} \cdot 5^2 - 24k - 49$

$= (576\lambda + 24k + 25) \cdot 25 - 24k - 49$ [from (i)]

$= 576.25\lambda + 600k + 625 - 24k - 49$

$= 576.25\lambda + 576k + 576$

$= 576\{25\lambda + k + 1\}$, which is divisible by 576.

$\therefore$ P(k + 1) is true whenever P(k) is true.

So, P(n) is true for all $n \in N$.

**36. (b)** The product of r consecutive natural numbers is divisible by r! and not by (r + 1)!

**37. (d)** Let P(n) : $\frac{4^n}{n+1} < \frac{(2n)!}{(n!)^2}$

For n = 1,
P(n) is not true.
For n = 2,

P(2) : $\frac{4^2}{2+1} < \frac{4!}{(2)^2} \Rightarrow \frac{16}{3} < \frac{24}{4}$ which is true.

Let for n = m > 2, P(n) is true, i.e.

$$\frac{4^m}{m+1} < \frac{(2m)!}{(m!)^2}$$

Now, $\frac{4^{m+1}}{m+2} = \frac{4^m}{m+2} \cdot \frac{4(m+1)}{m+2} < \frac{(2m)!}{(m!)^2} \cdot \frac{4(m+1)}{(m+2)}$

$$= \frac{(2m)!(2m+1)(2m+2)4(m+1)(m+1)^2}{(2m+1)(2m+2)(m!)^2(m+1)^2(m+2)}$$

$$= \frac{[2(m+1)]!}{[(m+1)!]^2} \cdot \frac{2(m+1)^2}{(2m+1)(m+2)} < \frac{[2(m+1)]!}{[(m+1)!]^2}$$

$$\left[\because \frac{2(m+1)^2}{(2m+1)(m+2)} < 1 \; \forall \; m > 2\right]$$

Hence, for $n \geq 2$, P(n) is true.

**38. (c)** Let the statement P(n) be defined as

P(n) : $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2n+1)}{n^2}\right) = (n+1)^2$

**Step I :** For n = 1,

i.e. P(1) : $\left(1+\frac{3}{1}\right) = (1+1)^2 = 2^2 = 4 = \left(1+\frac{3}{1}\right)$,

which is true.

**Step II :** Let it is true for n = k,

i.e. $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2k+1}{k^2}\right) = (k+1)^2$ ... (i)

**Step III :** For n = k + 1,

$$\left\{\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2k+1}{k^2}\right)\right\}\left(1+\frac{2k+1+2}{(k+1)^2}\right)$$

$= (k+1)^2\left(1+\frac{2k+3}{(k+1)^2}\right)$ [using equation (i)]

$$= (k+1)^2\left[\frac{(k+1)^2+2k+3}{(k+1)^2}\right]$$

$= k^2 + 2k + 1 + 2k + 3$

$= (k+2)^2 = [(k+1)+1]^2$ [$\because (a+b)^2 = a^2 + 2ab + b^2$]

Therefore, P(k + 1) is true when P(k) is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

**39. (d)** Let the statement P(n) be defined as

P(n): $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is a natural number for all $n \in N$.

**Step I :** For n = 1,

P(1) : $\frac{1}{5} + \frac{1}{3} + \frac{7}{15} = 1 \in N$

Hence, it is true for n = 1.

**Step II :** Let it is true for n = k,

i.e. $\frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15} = \lambda \in N$ ... (i)

**Step III :** For n = k + 1,

$$\frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{7(k+1)}{15}$$

$$= \frac{1}{5}(k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1) + \frac{1}{3}(k^3 + 3k^2 + 3k + 1) + \frac{7}{15}k + \frac{7}{15}$$

$$= \left(\frac{k^5}{5} + \frac{k^3}{3} + \frac{7}{15}k\right) + (k^4 + 2k^3 + 3k^2 + 2k) + \frac{1}{5} + \frac{1}{3} + \frac{7}{15}$$

$= \lambda + k^4 + 2k^3 + 3k^2 + 2k + 1$

[using equation (i)]

which is a natural number, since $\lambda \; k \in N$.

Therefore, P(k + 1) is true, when P(k) is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

**40. (c)** Let P(n) : $S_n = \begin{cases} \dfrac{n(n+1)^2}{2}, & \text{when n is even} \\ \dfrac{n^2(n+1)}{2}, & \text{when n is odd} \end{cases}$

Also, note that any term $T_n$ of the series is given by

$$T_n = \begin{cases} n^2, & \text{if n is odd} \\ 2n^2, & \text{if n is even} \end{cases}$$

We observe that P(1) is true, since

$$P(1) : S_1 = 1^2 = 1 = \frac{1 \cdot 2}{2} = \frac{1^2 \cdot (1+1)}{2}$$

Assume that P(k) is true for some natural number k, i.e

**Case I :** When k is odd, then k + 1 is even. We have,

$$P(k+1) : S_{k+1} = 1^2 + 2 \times 2^2 + \ldots.. + k^2 + 2 \times (k+1)^2$$

$$= \frac{k^2(k+1)}{2} + 2 \times (k+1)^2$$

$$\left[\text{as k is odd, } 1^2 + 2 \times 2^2 + \ldots.. + k^2 = k^2\frac{(k+1)}{2}\right]$$

$$= \frac{(k+1)}{2}[k^2 + 4(k+1)]$$

$$= \frac{k+1}{2}[k^2 + 4k + 4]$$

$$= \frac{k+1}{2}(k+2)^2$$

$$= (k+1)\frac{[(k+1)+1]^2}{2}$$

So, P(k + 1) is true, whenever P(k) is true, in the case when k is odd.

**Case II :** When k is even, then k + 1 is odd.

$$\text{Now, } P(k+1) : S_{k+1} = 1^2 + 2 \times 2^2 + \ldots.. + 2 \cdot k^2 + (k+1)^2$$

$$= \frac{k(k+1)^2}{2} + (k+1)^2$$

$$\left[\text{as k is even, } 1^2 + 2 \times 2^2 + \ldots.. + 2k^2 = k\frac{(k+1)^2}{2}\right]$$

$$= \frac{(k+1)^2(k+2)}{2} = \frac{(k+1)^2((k+1)+1)}{2}$$

Therefore, P(k + 1) is true, whenever P(k) is true for the case when k is even.

Thus, P(k + 1) is true whenever P(k) is true for any natural number k. Hence, P(n) true for all natural numbers n.

**41. (c)** $2^4 \equiv 1 \pmod 5 \Rightarrow (2^4)^{75} \equiv (1)^{75} \pmod 5$

i.e. $2^{300} \equiv 1 \pmod 5 \Rightarrow 2^{300} \times 2 \equiv (1.2) \pmod 5$

$\Rightarrow 2^{301} \equiv 2 \pmod 5$

$\therefore$ Least positive remainder is 2.