Determinants
Class 12th

Q.1)	Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc + bc + ca + ab$
Sol.1)	We have $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$ taking a , b , c common from R_1 , R_2 & R_3 respectively $= abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$ $R_1 \to R_1 + R_2 + R_3$ $= abc \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$ $= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$ $c_2 \to c_2 - c_1 \text{and} c_3 \to c_3 - c_1$ $= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$ expanding along R_1 $= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \times 1$ $= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ = RHS $= abc + bc + ca + ab$ = RHS ans.
Q.2)	Show $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$
Sol.2)	We have $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$ taking a common from C_1 $= \begin{vmatrix} 1 & a+b & a+b+c \\ 2 & 3a+2b & 4a+3b+2c \\ 3 & 6a+3b & 10a+6b+3c \end{vmatrix}$ $R_2 \to R_2 - 2R_1 \text{and} R_3 \to R_3 - 3R_1$

	$= a\begin{vmatrix}1 & a+b & a+b+c\\ 0 & a & 2a+b\\ 0 & 3a & 7a+3b\end{vmatrix}$ $R_3 \rightarrow R_3 - 3\mathrm{R}_2$ $= a\begin{vmatrix}1 & a+b & a+b+c\\ 0 & a & 2a+b\\ 0 & 0 & a\end{vmatrix}$ expanding along R_1 $= a[a^2] = a^3$ ans.
Q.3)	If x , y , z are different and $\begin{vmatrix}x & x^2 & 1+x^3\\ y & y^2 & 1+y^3\\ z & z^2 & 1+z^3\end{vmatrix} = 0$ then show $xyz = -1$.
Sol.3)	We have $\begin{vmatrix}x & x^2 & 1+x^3\\ y & y^2 & 1+y^3\\ z & z^2 & 1+z^3\end{vmatrix} = 0$ applying sum property in C_3 $\Rightarrow \begin{vmatrix}x & x^2 & 1\\ y & y^2 & 1\\ z & z^2 & 1\end{vmatrix} + \begin{vmatrix}x & x^2 & x^3\\ y & y^2 & y^3\\ z & z^2 & z^3\end{vmatrix} = 0$ taking x , y , z common R_1 , R_2 , R_3 resp. $\Rightarrow \begin{vmatrix}x & x^2 & 1\\ y & y^2 & 1\\ z & z^2 & 1\end{vmatrix} + xyz\begin{vmatrix}1 & x^2 & x^3\\ 1 & y^2 & y^3\\ 1 & z^2 & z^3\end{vmatrix} = 0$ $c_2 \leftrightarrow c_3$ $\Rightarrow -\begin{vmatrix}x & 1 & x^2\\ y & 1 & y^2\\ z & 1 & z^2\end{vmatrix} + xyz\begin{vmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{vmatrix} = 0$ $c_1 \leftrightarrow c_2$ $\Rightarrow \begin{vmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{vmatrix} + xyz\begin{vmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{vmatrix} = 0$ $\Rightarrow \begin{vmatrix}x & x^2 & 1\\ y & y^2 & 1\\ z & z^2 & 1\end{vmatrix}(1+xyz) = 0$ $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$ $\Rightarrow \begin{vmatrix}1 & x & x^2\\ 0 & y-x & y^2-x^2\\ 0 & z-x & z^2-x^2\end{vmatrix}(1+xyz) = 0$ $\Rightarrow (y-x)(z-x)\begin{vmatrix}x & x^2 & 1\\ y & y^2 & 1\\ z & z^2 & 1\end{vmatrix}(1+xyz) = 0$ expanding along R_1 $\Rightarrow (y-z)(z-x)[z+x-y-x](1+xyz) = 0$ $\Rightarrow (y-x)(z-x)(z-y)(1+xyz) = 0$ but $\Rightarrow y-x \neq 0$ $z-x \neq 0$ since$x \neq y \neq z$ given $z-y \neq 0$

$\therefore$ only $1 + xyz = 0$

$\Rightarrow \; xyz = -1$ Proved

Q.4)

Show $\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+a & b-ab \end{vmatrix} = (ab+bc+ca)^3$

Sol.4)

$R_1 \to aR_1; R_2 \to bR_2 \text{and} R_3 \to cR_3$

$$= \frac{1}{abc}\begin{vmatrix} -abc & ab^2+abc & ac^2+abc \\ a^2b+abc & -abc & c^2b+abc \\ a^2c+abc & b^2c+abc & -abc \end{vmatrix}$$

taking a , b , c common from $c_1, c_2 \text{and} c_3$

$$= \frac{abc}{abc}\begin{vmatrix} -\mathrm{bc} & ab+ac & ac+ab \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} \mathrm{ab+bc+ac} & ab+bc+ac & ab+bc+ac \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

taking $(ab + bc + ca)$ common from R_1

$$= (ab+bc+ca)\begin{vmatrix} 1 & 1 & 1 \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

$c_2 \to c_2 - c_1 \text{and} c_3 \to c_3 - c_1$

$$= (ab+bc+ca)\begin{vmatrix} 1 & 0 & 0 \\ ab+bc & -ab-bc-ac & 0 \\ ac+bc & 0 & -ab-bc-ca \end{vmatrix}$$

taking (ab + bc + ca) common from $c_2 \text{and} c_3$ both

$$= (ab+bc+ca)(ab+bc+ca)^2\begin{vmatrix} 100 \\ ab+bc-10 \\ ac+bc0-1 \end{vmatrix}$$

expanding along R_1

$= (ab+bc+ca)^3(1) = (ab+bc+ca)^3$ = RHS

Q.5)

Show $\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+b^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$

Sol.5)

We have $\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+b^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$

$R_1 \to aR_1; R_2 \to bR_2 \text{and} R_3 \to cR_3$

$$= \frac{1}{abc}\begin{vmatrix} a(b^2+c^2) & a^2b & a^2c \\ ab^2 & b(c^2+b^2) & b^2c \\ c^2a & c^2b & c(a^2+b^2) \end{vmatrix}$$

taking a , b , c common from c_1, c_2, c_3 resp.

$$= \frac{abc}{abc}\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

	$= \begin{vmatrix} 2(b^2+c^2) & 2(c^2+a^2) & 2(a^2+b^2) \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$ 2 common from R_1 $= \begin{vmatrix} b^2+c^2 & c^2+a^2 & a^2+b^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$ $R_2 \to R_2 - R_1 \text{and} R_3 \to R_3 - R_1$ $= 2\begin{vmatrix} b^2+c^2 & c^2+a^2 & a^2+b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix}$ $R_1 \to R_1 + R_2 + R_3$ $= 2\begin{vmatrix} 0 & c^2 & b^2 \\ -c^2 & 0 & -b^2 \\ -b^2 & -a^2 & 0 \end{vmatrix}$ $= 2\begin{vmatrix} 0 & c^2 & b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix}$ expanding $= 2[-c^2(-a^2b^2) + b^2(a^2c^2)]$ $= 2(a^2b^2c^2 + a^2b^2c^2) = 4\text{a}^2b^2c^2$ ans.
Q.6)	Show that $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$
Sol.6)	We have $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$ $c_1 \to c_1 + c_2 + c_3$ $= \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & r+p & p+q \\ 2(x+y+z) & z+x & x+y \end{vmatrix}$ $= 2\begin{vmatrix} a+b+c & c+a & a+b \\ p+q+r & r+p & p+q \\ x+y+z & z+x & x+y \end{vmatrix}$ $c_2 \to c_2 - c_1 \text{and} c_3 \to c_3 - c_1$ $= 2\begin{vmatrix} a+b+c & -b & -c \\ p+q+r & -q & -r \\ x+y+z & -y & -z \end{vmatrix}$ Now, $c_1 \to c_1 + c_2 + c_3$ $= 2\begin{vmatrix} a & -b & -c \\ p & -q & -r \\ x & -y & -z \end{vmatrix}$ taking (–) sign from c_1 & c_3 both $= 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$ = RHS

Q.7)	Show $\begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} = (1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$
Sol.7)	We have $\begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$ $R_1 \rightarrow R - 1 - xR_2$ = $\begin{vmatrix} a-ax^2 & c-cx^2 & p-px^2 \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$ = $\begin{vmatrix} a(1-x^2) & c(1-x^2) & p(1-x^2) \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$ taking $(1-x^2)$common from R_1 = $(1-x^2)\begin{vmatrix} a & c & p \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$ = $R_2 \rightarrow R_2 - xR - 1$ = $(1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$ = RHS
Q.8)	Show that the value of the determinants $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative.
Sol.8)	let $\Delta \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ $c_1 \rightarrow c_1 + c_2 + c_3$ = $\begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a^2+b+c & a & b \end{vmatrix}$ (a + b + c) common from C_1 = $(a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$ $R_2 \rightarrow R_2 - R_1 \text{and} R_3 \rightarrow R_3 - R_1$ = $(a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$ expanding along R_1 = $(a+b+c)(-a^2-b^2-c^2+ab+bc+ca)$ = $-(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$ multiply & divide by 2 = $-\frac{1}{2}(a+b+c)(2a^2+2b^2+2c^2-2ab-2bc-2ca)$ = $-\frac{1}{2}(a+b+c)((a-b)^2+(b-c)^2+(c-a)^2)$ clearly the value of determinant is –ve ans.

Q.9) If a , b , c are real number such that $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$then show that either a + b + c = 0 (or) a = b = c.

Sol.9) We have $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$

$c_1 \rightarrow c_1 + c_2 + c_3$

$$\Rightarrow \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix} = 0$$

$$\Rightarrow 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 1 & a+b & b+c \\ 1 & b+c & c+b \end{vmatrix} = 0$$

$R_2 \rightarrow$ and$R_3 \rightarrow R_3 - R_1$

$$\Rightarrow 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 0 & b-c & c-a \\ 0 & b-a & c-b \end{vmatrix} = 0$$

expanding along R_1

$$\Rightarrow 2(a+b+c)(-a^2 - b^2 - c^2 + ab + bc + ca) = 0$$

$$\Rightarrow -2(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$$

multiply and divide by 2

$$\Rightarrow -\frac{2}{2}(a+b+c)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca) = 0$$

$$\Rightarrow -(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$$

$$\Rightarrow (a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$$

$$\Rightarrow \text{either } a+b+c = 0$$

(or) $(a-b)^2 + (b-c)^2 + (c-a)^2 = 0$

this is possible only when

$a - b = 0 \Rightarrow a = b$

$b - c = 0 \Rightarrow b = c$

$c - a = \quad \Rightarrow c = a$

$\Rightarrow a = b = c$

$\therefore$ either $a + b + c = 0$ (or) $a = b = c$ ans.

Q.10) Show that $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$

Sol.10) let $\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$

$R_1 \rightarrow cR_1; R_2 \rightarrow bcR_2$ and $R_3 \rightarrow aR_3$

$$= \frac{1}{abc}\begin{vmatrix} 0 & ac & -bc \\ -a & b0 & -bc \\ ab & ac & 0 \end{vmatrix}$$

taking ab , ac , bc common from C_1 , C_2 & C_3

$$= \frac{(ab)(ac)(bc)}{abc}\begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix}$$

	$= abc\begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix}$ expanding $= abc(0) = 0$= RHS ans.