Chapter—8

FACTORS AND FACTORIZATION OF ALGEBRAIC EXPRESSIONS

Introduction

The teacher was distributing toffees to the students of class VIII on the occassion of Republic Day. All the students are careful about the fact that none of their friends got any extra toffee. They wanted the toffees to be equally distributed among themselves. Lata started thinking that her class had 10 students and if every student got 6 toffees no toffee would be left. Had there been 15 students in her class, each one would have got 4 toffees. Lata took out her notebook and started wrting down the calculation. To how many students can 60 toffees be equally distributed so that no toffee is left and in how many ways? She found that 60 toffees can be distributed in such a way that no toffee would be left.

Rama was watching Lata's activities. She told Lata that all the numbers that she had written in her notebook are completely divisible to 60. Therefore, all these numbers are factors of 60 and are also the factors possible for 60.

How many multiple factor ?

Umesh aksed lata whether factors for the algebraic expression 5 ab can also be determined in a similar manner ? Lata wrote in her notebook and showed that 5, a and b are completely divisible to 5 ab. Rama was listening to the conversation and she remembered that they have already studied this rule that every number is completely divisible by 1 and by it self, so 1and 5ab willl also be divisible factor along with 5a and 5b that would completety divide 5ab Thus :

The multiple factrors of 5ab = 1, 5, a, b, 5a, 5b and 5ab. Rekha said, "You have left one multiple factor out Everyone started thinking. Then Ramesh said, "Yes, ab willalso be one, multiple factor. Thus, the factors for 5ab are 1, 5, a, b, ab, 5a, 5b and 5ab.

Rama, suggested, "Let us play a game of finding out factors, Rohit wrote the factors of $12x^2$ in his notebook and found that the factors of $12x^2$ = 1, 2, 3, 4, 6, $2x^2$, $3x^2$, $4x^2$, $6x^2$ and $12x^2$. But Rama told Rohit that all the factors of $12x^2$ have not been written yet. Rama first wrote the factors for the factors and then for all the algebraic expression to get the complete factors of $12x^2$ like this :

Factors of 12 = 1, 2, 3, 4, 6 and 12

Factors of x^2 = 1, x, x^2

Therefore, total factors for $12x^2$ = 1, 2, 3, 4, 6, 12, x, $2x$, $3x$, $4x$, $6x$, $12x$, x^2, $2x^2$, $3x^2$, $4x^2$, $6x^2$, and $12x^2$.

How to recognize all the factors ?

Rohit said, "alright, but there must be some way to write many factors for a monomial algebraic expression that verifies that all the factors have been actually written". Lata said, "Let us go and ask our Maths teacher about it and also get the factors of $12x^2$ that Rama has written down cheked by her."

The teacher said, "you have rightly written all the factors of $12x^2$. Very good. Now learn to write all the factors of such algebraic expressions, we write all the factors for the constant in horizontal rows and the factors of the variable in vertical columns. Thus we shall get a multiplicative table when we fill up this table, we get all the factors of a monomial." The teacher made a table like thus :

Table 8.1

x	1	2	3	4	6	12
1	1	2	3	4	6	12
x	x	$2x$	$3x$	$4x$	$6x$	$12x$
x^2	x^2	$2x^2$	$3x^2$	$4x^2$	$6x^2$	$12x^2$

The teacher asked the students whether all the expressions mentioned in the table are completely divisible to $12x^2$? Verify yourself whether the table contains all the factors that Rama wrote for $12x^2$.

Some more example

The teacher asked Lata to findout the factors for $10ab^2$ in the above mentioned manner. Lata made the table for $10ab^2$ on the black board and asked all the students to verify it.

The factors of 10 = 1, 2, 5, 10.

Factors of ab^2 = 1, a, b, b^2, ab, ab^2.

Therefore $10ab^2$ will have the following possible factors :

Table 8.2

x	1	2	5	10
1	1	2	5	10
a	a	2a	5a	10a
b	b	2b	5b	10b
b^2	b^2	$2b^2$	$5b^2$	$10b^2$
ab	ab	2ab	5ab	10ab
ab^2	ab^2	$2ab^2$	$5ab^2$	$10ab^2$

All the expression in the table are completely divisible to $10ab^2$ and the remainder is zero. Now the students got one method for finding the factors of monomials.

Activity 1

Write the factors of the algebraic expressions given below in your notebook.

$8x$, $4a^2$, $6ab$, xy^2, $3x^2y$, $6y^2$

Indetify the similar common factors

Rohit wrote the factors of 6ab the teacher wrote them on the blackboard :

6ab = 1, 2, 3, 6, a, 2a, 3a, 6a, b, 2b, 3b, 6b, ab, 2ab, 3ab, 6ab.

The teacter wrote the factors of $4a^2$ that Rajesh found out on the blackboard below the factors of 6ab.. Hence the factors of

$4a^2 = 1, 2, 4, a, 2a, 4a, a^2, 2a^2, 4a^2$.

Now, the teacher asked the class whether there was any similarity between the factors of both the algabraic expressions. Rama said that the total number of factor for 6ab and $4a^2$ are different, but some factors are similar in both the expressions. There are common factors 1, 2, *a*, 2*a*.

The teacher said, "right, and among these 2a is the largest or greatest common factor. Since you know that the greatest common factor is the Highest common factor, 2a will be known as the highest common factor for 6ab and $4a^2$.

The Highest common factor for two or more than two monomials is that greatest algebraic expression that is completely divisible to each of the given algebraic expressions.

Now the teacher wrote all the factors for $3x^2y$ and $6y^2$ on the blackboard and asked Praveen to identify the common factors.

The factors of $3x^2y$ = 1, 3, x, $3x$, x^2, $3x^2$, y, 3y, xy, $3xy$, x^2y, $3x^2y$.

The factors of $6y^2$ = 1, 2, 3, 6, y, 2y, 3y, 6y, y^2, $2y^2$, $3y^2$, $6y^2$.

Praveen wrote the common multiples as : 1, 3, y, 3y.

And everyone could see that the highest common factors for expression one was 3y.

Another Method

The students found this method a bit lenghtly. So Rama asked the teacher if there is any other short method to find the Highest common factor for two or more than two monomials ? The teacher said, well, look at this : Suppose we need to find the Highest common factor for $6x^2y$ and $8xy^2$. First we find the highest common factors (H.C.F.) for the coffcients 6 and 8.

The H.C.F. for Coefficients 6 & 8 = 2

The H.C.F. for x and x^2 = x (Item with the lowest exponent for x)

The H.C.F. for y and y^2 = y (Item with the lowest exponent for y)

Therefore $6x^2$ and $8xy^2$ = $2xy$ (the product of all the H.C.F. found above)

Example 1

Find the H.C.F. of $12\, s^3 t^2 u^3$ and $6 s^4 t\, u^2$.

Solution

H.C.F. for 12 and 16 = 4

H.C. F. for s^3 and s^4 = s^3

H.C. F. for t^2 and t = t

H.C. F. for u^3 and u^2 = u^2

Therefore the H. C. F. for $12s^3 t^2 u^3$ and $16s^4 t u^2$ = $4s^3 t u^2$.

Example 2

Find the H.C.F. of $20a^2b$, ab^3c.

Solution

Here the coeffecient for the given algebraic expressions are 20 and 1 respectively.

Thus, H. C. F. for 20 and 1 = 1

H. C. F. for a^2 and a = a

H. C. F. for b and b^3 = b

Here c occures only in the second expression and not in the first.

Therefore, H. C. F. for $20a^2b$ and ab^3c = l. a. b = ab.

Exercise 8.1

1. Find out all the factors for the given algebraic expressions :

   (i) $5t^2$ (ii) $7xy$ (iii) $14l^2m$ (iv) $39\,lmn$.

2. Write down all the factors for the given algebraic expriession and find out their H.C.F.

   (i) $5s$, $2s^2$ (ii) $9m^2$, $3t$ (iii) $6a^2$, $8ab$ (iv) $7m^2$, $6m$

3. Find out the highest common factors for the following :

   (i) $6m^2l, 12ml$ (ii) $24a^2bc, 20bc^2$ (iii) $xy^3z, 10x^2y$ (iv) $14x^3y, 21xz^5$

   (v) $22p^2q^2r, 33pq^2r^2$ (vi) $3xy, 23x^2z$ (vii) 6pqr, 23 xyz

Factors of binomials

You have lerant how to find the factors monmials Now on the basis of your experienes can you find the factors of binomials ? For example, If thrice the number of boys in a class is added to thrice the number of girls, what will be the sum ? will this sum be equal to 3 times the sum of total number of boys and girls ? Verify the answer by taking any number of boys and girls.

Suppose the number of boys is 15 and the number of girls is 18. Thrice the number of boys is 45 and thrice the number of girls is 54. Thus the total becomes 99. While the original number of boys and girls together is 33. And you can see that 99 is 3 times 33.

This means , if the number of boys be taken as x and the number of girls be taken as y, then thrice the number of boys would be 3x and thrice the number of girls would be 3y.

The sum of both would be $3x+3y$. The total number of boys and girls would be $x+y$ and thrice this sum would be $3(x+y)$.

Also, $3(x+y)=3x+3y$

Here in $3x$ & $3y$, 3 is common.

Let us check this for the expression $9+3y$.

Therefore $9+3y=3\times 3+3\times y$

$$=3(3+y)$$

Thus, the factors for the expression $3x+3y$ above are 3, $(x+y)$ and $3x+3y$ but 3 and $(x+y)$ are such factors whose product is equal to $3x+3y$. Similarly, the factors for 9+3y are 3, (3+y) and 9+3y , but 3 and 3+y are two such factors whose product is (9 + 3y) again.

Do this also

Can you write down the two factors of 12 + 18y as products ?

Here the factors for 12 & 18y, 2, 3 & 6 are common factors.

1. Taking 2 common $12+18y=2\times 6+2\times 9y=2(6+9y)$. But here 3 can again be found as a common factor from $6+9y$, Therefore

$$12+18y=2\{3(2+3y)\}$$

OR

$$12+18y=6(2+3y)$$

2. If in $12+18y$, 3 is the common factor, then

$12+18y=3(4+6y)$ (Here again 2 is common in $4+6y$)

$$\therefore \quad 12+18y=3\{2(2+3y)\}$$

$$=6(2+3y)$$

Here out of 2, 3, 6 as common factors for $12+18y$, 6 is the highest common factor.

Activity 2

Find out the Highest common factor for the binomials given below and complete the table as shown in the example.

S.No.	Binomial	Writing both the items separately	The gretest common factor between two terms	On writing the binomial as the product of the common factors
1.	$36x+27y$	$36x$ & $27y$	9	$9(4x+3y)$
2.	$33y^2-11xy$			
3.	$15xz-90x^2$			
4.	$8ab+9ac$			

Make such problems yourself and ask your friends to solve them.

Factorization

To write binomials and polynomials as the product of its common factors, we write the greatest common factor (H.C.F.) for each of the expression in the polynomial outside the brackets. The written expression of any algebraic expression as the product of its factors is known as factorization.

For the highest common factor for $2ab+2ac=2a$

$$2ab+2ac=2a\times b+2a\times c=2a(b+c)$$

Thus, On factorizing $2ab+2ac$ we get $2a$ and $(b+c)$ whose product would be $2ab+2ac$.

Example 3

Factorize $4x^2y^2-18xy$

Solution

Here the highest common factor for $4x^2y^2-18xy=2xy\times 2xy-2xy\times 9$.

$$=2xy(2xy-9)$$

Example 4

Factorize the experssion $6ab^2+9a^2b^3+12a^2b^2$

Solution

Here the highest common factor

$$\therefore \quad 6ab^2+9a^2b^3+12a^2b^2=3ab^2\times(2+3ab+3ab^2\times 4a)$$

$$=3ab^2(2+3ab+4a)$$

Factorization of polynomials

Rama had learnt Factorization of binomials quite well. She was thinking how the Factorization of an algebraic expression with many would be determined ?

Do you have an answer to Rama's question ? The teacher explained : To factorize such algebraic expression, we make groups of the compoments. After making suitable groups of the algebraic expressions, we find the common factor. Then, we write them as the product of the factors.

For example : Factorize $ax+by+ay+bx$

Here it would be convenient to put the components a and the components with b in separate group. On writing them together we get :

$$=ax+ay+bx+by$$

$=a(x+y)+b(x+y)$ (Here $x+y$ is common in both the items)

$$=(x+y)(a+b)$$

The expression can be Factorized by writing the multiple of items containing x and y separately. Do this yourself. Did you get the same answer ?

Example 5

Facorize $2x^2-6y+4x^2y-12y^2$

Solution

In $2x^2-6y+4x^2y-12y^2$, it would be easy to take $2x^2$ & $4x^2y$ together.

Here, $2x^2+4x^2y$ have $2x^2$ common

$$2x^2+4x^2y=2x^2(1+2y)$$

similarly in $-6y-12y^2$ taking $-6y$ as common we get $-6y-12y=-6y(1+2y)$ and $(1+2y)$ is common is both the expressions.

Therfore the facrors are $(1+2y)(2x^2-6y)$.

But can we have factors of $2x^2-6y$ also.

Here 2 is common. Therefore its factors are 2 and (x^2-3y)

Therefore

$$2x^2-6y+4x^2y-12y^2$$

$$=2(x^2-3y)(1+2y)$$

Example 6

Factorize $2xy + y + 4x + 2$

Solution

$$2xy + y + 4x + 2$$

$$= y(2x+1)+2(2x+1)$$

$$=(2x+1)(y+2) \text{ [common factor } (2x+1) \text{]}$$

Now solve this question by taking the first and third component and the second & fourth component together.

Exercise 8.2

Q.1. Fill in the blanks :

i. $x^2 + 5x^3 =$ ____________ $(1+5x)$

ii. $10a^2 - 12b^2 = 2($_____$- 6b^2)$

iii. $27ab^2 + 18abc = 9ab(3b +$ _____$)$

iv. $16xz - 9z^2 - z = ($_____$-$_____$)$

v. $12ab^2c + 8abc^2 - 10a^2c = 2ac($_____$+$_____$-$_____$)$

Q.2. Factorize

i. $4ax + 6a^2y$

ii. $a^5y + ab^3$

iii. $pq^2r - 2q^2t$

iv. $-5lm^2 - 10l^2mn$

v. $5m^2 - 5n^2$

Q.3. Factorizing the group method

i. $2x^2y + 6xy^2 + 4x + 12y$

ii. $5m^2n - 10mn^2 + 12m - 24n$

iii. $6x^3 + 8x^2 + 9xy + 12y$

iv. $15x^4 + 10x^2y^2 + 12x^2y + 8y^3$

v. $x(x+3) + 8(x+3)$

vi. $3x(x-4)-5(x-4)$

vii. $2m(l-m)+3(l-m)$

Q.4. Solve the following :

i. $x(1-3y^2) = \_\_\_\_ - \_\_\_\_$

ii. $-17x^2(3x-9) = \_\_\_\_ + \_\_\_\_$

iii. $2a^2(3a-4a^2) = \_\_\_\_ - \_\_\_\_$

iv. $9m(m-n) = \_\_\_\_ - \_\_\_\_$

v. $9t^2(t-7t^3) = \_\_\_\_ - \_\_\_\_$

We have learnt

1. If an algebraic expression is written as a product of two or more algebraic expressions, then the expressions obtained are known as the factors of the given expression and the way of wrting any algebraic expression in this manner is called factorization.
2. The factorization of any two binomials is done by taking out the common comment of its highest common factors.
3. The H.C.F. of algebraic expressions is the greatest divisor of these expressions.
4. The factorization of algebraic expressions with more than 3 components is done by group method.