DETERMINANT

1. MINORS :

The minor of a given element of determinant is the determinant of the elements which remain after deleting the row & the column in which the given element stands.

For example, the minor of a_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ & the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

Hence a determinant of order three will have " 9 minors".

2. COFACTORS :

If M_{ij} represents the minor of the element belonging to i^{th} row and j^{th} column then the cofactor of that element : $C_{ij} = (-1)^{i+j}. M_{ij}$;

Important Note :

Consider $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Let A_1 be cofactor of a_1, B_2 be cofactor of b_2 and so on, then,

(i) $a_1A_1 + b_1B_1 + c_1C_1 = a_1A_1 + a_2A_2 + a_3A_3 = \ldots\ldots\ldots\ldots\ldots = \Delta$

(ii) $a_2A_1 + b_2B_1 + c_2C_1 = b_1A_1 + b_2A_2 + b_3A_3 = \ldots\ldots\ldots\ldots = 0$

3. PROPERTIES OF DETERMINANTS:

(a) The value of a determinants remains unaltered, if the rows & corresponding columns are interchanged.

(b) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. e.g.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D' = -D$.

(c) If a determinant has any two rows (or columns) identical or in same proportion, then its value is zero.

(d) If all the elements of any row (or column) be multiplied by the same number, then the determinant is multiplied by that number.

(e) $$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(f) The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column) e.g.

Let $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix}$$. Then $D' = D$.

Note : While applying this property **ATLEAST ONE ROW (OR COLUMN)** must remain unchanged.

(g) If the elements of a determinant Δ are rational function of x and two rows (or columns) become identical when $x = a$, then $x - a$ is a factor of Δ.

Again, if r rows become identical when a is substituted for x, then $(x-a)^{r-1}$ is a factor of Δ.

(h) If $D(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix}$, where f_r, g_r, h_r; r = 1, 2, 3 are three differentiable functions.

then $$\frac{d}{dx}D(x) = \begin{vmatrix} f'_1 & f'_2 & f'_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g'_1 & g'_2 & g'_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h'_1 & h'_2 & h'_3 \end{vmatrix}$$

4. MULTIPLICATION OF TWO DETERMINANTS :

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1 m_1 + b_1 m_2 \\ a_2 l_1 + b_2 l_2 & a_2 m_1 + b_2 m_2 \end{vmatrix}$$

Similarly two determinants of order three are multiplied.

(a) Here we have multiplied row by column. We can also multiply row by row, column by row and column by column.

(b) If D' is the determinant formed by replacing the elements of determinant D of order n by their corresponding cofactors then $D' = D^{n-1}$

5. SPECIAL DETERMINANTS :

(a) Symmetric Determinant :

Elements of a determinant are such that $a_{ij} = a_{ji}$.

e.g. $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

(b) Skew Symmetric Determinant :

If $a_{ij} = -a_{ji}$ then the determinant is said to be a skew symmetric determinant. Here all the principal diagonal elements are zero. The value of a skew symmetric determinant of odd order is zero and of even order is perfect square.

e.g. $$\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$$

(c) Other Important Determinants :

(i) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

(ii) $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$$

6. SYSTEM OF EQUATION :

(a) System of equation involving two variable :

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

Consistent (System of equation has solution)

- unique solution
  $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$
  or $a_1b_2 - a_2b_1 \neq 0$
  (Equations represents intersecting lines)

- Infinite solution
  $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$
  (Equations represents coincident lines)

Inconsistent (System of equation has no solution)

$\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$
(Equations represents parallel disjoint lines)

If $\Delta_1 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}$, $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, then $x = \dfrac{\Delta_1}{\Delta}$, $y = \dfrac{\Delta_2}{\Delta}$

(b) System of equations involving three variables :

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

To solve this system we first define following determinants

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \qquad \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Now following algorithm is used to solve the system.

Check value of Δ

- $\Delta \neq 0$ → Consistent system and has unique solution
  $$x = \frac{\Delta_1}{\Delta},\ y = \frac{\Delta_2}{\Delta},\ z = \frac{\Delta_3}{\Delta}$$
- $\Delta = 0$ → Check the values of Δ_1, Δ_2 and Δ_3
  - Atleast one of Δ_1, Δ_2 and Δ_3 is not zero → Inconsistent system
  - $\Delta_1 = \Delta_2 = \Delta_3 = 0$ → Put z = t and solve any two equations to get the values of x & y in terms of t
    - If these values of x, y and z in terms of t satisfy third equation → Consistent system (System has infinite solutions)
    - The values of x, y, z doesn't satisfy third equation → Inconsistent system

Note :

(i) **Trivial solution :** In the solution set of system of equation if all the variables assumes zero, then such a solution set is called Trivial solution otherwise the solution is called non-trivial solution.

(ii) If $d_1 = d_2 = d_3 = 0$ then system of linear equation is known as system of Homogeneous linear equation which always posses atleast one solution (0, 0, 0).

(iii) If system of homogeneous linear equation posses non-zero/non-trivial solution then $\Delta = 0$.

In such case given system has infinite solutions.