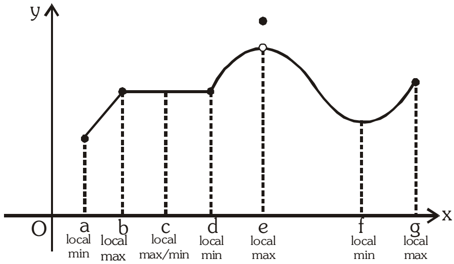
MAXIMA-MINIMA

1. INTRODUCTION :

MAXIMA & MINIMA :

(a) Local Maxima /Relative maxima :

A function $f(x)$ is said to have a local maxima at x = a



if $f(a) \geq f(x) \ \forall \ x \in (a - h, a + h) \cap D_{f(x)}$

Where h is some positive real number.

(b) Local Minima/Relative minima :

A function $f(x)$ is said to have a local minima at x = a if $f(a) \leq f(x) \ \forall \ x \in (a - h, a + h) \cap D_{f(x)}$

Where h is some positive real number.

(c) Absolute maxima (Global maxima) :

A function f has an absolute maxima (or global maxima) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number f(c) is called the maximum value of f on D.

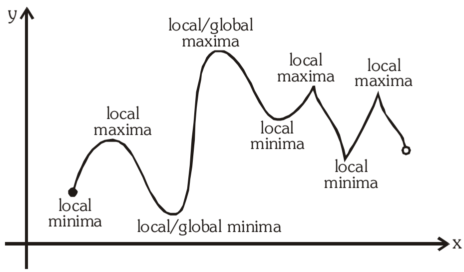
(d) Absolute minima (Global minima) :

A function f has an absolute minima at c if $f(c) \leq f(x)$ for all x in D and the number f(c) is called the minimum value of f on D.

Note :

(i) The term 'extrema' is used for both maxima or minima.

(ii) A local maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.



(iii) A function can have several extreme values such that local minimum value may be greater than a local maximum value.

(iv) It is not necessary that $f(x)$ always has local maxima/minima at end points of the given interval when they are included.

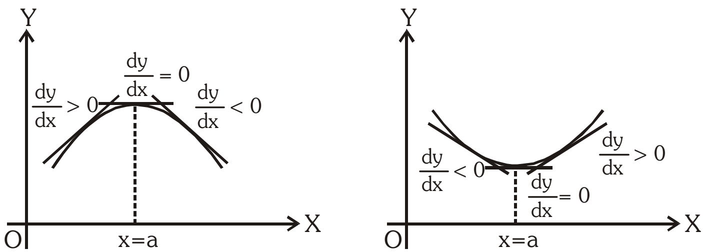
2. DERIVATIVE TEST FOR ASCERTAINING MAXIMA/MINIMA :

(a) First derivative test :

If $f'(x) = 0$ at a point (say $x = a$) and

(i) If $f'(x)$ changes sign from positive to negative in the neighbourhood of $x = a$ then $x = a$ is said to be a point **local maxima.**

(ii) If $f'(x)$ changes sign from negative to positive in the neighbourhood of $x = a$ then $x = a$ is said to be a point **local minima.**



Note : If $f'(x)$ does not change sign i.e. has the same sign in a certain complete neighbourhood of a, then $f(x)$ is either increasing or decreasing throughout this neighbourhood implying that $x=a$ is not a point of extremum of f.

(b) Second derivative test :

If $f(x)$ is continuous and differentiable at $x = a$ where $f'(a) = 0$ (stationary points) and $f''(a)$ also exists then for ascertaining maxima/minima at $x = a$, 2^{nd} derivative test can be used -

(i) If $f''(a) > 0$ $\Rightarrow x = a$ is a point of local minima

(ii) If $f''(a) < 0$ $\Rightarrow x = a$ is a point of local maxima

(iii) If $f''(a) = 0 \Rightarrow$ second derivative test fails. To identify maxima/minima at this point either first derivative test or higher derivative test can be used.

(c) n^{th} derivative test :

Let $f(x)$ be a function such that $f'(a) = f''(a) = f'''(a) = \ldots = f^{n-1}(a) = 0$ & $f^n(a) \neq 0$, then

(i) If n is even &

$$\begin{cases} f^n(a) > 0 \Rightarrow \text{Minima} \\ f^n(a) < 0 \Rightarrow \text{Maxima} \end{cases}$$

(ii) If n is odd then neither maxima nor minima at $x = a$.

3. USEFUL FORMULAE OF MENSURATION TO REMEMBER:

(a) Volume of a cuboid = ℓbh.

(b) Surface area of a cuboid = $2(\ell b + bh + h\ell)$.

(c) Volume of a prism = area of the base x height.

(d) Lateral surface area of prism = perimeter of the base x height.

(e) Total surface area of a prism = lateral surface area + 2 area of the base (Note that lateral surfaces of a prism are all rectangles).

(f) Volume of a pyramid = $\frac{1}{3}$ area of the base x height.

(g) Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) x slant height.
(Note that slant surfaces of a pyramid are triangles).

(h) Volume of a cone = $\frac{1}{3}\pi r^2 h$.

(i) Curved surface area of a cylinder = $2\pi rh$.

(j) Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.

(k) Volume of a sphere = $\frac{4}{3}\pi r^3$.

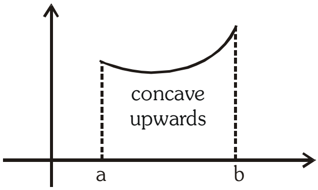
(l) Surface area of a sphere = $4\pi r^2$.

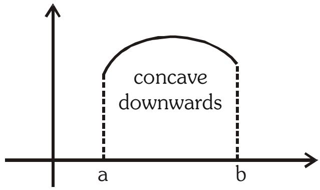
(m) Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.

(n) Perimeter of circular sector = $2r + r\theta$.

4. SIGNIFICANCE OF THE SIGN OF 2ND ORDER DERIVATIVE :

The sign of the 2^{nd} order derivative determines the concavity of the curve.
i.e. If $f''(x) \geq 0 \ \forall \ x \in (a, b)$ then graph of f(x) is concave upward in (a, b).
Similarly if $f''(x) \leq 0 \ \forall \ x \in (a, b)$ then graph of f(x) is concave downward in (a, b).





5. SOME SPECIAL POINTS ON A CURVE :

(a) Stationary points: The stationary points are the points of domain where $f'(x) = 0$.

(b) Critical points : There are three kinds of critical points as follows :

(i) The point at which $f'(x) = 0$

(ii) The point at which $f'(x)$ does not exists

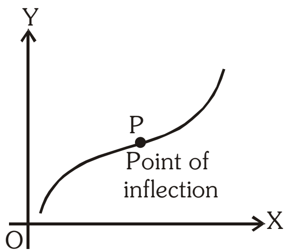
(iii) The end points of interval (if included)

These points belongs to domain of the function.

Note : Local maxima and local minima occurs at critical points only but not all critical points will correspond to local maxima/ local minima.

(c) Point of inflection :

A point where the graph of a function has a tangent line and where the strict concavity changes is called a point of inflection. For finding point of inflection of any function, compute the points



(x-coordinate) where $\frac{d^2y}{dx^2} = 0$ or $\frac{d^2y}{dx^2}$ does not exist. Let the solution is x = a, if $\frac{d^2y}{dx^2} = 0$ at x = a and sign of $\frac{d^2y}{dx^2}$ changes about this point then it is called point of inflection.

if $\frac{d^2y}{dx^2}$ does not exist at x = a and sign of $\frac{d^2y}{dx^2}$ changes about this point and tangent exist at this point then it is called point of inflection.

6. SOME STANDARD RESULTS :

(a) Rectangle of largest area inscribed in a circle is a square.

(b) The function $y = \sin^m x \cos^n x$ attains the max value at $x = \tan^{-1}\sqrt{\frac{m}{n}}$

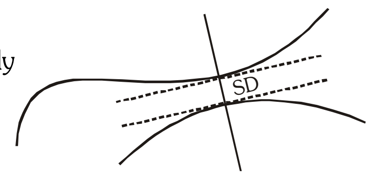
(c) If $0 < a < b$ then $|x-a| + |x-b| \geq b-a$ and equality hold when $x \in [a, b]$.

If $0 < a < b < c$ then $|x-a| + |x-b| + |x-c| \geq c-a$ and equality hold when x = b

If $0 < a < b < c < d$ then $|x-a| + |x-b| + |x-c| + |x-d| \geq d-a$ and equality hold when $x \in [b,c]$.

7. LEAST/GREATEST DISTANCE BETWEEN TWO CURVES :

Least/Greatest distance between two non-intersecting curves usually lies along the common normal. (Wherever defined)



Note : Given a fixed point A(a, b) and a moving point P(x, f(x)) on the curve y = f(x). Then AP will be maximum or minimum if it is normal to the curve at P.