19 Indefinite Integration

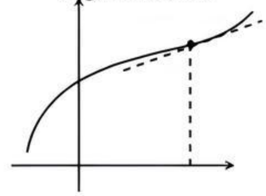
NITTY-GRITTY

You want to rotate a function around a vertical line, but do all your integrating in terms of x and $f(x)$, then the shell method is your new friend. It is similarly fantastic when you want to rotate around a horizontal line but integrate in terms of y. Applications of the Indefinite Integral shows how to find displacement (from velocity) and velocity (from acceleration) using the indefinite integral. There are also some electronics applications. In primary school, we learnt how to find areas of shapes with straight sides (e.g. area of a triangle or rectangle). But how do you find areas when the sides are curved? e.g.

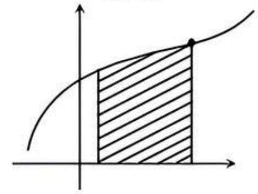
- Area under a Curve and Area in between the two curves. Answer is by Integration.
- Volume of Solid of Revolution explains how to use integration to find the volume of an object with curved sides, e.g. wine barrels.
- Centroid of an Area means the centre of mass. We see how to use integration to find the centroid of an area with curved sides.
- Moments of Inertia explain how to find the resistance of a rotating body. We use integration when the shape has curved sides.
- Work by a Variable Force shows how to find the work done on an object when the force is not constant.
- Electric Charges have a force between them that varies depending on the amount of charge and the distance between the charges. We use integration to calculate the work done when charges are separated.
- Average Value of a curve can be calculated using integration.

Integral Calculus is one of the two fundamental branches of Calculus, the other being Differential Calculus, and enables us to calculate the areas under arbitrary curves, as Differential Calculus helps us to find the slopes of arbitrary curves. This association should always be perfectly clear in your mind:

Differentiation relates to slopes of tangents to curves



Integration relates to areas under curves



Note

The principles of integration were formulated independently by Isaac Newton and Gottfried Leibniz in the late 17th century, who thought of the integral as an infinite sum of rectangles of infinitesimal width. A rigorous mathematical definition of the integral was given by Bernhard Riemann. It is based on a limiting procedure which approximates the area of a curvilinear region by breaking the region into thin vertical slabs. Beginning in the nineteenth century, more sophisticated notions of integrals began to appear, where the type of the function as well as the domain over which the integration is performed has been generalised. A line integral is defined for functions of two or three variables, and the interval of integration $[a, b]$ is replaced by a certain curve connecting two points on the plane or in the space. In a surface integral, the curve is replaced by a piece of a surface in the three-dimensional space.

Primitive functions, antiderivatives, indefinite integration

- If $\frac{df(x)}{dx} = F(x)$ then the derivative of $f(x)$ is $F(x)$, i.e. $f'(x) = F(x)$. Equivalently, $f(x)$ is the primitive function or antiderivative of $F(x)$.
- The indefinite integration of $f'(x)$ is $f(x)+c$, symbolically, $\int f'(x)dt = f(x)+c$, when c is an arbitrary constant, called constant of integration. $\int \phi(x)dx$ is an integral and $\phi(x)$ is the integrand.

Standard integrals of elementary functions: Standard integrals are as follows (without writing the constant of integration):

- $\int x^n dx = \frac{x^{n+1}}{n+1} (n \neq -1)$
- $\int \frac{1}{x} dx = \log_e x$ (where x is positive)
- $\int e^x dx = e^x$
- $\int a^x dx = \frac{a^x}{\log_e a}$
- $\int \sin x dx = -\cos x$
- $\int \cos x dx = \sin x$
- $\int \tan x dx = \log \sec x$
- $\int \cot x dx = \log \sin x$
- $\int \sec x dx = \log(\sec x + \tan x) \text{ or } \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$
- $\int \text{cosec} x dx = \log \tan \frac{x}{2} \text{ or } -\log(\text{cosec} x + \cot x)$
- $\int \sec x \cdot \tan x dx = \sec x$
- $\int \text{cosec} x \cdot \cot x dx = -\text{cosec} x$
- $\int \sec^2 x dx = \tan x$
- $\int \text{cosec}^2 x dx = -\cot x$
- $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$
- $\int \frac{dx}{1+x^2} = \tan^{-1} x$
- $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x$

Derivative of indefinite integral, integral of derivative

- $\frac{d}{dx} \int f(x) dx = f(x)$
- $\int \frac{d}{dx} f(x) dx = f(x) + c$
- $\int \{f(x) \cdot \phi'(x) + f'(x) \cdot \phi(x)\} dx = f(x) \cdot \phi(x) + c$
- $\int \frac{f'(x) \cdot \phi(x) - f(x) \cdot \phi'(x)}{\{\phi(x)\}^2} dx = \frac{f(x)}{\phi(x)} + c$

Rules of integration

- $\int \{f_1(x) \pm f_2(x) \pm ..... \pm f_n(x)\} dx$
  $= \int f_1(x) dx \pm \int f_2(x) dx \pm ... \pm \int f_n(x) dx$
  (rule of term-by-term integration)
- $\int kf(x) dx = k \int f(x) dx$, where k is a constant
- $\int f'(ax+b) dx = \frac{f(ax+b)}{a}$

Methods of integration

Given the integrand our primary objective is to change the integral into algebraics sum of standard integrands. This objective can be achieved in any of the following three methods:

- **Simplification or transformation-**We change the integral into algebraic sum of standard integrands by simplification, using algebraic or trigonometrical simplifications.
- **Substitution-** We make suitable substitution for the given variable in terms of some other variable so that the integral changes into a standard integral or algebraic sum of standard integrands in the new variable.
- By parts –
  $\int f(x) \cdot \phi(x) dx = f(x) \cdot \int \phi(x) dx - \int [\int \phi(x) dx] \cdot f'(x) dx$
  This method is applicable when the integral can be put as the product of two functions of which one can be integrated easily.

Forms of integrands suitable for specific substitution: There is no fixed rule for selecting a function as the new variable. While selecting it, we always remember that the resulting integrand after substitution must change the original integrand in algebraic sum of standard integrands in the new variable. However, there are certain forms of the integrand which indicate the appropriate substitution.

- Form $\int f\{\phi(x)\} \cdot \phi'(x) dx$, substitute $\phi(x) = z$
  $\int \frac{\phi'(x)}{\phi(x)} dx$, substitute $\phi(x) = z$
- Form $\int \sin^p x \cdot \cos^{2n-1} x dx, n \in N$, substitute $\sin x = z$
  $\int \sin^p x \cdot \cos^{2n-1} x dx, n \in N$, substitute $\cos x = z$
  $\int \sin^p x \cdot \cos^q x dx$, where $p + q$ = negative even integer, substitute then $x = z$.
- Form $\int f(x, \sqrt{a^2 - x^2}) dx$ substitute $x = a\sin\theta$ or $a\cos\theta$
  $\int f(x, \sqrt{x^2 - a^2}) dx$ substitute $x = a\sec\theta$ or $a\,\text{cosec}\,\theta$
  $\int f(x, \sqrt{x^2 + a^2}) dx$ substitute $x = a\tan\theta$ or $a\cot\theta$
  $\int f(x^2, \sqrt{a^2 - x^2}) dx$ substitute $x^2 = a^2 \cos 2\theta$

Choice of u, v in "by parts"

The method of "by parts" $\int uvdx = u\int vdx - \int\left[\int vdx\right]\frac{du}{dx}dx$, is used when the integrand is through of as a product of two functions. While using this method, one has to take care of the following:

(i) the second part v must be a standard integrand or can be easily integrated by simplification or substitution.

(ii) the integral $\int\left[\int vdx\right]\frac{du}{dx}dx$ must not be more complicated than the original integrand.

Using integration by parts, we can in principle calculate the integral of the product of any two arbitrary functions. You should be very thorough with the use of this technique, since it will be extensively required in solving integration problems. Let $u = f(x)$ and $v = g(x)$ be two arbitrary functions. We need to evaluate $\int f(x)g(x)dx$. The rule for integration by parts says that: $\int f(x)g(x)dx = f(x)\int g(x)dx - \int\{f'(x)\int g(x)dx\}dx$

Translated into words (which makes it easier to remember!), this rule says that: The integral of the product of two functions = (First function) × (Integral of second function) – Integral of (Derivative of the first function) × (Integral of the second function)}.

Theoretically, we can choose any of the two functions in the product as the first function and the other as the second function. However, a little observation of the expression above will show you that since we need to deal with the integral of the second function ($\int g(x)dx$, above)), we should choose the second function in such a way so that it is easier to integrate; consequently, the first function should be the one that is more difficult to integrate out of the two functions. We can thus define a priority list pertaining to the choice of the first function, corresponding to the degree of difficulty in integration:

I – inverse trigonometric functions
L – logarithmic function
A – algebraic functions
T – trigonometric functions
E – exponential function

Decreasing order of difficulty in carrying out integration. For example, inverse trigonometric functions are the most difficult to integrate while exponential functions are the easiest. Thus, we should choose the first function in this order.

The boxed letters should make it clear to you why this rule of thumb for the selection of the first function is referred to as the ILATE rule.

It is important to realize that the ILATE rule is just a guide that serves to facilitate the process of integration by parts; it is not a rule that always has to be followed; you can choose your first function contrary to the ILATE rule also if you wish to (and if you are able to integrate successfully with your choice). However, the ILATE rule works in most of the cases and is therefore widely used.

An important result

- $\int e^x\{f(x) + f'(x)\}dx = e^x \cdot f(x)$.

Integration of some standard rational and irrational functions (fractions)

Standard integrals are as follows (without writing the constant of integration) :

- $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$
- $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}, (x > a)$
- $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}, (x < a)$
- $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log(x + \sqrt{x^2 - a^2})$

  $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}$

  $\int \frac{dx}{\sqrt{a^2 + x^2}} = \log(x + \sqrt{a^2 + x^2})$
- $\int \sqrt{a^2 + x^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log(x + \sqrt{x^2 + a^2})$

  $\int \sqrt{a^2 - x^2}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2 - x^2}}{2}$

  $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log(x + \sqrt{x^2 - a^2})$

Integration of rational fractions: Let L_1, L_2 etc., denote polynomial of the first degree and Q_1, Q_2, etc., denote polynomial of the second degree, and N denote the numerator. Usual methods of integration of rational fractions are as follows:

- Form $\int \frac{dx}{L_1^n}$ Substitute $L_1 = z$
- Form $\int \frac{dx}{Q_1}$ Put $Q_1 = \pm a^2 \pm (L_1)^2$ and apply the suitable result from Section 1.

Use of partial fractions in the integration of rational fractions

- Form

  $\int \frac{Ndx}{Q_1}$, where $Q_1 = L_1 \cdot L_2$ and degree of $N \le 1$

  $\int \frac{Ndx}{L_1 \cdot Q_1}$, where Q_1 cannot be factorized and degree of $N \le 2$

  $\int \frac{Ndx}{L_1^2 \cdot Q_1}$, where Q_1 cannot be factorized and degree of $N \le 3$

  $\int \frac{Ndx}{Q_1 \cdot Q_2}$, where Q_1, Q_2 cannot be factorized and degree of $N \le 3$

  $\int \frac{Ndx}{L_1 \cdot Q_1^2}$, where Q_1 cannot be factorized and degree of $N \le 4$

  In each of these we use partial fractions to change the integrand in algebraic sum of standard integrands.

- Method of changing $\frac{N}{Q_1}$ in partial fractions, where $Q_1 = L_1 \cdot L_2$ and the degree of $N \le 1$:

  $\frac{N}{Q_1} = \frac{N}{L_1 \cdot L_2} = \frac{A}{L_1} + \frac{B}{L_2}$ where A, B are constants

  $\Rightarrow \ N = AL_2 + BL_1$

  Equate the coefficients of similar powers of the variable on both sides and obtain two equations in A, B to obtain their values. Or, put two suitable values for the variable on both sides successively and obtain two equations in A, B to find their values.

- Method of changing $\frac{N}{L_1 \cdot Q_1}$ in partial fractions, where Q_1 cannot be factorized and the degree of $N \le 2$:

  $\frac{N}{L_1 \cdot Q_1} = \frac{A}{L_1} + \frac{Bx + C}{Q_1}$

  where A, B, C are constants and x is the variable

  $\Rightarrow \ N = AQ_1 + (Bx + C)L_1$. Obtain three equations in A, B, C in any one of the two ways as given in the above point and solve.

- Method of changing $\frac{N}{L_1^2 \cdot Q_1}$ in partial fractions, where Q_1 cannot be factorized and the degree of $N \le 3$:

  $\frac{N}{L_1^2 \cdot Q_1} = \frac{A}{L_1} + \frac{B}{L_1^2} + \frac{Cx + D}{Q_1}$

  Find A, B, C, D as in the above points.

- Method of changing $\frac{N}{L_1 \cdot Q_1^2}$ in partial fractions, where Q_1 cannot be factorized and the degree of $N \le 4$:

  $\frac{N}{L_1 \cdot Q_1^2} = \frac{A}{L_1} + \frac{Bx + C}{Q_1} + \frac{Dx + E}{Q_1^2}$

  Find A, B, C, D, E as in the above points.

Note

If the fraction is $\frac{N}{D}$ where the degree of $N \ge$ the degree of D then write $\frac{N}{D} = P + \frac{N_1}{D}$ where the degree of $N_1 <$ the degree of D. Then express $\frac{N_1}{D}$ in partial fractions.

- Method of integrating $\int \frac{N}{D} dx$, where $N = AD + B\frac{dD}{dx} + N_1$ and A, B are constants, and the degree of $N_1 <$ the degree of D: $\int \frac{N}{D} dx = \int \frac{AD + B\frac{dD}{dx} + N_1}{D} dx$

  $= A\int dx + B\int \frac{dD}{D} + \int \frac{N_1}{D} dx$

  $= Ax + B\log D + \int \frac{N_1}{D} dx$

Where $\int \frac{N_1}{D} dx$ can be integrated by using partial fractions.

Integration of irrational fractions

- Form $\int \frac{dx}{\sqrt{L_1}}$ Substitute $L_1 = z$
- Form $\int \frac{dx}{\sqrt{Q_1}}$ Put $Q_1 = \pm a^2 \pm (L_1)^2$ and apply the suitable result from Section 1
- Form $\int \frac{L_1 dx}{\sqrt{Q_1}}$ Express $L_1 = A\frac{dQ_1}{dx} + B$ where A, B are constants.

- Form $\int \frac{Q_2 dx}{\sqrt{Q_1}}$ Express $Q_2 = AQ_1 + B\frac{dQ_1}{dx} + C$ where A, B, C are constants.
- Form $\int \frac{dx}{L_1\sqrt{L_2}}$ Substitute $\sqrt{L_2} = t$
- Form $\int \frac{dx}{L_1\sqrt{Q_1}}$ Substitute $L_1 = \frac{1}{t}$
- Form $\int \frac{dx}{Q_1\sqrt{L_2}}$ Substitute $\sqrt{L_2} = t$
- Form $\int \frac{dx}{Q_2\sqrt{Q_1}}$ Substitute $\sqrt{\frac{Q_1}{Q_2}} = t$

Illustrations

Illustration 1: If $I = \int_0^1 \frac{dx}{1+x^{\pi/2}}$, then?

Solution: As $0 < x < 1$

$\Rightarrow \quad x^2 < x^{\pi/2} < x$

$\Rightarrow \quad \frac{1}{1+x} < \frac{1}{1+x^{\pi/2}} < \frac{1}{1+x^2}$

$\Rightarrow \quad \int_0^1 \frac{dx}{1+x} < \int_0^1 \frac{dx}{1+x^{\pi/2}} < \int_0^1 \frac{dx}{1+x^2}$

$\Rightarrow \quad \log 2 < I < \frac{\pi}{2}$

Illustration 2: $\int \frac{dx}{x^2\left(1+x^4\right)^{3/4}}$ is equal to?

Solution: $\int \frac{dx}{x^2(1+x^4)^{3/4}} = \int \frac{dx}{x^5(1+\frac{1}{x^4})^{3/4}}$

$$= \frac{-1}{4}\int \frac{dt}{t^{3/4}} \quad \left(\text{Putting } 1+\frac{1}{x^4} = t \Rightarrow \frac{dx}{x^5} = \frac{-1}{4}dt\right)$$

$$= \frac{-1}{4}\cdot\frac{t^{1/4}}{1/4} + C = -\left(1+\frac{1}{x^4}\right)^{1/4} + C$$

$$= -\frac{(1+x^4)^{1/4}}{x} + C$$

Illustration 3: If $\int x^{13/2}\cdot(1+x^{5/2})^{1/2}\,dx$

$= A(1+x^{5/2})^{7/2} + B(1+x^{5/2})^{5/2} + C(1+x^{5/2})^{3/2}$, then?

Solution: $\int x^{13/2}\cdot(1+x^{5/2})\,dx$

$$= \int x^5\cdot x^{3/2}\cdot(1+x^{5/2})^{1/2}\,dx = \int x^5\cdot\frac{4}{5}z.z\,dz$$

$$\left[\text{Putting } 1+x^{5/2} = z^2 \Rightarrow \frac{5}{2}x^{3/2}dx = 2z\,dz \quad i.e., x^{3/2}dx = \frac{4}{5}z\,dz\right]$$

$$= \frac{4}{5}\int z^2\cdot(z^2-1)^2\,dz = \frac{4}{5}\int z^2(z^4-2z^2+1)dz$$

$$= \frac{4}{5}\left[\frac{z^7}{7} - \frac{2z^5}{5} + \frac{z^3}{3}\right] + C$$

$$= \frac{4}{35}(1+x^{5/2})^{7/2} - \frac{8}{25}(1+x^{5/2})^{5/2} + \frac{4}{15}(1+x^{5/2})^{3/2} + C$$

$$A = \frac{4}{35}, B = -\frac{8}{25} \text{ and } C = \frac{4}{15}$$

Illustration 4: $\int \frac{dx}{\cos^3 x\sqrt{\sin 2x}}$ is equal to?

Solution: $\int \frac{dx}{\cos^3 x\sqrt{\sin 2x}} = \int \frac{dx}{\cos^3\sqrt{\frac{2\tan x}{1+\tan^2 x}}}$

$$= \int \frac{\sec^4 x}{\sqrt{2\tan x}}dx = \frac{1}{12}\int \frac{(1+z^4).2z\,dz}{z}$$

(Putting $\tan x = z^2 \Rightarrow \sec^2 x\,dx = 2z\,dz$)

$$= \sqrt{2}\left(z + \frac{z^5}{5}\right) + C = \sqrt{2}\left(\sqrt{\tan x} + \frac{1}{5}\tanh^{5/2} x\right) + C.$$

Illustration 5: $\int \frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}$ is equal to?

Solution: $\int \frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}$

$$= \int \frac{2\sin\theta\cos\theta d\theta}{(1+\sin\theta)\sqrt{\sin^2\theta - \sin^4\theta}}$$

(Putting $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$)

$$= 2\int \frac{d\theta}{1+\sin\theta} = 2\int \frac{1-\sin\theta}{\cos^2\theta}d\theta = 2(\tan\theta - \sec\theta)$$

$$= 2\left(\frac{\sqrt{x}}{\sqrt{1-x}} - \frac{1}{\sqrt{1-x}}\right) + C\frac{2\left(\sqrt{x}-1\right)}{\sqrt{1-x}} + C$$

Illustration 6: $\int \frac{\sin^3 x dx}{(1+\cos^2 x)\sqrt{1+\cos^2 x+\cos^4 x}}$ is equal to?

Solution: $\int \frac{\sin^3 x}{(1+\cos^2 x)\sqrt{+\cos^2 x+\cos^4 x}}dx$

$$= \int \frac{\sin^3 x}{\cos x(\sec x+\cos x)\cos x\sqrt{\sec^2 x+1+\cos^2 x}}dx$$

$$= \int \frac{\sin^3 x dx}{\cos^2 x(\sec x + \cos x)\sqrt{(\sec x + \cos x)^2 - 1}} = \int \frac{dz}{z\sqrt{z^2 - 1}}$$

$$\left(\text{Putting } \sec x + \cos x = z \Rightarrow \frac{\sin^3 x}{\cos^2 x} dx = dz\right)$$

$$= \sec^{-1} z + c = \sec^{-1}(\sec x + \cos x) + C$$

Illustration 7: $\int \frac{1+x}{1+\sqrt[3]{x}} dx$ is equal to?

Solution: Put $x = z^3 \Rightarrow dx = 3z^2 dz$

$$\therefore \quad \int \frac{1+x}{1+\sqrt[3]{x}} dx = \int \frac{(1+z^3)3z^2}{1+z} dz$$

$$= 3\int z^2(z^2 - z + 1) dz$$

$$= 3\int (z^4 - z^3 + z^2) dz$$

$$= 3\int (\frac{z^5}{5} - \frac{z^4}{4} + \frac{z^3}{3}) + C$$

$$= \frac{3}{5} x^{5/3} - \frac{3}{4} x^{4/3} + x + C$$

Illustration 8: $\int \frac{\sin^3 x dx}{(\cos^4 x + 3\cos^2 x + 1)\tan^{-1}(\sec x + \cos x)} =$

Solution: $I = \int \frac{\sin^3 x\, dx}{(\cos^4 x + 3\cos^2 x + 1)\tan^{-1}(\sec x + \cos x)}$

Let $\tan^{-1}(\sec x + \cos x) = t$

$$\Rightarrow \quad \frac{1}{1+(\sec x + \cos x)^2}(\sec x \tan x - \sin x)\, dx = dt$$

$$\Rightarrow \quad \frac{\sin^3 x dx}{\cos^4 x + 3\cos^2 x + 1} = dt$$

$$\therefore \quad I = \int \frac{dt}{t} = \log|t| + c$$

$$= \log|\tan^{-1}(\sec x + \cos x)| + C$$

Illustration 9: $\int \frac{x + \sqrt[3]{x^2} + \sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)} dx$ is equal to?

Solution: Put $x = z^6 \Rightarrow dx = 6z^5 dz$

$$\therefore \quad \int \frac{x + \sqrt[3]{x^2} + \sqrt[6]{x}}{x(1+\sqrt[3]{x})} dx = \int \frac{\left(z^6 + z^4 + z\right)6z^5 dz}{z^6(1+z^2)}$$

$$= 6\int \frac{z^5 + z^3 + 1}{z^2 + 1} dz$$

$$= 6\int \left(z^3 + \frac{1}{z^2+1}\right) dz$$

$$= \frac{3}{2} z^4 + \tan^{-1} z + C = \frac{3}{2} x^{2/3} + 6\tan^{-1} x^{1/6} + C$$

Illustration 10: $\int \frac{\sqrt{1+\sqrt{x}}}{x} dx$ is equal to?

Solution: Put $1 + \sqrt{x} = z^2 \Rightarrow dx = 4z\sqrt{x}\, dz$

$$\therefore \quad \int \frac{\sqrt{1+\sqrt{x}}}{x} dx = \int \frac{z}{x}(4z\sqrt{x})\, dz$$

$$= 4\int \frac{z^2}{z^2 - 1} dz = 4\int \left(1 + \frac{1}{z^2 - 1}\right) dz$$

$$= 4\left(z + \frac{1}{2}\log\frac{z-1}{z+1}\right) + C$$

$$= 4\sqrt{1+\sqrt{x}} + 2\log\left(\frac{\sqrt{1+\sqrt{x}} - 1}{\sqrt{1+\sqrt{x}} + 1}\right) + C$$

Illustration 11: If $I_n = \int \tan^n x\, dx$, then

$I_0 + I_1 + 2(I_2 + ... + I_8) + I_9 + I_{10}$ is equal to?

Solution: We have, $I_n = \int \tan^n x\, dx = \int \tan^{n-2} x \cdot \tan^2 x\, dx$

$$= \int \tan^{n-2} x(\sec^2 x - 1)\, dx$$

$$= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx$$

$$= \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$\therefore \quad I_n + I_{n-2} = \frac{\tan^{n-1} x}{n-1}, n \geq 2$$

$$\therefore \quad I_0 + I_1 + 2(I_2 + I_3 + ... + I_8) + I_9 + I_{10}$$

$$= (I_2 + I_0) + (I_3 + I_1) + (I_4 + I_2) + (I_5 + I_3) + (I_6 + I_4)$$

$$+ (I_7 + I_5) + (I_8 + I_6) + (I_9 + I_7) + (I_{10} + I_8)$$

$$= \left(\frac{\tan x}{1} + \frac{\tan^2 x}{2} + ... + \frac{\tan^9 x}{9}\right)$$

Illustration 12: If $\int f(x)dx = F(x)$, then $\int x^3 f(x^2)dx$ is equal to?

Solution: $\int x^3 f(x^2)dx = \int x^2 f(x^2) \cdot x dx$

$$= \frac{1}{2}\int z f(z) dz$$

$$\left[\text{Putting } x^2 = z \Rightarrow x dx = \frac{1}{2} dz\right]$$

$$= \frac{1}{2}\left[zF(z) - \int 1 \cdot F(z) dz\right]$$

$$\left[\therefore \int f(x)dx = F(x)\right]$$

$$= \frac{1}{2} zF(z) - \frac{1}{2}\int F(z) dz$$

$$= \frac{1}{2} x^2 F(x^2) - \frac{1}{2}\int F(x^2) d(x^2).$$

Illustration 13: $\int \frac{dx}{x\sqrt{1-x^3}} = ?$

Solution: $I = \int x^{-1}(1-x^3)^{-1/2}dx$

Let $1-x^3 = t^2 \Rightarrow -3x^2dx = 2t\,dt$

$\Rightarrow \quad x^{-1}dx = -\frac{2}{3}\frac{tdt}{x^3} = -\frac{2}{3}\frac{tdt}{1-t^2}$

$\therefore \quad I = \int (t^{-1})\left(\frac{-2}{3}\right)\frac{tdt}{l-t^2} = -\frac{2}{3}\int \frac{dt}{l-t^{-2}}$

$= \frac{2}{3}\int \frac{dt}{t^2-1} = \frac{2}{3}\cdot\frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + c = \frac{1}{3}\text{In}\left(\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right|\right) + c.$

Illustration 14: The anti-derivative of $\frac{\cos 5x + \cos 4x}{1-2\cos 3x}$ is?

Solution: The given anti-derivative $= \int \frac{\cos 5x + \cos 4x}{1-2\cos 3x}dx$

$= \int \frac{2\cos\frac{9x}{2}\cos\frac{x}{2}}{1-2\left(2\cos^2\frac{3x}{2}-1\right)}dx = \int \frac{2\cos\frac{9x}{2}\cos\frac{x}{2}}{3-4\cos^2\frac{3x}{2}}dx$

$= \int \frac{2\cos\frac{9x}{2}\cos\frac{x}{2}\cos\frac{3x}{2}}{3\cos\frac{3x}{2}-4\cos^3\frac{3x}{2}}dx$

(Multiplying and dividing by $\cos\frac{3x}{2}$)

$= \int \frac{2\cos\frac{9x}{2}\cos\frac{3x}{2}\cos\frac{x}{2}}{-\cos\frac{9x}{2}}dx$ $(\cos 3x = 4\cos^3 x - 3\cos x)$

$= -\int 2\cos\frac{3x}{2}\cos\frac{x}{2}dx$

$= -\int(\cos 2x + \cos x)dx = -\frac{\sin 2x}{2} - \sin x + c$

Illustration 15: If $\int \tan^4 x\,dx = K\tan^3 x + L\tan x + f(x)$, then ?

Solution: Let $I = \int \tan^4 x dx$

$= \int \tan^2 x(\sec^2 x - 1)dx$

$= \int \tan^2 x\sec^2 x dx - \int \tan^2 x dx$

$= \int \tan^2 x d(\tan x) - \int(\sec^2 x - 1)dx$, Where $t = \tan x$

$= \frac{\tan^3 x}{3} - \tan x + x + C = K\tan^3 x + L\tan x + f(x)$

$\Rightarrow \quad K = \frac{1}{3}; L = -1; f(x) = x + C$

Illustration 16: $\int\left(\frac{\ln x - 1}{(\ln x)^2+1}\right)dx$ is equal to?

Solution: Put $\ln x = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$

$\therefore \quad I = \int e^t\left(\frac{t-1}{t^2+1}\right)^2 dt = \int e^t\left(\frac{1}{t^2+1} - \frac{2t}{(t^2+1)^2}\right)dt$

$= \frac{e^t}{t^2+1} + c = \frac{x}{(\ln x)^2+1} + c.$

Illustration 17: $\int \frac{\sqrt{x}}{\sqrt{x^3+4}}dx$ equals?

Solution: $I = \int \frac{\sqrt{x}}{\sqrt{x^3+4}}dx$

Let $x^{3/2} = 2\tan\theta \Rightarrow \frac{3}{2}x^{1/2} = 2\sec^2\theta\frac{d\theta}{dx}$

$\Rightarrow \quad x^{1/2}dx = \frac{4}{3}\sec^2\theta d\theta$

$I = \int \frac{\frac{4}{3}\sec^2\theta d\theta}{\sqrt{4\tan^2\theta+4}} = \frac{2}{3}\int \sec\theta d\theta$

$= \frac{2}{3}\text{In}(\sec\theta + \tan\theta) + c$

$= \frac{2}{3}\text{In}\left(\sqrt{\frac{x^3-4}{4}} + \frac{x^{3/2}}{2}\right) + c = \frac{2}{3}\text{In}\left(\frac{\sqrt{x^3}+\sqrt{x^3-4}}{2}\right) + c$

$= \frac{2}{3}\text{In}\left[\frac{x^3-(x^3-4)}{2\left(\sqrt{x^3}-\sqrt{x^3-4}\right)}\right] + c = \frac{2}{3}\text{In}\left(\frac{2}{\sqrt{x^3}-\sqrt{x^3-4}}\right) + c$

Illustration 18: $\int \frac{(x^2-2)dx}{(x^4+5x^2+4)\tan^{-1}\left(\frac{x^2+2}{x}\right)}$ is?

Solution: Put $\frac{x^2+2}{x} = y. \therefore dy = \left(1-\frac{2}{x^2}\right)dx$

$\therefore \quad \int \frac{(x^2-2)dx}{(x^4+5x^2+4)\tan^{-1}\left(\frac{x^2+2}{x}\right)}$

$= \int \frac{x^2\left(1-\frac{2}{x}\right)dx}{[(x^2+2)^2+x^2]\tan^{-1}\left(\frac{x^2+2}{x}\right)}$

$= \int \frac{dy}{(y^2+1)\tan^{-1}y}$

(Putting $x + \frac{2}{x} = t$) $= \log|\tan^{-1}y| = \log\left|\tan^{-1}\left(x+\frac{2}{x}\right)\right| + C$