3 Complex Number

QUICK LOOK

Algebra of Complex Numbers: $z = x + iy$ is a complex number where $x \in R, y \in R, i = \sqrt{-1}$, i.e., $i^2 = -1$; real part of $z = \text{Re}(x) = x$, imaginary part of $z = \text{Im}(z) = y$.

- If $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ then,

  $z_1 + z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$ and

  $z_1.z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$

  $$\frac{1}{z_2} = \frac{1}{x_2 + iy_2} = \frac{x_2 - iy_2}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_2 - iy_2}{x_2^2 - i^2y_2^2}$$

  $$= \frac{x_2 - iy_2}{x_2^2 + y_2^2} = \frac{x_2}{x_2^2 + y_2^2} - i\frac{y_2}{x_2^2 + y_2^2}$$

- $x_1 + iy_1 = x_2 + iy_2 \Leftrightarrow x_1 + x_2, y_1 = y_2$. Thus, one complex equation is equivalent to two real equations.
- $i^n = 1, i, -1, -i$ according as $n = 4m, 4m+1, 4m+2, 4m+3$.

Note

The values of different integral powers of i are i or –1 or –i and 1. The digit in the units place of the value of a positive integral power of a digit also follows a sequence of digits. The digits in unit places of $7^1, 7^2, 7^3, 7^4, 7^5$ etc., are 7, 9, 3, 1, 7 etc. Using this fact we can determine the digit in the unit place of a power of a natural number.

For example: What is the digit in the unit place of $(193)^{50}$?

Consider the value of $3^1, 3^2, 3^3, 3^4, 3^5, 3^6$ etc. The digit in the unit place will be in the sequence 3, 9, 7, 1, 3, 9, 7, 1,..The 50th term in it is 9. So the digit in the unit place of $(193)^{50}$ is 9.

Representation of complex numbers in Argand plane

- The complex number $z = x + iy$ is represented in a plane by the point (*x, y*). The plane in which complex numbers are represented by point is called the Argand plane. If $x > 0, y > 0$ then the complex number will be represented by a point in the first quadrant. Similarly for other possible signs of *x* and *y* the location of the point is as shown in the figure.

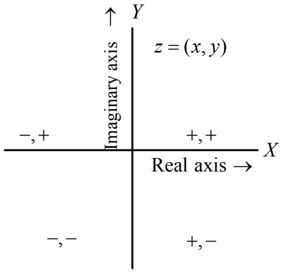


Figure: 3.1

Modulus, Amplitude (Argument), Conjugate of a Complex Number: If $z = x + iy$ then

- modulus of $z = |z| = +\sqrt{x^2 + y^2}$
- amplitude of $z = amp\, z (or \arg z) = \tan^{-1}\frac{y}{x}$. We know that $\tan^{-1}\frac{y}{x}$ has many values. The smallest numerical value falling in the quadrant of the complex number is called the fundamental amplitude (or simply amplitude).The general value is $2\pi r + \tan^{-1}\frac{y}{x}$ where $\tan^{-1}\frac{y}{x}$ is the fundamental amplitude, and this value is called the general amplitude. The amp *z* lies between –*p* and *p*, i.e., $0 \geq ampz \leq \pi$ or $-\pi < ampz < 0$. If *z* belongs to the first quadrant then its amplitude is between 0 and $\frac{\pi}{2}$; if *z* belongs to the second quadrant then the amplitude is between $\frac{\pi}{2}$ and π; if *z* belongs to the third quadrant then the amplitude is between –*p* and $\frac{-\pi}{2}$ and if *z* belongs to the fourth quadrant then the amplitude is between $\frac{-\pi}{2}$ and 0.
- Method of calculating amp *z* is as follows.

  Calculate $\tan^{-1}\left|\frac{y}{x}\right| (= \alpha)$ in the first quadrant.

  If *z* is in the first quadrant, amp $z = \alpha$

  If *z* is in the second quadrant, amp $z = \pi - \alpha$

  If *z* is in the third quadrant, amp $z = \alpha - \pi$

  If *z* is in the fourth quadrant, amp $z = -\alpha$.

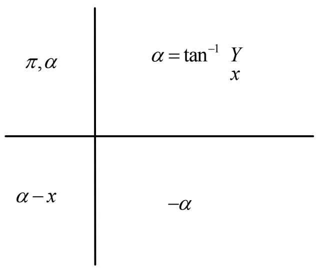


Figure: 3.2

Note

The complex number $z = 0$ has indeterminate amplitude.

- Conjugate of $z = \overline{z} = x - iy$.
- The trigonometrical form (or polar form) of a complex

number is $z = |z| \{(\cos(\text{amp}\, z) + i\sin(\text{amp}\, z)\}$, i.e., $z = r(\cos + i\sin\theta)$ where r = |z| and q = amp z.

- Unimodular complex number z is such that $|z| = 1$ and hence unimodular complex number. $z = \cos\theta + i\sin\theta$ where $\theta = amp\, z$.

While taking a complex number z in working out a problem or solving an equation we take $z = x + iy$ (in algebraic form) or $z = r(\cos\theta + i\sin\theta)$ (in trigonometrical form). If the modulus or amplitude of the complex number is known, it is always convenient to take z in the trigonometrical form.

(i) $\frac{z}{|z|}$ is always a unimolecular complex number if $z \neq 0$.

(ii) If amp $z = \frac{\pi}{2}$ or $-\frac{\pi}{2}, z$ is purely imaginary; if amp $z = 0$ or π, z is purely real.

To Express Real Part and Imaginary Part in Terms of the Complex Number

- Let $z = x + iy$; then $\bar{z} = x - iy$

  Adding these, $z + \bar{z} = 2x$;

$\therefore\ x = \frac{1}{2}(z + \bar{z})$ Subtracting these, $z - \bar{z} = 2iy$

$\therefore\ y = \frac{1}{2i}(z - \bar{z})$

Note

(i) $z + \bar{z}$ is always real and $z - \bar{z}$ is always imaginary.
(ii) $z\bar{z}$ is always real.

Properties of Conjugate, Amplitude and Modulus

- $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$ $\overline{z_1 . z_2} = \bar{z}_1 . \bar{z}_2$ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

- $amp(z_1 . z_2) = ampz_1 + ampz_2$

  $amp\left(\frac{z_1}{z_2}\right) = amp\, z_1 - amp\, z_2$

  $amp\frac{z}{\bar{z}} = 2amp\, z$

  $amp\, z^2 = 2\, amp\, z.$

Note

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then $z_1 . z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = r_1 r_2$ $\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

and $\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$

$\therefore\ amp\,(z_1 z_2) = amp\, z_1 + amp\, z_2$

and $amp\frac{z_1}{z_2} = amp\, z_1 - amp\, z_2$

Clearly, these results on amplitude hold when we take fundemantal amplitudes only.

- $|z_1 . z_2| = |z_1| . |z_2|$ $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ $|z^n| = |z|^n$

- $|z_1 + z_2| \leq |z_1| + |z_2|$

  equality holding if $z = 0$, z_1, z_2 are collinear with $z = 0$ at one end.

  $|z_1 - z_2| \geq ||z_1| - |z_2||$

- $z \cdot \bar{z} = |z|^2$ $\left|\frac{z}{\bar{z}}\right| = 1$

Power of a Complex Number (De Moivre's Theorem)

- If $z_1 = \cos\theta_1 + i\sin\theta_1, z_2 = \cos\theta_2 + i\sin\theta_2$, etc., then

  $z_1 . z_2 = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

  $= \cos(\theta_1 + \theta_2) = i\sin(\theta_1 + \theta_2)$

  $z_1 . z_2 . z_3 .... = \cos(\theta_1 + \theta_2 + \theta_3 + ...) + i\sin(\theta_1 + \theta_2 + \theta_3 + ...)$

- $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, where n is a positive integer.

  $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

  $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$

- $(\cos\theta + i\sin\theta)^{p/q} = \{\cos(2r\pi + \theta) + i\sin(2r\pi + \theta)\}^{p/q}$

  $= \cos\frac{(2r\pi + \theta)p}{q} + i\sin\frac{(2r\pi + \theta)p}{q}$,

  where $r = 0, 1, 2, ..., q - 1$.

- nth roots of unity $= 1^{1/n} = (\cos 0 + i\sin 0)^{1/n}$

  $= (\cos 2r\pi + i\sin 2r\pi)^{1/n} = \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}$,

  where $r = 0, 1, 2, ..., n - 1$.

If $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = z_0$ then the nth roots of unity are $1, z_0, z_0^2, z_0^3, ..., z_0^{n-1}$ which are in G.P.

Cube Roots of Unity

- Cube roots of unity

  $= 1^{1/3} = (\cos 0 + i\sin 0)^{1/3} = (\cos 2r\pi + i\sin 2r\pi)^{1/3}$

$= \cos\frac{2r\pi}{3} + i\sin\frac{2r\pi}{3}$,

where $r = 0, 1, 2$

$= 1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

- If one of the non-real complex roots be w then the other non-real complex root will be w^2.
- $3\sqrt{1} = 1, w, w^2$ where $w^3 = 1$ and $1 + w + w^2 = 0$.
- The value of $1 + w^n + w^{2n} = 3$ if $n = 3m$, i.e., n is divisible by 3 = 0 if $n \neq 3m$, i.e., n is not divisible by 3

Note

Any complex number for which

$\left|\frac{real\ part}{imaginary\ part}\right| = 1 : \sqrt{3}\ or\ \sqrt{3} : 1$,

can be expressed in terms of w and i.

Application of Complex Numbers in Geometrical Problems

The geometrical meaning of complex expression, equations and inequations are as follows:

- $z = x + iy \Rightarrow z$ is a point whose coordinates are (x, y)

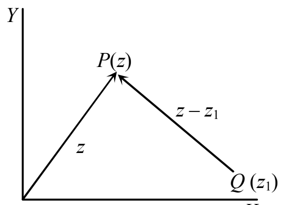


Figure: 3.3

Note

The complex number z is also represented by a vector. If P represents z in the Argand plane we say $\overrightarrow{OP}$ represents the complex number z, O being the origin. If P and Q represents complex numbers z and $z1$ respectively then $\overrightarrow{QP} = z - z_1$.

- $|z| =$ distance between the origin and the point z $|z - z_1| =$ distance between the points z and z_1.
- $amp\ z = \angle ZOX$, where Z represents z

  $amp = \frac{z}{z_1} = \angle ZOZ_1$.

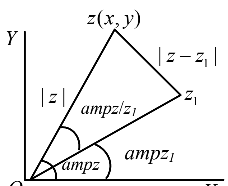


Figure: 3.4

- If z_3, z_1, z_2 are three points taken in the anticlockwise sense then $amp = \frac{z_3 - z_1}{z_2 - z_1}$

  $= \angle Z_3Z_1Z_2$
- The angle between two line segment joining the points z_1, z_2 and z_3, z_4 is

  $amp = \frac{z_1 - z_2}{z_3 - z_4}$

  or $\pi - amp = \frac{z_1 - z_2}{z_3 - z_4}$

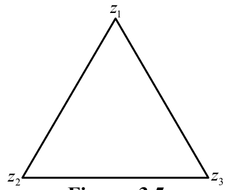


Figure: 3.5

- Complex numbers z satisfying $|z - z_0| = \rho$ represents points on the circle whose centre is z_0 and radius $= \rho$.

  Complex numbers z satisfying $|z - z_0| < \rho$ represents points inside the circle whose centre is z_0 and radius $= \rho$.

  Complex numbers z satisfying $|z - z_0| > \rho$ represents points outside the circle whose centre is z_0 and radius $= \rho$.

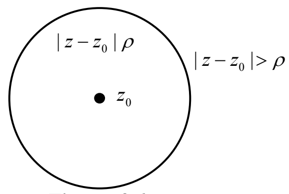


Figure: 3.6

- If z is on the circle $|z| = \rho$ then iz is also on the circle, the radius vector being shifted by $\pi/2$ in the anticlockwise sense.

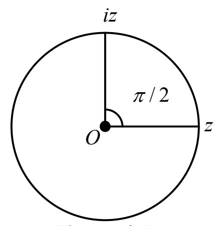


Figure: 3.7

- The line segment joining the complex numbers z_1,z_2 is divided by the complex number z in the ratio m:n if $z = \frac{mz_2 + nz_1}{m + n}$
- General equation of a line is $\bar{\alpha}z + \alpha\bar{z} + \beta = 0$ where β is a real constant and α is a non-real complex constant.

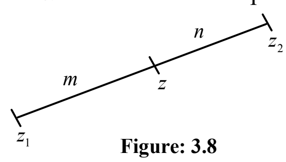


Figure: 3.8

- General equation of a circle is $z\bar{z} + \alpha\bar{z} + \bar{\alpha}z + \beta = 0$ where â is a real constant and α is a non-real complex constant.

MULTIPLE CHOICE QUESTIONS

Basic Concepts of Complex Number

1. If $i^2 = -1$, then the value of $\sum_{n=1}^{200} i^n$ is:

   a. 50 **b.** – 50 **c.** 0 **d.** 100

2. If $i = \sqrt{-1}$ and n is a positive integer, than $i^n + i^{n+1} + i^{n+2} + i^{n+3} = ?$

   a. 1 **b.** i
   c. i^n **d.** 0

3. If $x = 3+i$, then $x^3 - 3x^2 - 8x + 15 = ?$

   a. 6 **b.** 10
   c. –18 **d.** –15

4. The complex number $\frac{2^n}{(1-i)^{2n}} + \frac{(1+i)^{2n}}{2^n}$, $(n \in Z)$is equal to:

   a. 0 **b.** 2
   c. $[1 + (-1)^n].i^n$ **d.** None of these

Algebraic Operations with Complex Numbers

5. $\frac{1-2i}{2+i} + \frac{4-i}{3+2i} = ?$

   a. $\frac{24}{13} + \frac{10}{13}i$ **b.** $\frac{24}{13} - \frac{10}{13}i$ **c.** $\frac{10}{13} + \frac{24}{13}i$ **d.** $\frac{10}{13} - \frac{24}{13}i$

6. $\left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$?

   a. $\frac{1}{2} + \frac{9}{2}i$ **b.** $\frac{1}{2} - \frac{9}{2}i$
   c. $\frac{1}{4} - \frac{9}{4}i$ **d.** $\frac{1}{4} + \frac{9}{4}i$

7. Which of the following is correct?

   a. $6+i > 8-i$ **b.** $6+i > 4-i$
   c. $6+i > 4+2i$ **d.** None of these

8. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then:

   a. $x = 3, y = 1$ **b.** $x = 1,\ y = 3$
   c. $x = 0,\ y = 3$ **d.** $x = 0, y = 0$

9. The real values of x and y for which the equation $(x^4 + 2xi) - (3x^2 + yi) = (3-5i) + (1+2yi)$ is satisfied, are:

   a. $x = 2, y = 3$ **b.** $x = -2, y = \frac{1}{3}$
   c. Both (a) and (b) **d.** None of these

10. $\sqrt{-2}\,\sqrt{-3} = ?$

    a. $\sqrt{6}$ **b.** $-\sqrt{6}$
    c. $i\sqrt{6}$ **d.** None of these

11. If n is a positive integer, then $\left(\frac{1+i}{1-i}\right)^{4n+1}$ =?

    a. 1 **b.** – 1 **c.** i **d.** $-i$

12. If $(1-i)^n = 2^n$, then $n = ?$

    a. 1 **b.** 0
    c. -1 **d.** None of these

13. The smallest positive integer n for which $(1+i)^{2n} = (1-i)^{2n}$ is

    a. 1 **b.** 2
    c. 3 **d.** 4

14. If $z_1 = (4,5)$ and $z_2 = (-3,2)$ then $\frac{z_1}{z_2}$ equals?

    a. $\left(\frac{-23}{12}, \frac{-2}{13}\right)$ **b.** $\left(\frac{2}{13}, \frac{-23}{13}\right)$
    c. $\left(\frac{-2}{13}, \frac{-23}{13}\right)$ **d.** $\left(\frac{-2}{13}, \frac{23}{13}\right)$

15. The complex number $\frac{1+2i}{1-i}$ lies in which quadrant of the complex plane:

    a. First **b.** Second
    c. Third **d.** Fourth

16. If $z = x + iy, z^{1/3} = a - ib$ and $\frac{x}{a} - \frac{y}{b} = k(a^2 - b^2)$ then value of k equals:

    a. 2 **b.** 4
    c. 6 **d.** 1

Conjugate of a Complex Number

17. If the conjugate of $(x+iy)(1-2i)$ be $1 + i$, then:

    a. $x = \frac{1}{5}$ **b.** $y = \frac{3}{5}$
    c. $x + iy = \frac{1-i}{1-2i}$ **d.** $x - iy = \frac{1-i}{1+2i}$

18. The conjugate of complex number $\frac{2-3i}{4-i}$ is:

    a. $\frac{3i}{4}$ **b.** $\frac{11+10i}{17}$
    c. $\frac{11-10i}{17}$ **d.** $\frac{2+3i}{4i}$

19. The real part of $(1-\cos\theta+2i\sin\theta)^{-1}$ is

a. $\frac{1}{3+5\cos\theta}$ b. $\frac{1}{5-3\cos\theta}$

c. $\frac{1}{3-5\cos\theta}$ d. $\frac{1}{5+3\cos\theta}$

20. The reciprocal of $3+\sqrt{7}i$ is

a. $\frac{3}{4}-\frac{\sqrt{7}}{4}i$ b. $3-\sqrt{7}i$

c. $\frac{3}{16}-\frac{\sqrt{7}}{16}i$ d. $\sqrt{7}+3i$

Conjugate, Modulus and Argument of Complex Numbers

21. $\left|(1+i)\frac{(2+i)}{(3+i)}\right|=$

a. – 1/2 b. 1/2 c. 1 d. – 1

22. For $x_1, x_2, y_1, y_2 \in R$, if $0<x_1<x_2, y_1=y_2$ and $z_1=x_1+iy_1$, $z_2=x_2+iy_2$ and $z_3=\frac{1}{2}(z_1+z_2)$, then z_1, z_2 and z_3 satisfy

a. $|z_1|=|z_2|=|z_3|$ b. $|z_1|<|z_2|<|z_3|$

c. $|z_1|>|z_2|>|z_3|$ d. $|z_1|<|z_3|<|z_2|$

23. Amplitude of $\left(\frac{1-i}{1+i}\right)$ is:

a. $-\pi/2$ b. $\pi/2$

c. $\pi/4$ d. $\pi/6$

24. The amplitude of $\sin\frac{\pi}{5}+i\left(1-\cos\frac{\pi}{5}\right)$?

a. $\pi/5$ b. $2\pi/5$

c. $\pi/10$ d. $\pi/15$

25. If $|z|=4$ and $arg\ z=\frac{5\pi}{6}$, then $z=$

a. $2\sqrt{3}-2i$ b. $2\sqrt{3}+2i$

c. $-\sqrt{3}+i$ d. $-2\sqrt{3}+2i$

26. If z and ω are to non-zero complex numbers such that $|z\omega|=1$ and $arg(z)-arg(\omega)=\frac{\pi}{2}$, then $\bar{z}\omega$ is equal to

a. 1 b. –1 c. i d. $-i$

27. The complex numbers $\sin x+i\cos 2x$ and $\cos x-i\sin 2x$ are conjugate to each other for:

a. $x=n\pi$ b. $x=\left(n+\frac{1}{2}\right)\pi$

c. $x=0$ d. No value of x

28. There exists no value of x common in (i) and (ii). Therefore there is no value of x for which the given complex numbers are conjugate.

If z is a complex number such that $z^2=(\bar{z})^2$, then

a. z is purely real

b. z is purely imaginary

c. Either z is purely real or purely imaginary

d. None of these

29. The number of solutions of the equation $z^2+\bar{z}=0$ is

a. 1 b. 2

c. 3 d. 4

30. The conjugate of $\frac{(2+i)^2}{3+i}$, in the form of $a+ib$, is

a. $\frac{13}{2}+i\left(\frac{15}{2}\right)$ b. $\frac{13}{10}+i\left(\frac{-15}{2}\right)$

c. $\frac{13}{10}+i\left(\frac{-9}{10}\right)$ d. $\frac{13}{10}+i\left(\frac{9}{10}\right)$

31. If z_1 and z_2 are any two complex numbers then $|z_1+z_2|^2+|z_1-z_2|^2$ is equal to

a. $2|z_1|^2|z_2|^2$ b. $2|z_1|^2+2|z_2|^2$

c. $|z_1|^2+|z_2|^2$ d. $2|z_1||z_2|$

32. The maximum value of $|z|$ where z satisfies the condition $\left|z+\frac{2}{z}\right|=2$ is

a. $\sqrt{3}-1$ b. $\sqrt{3}+1$

c. $\sqrt{3}$ d. $\sqrt{2}+\sqrt{3}$

33. If z_1 and z_2 be complex numbers such that $z_1\neq z_2$ and $|z_1|=|z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $\frac{(z_1+z_2)}{(z_1-z_2)}$ may be

a. Purely imaginary b. Real and positive

c. Real and negative d. None of these

34. The product of two complex numbers each of unit modulus is also a complex number, of:

a. Unit modulus b. Less than unit modulus

c. Greater than unit modulus d. None of these

35. For any complex number $z, \bar{z}=\left(\frac{1}{z}\right)$ if and only if

a. z is a pure real number

b. $|z|=1$

c. z is a pure imaginary number

d. $z=1$

36. Let z be a complex number (not lying on X-axis of maximum modulus such that $\left|z+\frac{1}{z}\right|=1$. Then:

a. $\text{Im}(z) = 0$ **b.** $\text{Re}(z) = 0$
c. $\text{amp}(z) = \pi$ **d.** None of these

37. Modulus of $\left(\frac{3+2i}{3-2i}\right)$ is:

a. 1 **b.** 1/2 **c.** 2 **d.** $\sqrt{2}$

38. If $|z|=1$ and $\omega=\frac{z-1}{z+1}$ (where $z\neq -1$), then $\text{Re}(\omega)$ is:

a. 0 **b.** $-\frac{1}{|z+1|^2}$
c. $\left|\frac{z}{z+1}\right|.\frac{1}{|z+1|^2}$ **d.** $\frac{\sqrt{2}}{|z+1|^2}$

39. If z_1 and z_2 are two non-zero complex numbers such that $|z_1+z_2|=|z_1|+|z_2|$, then $arg\ (z_1)-arg\ (z_2)$ is equal to:

a. $-\pi$ **b.** $-\frac{\pi}{2}$ **c.** $\frac{\pi}{2}$ **d.** 0

Square Root of a Complex Number

40. The square root of $3 - 4i$ are

a. $\pm(2-i)$ **b.** $\pm(2+i)$
c. $\pm(\sqrt{3}-2i)$ **d.** $\pm(\sqrt{3}+2i)$

41. $\sqrt{2i}$ equals?

a. $1 + i$ **b.** $1 - i$
c. $-\sqrt{2}i$ **d.** None of these

Representation of Complex Number

42. If $-1+\sqrt{-3}=re^{i\theta}$, then θ is equal to

a. $\frac{\pi}{3}$ **b.** $-\frac{\pi}{3}$
c. $\frac{2\pi}{3}$ **d.** $-\frac{2\pi}{3}$

43. Real part of $e^{e^{i\theta}}$ is

a. $e^{\cos\theta}[\cos(\sin\theta)]$ **b.** $e^{\cos\theta}[\cos(\cos\theta)]$
c. $e^{\sin\theta}[\sin(\cos\theta)]$ **d.** $e^{\sin\theta}[\sin(\sin\theta)]$

44. If $\frac{1}{x}+x=2\cos\theta$, then $x^n+\frac{1}{x^n}$ is equal to

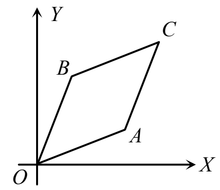
a. $2\cos n\theta$ **b.** $2\sin n\theta$
c. $\cos n\theta$ **d.** $\sin n\theta$

45. i^i is equal to:

a. $e^{\pi/2}$ **b.** $e^{-\pi/2}$
c. $-\pi/2$ **d.** None of these

Complex Numbers in Co-ordinate Geometry

46. If in the diagram, A and B represent complex number z_1 and z_2 respectively, then C represents:



a. $z_1 + z_2$ **b.** $z_1 - z_2$
c. $z_1 . z_2$ **d.** z_1 / z_2

47. If the complex number z_1,z_2 and the origin form an equilateral triangle then $z_1^2+z_2^2=$

a. z_1z_2 **b.** $z_1\bar{z}_2$
c. $\bar{z}_2z_1$ **d.** $|z_1|^2=|z_2|^2$

Rotation Theorem

48. In the arg and diagram, if O ,P and Q represents respectively the origin, the complex numbers z and $z + iz$, then the angle $\angle OPQ$ is

a. $\pi/4$ **b.** $\pi/3$
c. $\pi/2$ **d.** $2\pi/3$

49. If complex numbers z_1,z_2 and z_3 represent the vertices A, B and C respectively of an isosceles triangle ABC of which $\angle C$ is right angle, then correct statement is

a. $z_1^2+z_2^2+z_3^2=z_1z_2z_3$
b. $(z_3-z_1)^2=z_3-z_2$
c. $(z_1-z_2)^2=(z_1-z_3)(z_3-z_2)$
d. $(z_1-z_2)^2=2(z_1-z_3)(z_3-z_2)$

Triangle Inequalities

50. The points $1+3i$, $5+i$ and $3 + 2i$ in the complex plane are

a. Vertices of a right angled triangle
b. Collinear
c. Vertices of an obtuse angled triangle
d. Vertices of an equilateral triangle

51. If $z = x + iy$, then area of the triangle whose vertices are points z, iz and $z + iz$ is

a. $2|z|^2$ **b.** $\frac{1}{2}|z|^2$ **c.** $|z|^2$ **d.** $\frac{3}{2}|z|^2$

Standard Loci in the Argand Plane

52. The locus of the points z which satisfy the condition *arg* $\left(\frac{z-1}{z+1}\right)=\frac{\pi}{3}$ is:

a. A straight line **b.** A circle

c. A parabola **d.** None of these

53. The locus of z satisfying the inequality $\log_{1/3}|z+1| > \log_{1/3}|z-1|$ is:

a. $R(z)<0$ **b.** $R(z)>0$

c. $I(z)<0$ **d.** None of these

54. If $\alpha+i\beta=\tan^{-1}(z), z=x+iy$ and α is constant, the locus of 'z' is:

a. $x^2+y^2+2x\cot 2\alpha=1$ **b.** $\cot 2\alpha(x^2+y^2)=1+x$

c. $x^2+y^2+2x\sin 2\alpha=1$ **d.** $x^2+y^2+2y\tan 2\alpha=1$

De' Moivre's Theorem and it's Applications

55. If $\left(\frac{1-i}{1+i}\right)^{100}=a+ib$, then:

a. $a=2, b=-1$ **b.** $a=1, b=0$

c. $a=0, b=1$ **d.** $a=-1, b=2$

56. If $x_r=\cos\left(\frac{\pi}{2^r}\right)+i\sin\left(\frac{\pi}{2^r}\right)$, then $x_1.x_2.x_3...\infty$ is:

a. –3 **b.** –2

c. –1 **d.** 0

Roots of a Complex Number

57. If ω is the cube root of unity, then $(3+5\omega+3\omega^2)^2+(3+3\omega+5\omega^2)^2=?$

a. 4 **b.** 0

c. – 4 **d.** 5

58. If $i=\sqrt{-1}$,then $4+5\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{334}+3\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{365}$ is equal to:

a. $1-i\sqrt{3}$ **b.** $-1+i\sqrt{3}$

c. $i\sqrt{3}$ **d.** $-i\sqrt{3}$

59. Let ω is an imaginary cube root of unity then the value of $2(\omega+1)(\omega^2+1)+3(2\omega+1)(2\omega^2+1)+.....+(n+1)(n\omega+1)(n\omega^2+1)$ is:

a. $\left[\frac{n(n+1)}{2}\right]^2+n$ **b.** $\left[\frac{n(n+1)}{2}\right]^2$

c. $\left[\frac{n(n+1)}{2}\right]^2-n$ **d.** None of these

60. The roots of the equation $x^4-1=0$, are:

a. $1, 1, i, -i$

b. $1, -1, i, -i$

c. $1, -1, \omega, \omega^2$

d. None of these

Logarithm of Complex Numbers

61. If $(1+i\sqrt{3})^9=a+ib$, then b is equal to:

a. 1 **b.** 256

c. 0 **d.** 9^3

62. The amplitude of $e^{e^{-i\theta}}$ is equal to:

a. $\sin\theta$ **b.** $-\sin\theta$

c. $e^{\cos\theta}$ **d.** $e^{\sin\theta}$

63. If $z=\frac{1+i\sqrt{3}}{\sqrt{3}+i}$, then $(\bar{z})^{100}$ lies in:

a. I quadrant **b.** II quadrant

c. III quadrant **d.** IV quadrant

64. If $x+\frac{1}{x}=\sqrt{3}$, then $x=?$

a. $\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$

b. $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$

c. $\sin\frac{\pi}{6}+i\cos\frac{\pi}{6}$

d. $\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$

65. The imaginary part of $\tan^{-1}\left(\frac{5i}{3}\right)$ is:

a. 0 **b.** ∞

c. log 2 **d.** log 4

66. $i\log\left(\frac{x-i}{x+i}\right)$ is equal to:

a. $\pi+2\tan^{-1}x$

b. $\pi-2\tan^{-1}x$

c. $-\pi+2\tan^{-1}x$

d. $-\pi-2\tan^{-1}x$

67. If $e^{i\theta}=\cos\theta+i\sin\theta$, then in ΔABC value of $e^{iA}.e^{iB}.e^{iC}$ is:

a. $-i$ **b.** 1

c. –1 **d.** None of these

68. If $z=\frac{7-i}{3-4i}$ then $z^{14}=?$

a. 2^7 **b.** $2^7 i$ **c.** $2^{14} i$ **d.** $-2^7 i$

Shifting the Origin and Inverse Points

69. Inverse of a point a with respect to the circle $|z-c| = R$ (a and c are complex numbers, centre C and radius R) is the point $c+\frac{R^2}{\bar{a}-\bar{c}}$

a. $c+\frac{R^2}{\bar{a}-\bar{c}}$ **b.** $c-\frac{R^2}{\bar{a}-\bar{c}}$

c. $c+\frac{R}{\bar{c}-\bar{a}}$ **d.** None of these

Dot and Cross Product

70. If $z_1 = 2+5i$, $z_2 = 3-i$ then projection of z_2 on z_2 is

a. $1/10$ **b.** $1/\sqrt{10}$

c. $-7/10$ **d.** None of these

NCERT EXEMPLAR PROBLEMS

More than One Answer

71. If $z_1 = a+ib$ and $z_2 = c+ib$ are complex numbers such that $|z_1|=|z_2|=1$ and $\operatorname{Re}(z_1\bar{z}_2)=0$, then the pair of complex numbers $w_1 = a+ic$ and $w_2 = b+id$ satisfies**?**

a. $|w_1| = 1$ **b.** $|w_2| = 1$

c. $\operatorname{Re}|w_1\bar{w}_2| = 0$ **d.** None of these

72. Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1|=|z_2|$, If z_1 has positive real part and z_2 has negative imaginary part, then $\frac{z_1+z_2}{z_1-z_2}$ may be:

a. zero **b.** real and positive

c. 0 **d.** 0

73. Let z_1 and z_2 be two distinct complex number and let $z = (1-t)\, z_1 = tz_2$ for some real number t with $0 < t < 1$. If arg (w) denotes the principle argument of a non-zero complex number w, then:

a. $|z-z_1| + |z-z_2| = |z_1-z_2|$

b. $\arg(z-z_1) = \arg(z-z_2)$

c. $\begin{vmatrix} z-z_1 & \bar{z}-\bar{z}_1 \\ z_2-z_1 & \bar{z}_2-\bar{z}_1 \end{vmatrix} = 0$

d. $\arg(z-z_1) = \arg(z_2-z_1)$

74. Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then, $P^2 \neq 0$ when n is equal to:

a. 57 **b.** 55

c. 58 **d.** 56

75. If z_1, z_2, z_3, z_4 are the four complex numbers represented by the vertices of a quadrilateral taken in order such that $z_1-z_4=z_2-z_3$ and $\operatorname{amp}\left(\frac{z_4-z_1}{z_2-z_1}\right)=\frac{\pi}{2}$, then the quadrilateral is:

a. rhombus **b.** square

c. rectangle **d.** cyclic quadrilateral

76. Let z_1, z_2 be two complex numbers represented by points on the circle $|z| = 1$ and $|z| = 2$ respectively, then :

a. max $|2z_1 + z_2| = 4$ **b.** min $|z_1 - z_2| = 1$

c. $\left|z_2+\frac{1}{z_1}\right| \geq 3$ **d.** none of these

77. If α is a complex constant such that $\alpha z^2 + z + \bar{\alpha} = 0$ has a real root, then:

a. $\alpha + \bar{\alpha} = 1$

b. $\alpha + \bar{\alpha} = 0$

c. $\alpha + \bar{\alpha} = -1$

d. the absolute value of the real root is 1

78. If z_1, z_2, z_3, z_4 are roots of the equation $a_0z^4 + a_1z^3 + a_2z^2 + a_3z + a_4 = 0$ where a_0, a_1, a_2, a_3, and a_4 are real, then:

a. $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$ are also roots of the equation

b. z_1 is equal to at least one of $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$

c. $-\bar{z}_1, -\bar{z}_2, -\bar{z}_3, -\bar{z}_4$ are also roots of the equation

d. none of the above

79. The reflection of the complex number $\frac{2-i}{3+i}$ (where $i = \sqrt{-1}$) in the straight line $z(1+i) = \bar{z}(i-1)$ is:

a. $\frac{-1-i}{2}$ **b.** $\frac{-1+i}{2}$ **c.** $\frac{i(i+1)}{2}$ **d.** $\frac{-1}{1+i}$

80. The common roots of the equations $z^3 + (1+i)z^2 + (1+i)z + i = 0$ (where $i = \sqrt{-1}$) and $z^{1993} + z^{1994} + 1 = 0$ are:

a. 1 **b.** ω **c.** ω^2 **d.** ω^{981}

Assertion and Reason

Note: Read the Assertion (A) and Reason (R) carefully to mark the correct option out of the options given below:

a. If both assertion and reason are true and the reason is the correct explanation of the assertion.

b. If both assertion and reason are true but reason is not the correct explanation of the assertion.

c. If assertion is true but reason is false.

d. If the assertion and reason both are false.

e. If assertion is false but reason is true.

81. Assertion: If $z_1, z_2 z_3$ are such that $|z_1| + |z_2| = |z_3| = 1$, then maximum value of $|z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 - z_2|^2$ is 9.

Reason: If z_1, z_2, z_3 are such that $|z_1| = |z_2| = |z_3| = 1$, hen $\text{Re}(z_2\bar{z}_3 + z_3\bar{z}_1 + z_1\bar{z}_2) \geq 3/2$

82. Assertion: If z is a root of the equation $z^7 + 2x + 3 = 0$, then $1 \leq |z| < 3/2$.

Reason: If z lies in the annular region $1 < |z| \leq 3/2$, then z satisfies the $\frac{1}{z-1} + \frac{1}{z-\omega} + \frac{1}{z-\omega^2} = 1$ where $\omega \neq 1$ is a cube root of unity.

83. Assertion: If $\omega \neq 1$ is a cube root of unity, then $A^2 = O$, where $A \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix}$

Reason: If $\omega \neq 1$ is a cube root of unity, then

$$\Delta = \begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = x^3$$

84. Assertion: If $z^2 - z + 1 = 0$ and n is a natural number, then $\sum_{k=1}^{n} (z^k + z^{-k})^2 = n + 3 \left[\frac{n}{3}\right]$ where $[x]$ denotes the greatest integer $\leq x$.

Reason: If $\omega \neq 1$ is a cube root of unity, then

$$\omega^k + (\bar{\omega}) = \begin{cases} -1 & \text{if } k \text{ is not a multiple of } 3 \\ 2 & \text{if } k \text{ is a multiple of } 3. \end{cases}$$

85. Assertion: The maximum value of $f(\theta) = \left|\frac{2i}{3 - ie^{i\theta}}\right|$ is $\frac{1}{\sqrt{2}}$

Reason: The minimum value of $f(\theta) = \left|\frac{2i}{3 - ie^{i\theta}}\right|$ is 1.

Comprehension Based

Paragraph –I

Read the following passage and answer the questions. Let *A, B, C* be three sets of complex number as defined below

$A = \{z : \text{Im}\, z \geq 1\}$, $B = \{z : |z - 2 - i| = 3|\}$

$C = \{Z; \text{Re}((l - i)z) + \sqrt{2}\}$

86. The number of elements in the set $A \cap B \cap C$ is:

a. 0 **b.** 1 **c.** 2 **d.** ∞

87. Let z be any point in $A \cap B \cap C$. The $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between:

a. 25 and 29 **b.** 30 and 34 **c.** 35 and 39 **d.** 40 and 44

88. Let z be any point in $A \cap B \cap C$ and let w be any point satisfying $|w - 2 - i| < 3$. Then, $|z| - |w| + 3$ lies between:

a. -6 and 3 **b.** -3 and 6 **c.** -6 and 6 **d.** -3 and 9

Paragraph –II

Read the following passage and answer the questions. Let $S = S_1 \cap S_2 \cap S_3$, where $S_1 = \{z \in C : |z| < 4\}$,

$= \left\{z \in C : \text{Im}\left[\frac{z - 1 + \sqrt{3}i}{1 - \sqrt{3}i}\right] > 0\right\}$ and $S_3 : \{z \in C : \text{Re}\, z > 0\}$

89. Area of S is equal to:

a. $\frac{10\pi}{3}$ **b.** $\frac{20\pi}{3}$ **c.** $\frac{16\pi}{3}$ **d.** $\frac{32\pi}{3}$

90. $\min_{z \in s} |1 - 3i - z|$ is equal to:

a. $\frac{2 - \sqrt{3}}{2}$ **b.** $\frac{2 + \sqrt{3}}{2}$ **c.** $\frac{3 - \sqrt{3}}{2}$ **d.** $\frac{3 + \sqrt{3}}{2}$

Match the Column

91. Match the statements/expression given is Column-I with the values given in Column-II:

Column I	Column II
(A) In R_2, if the magnitude of the projection vector of $a\hat{i} + \beta\hat{j}$ no $\sqrt{3}\hat{i} + \hat{j}$ is $\sqrt{3}$ and if $\alpha = 2 + \sqrt{3}\beta$ then possible value (s) of $\lvert\alpha\rvert$ is (are)	**1.** 1
(B) Let a and b be real number such that the function $f(x) = \begin{cases} -3ax^2 - 2 & x < 1 \\ bx + a^2, & x \geq 1 \end{cases}$ is differentiable for all $x \in R$ Then possible value (s) of a is are	**2.** 2
(C) Let $\omega \neq 1$ be a complex cube root of unity. If	**3.** 3

$(3-3\omega+2\omega^2)^{4n+3}$ $+(2+3\omega-2\omega^2)^{4n+3}$ $+(-3+2\omega+3\omega^2)^{4n+3}=0$, then possible value ($s$) of n is (are)	
(D) Let the harmonic mean of two positive real numbers a and b be 4. If 2, is a positive real number such that a, 5, q, b is an arithmetic progression, then the value (s) of $\|q-a\|$ is are|	**4.** 4
	5. 5

a. A→ 2; B→ 4; C→ 2; D→ 3
b. A→ 3; B→ 1; C→ 5; D→ 4
c. A→ 4; B→ 3; C→ 5; D→1
d. A→ 1; B→ 2; C→ 3; D→ 4

92 Match the statements of Column I with these in Column II: [Note: Here z takes values in the complex plane and $\text{Im}(z)$ and $\text{Re}(z)$ denote respectively, the imaginary part and the real part of z]

Column I	Column II
(A) The set of points z satisfying $\|z-i\|z\|\| = \|z+i\|z\|\|$ is contained in or equal to	**1.** an ellipse with eccentricity 4/5
(B) The set of points z satisfying $\|z+4\|+\|z-4\| = 0$ is contained in or equal to	**2.** the set of points z satisfy Im(z) = 0
(C) If $\|w\|=2$ then the set of point $z=w-\frac{1}{w}$ is contained in or equal to	**3.** the set of points z satisfying $\|\text{Im}\, z\|\le 1$
(D) If $\|w\|=1$, then the set of point $z=w+\frac{1}{w}$ is contained in or equal to	**4.** the set of points satisfying $\|\text{Re}\, z\|\le 2$
	5. the set of points z satisfying $\|z\|\le 3$

a. A→ 2; B→ 1; C→ 5; D→ 3
b. A→ 1; B→ 2; C→ 3,5; D→1
c. A→ 4; B→ 2; C→ 3; D→ 2
d. A→ 1; B→ 2; C→ 3; D→ 4

93. Let $z_k = \cos\left(\frac{2k\pi}{10}\right)+i\sin\left(\frac{2k\pi}{10}\right)$; k = 1,2, …. , 9?

Column I	Column II
(A) For each z_k, there exists a z_j such that $z_k.z_j=1$	**1.** True
(B) There exists $ak\in\{1,2,...9\}$ such that $z_1.z=z_k$ has no solution z in the set of complex numbers	**2.** False
(C) $\frac{\|1-z_1\|\|1-z_2\|...\|1-z_9\|}{10}$ equals	**3.** 1
(D) $1-\sum_{k=1}^{9}\cos\left(\frac{2k\pi}{10}\right)$ equals	**4.** 2

a. A→ 2; B→ 1, C→ 2; D→ 1
b. A→ 1; B→ 2; C→ 3; D→ 4
c. A→ 4; B→ 2; C→ 3; D→ 2
d. A→ 1; B→ 2; C→ 4; D→ 3

Integer

94. If z is any complex number satisfying $|z-3-2i|\le 2$, then the maximum value of $|2z-6+5i|$is:

95. Let $\omega=e^{i\pi/3}$ and a,b,c,x,y,z be non - zero complex numbers such that $a+b+c=x, +b\omega+c\omega^2=y, a+b\omega^2+c\omega=z$. Then, the value of $\frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}$ is:

96. If $|z_1|=2, |z_2|=3, |z_3|=4$ and $|2z_1+3z_2+4z_3|=9$, then absolute value $8z_2z_3+27z_3z_1+64z_1z_2$ must be equal to:

97. If a,b,c, are distinct integers and $\omega\neq 1$ is a cube root of unity and if minimum value of $|a+b\omega+c\omega^2|+|a+b\omega^2+2\omega|$ $=n^{1/4}$ then the value of n must be equal to:

98. If the equation of all the circles which are orthogonal to $|z|=1$and $|z+1|=4$ is $|z+7-ib|=\sqrt{(\lambda+b^2)}$, $i=\sqrt{-1}$ and $b\in R$, then the value of λ must be equal to:

99. If $\sum_{p=1}^{32}(3p+2)\left\{\sum_{q=1}^{10}\left\{\sin\left(\frac{2q\pi}{11}\right)-i\cos\left(\frac{2q\pi}{11}\right)\right\}\right\}^{4p}=\lambda$ (where $i=\sqrt{-1}$), then the value of λ must be equal to:

100. If z_1 and z_2 are complex numbers, such that $|15z_1-13z_2|^2+|13z_1+15z_2|^2=\lambda(|z_1|^2+|z_2|^2)$, then the value of $\sqrt{\lambda\sqrt{\lambda\sqrt{\lambda\sqrt{\lambda\ldots\infty}}}}$ must be equal to:

ANSWER

1.	**2.**	**3.**	**4.**	**5.**	**6.**	**7.**	**8.**	**9.**	**10.**
c	d	d	d	d	d	d	d	c	b
11.	**12.**	**13.**	**14.**	**15.**	**16.**	**17.**	**18.**	**19.**	**20.**
c	b	b	c	b	b	c	b	c	c
21.	**22.**	**23.**	**24.**	**25.**	**26.**	**27.**	**28.**	**29.**	**30.**
c	d	a	c	c	d	d	c	d	c
31.	**32.**	**33.**	**34.**	**35.**	**36.**	**37.**	**38.**	**39.**	**40.**
b	b	a	a	b	b	a	a	d	a
41.	**42.**	**43.**	**44.**	**45.**	**46.**	**47.**	**48.**	**49.**	**50.**
a	c	a	a	b	a	a	c	d	b
51.	**52.**	**53.**	**54.**	**55.**	**56.**	**57.**	**58.**	**59.**	**60.**
b	c	a	a	b	c	c	c	a	b
61.	**62.**	**63.**	**64.**	**65.**	**66.**	**67.**	**68.**	**69.**	**70.**
c	b	c	d	c	b	c	d	a	b
71.	**72.**	**73.**	**74.**	**75.**	**76.**	**77.**	**78.**	**79.**	**80.**
a,b,c	a,d	a,c,d	c,d	c,d	a,b,c	a,c,d	a,b	b,c,d	b,c
81.	**82.**	**83.**	**84.**	**85.**	**86.**	**87.**	**88.**	**89.**	**90.**
a	c	b	a	d	b	c	d	b	c
91.	**92.**	**93.**	**94.**	**95.**	**96.**	**97.**	**98.**	**99.**	**100.**
a	a	b	5	3	216	144	48	1648	394

SOLUTION

Multiple Choice Questions

1. **(c)** $\sum_{n=1}^{200} i^n = i + i^2 + i^3 + ... + i^{200} = \frac{i(1-i^{200})}{1-i}$

(since G.P.) $= \frac{i(1-1)}{1-i} = 0.$

2. **(d)** $i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n(1 + i + i^2 + i^3)$

$= i^n(1 + i - 1 - i) = o.$

Since the sum of four consecutive powers of i is always zero.

$\Rightarrow \quad i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0, \; n \in I.$

3. **(d)** Given that; $x - 3 = i \Rightarrow (x-3)^2 = i^2$

$\Rightarrow \quad x^2 - 6x + 10 = 0$

Now, $x^3 - 3x^2 - 8x + 15 = x(x^2 - 6x + 10) + 3(x^2 - 6x + 10) - 15$

$= 0 + 0 - 15 = -15.$

4. **(d)** $(1+i)^{2n} = ((1+i)^2)^n = (1 + i^2 + 2i)^n = (1 - 1 + 2i)^n = 2^n i^n$

$(1-i)^{2n} = ((1-i)^2)^n = (1 + i^2 - 2i)^n = (1 - 1 - 2i)^n = (-2)^n i^n$

$\therefore \quad \frac{2^n}{(1-i)2^n} + \frac{(1+i)^{2n}}{2^n} = \frac{2^n}{(-2)^n i^n} + \frac{2^n i^n}{2^n} = \frac{1}{(-1)^n i^n} + i^n$

$= \frac{1 + (-1)^n i^{2n}}{(-1)^n i^n} = \frac{1 + (-1)^n (i^2)^n}{(-1)^n i^n}$

5. **(d)** $\frac{1-2i}{2+i} + \frac{4-i}{3+2i}$

$= \frac{(1-2i)(3+2i) + (4-i)(2+i)}{(2+i)(3+2i)}$

$= \frac{50-120i}{65} = \frac{10}{13} - \frac{24}{13}i.$

6. **(d)** $\left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$

$= \left[\frac{1+2i}{1^2+2^2} + \frac{3-3i}{1^1+1^2}\right]\left[\frac{6-16+12i+8i}{2^2+4^2}\right]$

$= \left(\frac{2+4i+15-15i}{10}\right)\left(\frac{-1+2i}{2}\right)$

$= \frac{(17-11i)(-1+2i)}{20} = \frac{5+45i}{20} = \frac{1}{4} + \frac{9}{4}i.$

7. **(d)** Because, inequality is not applicable for a complex number.

8. **(d)** $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix}$

Applying $C_2 \to C_2 + 3i\,C_3$

$\begin{vmatrix} 6i & 0 & 1 \\ 4 & 0 & -1 \\ 20 & 0 & i \end{vmatrix} = 0 = 0 + 0\,i,$

Equating real and imaginary parts $x = 0, y = 0$

9. **(c)** Given equation $(x^4 + 2xi) - (3x^2 + yi)$

$= (3 - 5i) + (1 + 2yi)$

$\Rightarrow \quad (x^4 - 3x^2) + i(2x - 3y) = 4 - 5i$

Equating real and imaginary parts, we get

$x^4 - 3x^2 = 4 \qquad ... (i)$

and $2x - 3y = -5 \qquad ... (ii)$

Form (i) and (ii), we get $x = \pm 2$ and $y = 3, \frac{1}{3}$.

Put $x = 2, y = 3$ and then $x = -2, y = \frac{1}{3}$,

We see that they both satisfy the given equation.

10. **(b)** $\sqrt{-2}\sqrt{-3} = i\sqrt{2}\,i\sqrt{3}$

$= i^2\sqrt{6} = -\sqrt{6}$

11. **(c)** Since $\frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = i$

Therefore $\left(\frac{1+i}{1-i}\right)^{4n+1} = i^{4n+1} = ii^{4n} = i$

$(\because i^{4n} = 1)$

12. **(b)** If $(1-i)^n = 2^n$ . . . (i)

We know that if two complex numbers are equal, their module must also be equal, therefore from (i), we have

$|(1-i)^n| = |2^n|$

$\Rightarrow$ $|1-i|^n = |2|^n$, $(\because 2^n > 0)$

$\Rightarrow$ $[\sqrt{1^2+(-1)^2}]^n = 2^n \Rightarrow (\sqrt{2})^n = 2^n$

$\Rightarrow$ $2^{n/2} = 2^n \Rightarrow \frac{n}{2} = n \Rightarrow n = 0$

By inspection, $(1-i)^0 = 2^0 \Rightarrow 1 = 1$

13. **(b)** We have $(1+i)^{2n} = (1-i)^{2n}$

$\Rightarrow$ $\left(\frac{1+i}{1-i}\right)^{2n} = 1 \Rightarrow (i)^{2n} = 1 \Rightarrow (i)^{2n} = (-1)^2$

$\Rightarrow$ $(i)^{2n} = (i^2)^2 \Rightarrow (i)^{2n} = (i)^4 \Rightarrow 2n = 4 \Rightarrow n = 2$.

14. **(c)** $\frac{z_1}{z_2} = \frac{4+5i}{-3+2i} \times \frac{-3-2i}{-3-2i} = \frac{-12-8i-15i+10}{9-(2i)^2}$

$\frac{z_1}{z_2} = \frac{-2}{13} - i\left(\frac{23}{13}\right) = \left(\frac{-2}{13}, \frac{-23}{13}\right)$

15. **(b)** $z = \frac{1+2i}{1-i} \Rightarrow z = \frac{1+2i}{1-i} \times \frac{1+i}{1+i} = \frac{-1}{2} + i\frac{3}{2}$

This complex number will lie in the II quadrant.

16. **(b)** $(x+iy)^{1/3} = a - ib$

$x + iy = (a-ib)^3 = (a^3 - 3ab^2) + i(b^3 - 3a^2b)$

$\Rightarrow$ $x = a^3 - 3ab^2, y = b^3 - 3a^2b$

$\Rightarrow$ $\frac{x}{a} = a^2 - 3b^2, \frac{y}{b} = b^2 - 3a^2$

$\therefore$ $\frac{x}{a} - \frac{y}{b} = a^2 - 3b^2 - b^2 + 3a^2$

$\frac{x}{a} - \frac{y}{b} = 4(a^2 - b^2) = k(a^2 - b^2)$

$\therefore$ $k = 4$.

17. **(c)** Given that $\overline{(x+iy)(1-2i)} = 1+i$

$\Rightarrow$ $x - iy = \frac{1+i}{1+2i} \Rightarrow x + iy = \frac{1-i}{1-2i}$

18. **(b)** $\frac{2-3i}{4-i} = \frac{(2-3i)(4+i)}{(4-i)(4+i)} = \frac{8+3-12i+2i}{16+1}$

$= \frac{11-10i}{17}$

$\Rightarrow$ Conjugate $= \frac{11+10i}{17}$.

19. **(c)** $\{(1-\cos\theta) + i.2\sin\theta\}^{-1}$

$= \left\{2\sin^2\frac{\theta}{2} + i.4\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\}^{-1}$

$= \left(2\sin\frac{\theta}{2}\right)^{-1}\left\{\sin\frac{\theta}{2} + i.2\cos\frac{\theta}{2}\right\}^{-1}$

$= \left(2\sin\frac{\theta}{2}\right)^{-1} . \frac{1}{\sin\frac{\theta}{2} + i.2\cos\frac{\theta}{2}} \times \frac{\sin\frac{\theta}{2} - i.2\cos\frac{\theta}{2}}{\sin\frac{\theta}{2} - i.2\cos\frac{\theta}{2}}$

$= \frac{\sin\frac{\theta}{2} - i.2\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}\left(\sin^2\frac{\theta}{2} + 4\cos^2\frac{\theta}{2}\right)}$

Hence, real part

$= \frac{\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}\left(1+3\cos^2\frac{\theta}{2}\right)} = \frac{1}{2\left(1+3\cos^2\frac{\theta}{2}\right)} = \frac{1}{5+3\cos\theta}$.

20. **(c)** $\frac{1}{3+\sqrt{7}i} = \frac{1}{3+\sqrt{7}i} . \frac{3-\sqrt{7}i}{3-\sqrt{7}i} = \frac{3-\sqrt{7}i}{9+7}$

$= \frac{3-\sqrt{7}i}{16} = \frac{3}{16} - \frac{\sqrt{7}}{16}i.$

21. **(c)** $z = \frac{(1+i)(2+i)}{(3+i)} = \frac{1+3i}{3+i} \times \frac{3-i}{3-i} = \frac{3+4i}{5} \Rightarrow |z| = 1$

Short Trick: $|z| = \frac{|z_1||z_2|}{|z_3|} = \frac{\sqrt{2}.\sqrt{5}}{\sqrt{10}} = 1$

22. **(d)** $0 < x_1 < x_2, y_1 = y_2$ (Given)

$\Rightarrow$ $|z_1| = \sqrt{x_1^2 + y_1^2}, |z_2| = \sqrt{x_2^2 + y_2^2}$

$\Rightarrow$ $|z_2| > |z_1| \Rightarrow |z_3| = \frac{|z_1 + z_2|}{2}$

$\Rightarrow$ $= \sqrt{\left(\frac{x_1+x_2}{2}\right)^2 + \left(\frac{y_1+y_2}{2}\right)^2}$

$\Rightarrow$ $\sqrt{\left(\frac{x_1+x_2}{2}\right)^2 + y_1^2} < |z_2| > |z_1|$.

Hence, $|z_1| < |z_3| < |z_2|$

23. **(a)** $z = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = -i$

$\Rightarrow$ $\theta = \tan^{-1}\left(\frac{-1}{0}\right) = -\frac{\pi}{2}$

(Since z lies on negative imaginary axis)

24. **(c)** $\sin\frac{\pi}{5}+i(1-\cos\frac{\pi}{5})$

$=2\sin\frac{\pi}{10}.\cos\frac{\pi}{10}+i\,2\sin^2\frac{\pi}{10}$

$=2\sin\frac{\pi}{10}\left(\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}\right)$

For amplitude, $\tan\theta=\frac{\sin\frac{\pi}{10}}{\cos\frac{\pi}{10}}=\tan\frac{\pi}{10}\Rightarrow\theta=\frac{\pi}{10}$.

25. **(c)** $|z|=4$ and $arg\ z=\frac{5\pi}{6}=150°$,

Let $z=x+iy$, then $|z|=r=\sqrt{x^2+y^2}=4$

and $\theta=\frac{5\pi}{6}=150°$

$\therefore\quad x=r\cos\theta=4\cos150°=-2\sqrt{3}$

and $y=r\sin\theta=4\sin150°=4\cdot\frac{1}{2}=2.$

$\therefore\quad z=x+iy=-2\sqrt{3}+2i.$

Since $arg\ z=\frac{5\pi}{6}=150°$, here the complex number must lie in second quadrant, So (a) and (b) rejected.

Also, $|z|=4$, which satisfies **c.** only.

26. **(d)** $|z|\omega|=1$ . . .(i)

and $arg\left(\frac{z}{\omega}\right)=\frac{\pi}{2}\Rightarrow\frac{z}{\omega}=i\Rightarrow\left|\frac{z}{\omega}\right|=1$ . . .(ii)

From equation (i) and (ii), $|z|=|\omega|=1$

and $\frac{z}{\omega}+\frac{\bar{z}}{\bar{\omega}}=0;\quad z\bar{\omega}+\bar{z}\omega=0$

$\Rightarrow\quad\bar{z}\omega=-z\bar{\omega}=\frac{-z}{\omega}\bar{\omega}\omega$

$\Rightarrow\ \bar{z}\omega=-i|\omega|^2=-i.$

27. **(d)** $\sin x+i\cos2x$ and $\cos x-i\sin2x$ are conjugate to each other if $\sin x=\cos x$ and $\cos2x=\sin2x$

or $\tan x=1$

$\Rightarrow\quad x=\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4},......$ . . . (i)

and $\tan2x=1$

$\Rightarrow\quad 2x=\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4},........$

or $x=\frac{\pi}{8},\frac{5\pi}{8},\frac{9\pi}{8}\ ....$ . . . (ii)

28. **(c)** Let $z=x+iy$, then its conjugate $\bar{z}=x-iy$

Given that $z^2=(\bar{z})^2$

$\Rightarrow\quad x^2-y^2+2ixy=x^2-y^2-2ixy\Rightarrow\ 4ixy=0$

If $x\neq0$ then $y=0$ and if $y\neq0$ then $x=0$

29. **(d)** Let $z=x+iy$, so that $\bar{z}=x-iy$, therefore

$z^2+\bar{z}=0\Leftrightarrow(x^2-y^2+x)+i(2xy-y)=0$

Equating real and imaginary parts, we get

$x^2-y^2+x=0$ . . . (i)

and $2xy-y=0\ \Rightarrow y=0$ or $x=\frac{1}{2}$

If $y=0$, then (i) gives $x^2+x=0\Rightarrow x=0$ or $x=-1$

If $x=\frac{1}{2}$,

Then $x^2-y^2+x=0\Rightarrow y^2=\frac{1}{4}+\frac{1}{2}=\frac{3}{4}\Rightarrow y=\pm\frac{\sqrt{3}}{2}$

Hence, there are four solutions in all.

30. **(c)** $z=\frac{(2+i)^2}{3+i}=\frac{3+4i}{3+i}\times\frac{3-i}{3-i}=\frac{13}{10}+i\frac{9}{10}$

Conjugate $=\frac{13}{10}-i\frac{9}{10}$.

31. **(b)** $|z_1+z_2|^2+|z_1-z_2|^2$

$=(x_1+x_2)^2+(y_1+y_2)^2+(x_1-x_2)^2+(y_1-y_2)^2$

$=2(x_1^2)+2(y_1^2)+2(x_2^2)+2(y_2^2)=2|z_1|^2+2|z_2|^2$

32. **(b)** $\left|z+\frac{2}{z}\right|=2\Rightarrow|z|-\frac{2}{|z|}\leq2\Rightarrow|z|^2-2|z|-2\leq0$

$|z|\leq\frac{2\pm\sqrt{4+8}}{2}\leq1\pm\sqrt{3}$.

Hence max. value of $|z|$ is $1+\sqrt{3}$

33. **(a)** Let $z_1=a+ib=(a,b)$ and $z_2=c-id=(c,-d)$

Where $a>0$ and $d>0$ . . . (i)

Then $|z_1|=|z_2|\Rightarrow a^2+b^2=c^2+d^2$

Now $\frac{z_1+z_2}{z_1-z_2}=\frac{(a+ib)+(c-id)}{(a+ib)-(c-id)}$

$=\frac{[(a+c)+i(b-d)][(a-c)-i(b+d)]}{[(a-c)+i(b+d)][(a-c)-i(b+d)]}$

$=\frac{(a^2+b^2)-(c^2+d^2)-2(ad+bc)i}{a^2+c^2-2ac+b^2+d^2+2bd}$

$\frac{-(ad+bc)i}{a^2+b^2-ac+bd}$ [using (i)]

$\therefore\quad\frac{(z_1+z_2)}{(z_1-z_2)}$ is purely imaginary.

However if $ad+bc=0$, then $\frac{(z_1+z_2)}{(z_1-z_2)}$ will be equal to zero. According to the conditions of the equation, we can have $ad+bc=0$

Assume any two complex numbers satisfying both conditions *i.e.*, $z_1 \ne z_2$ and $|z_1| = |z_2|$

Let $z_1 = 2+i, z_2 = 1-2i, \therefore \frac{z_1+z_2}{z_1-z_2} = \frac{3-i}{1+3i} = -i$

Hence the result.

34. **(a)** If $|z_1|=1$ and $|z_2|=1$, then $|z_1 z_2| = |z_1||z_2| = 1.1 = 1$

35. **(b)** Given that $\bar{z} = \frac{1}{z} \Rightarrow z\bar{z} = 1 \Rightarrow |z|^2 = 1 \Rightarrow |z| = 1$

36. **(b)** Let $z = r(\cos\theta + i\sin\theta)$.

Then $\left|z+\frac{1}{z}\right| = 1 \Rightarrow \left|z+\frac{1}{z}\right|^2 = 1$

$\Rightarrow \left|r(\cos\theta+i\sin\theta)+\frac{1}{r}(\cos\theta-i\sin\theta)\right|^2 = 1$.

$\Rightarrow \left(r+\frac{1}{r}\right)^2\cos^2\theta + \left(r-\frac{1}{r}\right)^2\sin^2\theta = 1$

$\Rightarrow r^2+\frac{1}{r^2}+2\cos 2\theta = 1$

Since $|z| = r$ is maximum, therefore $\frac{dr}{d\theta} = 0$

Differentiating (i) w.r.t. θ, we get

$2r\frac{dr}{d\theta} - \frac{2}{r^3}\frac{dr}{d\theta} - 4\sin 2\theta = 0$

Putting $\frac{dr}{d\theta} = 0$, we get $\sin 2\theta = 0 \Rightarrow \theta = 0$ or $\frac{\pi}{2}$

$\Rightarrow$ z is purely imaginary or purely real.

($\because \theta = 0$ is not given)

37. **(a)** $\left(\frac{3+2i}{3-2i}\right) = \left(\frac{3+2i}{3-2i}\right)\left(\frac{3+2i}{3+2i}\right)$

$= \frac{9-4+12i}{13} = \frac{5}{13} + i\left(\frac{12}{13}\right)$

Modulus $= \sqrt{\left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2} = 1$.

38. **(a)** $|z| = 1 \Rightarrow |x+iy| = 1 \Rightarrow x^2+y^2 = 1$

$\omega = \frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$

$= \frac{(x^2+y^2-1)}{(x+1)^2+y^2} + \frac{2iy}{(x+1)^2+y^2} = \frac{2iy}{(x+1)^2+y^2}$

$(\because x^2+y^2 = 1)$

$\therefore$ $\text{Re}(\omega) = 0$.

39. **(d)** Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore$ $|z_1+z_2| = [(r_1\cos\theta_1 + r_2\cos\theta_2)^2$

$+(r_2\sin\theta_1 + r_2\sin\theta_2)^2]^{1/2}$

$= [r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1-\theta_2)]^{1/2} = [(r_1+r_2)^2]^{1/2}$

$(\because |z_1+z_2| = |z_1| + |z_2|)$

Therefore $\cos(\theta_1-\theta_2) = 1 \Rightarrow \theta_1-\theta_2 = 0 \Rightarrow \theta_1 = \theta_2$

Thus $arg\,(z_1) - arg(z_2) = 0$.

$|z_1+z_2| = |z_1| + |z_2| \Rightarrow z_1, z_2$ lies on same straight line.

$\therefore$ $arg\, z_1 = arg\, z_2 \Rightarrow arg\, z_1 - arg\, z_2 = 0$

40 **(a)** $|z| = 5$, $\therefore \sqrt{3-4i}$

$= \pm\left(\sqrt{\frac{5+3}{2}} - i\sqrt{\frac{5-3}{2}}\right) = \pm\ (2-i)$

41 **(a)** $z = 2i = a+bi \Rightarrow a = 0, b = 2, |z| = 2$

$\therefore$ $\sqrt{z} = \pm\left(\sqrt{\frac{2+0}{2}} + i\sqrt{\frac{2-0}{2}}\right) = \pm(1+i)$

It is always better to square the options rather than finding the square root.

42. **(c)** Here $-1+\sqrt{-3} = re^{i\theta}$

$\Rightarrow -1+i\sqrt{3} = re^{i\theta} = r\cos\theta + ir\sin\theta$

Equating real and imaginary part,

We get $r\cos\theta = -1$ and $r\sin\theta = \sqrt{3}$

Hence $\tan\theta = -\sqrt{3}$

$\Rightarrow$ $\tan\theta = \tan\frac{2\pi}{3} \Rightarrow \theta = \frac{2\pi}{3}$.

43. **(a)** $e^{e^{i\theta}} = e^{(\cos\theta+i\sin\theta)} = e^{\cos\theta}.e^{i\sin\theta}$

$= e^{\cos\theta}[\cos(i\sin\theta) + i\sin(\sin\theta)]$

$\therefore$ Real part of $e^{e^{i\theta}}$ is $e^{\cos\theta}[\cos(\sin\theta)]$.

44. **(a)** Let $x = \cos\theta + i\sin\theta = e^{i\theta}$

then $x^n + \frac{1}{x^n} = e^{in\theta} + \frac{1}{e^{in\theta}} = e^{in\theta} + e^{-in\theta}$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta = 2\cos n\theta$.

45. **(b)** Let $A = i^i$ then $\log A = \log i^i = i\log i$

$\Rightarrow$ $\log A = i\log(0+i)$

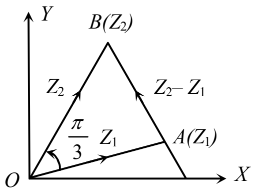
$\Rightarrow$ $\log A = i[\log 1 + i\pi/2]$ $(\because |i| = 1$ and $arg\,(i) = \pi/2)$

$\log A = i[0+i\pi/2] = -\pi/2$

$\Rightarrow$ $A = e^{-\pi/2}$.

46. **(a)** It is a fundamental concept.

47. (a) Let OA, OB be the sides of an equilateral $\Delta\ OAB$ and OA, OB represent the complex numbers or vectors z_1, z_2 respectively.



From the equilateral $\Delta\ OAB$, $\overrightarrow{AB} = Z_2 - Z_1$

$$\therefore \quad arg\left(\frac{z_2 - z_1}{z_2}\right) = arg(z_2 - z_1) - arg z_2 = \frac{\pi}{3}$$

and $arg\left(\frac{z_2}{z_1}\right) = arg(z_2) - arg(z_1) = \frac{\pi}{3}$

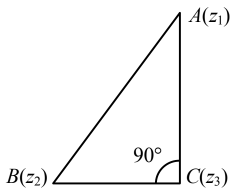
Also, $\left|\frac{z_2 - z_1}{z_2}\right| = 1 = \left|\frac{z_2}{z_1}\right|$, since triangle is equilateral.

Thus the vectors $\frac{z_2 - z_1}{z_2}$ and $\frac{z_2}{z_1}$ have same modulus and same argument, which implies that the vectors are equal, that is $\frac{z_2 - z_1}{z_2} = \frac{z_2}{z_1} \Rightarrow z_1 z_2 - z_1^2 = z_2^2$

$\Rightarrow \quad z_1^2 + z_2^2 = z_1 z_2$.

48. (c) It is a fundamental concept.

49. (d)



$BC = AC$ and $\angle C = \pi/2$

By rotation about C in anticlockwise sense $CB = CAe^{i\pi/2}$

$\Rightarrow \quad (z_2 - z_3) = (z_1 - z_3)e^{i\pi/2} = i(z_1 - z_3)$

$\Rightarrow \quad (z_2 - z_3)^2 = -(z_1 - z_3)^2$

$\Rightarrow \quad z_2^2 + z_3^2 - 2z_2 z_3 = -z_1^2 - z_3^2 + 2z_1 z_3$

$\Rightarrow \quad z_1^2 + z_2^2 - 2z_1 z_2 = 2z_1 z_3 + 2z_2 z_3 - 2z_3^2 - 2z_1 z_2$

$\Rightarrow \quad (z_1 - z_2)^2 = 2[(z_1 z_3 - z_3^2) - (z_1 z_2 - z_2 z_3)]$

$\Rightarrow \quad (z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

50. (b) Let $z_1 = 1 + 3i$, $z_2 = 5 + i$ and $z_3 = 3 + 2i$.

Then area of triangle $A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 0$,

Hence z_1, z_2 and z_3 are collinear.

51. (b) Let $z = x + iy$ $z + iz = (x - y) + i(x + y)$, $iz = -y + ix$

If A denotes the area of the triangle formed by z, $z + iz$ and iz,

then $A = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ x - y & x + y & 1 \\ -y & x & 1 \end{vmatrix}$

(Applying transformation $R_2 \to R_2 - R_1 - R_3$)

We get $A = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 0 & 0 & -1 \\ -y & x & 0 \end{vmatrix}$

$= \frac{1}{2}(x^2 + y^2) = \frac{1}{2}|z|^2$.

52. (c) We have $\frac{z - 1}{z + 1} = \frac{x + iy - 1}{x + iy + 1} = \frac{(x^2 + y^2 - 1) + 2iy}{(x + 1)^2 + y^2}$

$\Rightarrow \quad arg\ \frac{z - 1}{z + 1} = \tan^{-1}\frac{2y}{x^2 + y^2 - 1}$

Hence $\tan^{-1}\frac{2y}{x^2 + y^2 - 1} = \frac{\pi}{3}$

$\Rightarrow \quad \frac{2y}{x^2 + y^2 - 1} = \tan\frac{\pi}{3} = \sqrt{3}$

$\Rightarrow \quad x^2 + y^2 - 1 = \frac{2}{\sqrt{3}}y$

$\Rightarrow \quad x^2 + y^2 - \frac{2}{\sqrt{3}}y - 1 = 0$, which is obviously a circle.

53. (a) $\log_{1/3}|z + 1| > \log_{1/3}|z - 1|$

$\Rightarrow \quad |z + 1| < |z - 1|$

$\Rightarrow \quad x^2 + 1 + 2x + y^2 < x^2 + 1 - 2x + y^2$

$\Rightarrow \quad x < 0$

$\Rightarrow \quad$ Re$(z) < 0$.

54. (a) $\tan(\alpha + i\beta) = x + iy$

$\therefore \quad \tan(\alpha - i\beta) = x - iy$ (conjugate)

α is a constant and β is known to be eliminated

$\tan 2\alpha = \tan(\overline{\alpha + i\beta} + \overline{\alpha - i\beta})$

$\Rightarrow \quad \tan 2\alpha = \frac{x + iy + x - iy}{1 - (x^2 + y^2)}$

$\Rightarrow \quad 1 - (x^2 + y^2) = 2x\cot 2\alpha$

$\therefore \quad x^2 + y^2 + 2x\cot 2\alpha = 1$.

55. (b) $\frac{1 - i}{1 + i} \times \frac{1 - i}{1 - i} = -i = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$

$\Rightarrow \quad (-i)^{100} = \cos(-50\pi) + i\sin(-50\pi) = 1 + i(0)$

$\Rightarrow \quad a = 1,\ b = 0$

56. **(c)** $x_1.x_2.x_3.....\text{upto}\,\infty = \left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

$$\left(\cos\frac{\pi}{2^2}+i\sin\frac{\pi}{2^2}\right)....\ \cos\left(\frac{\pi}{2}+\frac{\pi}{2^2}+...\right)+i\sin\left(\frac{\pi}{2}+\frac{\pi}{2^2}+....\right)$$

$$= \cos\left(\frac{\frac{\pi}{2}}{1-\frac{1}{2}}\right)+i\sin\left(\frac{\frac{\pi}{2}}{1-\frac{1}{2}}\right) = \cos\pi+i\sin\pi=-1$$

57. **(c)** $(3+5\omega+3\omega^2)^2\ +(3+3\omega+5\omega^2)^2$

$$=(3+3\omega+3\omega^2+2\omega)^2+(3+3\omega+3\omega^2+2\omega^2)^2$$

$$=(2\omega)^2+(2\omega^2)^2=4\omega^2+4\omega^4=4(-1)=-4$$

$(\because 1+\omega+\omega^2=0, \omega^3=1)$

58. **(c)** Given equation is

$$4+5\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{334}+3\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{365}$$

$$=4+5\omega^{334}+3\omega^{365}\ =4+5\omega+3\omega^2$$

$$=1+2w=1+2\left(\frac{-1+i\sqrt{3}}{2}\right)\ =i\sqrt{3}$$

59. **(a)** $2(\omega+1)(\omega^2+1)+3(2\omega+1)(2\omega^2+1)+.....$

$$+(n+1)(n\omega+1)(n\omega^2+1)\ =\sum_{r=1}^{n}(r+1)(r\omega+1)(r\omega^2+1)$$

$$=\sum_{r=1}^{n}(r+1)(r^2\omega^3+r\omega+r\omega^2+1)$$

$$=\sum_{r=1}^{n}(r+1)(r^2-r+1)$$

$$=\sum_{r=1}^{n}(r^3-r^2+r+r^2-r+1)$$

$$=\sum_{r=1}^{n}(r^3)+\sum_{r=1}^{n}(1)=\left[\frac{n(n+1)}{2}\right]^2+n.$$

60. **(b)** Given equation $x^4-1=0$

$\Rightarrow\ (x^2-1)(x^2+1)=0$

$\Rightarrow\ x^2=1$ and $x^2=-1 \Rightarrow x=\pm1,\pm i$

61. **(c)** $1+i\sqrt{3}\ =2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=2\left[\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right]=2e^{i\pi/3}$

$\therefore\ (1+i\sqrt{3})^9=(2e^{i\pi/3})^9\ =2^9.e^{i(3\pi)}$

$=2^9(\cos 3\pi+i\sin 3\pi)=-2^9$

$\therefore\ a+ib=(1+i\sqrt{3})^9\ =-2^9$

$\therefore\ b=0$.

62. **(b)** Let $z=e^{e^{-i\theta}}\ =e^{\cos\theta-i\sin\theta}\ =e^{\cos\theta}e^{-i\sin\theta}$

$z=e^{\cos\theta}[\cos(\sin\theta)-i\sin(\sin\theta)]$

$z=e^{\cos\theta}\cos(\sin\theta)-ie^{\cos\theta}\sin(\sin\theta)$

$amp(z)=\tan^{-1}\left[-\frac{e^{\cos\theta}\sin(\sin\theta)}{e^{\cos\theta}\cos(\sin\theta)}\right]$

$=\tan^{-1}[\tan(-\sin\theta)]=-\sin\theta$.

63. **(c)** $z=\frac{1+i\sqrt{3}}{\sqrt{3}+i}\ \Rightarrow z=\frac{1+i\sqrt{3}}{\sqrt{3}+i}\times\frac{\sqrt{3}-i}{\sqrt{3}-i}$

$\Rightarrow\ z=\frac{\sqrt{3}+3i-i+\sqrt{3}}{3+1}\ =\frac{2(\sqrt{3}+i)}{4}$

$\Rightarrow\ z=\frac{\sqrt{3}+i}{2}=\left[\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right]$

Now $\bar{z}=\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}$

$\Rightarrow\ (\bar{z})^{100}\ =\left[\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right]^{100}$

$\Rightarrow\ (\bar{z})^{100}\ =\cos\frac{50\pi}{3}-i\sin\frac{50\pi}{3}\ =\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}$

$(\bar{z})^{100}$ lies in III quadrant.

64. **(d)** $x^2-\sqrt{3}x+1=0\ \ \Rightarrow\ x=\frac{\sqrt{3}\pm\sqrt{3-4}}{2}$

$\Rightarrow\ x=\frac{\sqrt{3}\pm i}{2}=\frac{\sqrt{3}}{2}\pm\frac{i}{2}$

$\Rightarrow\ x=\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)$ [Taking $+ve$ sign]

65. **(c)** $\tan^{-1}\left(\frac{5i}{3}\right)=i\tan^{-1}\left(\frac{5}{3}\right)=\frac{i}{2}\log\left(\frac{\frac{5}{3}+1}{\frac{5}{3}-1}\right)$

$\text{Im}\left(\tan^{-1}\left(\frac{5i}{3}\right)\right)=\frac{1}{2}\log 4=\frac{1}{2}.2\log 2=\log 2$.

66. **(b)** Let $z=i\log\left(\frac{x-i}{x+i}\right)\Rightarrow\frac{z}{i}=\log\left(\frac{x-i}{x+i}\right)$

$\Rightarrow\ \frac{z}{i}=\log\left[\frac{x-i}{x+i}\times\frac{x-i}{x-i}\right]=\log\left[\frac{x^2-1-2ix}{x^2+1}\right]$

$\Rightarrow\ \frac{z}{i}=\log\left[\frac{x^2-1}{x^2+1}-i\frac{2x}{x^2+1}\right]$ . . . (i)

$\because\ \log(a+ib)=\log(re^{i\theta})=\log r+i\theta$

$=\log\sqrt{a^2+b^2}+i\tan^{-1}(b/a)$

Hence, $\frac{z}{i}=\log\sqrt{\left(\frac{x^2-1}{x^2+1}\right)^2+\left(\frac{-2x}{x^2+1}\right)^2}+i\tan^{-1}\left(\frac{-2x}{x^2-1}\right)$

[by eq^n. (i)]

$\frac{z}{i} = \log \frac{\sqrt{x^4 + 1 - 2x^2 + 4x^2}}{(x^2+1)^2} + i\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$= \log 1 + i(2\tan^{-1} x)$

$= 0 + i(2\tan^{-1} x)$

$\therefore \quad z = i^2 2\tan^{-1} x = -2\tan^{-1} x = \pi - 2\tan^{-1} x$.

67. **(c)** $e^{iA}.e^{iB}.e^{iC} = e^{iA+iB+iC} = e^{i(A+B+C)} = e^{i\pi}$

$[\therefore A + B + C = \pi]$

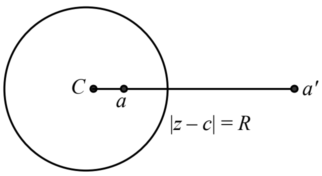
$= \cos\pi + i\sin\pi = (-1) + i(0) = -1$.

68. **(d)** $z = \frac{7-i}{3-4i} \times \frac{3+4i}{3+4i}$

$= \frac{21 + 25i + 4}{16+9} = \frac{25(1+i)}{25} = (1+i)$

$z^{14} = (1+i)^{14} = [(1+i)^2]^7 = (2i)^7 = 2^7 i^7 = -2^7 i$.

69. **(a)**



Let a' be the inverse point of a with respect to the circle $|z-c| = R$, then by definition the points c, a, a' are collinear.

We have, $arg(a'-c) = arg(a-c) = -arg(\overline{a} - \overline{c})$

$(\because arg\overline{z} = -arg z)$

$\Rightarrow \quad arg(a'-c) + arg(\overline{a} - \overline{c}) = 0$

$\Rightarrow \quad arg\{(a'-c)(\overline{a} - \overline{c})\} = 0$

$\therefore \quad (a'-c)(\overline{a} - \overline{c})$ is purely real and positive.

By definition $|a'-c||a-c| = R^2$ $\quad (\because CP.CQ = r^2)$

$\Rightarrow \quad |a'-c||\overline{a} - \overline{c}| = R^2$ $\quad (\because |z| = |\overline{z}|)$

$\Rightarrow \quad |(a'-c)(\overline{a} - \overline{c})| = R^2$

$\Rightarrow \quad (a'-c)(\overline{a} - \overline{c}) = R^2$

$\{\because (a'-c)(\overline{a} - \overline{c})$ is purely real and positive$\}$

$\Rightarrow \quad a' - c = \frac{R^2}{\overline{a} - \overline{c}}$. Therefore, the inverse point a' of a point a, $a' = c + \frac{R^2}{\overline{a} - \overline{c}}$.

70. **(b)** Projection of z_1 on $z_2 = \frac{z_1 o z_2}{|z_2|}$

$= \frac{a_1a_2 + b_1b_2}{\sqrt{a_2^2 + b_2^2}} = \frac{1}{\sqrt{10}}$.

NCERT Exemplar Problems

More than One Answer

71. **(a, b, c)** Since, $z_1 = a + ib$ and $z_2 = c + id$

$\Rightarrow \quad |z_1|^2 = a^2 + b^2 = 1$ and $|z_2|^2 = c^2 + d^2 = 1$ $\quad \ldots(i)$

$(\because |z_1| = |z_2| = 1)$

Also, $\text{Re}(z_1\overline{z_2}) = 0 \Rightarrow ac + bd = 0$

$\Rightarrow \quad \frac{a}{b} = -\frac{d}{c} = \lambda$ (say) $\quad \ldots(ii)$

From Eqs. (*i*) and (*ii*), $b^2\lambda^2 + b^2 = c^2 + \lambda^2c^2$

$\Rightarrow \quad b^2 = c^2$ and $a^2 = d^2$

Also, given $w_1 = a + ic$ and $w_2 = b + id$

Now, $|w_1| = \sqrt{a^2 + c^2} = \sqrt{a^2 + b^2} = 1$

$|w_2| = \sqrt{b^2 + d^2} = \sqrt{a^2 + b^2} = 1$

and $\text{Re}(w_1\overline{w}_2) = ab + cd = (b\lambda)b + c(-\lambda c)$ [From Eq. (*i*)]

$= \lambda(b^2 - c^2) = 0$

72. **(a, d)** Given, $|z_1| = |z_2|$,

Now, $\frac{z_1 + z_2}{z_1 - z_2} \times \frac{\overline{z}_1 - \overline{z}_2}{z_1 - z_2} = \frac{z_1\overline{z}_1 - z_1\overline{z}_2 + z_2\overline{z}_1 - z_2\overline{z}_2}{|z_1 - z_2|^2}$

$= \frac{|z_1|^2 + (z_2\overline{z}_1 - z_1\overline{z}_2) - |z_2|^2}{|z_1 - z_2|^2} = \frac{z_2\overline{z}_1 - z_1\overline{z}_2}{|z_1 - z_2|^2}$

$(\because |z_1|^2 = |z_2|^2)$

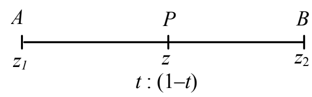
As, we know $z - \overline{z} = 2i\,\text{Im}(z)$

$\therefore \quad z_2\overline{z}_1 - z_1\overline{z}_2 = 2i\,\text{Im}(z_2\overline{z}_1)$ $\quad z_2\overline{z}_1 - z_1\overline{z}_2 = 2i\,Im(z_2\overline{z}_1)$

$\therefore \quad \frac{z_1 + z_2}{z_1 - z_2} = \frac{2i\,\text{Im}(z_2\overline{z}_1)}{|z_1 - z_2|^2}$

Which is purely imaginary or zero.

73. **(a, c, d)** Given, $z = \frac{(1-t)z_1 + tz_2}{(l-t) + t}$



Clearly, z divides z_1 and z_2 in the ration of $t : (1-t), 0 < t < 1$

$\Rightarrow \quad AP + BP = AB$ ie, $|z - z_1| + |z - z_2| = |z_1 - z_2|$

$\Rightarrow$ Option (a) is true.

and $\arg(z - z_1) = \arg(z_2 - z) = \arg(z_2 - z_1)$

$\Rightarrow$ (b) is false and (d) is true. Also, $\arg(z - z_1) = \arg(z_2 - z_1)$

$\Rightarrow \quad \arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0$

$\therefore \quad \frac{z-z_1}{z_2-z_1}$ is purely real.

$\Rightarrow \quad \frac{z-z_1}{z_2-z_1}=\frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}$ or $\begin{vmatrix} z-z_1 & \bar{z}-\bar{z}_1 \\ z_2-z_1 & \bar{z}_2-\bar{z}_1 \end{vmatrix}=0$

74. **(c, d)** It is the simple representation of points on argand plane and to find the angle between the points.

Here, $P=W^n=\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^n=\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6}$

$H_1=\left\{Z\in C:\text{Re}(z)>\frac{1}{2}\right\}$

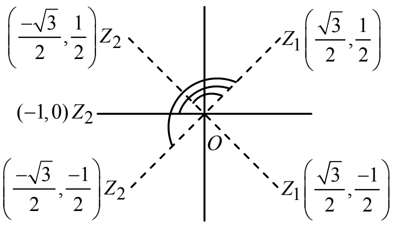
$\therefore \quad P\cap H_1$ represents those points for which $\cos\frac{n\pi}{6}$ is +ve

$\therefore \quad$ It belongs to I or IV quadrant

$\Rightarrow \quad z_1=P\cap H_1=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$

Or $\quad \cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}$

$\therefore \quad z_1=\frac{\sqrt{3}}{2}+\frac{i}{2}$ or $\frac{\sqrt{3}}{2}-\frac{i}{2}$ $\quad \ldots(i)$



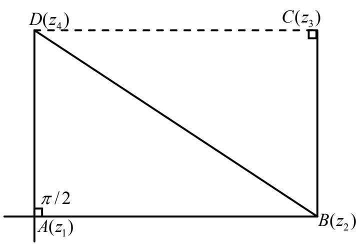
Similarly, $z_2=P\cap H_2$ *i.e.*, those points for which $\cos\frac{n\pi}{6}<0$

$\therefore \quad z_2=\cos\pi+i\sin\pi,\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6},\frac{\cos 7\pi}{6}+i\sin\frac{7x}{6}$

$\Rightarrow \quad z_2=-1,\frac{-\sqrt{3}}{2}+\frac{i}{2},\frac{-\sqrt{3}}{2}-\frac{i}{2}$

Thus, $\angle z_1Oz_2=\frac{2\pi}{3},\frac{5\pi}{6},\pi$

75. **(c, d)** $amp\left(\frac{z_4-z_1}{z_2-z_1}\right)=\frac{\pi}{2}$ but $z_1-z_4=z_2-z_3$



and $\quad z_3-z_4=z_2-z_1$ $amp\left(\frac{-(z_2-z_3)}{z_3-z_4}\right)=\frac{\pi}{2}$

$\Rightarrow \quad amp\left(\frac{z_2-z_3}{z_4-z_3}\right)=\frac{\pi}{2}.$

$\therefore \quad ABCD$ is rectangle and cyclic quadrilateral ($\because A+C=180°$)

76. **(a, b, c)** Since z_1 and z_2 lie on $|z|=1$ and $|z|=2$,

then $|z_1|=1$ and $|z_2|=2$

(a) $|2z_1+z_2|\le 2|z_1|+|z_2|=2\cdot 1+2\le 4$

max $|2z_1+z_2|=4$

(b) $|z_1-z_2|\ge||z_1|-|z_2||=|1-2|=1$

$\therefore \quad |z_1-z_2|\ge 1$ min $|z_1-z_2|=1$

(c) $\left|z_2+\frac{1}{z_1}\right|\ge|z_2|+\left|\frac{1}{z_1}\right|=|z_2|+\frac{1}{|z_1|}=2+1=3$

$\therefore \quad \left|z_2+\frac{1}{z_1}\right|\le 3.$

77. **(a, c, d)** $\alpha z^2+z+\bar{\alpha}=0$

Let $z=\alpha$ be a real root,

Then $\alpha a^2+a+\bar{\alpha}=0$ $\quad \ldots(i)$

and let $\alpha=p+iq$

$\therefore \quad (p+iq)a^2+\alpha+p-iq=0$

$\Rightarrow \quad pa^2+a+p=0$ and $a^2q-q=0$

$\therefore \quad a=\pm 1$ ($\because q\ne 0$)

$\therefore \quad$ From (i) $\alpha\pm 1+\bar{\alpha}=0$ Also $|a|=1$.

78. **(a, b)** $a_0z^4+a_1z^3+a_2z^2+a_3z+a_4=0$

Taking conjugate on both sides.

$a_0(\bar{z})^4+a_1(\bar{z})^3+a_2(\bar{z})^2+a_3(\bar{z})+a_4=0$

$\therefore \quad \bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4$ are the roots of the equation if z_1 is real, then $z_1=\bar{z}_1$ and is z_1 if non real, then $\bar{z}_1$ is also root because imaginary roots occur in conjugate pair.

79. **(b, c, d)** $\frac{2-i}{3+i}=\frac{(2-i)(3-i)}{10}=\frac{5+5i}{10}=\frac{1}{2}-\frac{i}{2}$

ie, $\left(\frac{1}{2},-\frac{1}{2}\right)$ and $z(1+i)=\bar{z}(i-1)$

$\Rightarrow \quad (z+\bar{z})+i(z-\bar{z})=0$

$\Rightarrow \quad \left(\frac{z+\bar{z}}{2}\right)+i\left(\frac{z-\bar{z}}{2}\right)=0$

$\Rightarrow \quad x+i(iy)=0$ $(z=x+iy)$

$\Rightarrow \quad x-y=0$

$\therefore$ Reflection of $\left(\frac{1}{2}, -\frac{1}{2}\right)$ w.r.t. $y = x$ is $\left(-\frac{1}{2}, \frac{1}{2}\right)$

ie. $-\frac{1}{2} + \frac{i}{2} = \frac{-1+i}{2}$ [Alternate (b)]

$= \frac{i^2 + i}{2} = \frac{i(i+1)}{2}$ [Alternate (c)]

$= \frac{i(1+i)^2}{2(1+i)} = \frac{i(1+i^2+2i)}{2(1+i)}$

$= \frac{i^2}{1+i} = \frac{-1}{1+i}$ [Alternate (d)]

80. **(b, c)** $z^3 + (1+i)z^2 + (1+i)z + i = 0$

$\Rightarrow$ $z^2(z+i) + (z+i) + (z+i) = 0$

$\Rightarrow$ $(z+i)(z^2 + z + 1) = 0$

$\Rightarrow$ $(z+i)(z-\omega)(z-\omega^2) = 0$

$\therefore$ $z = -i, \omega, \omega^2$

Now in $z^{1993} + z^{1994} + 1$ put $z = -i, \omega, \omega^2$

Then $(-i)^{1993} + (-i)^{1994} + 1 = -i - 1 + 1 = -i \neq 0$

and $(\omega)^{1993} + \omega^{1994} + 1 = \omega + \omega^2 + 1 = 0$

and $(\omega^2)^{1993} + (\omega^2)^{1994} + 1 = \omega^2 + \omega + 1 = 0$

Hence ω and ω^2 are common roots.

Assertion and Reason

81. **(a)** $0 \leq |z_1 + z_2 + z_3|^2$

$\Rightarrow$ $0 \leq |z_1|^2 + |z_2|^2 + |z_3|^2 + 2\ \text{Re}(z_2\bar{z}_3 + z_3\bar{z}_1 + z_1\bar{z}_2)$

$\Rightarrow$ $\text{Re}(z_2\bar{z}_3 + z_3\bar{z}_1 + z_1\bar{z}_2) \geq 3/2$ $[\because |z_1| = |z_2| = |z_3| = 1]$

Next, $|z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 - z_2|^2$

$= 2(|z_1|^2 + |z_2|^2 + |z_3|^2) - 2\text{Re}(z_2\bar{z}_3 + z_3\bar{z}_1 + z_1\bar{z}_2)$

$\leq 2(1+1+1) + 2(3/2) = 9$

Maximum value is obtained when $z_1 = 1, z_2 = \omega, z_3 = \omega^2$, where ω is a cube root of unity.

82. **(c)** Suppose $|z| < 1$ and $z^7 + 2x + 3 = 0$,

Then $3 = |-3| = |z^7 + 2z| \leq |z^7| + 2|z|$

$\Rightarrow$ $3 \leq |z|^7 + 2|z| < 1 + 2(1) = 3.$

A contradiction. Next, suppose that $|z| \geq 3/2$ and $z^6 + 2z + 3 = 0$ then ω $1/z$ satisfies the equation $1 + 2\omega^6 + 3\omega^7 = 0$

Now, $1 = |-1| = |2\omega^6 + 3\omega^7| \geq 2|\omega|^6 + 3|\omega|^7$. A contradiction.

Thus if z satisfies the equation $z^7 + 2z + 3 = 0$ then $1 \leq |z| < 3/2$. eason is false, as the given relation implies $z^3 - 3z^2 + 1 = 0$ which is satisfied by just 3 values of z where as $1 < |z| \leq 3/2$ contains infinite number of points.

83. **(b)** It is easy to show $A^2 = \begin{pmatrix} y & y & y \\ y & y & y \\ y & y & y \end{pmatrix} = O$

Where $y = 1 + \omega + \omega^2 = 0$.

$\therefore$ Assertion is true. Using $C_1 \to C_1 + C_2 + C_3$,

We get $\Delta = x\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & x+\omega^2 & 1 \\ 1 & 1 & x+\omega \end{vmatrix} = x\begin{vmatrix} 1 & \omega & \omega^2 \\ 0 & x+\omega^2-\omega & 1-\omega^2 \\ 0 & 1-\omega & x+\omega-\omega^2 \end{vmatrix}$

$= x[(x+\omega^2-\omega)(x+\omega-\omega^2) - (1-\omega)(1-\omega^2)]$

$= x[x^2 - (\omega-\omega^2)^2 - \{1-\omega-\omega^2+1\}] = x^3$

$\therefore$ Reason is also true but is not the correct reason for the Assertion.

84. **(a)** For Reason, note that $\omega^k = \bar{\omega}^k = 1$ if k is a multiple of 3, and if k is not a multiple of 3, then one of $\omega^k, \bar{\omega}^k$ equal ω and other equals ω^2. Next, note that roots of $z^2 - z + 1 = 0$ are $-\omega$ and $-\omega^2 = -\bar{\omega} = -1/\omega$.

Now, use $(z^k + z^{-k})^2 = [(-\omega)^k + (\bar{\omega}^k)]^2$

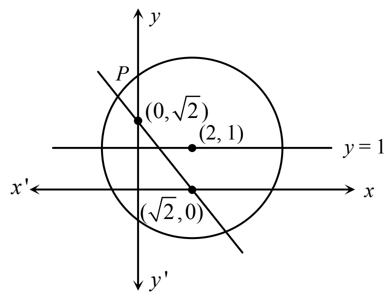
$= \begin{cases} 1 & \text{if } k \text{ is not a multiple of 3} \\ 4 & \text{if } k \text{ is a multiple of 3} \end{cases}$

85. **(d)** $f(\theta)^2 = \frac{4}{(3+\sin\theta)^2 + \cos^2\theta} = \frac{4}{10 + 6\sin\theta} = \frac{2}{5 + 3\sin\theta}$

$\therefore$ $f(\theta)^2$ is maximum when $\cos\theta = -1$, and minimum when $\cos\theta = 1$. Thus, $1/2 \leq f(\theta) \leq 1$.

Comprehension Based

86. **(b)**



Let $z = x + iy$

Set A corresponds to the region $y \geq 1$ . . .(*i*)

Set B consists of points lying on the circle, centre at (2, 1) and radius 3.

i.e., $x^2 + y^2 - 4x - 2y = 4$ . . .(*ii*)

Set C consists of points lying on the $x + y = \sqrt{2}$ . . .(*iii*)

Clearly, there is only one point of intersection of the line $x + y = \sqrt{2}$ and circle $x^2 + y^2 - 4x - 2y = 4$

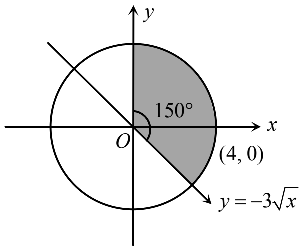
87. **(c)** $|z+1-i|^2+|z-5-i|^2$

$=(x+1)^2+(y-1)^2+(x-5)^2+(y-1)^2$

$=2(x^2+y^2-4x-2y)+28$

$=2(4)+28\ (\because x^2+y^2-4x-2y=4)\ =36$

88. **(d)**



Since, $|w-(2+i)|<3$

$\Rightarrow\ |w|-|2+i|<3$

$\Rightarrow\ -3+\sqrt{5}<|w|<3+\sqrt{5}$

$\Rightarrow\ -3-\sqrt{5}<-|w|<3-\sqrt{5}$ Also, $|z-(2+i)|=3$

$\Rightarrow\ -3+\sqrt{5}\le|z|\le 3+\sqrt{5}$

$\therefore\ -3<|z|-|w|+3<9$

89. **(b)** Since, $S=S_1\cap S_2\cap S_3$

Clearly, the shaded region represents the area of sector

$\therefore\ S=\frac{1}{2}r^2\theta=\frac{1}{2}\times 4^2\times\frac{5\pi}{6}=\frac{20\pi}{3}$

90. **(c)** $\min_{Z\in S}|1-3i-z|=$ perpendicular distance of point (1, –3)

From the line $\sqrt{3}x+y=0$

$\Rightarrow\ \frac{|\sqrt{3}-3|}{\sqrt{3}+1}=\frac{3-\sqrt{3}}{2}$

Match the Column

91. **(a)** **(A)** $|z-1|=|z-i|$

Hence, it lies on the perpendicular bisector of the line joining (1,0) and (0,1) which is a straight line passing through the origin.

(B) $|z+\bar{z}|+|z-\bar{z}|=2$

$\Rightarrow\ |x|+|y|=1$

Hence z lie on a square

(C) Let $z=x+iy\ |z+\bar{z}|+|z-\bar{z}|\Rightarrow|2x|+|2iy|$

$\Rightarrow\ |x|+|y|\Rightarrow x=\pm y$

Hence, the locus of z is a pair of straight lines

(D) Let $z=2/z$

Then $|z|=\left|\frac{2}{z}\right|=\frac{2}{|z|}=\frac{2}{1}=2$

92. **(a)** **(A)** z is equidistant from the points $i|z|$ and $-i|z|$, whose perpendicular bisector is $\text{Im}(z)=0$.

(B) Sum of distance of z from (4, 0) and (–4, 0) is a constant 10, hence locus of z is ellipse with semi-major axis 5 and focus at $(\pm 4,0)$, $ae=4$.

$\therefore\ e=4/5$

(C) $|z|\le|w|+\left|\frac{1}{w}\right|=\frac{5}{2}<3$

(D) $|z|\le|w|+\left|\frac{1}{w}\right|=2$

$\therefore\ \text{Re}(z)\le|z|\le 2$

93. **(b)** **(A)** Plan $e^{i\theta}\cdot e^{i\alpha}=e^{i(\theta+\alpha)}$

Given $Z_k=e^{i\frac{2k\pi}{10}}\Rightarrow Z_k\cdot Z_j=e^{i\left(\frac{2\pi}{10}\right)(k+j)}$

Z_k is 10^{th} root of unity.

$\Rightarrow\ \bar{Z}_k$ will also be 10^{th} root of unity.

Taking Z_j as $\bar{Z}_k$, we have $Z_k\cdot Z_2=1$ (True)

(B) Plan $\frac{e^{i\theta}}{e^{i\alpha}}=e^{i(\theta-\alpha)}$

$z=z_k/z_1=e^{i\left(\frac{2k\pi}{10}-\frac{2\pi}{10}\right)}=e^{i\frac{\pi}{5}(k-1)}$

For $k=2; z=e^{i\frac{\pi}{5}}$ which is in the given set (False)

(C) Plan

$1-\cos 2\theta=2\sin^2\theta$

$\sin 2\theta=2\sin\theta\cos\theta$ and

$\cos 36°=\frac{\sqrt{5}-1}{4}$

$\cos 108°=\frac{\sqrt{5}+1}{4}$

$\frac{|1-z_1||1-z_2|\ldots|1-z_9|}{10}$

Note: $|1-z_k|=\left|1-\cos\frac{2\pi k}{10}-i\sin\frac{2\pi k}{10}\right|$

$=\left|2\sin\frac{\pi k}{10}\right|\left|\sin\frac{\pi k}{10}-i\cos\frac{\pi k}{10}\right|$

$=2\left|\sin\frac{\pi k}{10}\right|$

Now, required product is

$\frac{2^9\sin\frac{\pi}{10}\cdot\sin\frac{2\pi}{10}\cdot\sin\frac{3\pi}{10}\ldots\sin\frac{8\pi}{10}\cdot\sin\frac{9\pi}{10}}{10}$

$$=\frac{2^9\left(\sin\frac{\pi}{10}\sin\frac{2\pi}{10}\sin\frac{3\pi}{10}\sin\frac{4\pi}{10}\right)^2\sin\frac{5\pi}{10}}{10}$$

$$=\frac{2^9\left(\sin\frac{\pi}{10}\cos\frac{\pi}{10}\cdot\sin\frac{2\pi}{10}\cos\frac{2\pi}{10}\right)^2\cdot 1}{10}$$

$$=\frac{2^9\left(\frac{1}{2}\sin\frac{\pi}{5}\cdot\frac{1}{2}\sin\frac{2\pi}{5}\right)^2}{10}$$

$$=\frac{2^5(\sin 36°\cdot\sin 72°)^2}{10}$$

$$=\frac{2^5}{2^2\times 10}(2\sin 36°\sin 72°)^2$$

$$=\frac{2^2}{5}(\cos 36°-\cos 108°)^2$$

$$=\frac{2^2}{5}\left[\left(\frac{\sqrt{5}-1}{4}\right)+\left(\frac{\sqrt{5}+1}{4}\right)\right]^2$$

$$=\frac{2^2}{5}\cdot\frac{5}{4}=1$$

(D) Sum of *n*th roots of unity = 0

$$1+\alpha+\alpha^2+\alpha^3+...+\alpha^9=0$$

$$1+\sum_{k=1}^{9}\alpha^k=0$$

$$1+\sum_{k=1}^{9}\left(\cos\frac{2k\pi}{10}+i\sin\frac{2k\pi}{10}\right)=0$$

$$1+\sum_{k=1}^{9}\cos\frac{2k\pi}{10}=0$$

So, $1-\sum_{k=1}^{9}\cos\frac{2k\pi}{10}=2$

Integer

94. **(5)** Given, $|z-3-2i|\le 2$ . . .(*i*)

To find minimum of $|2z-6+5i|$

or $2\left|z-3+\frac{5}{2}i\right|$, using triangle inequality

i.e., $||z_1|-|z_2||\le|z_1+z_2|$

$\therefore$ $\left|z-3+\frac{5}{3}i\right|=\left|z-3-2i+2i+\frac{5}{2}i\right|$

$$=\left|(z-3-2i)+\frac{9}{2}i\right|$$

$$\ge\left||z-3-2i|-\frac{9}{2}\right|\ge\left|2-\frac{9}{2}\right|\ge\frac{5}{2}$$

$\Rightarrow$ $\left|z-3+\frac{5}{2}i\right|\ge\frac{5}{2}$

or $|2z-6+5i|\ge 5$

95. **(3)** Printing error, $\omega=e^{i\frac{2\pi}{3}}$

Then, $\frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}=3$

Note: Here, $\omega=e^{i2\pi/3}$, then only integer solution exists.

96. **(216)** The absolute value of $8z_2z_3+27z_3z_1+64z_1z_2$

$$=|8z_2z_3+27z_3z_1+64z_1z_2|$$

$$=|z_1z_2z_3|\left|\frac{8}{z_1}+\frac{27}{z_2}+\frac{64}{z_3}\right|$$

$$=|z_1||z_2||z_3|\left|\frac{8\bar{z}_1}{z_1\bar{z}_1}+\frac{27\bar{z}_2}{z_2\bar{z}_2}+\frac{64\bar{z}_3}{z_3\bar{z}_3}\right|$$

$$=|z_1||z_2||z_3|\left|\frac{8\bar{z}_1}{|z_1|^2}+\frac{27\bar{z}_2}{|z_2|^2}+\frac{64\bar{z}_3}{|z_3|^3}\right|$$

$$=|z_1||z_2||z_3|\left|\frac{8\bar{z}_1}{4}+\frac{27\bar{z}_2}{9}+\frac{64\bar{z}_3}{16}\right|$$

$$=|z_1||z_2||z_3||\overline{2\bar{z}_1+3\bar{z}_2+4\bar{z}_3}|$$

$$=|z_1||z_2||z_3||2z_1+3z_2+4z_3|$$

$$=2\cdot 3\cdot 4\cdot 9=216$$

97. **(144)** Let $z=(a+b\omega+c\omega^2)$

$\therefore$ $\bar{z}=\overline{(a+b\omega+c\omega^2)}=(a+b\bar{\omega}+c\overline{\omega^2})=(a+b\omega^2+c\omega)$

and $z\bar{z}=(a+b\omega+c\omega^2)=(a+b\omega^2+c\omega)$

$$=(a^2+b^2+c^2-ab-bc-ca)$$

$$=\frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$$

or $|z|^2=\frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$

$\therefore$ $|a+b\omega+c\omega^2|+|a+b\omega^2+c\omega|=|z|+|\bar{z}|=|z|+|z|=2|z|$

$$=2\cdot\frac{1}{2}\sqrt{(a-b)^2+(b-c)^2+(c-a)^2}$$

$$=\sqrt{2}\sqrt{(a-b)^2+(b-c)^2+(c-a^2)}\ge\sqrt{2}\sqrt{(1^2+1^2+2^2)}$$

$$=\sqrt{12}=(144)^{1/2}$$

($\because$ *a, b, c* are distinct integers, minimum value of $(a-b)^2+(b-c)^2+(c-a)^2=1^2+1^2+2^2=6$)

$\therefore \quad \left|a+b\omega+c\omega^2\right|+\left|a+b\omega^2+c\omega\right| \ge (144)^{1/4}$

$\therefore$ Minimum value of $\left|a+b\omega+c\omega^2\right|+\left|a+b\omega^2+c\omega\right| = (144)^{1/4}$

$\therefore \quad n = 144$

98. **(48)** Let $|z-\alpha| = k$ . . .(i)

(where $\alpha = a + ib$ and $a,b,k \in R$) be a circle which cuts the circle $|z| = 1$ . . .(ii)

and $|z-1| = 4$ . . .(iii)

orthogonally, then $k^2+1 = |\alpha - 0|^2 = \alpha\bar{\alpha}$

and $k^2+16 = |\alpha-1|^2 = (\alpha-1)(\bar{\alpha}-1) = \alpha\bar{\alpha}-(\alpha+\bar{\alpha})+1$

or $1-(\alpha+\bar{\alpha})-15 = 0$ or $\alpha+\bar{\alpha} = -14$ or $2a = -14$

$\Rightarrow \quad a = -7$

$\Rightarrow \quad \alpha = a+ib = -7+ib$

Also $k^2 = |\alpha^2| - 1 = 49 + b^2 - 1 = 48 + b^2$

$\Rightarrow \quad k = \sqrt{(48+b^2)}$

Therefore, required family of circles is given by

$|z+7-ib| = \sqrt{(48+b^2)}$

$\therefore \quad \lambda = 48$

99. **(1648)** $\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)$

$= -i\sum_{q=1}^{10}\left(\cos\left(\frac{2q\pi}{11}\right)+i\sin\left(\frac{2q\pi}{11}\right)\right)$

$= -i\sum_{q=1}^{10} e^{2q\pi/11} = -i\left(\sum_{q=1}^{10} e^{2q\pi i/11} - 1\right)$

$= -i$ (sum of 11, 11$^{\text{th}}$ roots of unity -1) $= -i(0-1) = i$

$\therefore \quad \sum_{p=1}^{32}(3p+2)\left(\sum_{q=1}^{10}\left(\sin\left(\frac{2q\pi}{11}\right)-i\cos\left(\frac{2q\pi}{11}\right)\right)\right)^{4p}$

$= \sum_{p=1}^{32}(3p+2)(i)^{4p} = \sum_{p=1}^{32}(3p+2)$

$= 3\sum_{p=1}^{32} p + 2\sum_{p=1}^{32} 1 = \frac{3\cdot 32\cdot 23}{2} + 2\cdot 32 = 1648$

100. **(394)** $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2+b^2) + (|z_1|^2 + |z_2|^2)$

Then $\lambda = a^2 + b^2 = (15)^2 + (13)^2 = 394$

$\therefore \quad \sqrt{\lambda\sqrt{\lambda\sqrt{\lambda\sqrt{\lambda\ldots\infty}}}} = \lambda^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\infty}$

$= \lambda^{\frac{1/2}{1-(1/2)}} = \lambda = 394$

* * *