Session 3

Number of Permutations Under Certain Conditions, Circular Permutations, Restricted Circular Permutations

Number of Permutations Under Certain Conditions

(i) Number of permutations of n different things, taken r at a time, when a particular thing is to be always included in each arrangement, is $r \cdot {}^{n-1}P_{r-1}$

Corollary Number of permutations of n different things, taken r at a time, when p particular things is to be always included in each arrangement, is

$$p!\,(r-(p-1))^{n-p}P_{r-p}.$$

(ii) Number of permutations of n different things, taken r at a time, when a particular thing is never taken in each arrangement, is

$${}^{n-1}P_r.$$

(iii) Number of permutations of n different things, taken all at a time, when m specified things always come together, is

$$m!\times(n-m+1)!$$

(iv) Number of permutations of n different things, taken all at a time, when m specified things never come together, is

$$n!-m!\times(n-m+1)!$$

Example 34. How many permutations can be made out of the letters of the word 'TRIANGLE' ? How many of these will begin with T and end with E ?

Sol. The word 'TRIANGLE' has eight different letters, which can be arranged themselves in 8! ways.

$\therefore$ Total number of permutations $= 8! = 40320$

Again, when T is fixed at the first place and E at the last place, the remaining six can be arranged themselves in $6!$ ways.

$\therefore$ The number of permutations which begin with T and end with

$$E = 6! = 720.$$

Example 35. In how many ways can the letters of the word 'INSURANCE' be arranged, so that the vowels are never separate?

Sol. The word 'INSURANCE' has nine different letters, combine the vowels into one bracket as (IUAE) and treating them as one letter we have six letters viz.

(IUAE) N S R N C and these can be arranged among themselves in $\dfrac{6!}{2!}$ ways and four vowels within the bracket can be arranged themselves in $4!$ ways.

$\therefore$ Required number of words $= \dfrac{6!}{2!} \times 4! = 8640$

Example 36. How many words can be formed with the letters of the word 'PATALIPUTRA' without changing the relative positions of vowels and consonants?

Sol. The word 'PATALIPUTRA' has eleven letters, in which 2P's, 3A's, 2T's, 1L, 1U, 1R and 1I. Vowels are AAIUA

These vowels can be arranged themselves in $\dfrac{5!}{3!} = 20$ ways.

The consonants are PTLPTR these consonants can be arranged themselves in $\dfrac{6!}{2!\,2!} = 180$ ways

$\therefore$ Required number of words

$$= 20 \times 180 = 3600 \text{ ways.}$$

Example 37. Find the number of permutations that can be had from the letters of the word 'OMEGA'

(i) O and A occuping end places.

(ii) E being always in the middle.

(iii) Vowels occuping odd places.

(iv) Vowels being never together.

Sol. There are five letters in the word 'OMEGA'.

(i) When O and A occuping end places

i.e., M E G (OA)

the first three letters (M, E, G) can be arranged themselves by $3! = 6$ ways and last two letters (O, A) can be arranged themselves by $2! = 2$ ways.

$\therefore$ Total number of such words

$$= 6 \times 2 = 12 \text{ ways.}$$

(ii) When E is the fixed in the middle, then there are four places left to be filled by four remaining letters O, M, G and A and this can be done in $4!$ ways.

$\therefore$ Total number of such words $= 4! = 24$ ways.

(iii) Three vowels (O, E, A) can be arranged in the odd places in $3!$ ways (1st, 3rd and 5th) and the two consonants (M, G) can be arranged in the even places in $2!$ ways (2nd and 4th)

$\therefore$ Total number of such words

$$= 3! \times 2! = 12 \text{ ways.}$$

(iv) Total number of words $= 5! = 120$

Combine the vowels into one bracket as (OEA) and treating them as one letter, we have

(OEA), M, G and these can be arranged themselves in $3!$ ways and three vowels with in the bracket can be arranged themselves in $3!$ ways.

$\therefore$ Number of ways when vowels come together

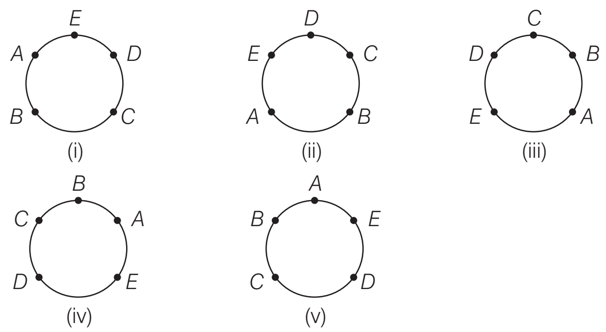
$$= 3! \times 3! = 36 \text{ ways.}$$

Hence, number of ways when vowels being never together $= 120 - 36 = 84$ ways.

Circular Permutations

(i) Arrangements round a circular table

Consider five persons A, B, C, D and E on the circumference of a circular table in order which has no head now, shifting A, B, C, D and E one position in anti-clockwise direction we will get arragements as follows



We see that, if 5 persons are sitting at a round table, they can be shifted five times and five different arrangements. Thus, obtained will be the same, because anti-clockwise order of A, B, C, D and E does not change.

But if A, B, C, D and E are sitting in a row and they are shifted in such an order that the last occupies the place of first, then the five arrangements will be different. Thus, if there are 5 things, then for each circular arrangement number of linear arrangements is 5.

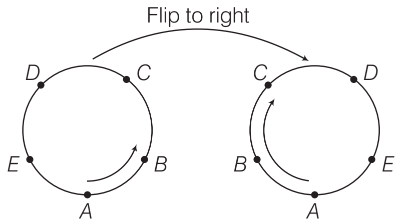
Similarly, if n different things are arranged along a circle for each circular arrangement number of linear arrangements is n.

Therefore, the number of linear arrangements of n different things $= n \times$ number of circular arrangements of n different things

(ii) Arrangements of beads or flowers (all different) around a circular necklace or garland

Consider five beads A, B, C, D and E in a necklace or five flowers A, B, C, D and E in a garland, etc. If the necklace or garland on the left is turned over, we obtain the arrangement on the right i.e. anti-clockwise and clockwise order of arrangement is not different we will get arrangements as follows:

We see that arrangements in figures are not different.



Then, the number of circular permutations of n different things taken all at a time is $\frac{1}{2}(n-1)!$, if clockwise and anti-clockwise orders are taken as not different.

Example 38. Find the number of ways in which 12 different beads can be arranged to form a necklace.

Sol. 12 different beads can be arranged among themselves in a circular order in $(12-1)! = 11!$ ways. Now, in the case of necklace, there is no distinction between clockwise and anti-clockwise arrangements. So, the required number of arrangements $= \frac{1}{2}(11!)$.

Example 39. Consider 21 different pearls on a necklace. How many ways can the pearls be placed in on this necklace such that 3 specific pearls always remain together?

Sol. After fixing the places of three pearls, treating 3 specific pearls = 1 unit. So, we have now

18 pearls + 1 unit = 19 and the number of arrangement will be $(19-1)! = 18!$

Also, the number of ways of 3 pearls can be arranged between themselves is $3! = 6$.

Since, there is no distinction between the clockwise and anti-clockwise arrangements.

So, the required number of arrangements $= \frac{1}{2} 18! \cdot 6 = 3(18!)$.

Restricted Circular Permutations

***Case* I** If clockwise and anti-clockwise orders are taken as different, then the number of circular permutations of n different things taken r at a time.

$$= \frac{{}^nP_r}{r} = \frac{1}{r} \cdot \frac{n!}{(n-r)!}$$

Note For checking correctness of formula, put $r = n$, then we get $(n-1)!$ [result (5) (i)]

Example 40. In how many ways can 24 persons be seated round a table, if there are 13 sets ?

Sol. In case of circular table, the clockwise and anti-clockwise orders are different, the required number of circular permutations $= \frac{{}^{24}P_{13}}{13} = \frac{24!}{13 \times 11!}$.

$\Rightarrow$ $n! = n \times$ number of circular arrangements of n different things

$\Rightarrow$ Number of circular arrangements of n different things

$$= \frac{n!}{n} = (n-1)!$$

Hence, the number of circular permutations of n different things taken all at a time is $(n-1)!$, if clockwise and anti-clockwise orders are taken as different.

Example 41. Find the number of ways in which three Americans, two British, one Chinese, one Dutch and one Egyptian can sit on a round table so that persons of the same nationality are separated.

Sol. The total number of persons without any restrictions is

$$n(U) = (8-1)!$$
$$= 7! = 5040$$

When, three Americans (A_1, A_2, A_3) are sit together,

$$n(A) = 5! \times 3!$$
$$= 720$$

When, two British (B_1, B_2) are sit together

$$n(B) = 6! \times 2!$$
$$= 1440$$

When, three Americans (A_1, A_2, A_3) and two British (B_1, B_2) are sit together $n(A \cap B) = 4! \times 3! \times 2! = 288$

$\therefore$ $n(A \cup B) = n(A) + nB) - n(A \cap B)$

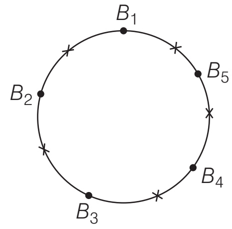
$$= 720 + 1440 - 288 = 1872$$

Hence, $n(A \cap B') = n(U) - n(A \cup B)$

$$= 5040 - 1872$$
$$= 3168$$

Example 42. In how many different ways can five boys and five girls form a circle such that the boys and girls alternate?

Sol. After fixing up one boy on the table, the remaining can be arranged in $4!$ ways but boys and girls are to alternate. There will be 5 places, one place each between two boys these five places can be filled by 5 girls in $5!$ ways.



Hence, by the principle of multiplication, the required number of ways $= 4! \times 5! = 2880$.

Example 43. 20 persons were invited to a party. In how many ways can they and the host be seated at a circular table ? In how many of these ways will two particular persons be seated on either side of the host?

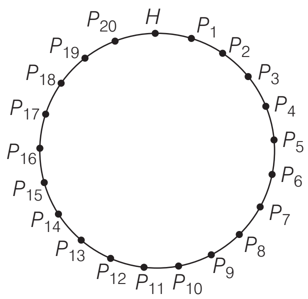
Sol. **I Part** Total persons on the circular table

$$= 20 \text{ guest} + 1 \text{ host} = 21$$

They can be seated in $(21-1)! = 20!$ ways.

II Part After fixing the places of three persons (1 host + 2 persons).

Treating (1 host + 2 persons) = 1 unit, so we have now {(remaining 18 persons + 1 unit) = 19} and the number of arrangement will be $(19-1)! = 18!$ also these two particular persons can be seated on either side of the host in $2!$ ways.



Hence, the number of ways of seating 21 persons on the circular table such that two particular persons be seated on either side of the host $= 18! \times 2! = 2 \times 18!$

***Case* II** If clockwise and anti-clockwise orders are taken as not different, then the number of circular permutations of n different things taken r at a time $= \frac{{}^nP_r}{2r} = \frac{1}{2r} \cdot \frac{n!}{(n-r)!}$

Note
For checking correctness of formula put $r = n$, then we get $\frac{(n-1)!}{2}$ [result (5) (ii)]

Example 44. ***How many necklace of 12 beads each can be made from 18 beads of various colours?***

Sol. In the case of necklace, there is no distinction between the clockwise and anti-clockwise arrangements, the required number of circular permutations.

$$= \frac{^{18}P_{12}}{2 \times 12} = \frac{18\,!}{6\,! \times 24} = \frac{18 \times 17 \times 16 \times 15 \times 14 \times 13\,!}{6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 24} = \frac{119 \times 13\,!}{2}$$

Exercise for Session 3

1. How many words can be formed from the letters of the word 'COURTESY' whose first letter is C and the last letter is Y ?

(a) 6! (b) 8! (c) 2(6)! (d) 2(7)!

2. The number of words that can be made by writing down the letters of the word 'CALCULATE' such that each word starts and ends with a consonant, is

(a) $\frac{3}{2}(7)!$ (b) 2(7)! (c) $\frac{5}{2}(7)!$ (d) 3(7)!

3. The number of words can be formed from the letters of the word 'MAXIMUM', if two consonants cannot occur together, is

(a) 4! (b) $3! \times 4!$

(c) 3! (d) $\frac{4!}{3!}$

4. All the letters of the word 'EAMCET' are arranged in all possible ways. The number of such arrangements in which two vowels are not adjacent to each other, is

(a) 54 (b) 72

(c) 114 (d) 360

5. How many words can be made from the letters of the word 'DELHI', if L comes in the middle in every word?

(a) 6 (b) 12 (c) 24 (d) 60

6. In how many ways can 5 boys and 3 girls sit in a row so that no two girls are sit together?

(a) $5! \times 3!$ (b) $^4P_3 \times 5!$ (c) $^6P_3 \times 5!$ (d) $^5P_3 \times 3!$

7. There are n numbered seats around a round table. Total number of ways in which n_1 $(n_1 < n)$ persons can sit around the round table, is equal to

(a) $^nC_{n_1}$ (b) $^nP_{n_1}$ (c) $^nC_{n_1-1}$ (d) $^nP_{n_1-1}$

8. In how many ways can 7 men and 7 women can be seated around a round table such that no two women can sit together?

(a) 7! (b) $7! \times 6!$ (c) $(6!)^2$ (d) $(7!)^2$

9. The number of ways that 8 beads of different colours be string as a necklace, is

(a) 2520 (b) 2880 (c) 4320 (d) 5040

10. If 11 members of a committee sit at a round table so that the President and secretary always sit together, then the number of arrangements, is

(a) $9! \times 2$ (b) 10! (c) $10! \times 2$ (d) 11!

11. In how many ways can 15 members of a council sit along a circular table, when the secretary is to sit on one side of the Chairman and the deputy secretary on the other side?

(a) $12! \times 2$ (b) 24 (c) $15! \times 2$ (d) 30

Answers

Exercise for Session 3

1. (a)	**2.** (c)	**3.** (a)	**4.** (b)	**5.** (c)	**6.** (c)
7. (b)	**8.** (b)	**9.** (a)	**10.** (a)	**11.** (a)	