Session 3

Discontinuity of a Function

A function f which is not continuous, said to be discontinuous functions, i.e. if there is a break in the function of any kind, then it is discontinuous functions.

There are two types of discontinuity

1. Removable discontinuity
2. Non-removable discontinuity

Removable Discontinuity

In this type of discontinuity, $\lim_{x \to a} f(x)$ necessarily exists, but is either not equal to $f(a)$ or $f(a)$ is not defined. Such function is said to have a removable discontinuity of the first kind.

In this case, it is possible to redefine the function in such a manner that $\lim_{x \to a} f(x) = f(a)$ and thus making function continuous.

These discontinuities can be further classified as

(i) Missing point discontinuity

(ii) Isolated point discontinuity

Examples of Missing Point Discontinuity

Here, $\lim_{x \to a} f(x)$ exists finitely but $f(a)$ is not defined. e.g.

(i) Let $f(x) = \dfrac{(x-1)(9-x^2)}{(x-1)}$,

clearly $f(1) \to \dfrac{0}{0}$ form

$\therefore$ $f(x)$ has missing point discontinuity. Shown as,

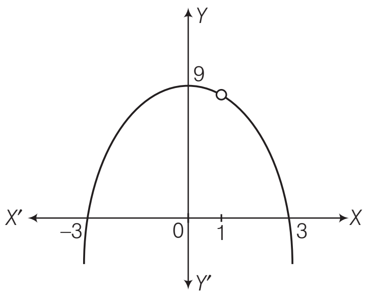

Figure 6.5

(ii) $f(x) = \dfrac{x^2 - 4}{x - 2}$ has missing point discontinuity at $x = 2$.

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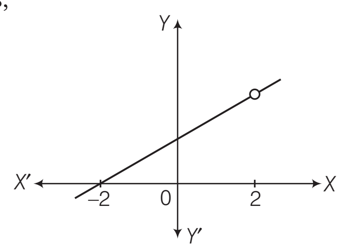

Figure 6.6

(iii) $f(x) = \dfrac{\sin x}{x}$ has missing point discontinuity at $x = 0$.

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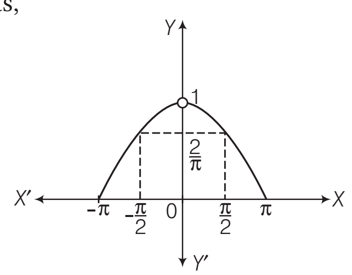


Figure 6.7

Examples of Isolated Point Discontinuity

Here, $\lim_{x \to a} f(x)$ exists and $f(a)$ also exists but $\lim_{x \to a} f(x) \neq f(a)$. e.g.

(i) Let $f(x) = [x] + [-x] \Rightarrow f(x) = \begin{cases} 0, \text{ if } x \in I \\ -1, \text{ if } x \notin I \end{cases}$

where $x =$ Integer, has isolated point discontinuity, can be shown as

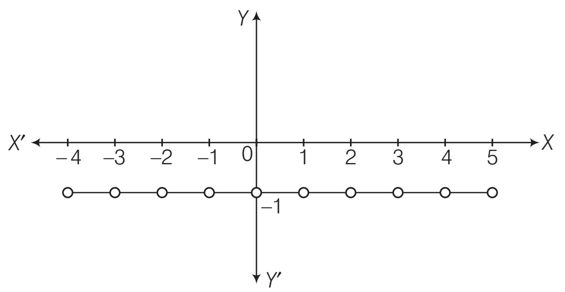


Figure 6.8

(ii) Let $f(x) = \text{sgn}(\cos 2x - 2\sin x + 3)$

$\Rightarrow$ $f(x) = \text{sgn}(1 - 2\sin^2 x - 2\sin x + 3)$

$= \text{sgn}(2(2 + \sin x)(1 - \sin x))$

$$= \begin{cases} 0, \text{ if } x = 2n\pi + \dfrac{\pi}{2} \\ 1, \text{ if } x \neq 2n\pi + \dfrac{\pi}{2} \end{cases}$$

$\therefore f(x)$ has an isolated point discontinuity at $x = 2n\pi + \dfrac{\pi}{2}$.

Shown as,

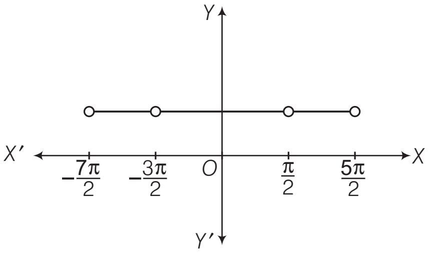


Figure 6.9

Non-removable Discontinuity

In this type of discontinuity $\lim_{x \to a} f(x)$ doesn't exist and therefore, it is not possible to redefine the function in any manner and make it continuous. Such function is said to have non-removable discontinuity or discontinuity of second kind.

Such discontinuities can further be classified into three types.

(i) Finite Discontinuity

If for a function $f(x) : \lim_{x \to a^-} f(x) = L_1$ and $\lim_{x \to a^+} f(x) = L_2$ and $L_1 \neq L_2$, then such function is said to have a finite discontinuity or a jump discontinuity.

In this case, the non-negative difference between the two limits is called the **Jump of Discontinuity**. A function having a finite number of jumps in a given interval is called a **Piecewise Continuous** or **Sectionally Continuous Function.**

Examples of Finite Discontinuity

(i) $f(x) \tan^{-1}\left(\dfrac{1}{x}\right)$ at $x = 0$

$$\left.\begin{aligned} &\text{RHL} \quad \text{i.e. } f(0^+) = \frac{\pi}{2} \\ &\text{LHL} \quad \text{i.e. } f(0^-) = -\frac{\pi}{2} \end{aligned}\right\} \text{jump} = \pi$$

(ii) $f(x) = \dfrac{|\sin x|}{x}$ at $x = 0$

$$\left.\begin{aligned} &\text{RHL} \quad \text{i.e. } f(0^+) = 1 \\ &\text{LHL} \quad \text{i.e. } f(0^-) = -1 \end{aligned}\right\} \text{jump} = 2$$

(ii) Infinite Discontinuity

If for a function $f(x) : \lim_{x \to a^-} f(x) = L_1$ and $\lim_{x \to a^+} f(x) = L_2$ and either L_1 or L_2 is infinity, then such function is said to have infinite discontinuity.

In other words, If $x = a$ is a vertical asymptote for the graph of $y = f(x)$, then f is said to have infinite discontinuity at a.

Examples of Infinite Discontinuity

(i) $f(x) = \dfrac{x}{1-x}$, at $x = 1$

RHL i.e. $f(1^+) = -\infty$

LHL i.e. $f(1^-) = \infty$

(ii) $f(x) = \dfrac{1}{x^2}$, at $x = 0$

RHL i.e. $f(0^+) = \infty$

LHL i.e. $f(0^-) = \infty$

(iii) Oscillatory Discontinuity

If for a function $f(x) : \lim_{x \to a} f(x)$ doesn't exist but oscillate between two finite quantities, then such function is said to have oscillatory discontinuity.

Examples of Oscillatory Discontinuity

(i) $f(x) = \sin\left(\dfrac{1}{x}\right)$

$\Rightarrow \lim_{x \to 0} f(x) =$ a value between -1 to 1.

$\therefore$ Limit doesn't exist, as it oscillates between -1 and 1 as $x \to 0$.

(ii) $f(x) = \cos\left(\dfrac{1}{x}\right)$

$\Rightarrow \lim_{x \to 0} \cos\left(\dfrac{1}{x}\right) =$ a value between -1 to 1.

$\therefore$ Limit doesn't exist, as it oscillates between -1 to 1 at $x \to 0$.

List of Continuous Functions

	Function $f(x)$	Interval in which $f(x)$ is continuous
1.	constant c	$(-\infty, \infty)$
2.	x^n, n is an integer ≥ 0	$(-\infty, \infty)$
3.	x^{-n}, n is a positive integer	$(-\infty, \infty)-\{0\}$
4.	$\|x-a\|$	$(-\infty, \infty)$
5.	$P(x) = a_0x^n + a_1x^{n-1} + ... + a_n$	$(-\infty, \infty)$
6.	$\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial in x	$(-\infty, \infty)-\{x : q(x) = 0\}$
7.	$\sin x$	$(-\infty, \infty)$
8.	$\cos x$	$(-\infty, \infty)$
9.	$\tan x$	$(-\infty, \infty)-\left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$
10.	$\cot x$	$(-\infty, \infty)-\{n\pi : n \in I\}$
11.	$\sec x$	$(-\infty, \infty)-\{(2n+1)\pi/2 : n \in I\}$
12.	$\text{cosec } x$	$(-\infty, \infty)-\{n\pi : n \in I\}$
13.	e^x	$(-\infty, \infty)$
14.	$\log_e x$	$(0, \infty)$

Example 11 ***Examine the function,*** $f(x) = \begin{cases} x-1, & x < 0 \\ 1/4, & x = 0 \\ x^2-1, & x > 0 \end{cases}$

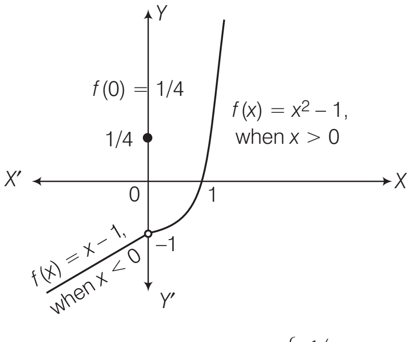
Discuss the continuity and if discontinuous remove the discontinuity.

Sol. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = -1$

But $f(0) = 1/4$. Thus, $f(x)$ has removable discontinuity and $f(x)$ could be made continuous by taking

$$f(0) = -1 \Rightarrow f(x) = \begin{cases} x-1, & x < 0 \\ -1, & x = 0 \\ x^2-1, & x > 0 \end{cases}$$

Graphically $f(x)$ could be plotted as showing



Example 12 ***Show the function,*** $f(x) = \begin{cases} \frac{e^{1/x}-1}{e^{1/x}+1}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$

has non-removable discontinuity at $x = 0$.

Sol. We have, $f(x) = \begin{cases} \frac{e^{1/x}-1}{e^{1/x}+1}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$

$\therefore$ RHL at $x = 0$, let $x = 0 + h$

$$\Rightarrow \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{e^{\frac{1}{0+h}}-1}{e^{\frac{1}{0+h}}+1} = \lim_{h \to 0} \frac{e^{\frac{1}{h}}-1}{e^{\frac{1}{h}}+1}$$

$$\Rightarrow \lim_{x \to 0^+} f(x) = \lim_{h \to 0} \frac{1-\frac{1}{e^{1/h}}}{1+\frac{1}{e^{1/h}}} \quad \Rightarrow \lim_{x \to 0^+} f(x) = \frac{1-0}{1+0} = 1$$

$$\left[\text{as } h \to 0;\ \frac{1}{h} \to \infty \Rightarrow e^{1/h} \to \infty;\ 1/e^{1/h} \to 0\right]$$

$$\therefore \quad \lim_{x \to 0^+} f(x) = 1$$

Again, LHL at $x = 0$, let $x = 0 - h$

$$\Rightarrow \quad \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \frac{e^{-1/h}-1}{e^{-1/h}+1} = \frac{0-1}{0+1} = -1$$

$$[\text{as } h \to 0;\ e^{-1/h} \to 0]$$

$$\lim_{x \to 0^-} f(x) = -1$$

$$\Rightarrow \quad \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x).$$

Thus, $f(x)$ has non-removable discontinuity.

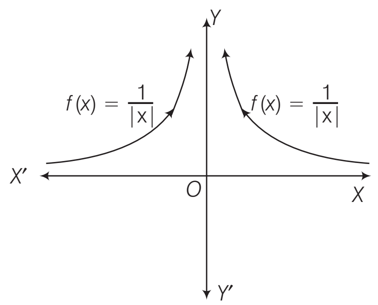
Example 13 ***Show*** $f(x) = \frac{1}{|x|}$ ***has discontinuity of second kind at*** $x = 0$.

Sol. Here, $f(0) = \frac{1}{|0|}$, which shows function has discontinuity of second kind.

Graphically Here, the graph is broken at $x = 0$ as

$$x \to 0$$

$$\Rightarrow \quad f(x) \to \infty$$



Therefore, $f(x)$ has discontinuity of second kind.

Exercise for Session 3

1. Which of the following function(s) has/have removable discontinuity at the origin?

(a) $f(x) = \dfrac{1}{1 + 2^{\cot x}}$ (b) $f(x) = \cos\left(\dfrac{|\sin x|}{x}\right)$

(c) $f(x) = x \sin\dfrac{\pi}{x}$ (d) $f(x) = \dfrac{1}{\ln|x|}$

2. Function whose jump (non-negative difference of LHL and RHL) of discontinuity is greater than or equal to one. is/are

(a) $f(x) = \begin{cases} \dfrac{(e^{1/x} + 1)}{(e^{1/x} - 1)}; & x < 0 \\ \dfrac{(1 - \cos x)}{x}; & x > 0 \end{cases}$ (b) $g(x) = \begin{cases} \dfrac{(x^{1/3} - 1)}{(x^{1/2} - 1)}; & x > 0 \\ \dfrac{\ln x}{(x - 1)}; & \dfrac{1}{2} < x < 1 \end{cases}$

(c) $u(x) = \begin{cases} \dfrac{\sin^{-1} 2x}{\tan^{-1} 3x}; & x \in \left(0, \dfrac{1}{2}\right] \\ \dfrac{|\sin x|}{x}; & x < 0 \end{cases}$ (d) $v(x) = \begin{cases} \log_3(x + 2); & x > 2 \\ \log_{1/2}(x^2 + 5); & x < 2 \end{cases}$

3. Consider the piecewise defined function $f(x) = \begin{cases} \sqrt{-x}, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 4 \\ x - 4, & \text{if } x > 4 \end{cases}$, choose the answer which best describes the continuity of this function.

(a) the function is unbounded and therefore cannot be continuous.
(b) the function is right continuous at $x = 0$.
(c) the function has a removable discontinuity at 0 and 4, but is continuous on the rest of the real line.
(d) the function is continuous on the entire real line.

4. If $f(x) = \text{sgn}(\cos 2x - 2\sin x + 3)$, where sgn () is the signum function, then $f(x)$

(a) is continuous over its domain.
(b) has a missing point discontinuity.
(c) has isolated point discontinuity.
(d) has irremovable discontinuity

5. $f(x) = \dfrac{2\cos x - \sin 2x}{(\pi - 2x)^2}$; $g(x) = \dfrac{e^{-\cos x} - 1}{8x - 4\pi}$

$h(x) = f(x)$ for $x < \pi/2 = g(x)$ for $x > \pi/2$, then which of the followings does not hold?

(a) h is continuous at $x = \pi/2$
(b) h has an irremovable discontinuity at $x = \pi/2$
(c) h has a removable discontinuity at $x = \pi/2$
(d) $f\left(\dfrac{\pi^+}{2}\right) = g\left(\dfrac{\pi^-}{2}\right)$

Answers

Exercise for Session 3

1. (b,c,d) **2.** (a,c,d) **3.** (d) **4.** (c) **5.** (a,c,d)