## **3D-COORDINATE GEOMETRY**

## 1. DISTANCE FORMULA :

The distance between two points A (x $_1,$  y $_1,$  z $_1)$  and B (x $_2,$  y $_2,$  z $_2)$  is

given by 
$$AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$$

## 2. SECTION FORMULAE :

Let  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  be two points and let R(x, y, z) divide PQ in the ratio  $m_1 : m_2$ . Then R is

$$(x, y, z) = \left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}, \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}\right)$$

If  $(m_1/m_2)$  is positive, R divides PQ internally and if  $(m_1/m_2)$  is negative, then externally.

Mid point of PQ is given by 
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

## 3. CENTROID OF A TRIANGLE :

Let A( $x_1$ ,  $y_1$ ,  $z_1$ ), B( $x_2$ ,  $y_2$ ,  $z_2$ ), C( $x_3$ ,  $y_3$ ,  $z_3$ ) be the vertices of a triangle ABC. Then its centroid G is given by

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

### 4. DIRECTION COSINES OF LINE :

If  $\alpha$ ,  $\beta$ ,  $\gamma$  be the angles made by a line with x-axis, y-axis & z-axis respectively then  $\cos\alpha$ ,  $\cos\beta$  &  $\cos\gamma$  are called direction cosines of a line, denoted by l, m & n respectively and the relation between  $\ell$ , m, n is given by  $\ell^2 + m^2 + n^2 = 1$ D. cosine of x-axis, y-axis & z-axis are respectively 1, 0, 0; 0, 1, 0; 0, 0, 1

## 5. DIRECTION RATIOS :

Any three numbers a, b, c proportional to direction cosines  $\ell$ , m, n are called direction ratios of the line.

i.e. 
$$\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$$

It is easy to see that there can be infinitely many sets of direction ratios for a given line.

## 6. RELATION BETWEEN D.C'S & D.R'S:

$$\begin{aligned} &\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c} \\ &\therefore \quad \frac{\ell^2}{a^2} = \frac{m^2}{b^2} = \frac{n^2}{c^2} = \frac{\ell^2 + m^2 + n^2}{a^2 + b^2 + c^2} \\ &\therefore \quad \ell = \frac{\pm a}{\sqrt{a^2 + b^2 + c^2}} ; \quad m = \frac{\pm b}{\sqrt{a^2 + b^2 + c^2}} ; \quad n = \frac{\pm c}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

## 7. DIRECTION COSINE OF AXES :

# Direction ratios and Direction cosines of the line joining two points :

Let A(x<sub>1</sub>, y<sub>1</sub>, z<sub>1</sub>) and B(x<sub>2</sub>, y<sub>2</sub>, z<sub>2</sub>) be two points, then d.r.'s of AB are x<sub>2</sub>-x<sub>1</sub>, y<sub>2</sub>-y<sub>1</sub>, z<sub>2</sub>-z<sub>1</sub> and the d.c.'s of AB are  $\frac{1}{r}(x_2-x_1)$ ,  $\frac{1}{r}(y_2-y_1)$ ,  $\frac{1}{r}(z_2-z_1)$  where  $r = \sqrt{[\Sigma(x_2-x_1)^2]} = |\overline{AB}|$ 

## 8. PROJECTION OF A LINE ON ANOTHER LINE :

Let PQ be a line segment with P(x<sub>1</sub>, y<sub>1</sub>, z<sub>1</sub>) and Q(x<sub>2</sub>, y<sub>2</sub>, z<sub>2</sub>) and let L be a straight line whose d.c.'s are  $\ell$ , m, n. Then the length of projection of PQ on the line L is  $|\ell|(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$ 

#### 9. ANGLE BETWEEN TWO LINES :

Let  $\theta$  be the angle between the lines with d.c.'s  $l_1$ ,  $m_1$ ,  $n_1$  and  $l_2$ ,  $m_2$ ,  $n_2$  then  $\cos \theta = l_1 l_2 + m_1 m_2 + n_1 n_2$ . If  $a_1$ ,  $b_1$ ,  $c_1$  and  $a_2$ ,  $b_2$ ,  $c_2$  be D.R.'s of two lines then angle  $\theta$  between them is given by

$$\cos \theta = \frac{(a_1 a_2 + b_1 b_2 + c_1 c_2)}{\sqrt{(a_1^2 + b_1^2 + c_1^2)} \sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

#### 10. PERPENDICULARITY AND PARALLELISM :

Let the two lines have their d.c.'s given by  $l_1$ ,  $m_1$ ,  $n_1$  and  $l_2$ ,  $m_2$ ,  $n_2$ respectively then they are perpendicular if  $\theta = 90^\circ$  i.e.  $\cos \theta = 0$ , i.e.  $l_1 \ l_2 + m_1 m_2 + n_1 n_2 = 0$ .

Also the two lines are parallel if  $\theta = 0$  i.e.  $\sin \theta = 0$ , i.e.  $\frac{\ell_1}{\ell_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$ 

#### Note:

If instead of d.c.'s, d.r.'s  $a_1$ ,  $b_1$ ,  $c_1$  and  $a_2$ ,  $b_2$ ,  $c_2$  are given, then the lines are perpendicular if  $a_1a_2 + b_1b_2 + c_1c_2 = 0$  and parallel if  $a_1/a_2 = b_1/b_2 = c_1/c_2$ .

# 11. EQUATION OF A STRAIGHT LINE IN SYMMETRICAL FORM :

(a) One point form : Let  $A(x_1, y_1, z_1)$  be a given point on the straight line and l, m, n the d.c's of the line, then its equation is

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r$$
 (say)

It should be noted that  $P(x_1 + lr, y_1 + mr, z_1 + nr)$  is a general point on this line at a distance r from the point  $A(x_1, y_1, z_1)$  i.e. AP = r. One should note that for AP = r; l, m, n must be d.c.'s not d.r.'s. If a, b, c are direction ratios of the line, then equation of the line

is 
$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = r$$
 but here AP  $\neq$  r

**(b)** Equation of the line through two points  $A(x_1, y_1, z_1)$  and  $B(x_2, y_2, z_2)$ 

is  $\frac{\mathbf{x} - \mathbf{x}_1}{\mathbf{x}_2 - \mathbf{x}_1} = \frac{\mathbf{y} - \mathbf{y}_1}{\mathbf{y}_2 - \mathbf{y}_1} = \frac{\mathbf{z} - \mathbf{z}_1}{\mathbf{z}_2 - \mathbf{z}_1}$ 

## 12. FOOT, LENGTH AND EQUATION OF PERPENDICULAR FROM A POINT TO A LINE :

Let equation of the line be

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r$$
 (say) .....(i)

and A ( $\alpha$ ,  $\beta$ ,  $\gamma$ ) be the point. Any point on the line (i) is

$$\begin{split} P(\ell r + x_1, \, mr + y_1, \, nr + z_1) & \dots \dots (ii) \\ \text{If it is the foot of the perpendicular, from A on the line, then AP is } \\ \text{to the line, so } \ell(\ell r + x_1 - \alpha) + m(mr + y_1 - \beta) + n(nr + z_1 - \gamma) = 0 \\ \text{i.e.} & r = (\alpha - x_1) \ell + (\beta - y_1) m + (\gamma - z_1) n \\ \text{since} & \ell^2 + m^2 + n^2 = 1 \end{split}$$

Putting this value of r in (ii), we get the foot of perpendicular from point A to the line.

**Length** : Since foot of perpendicular P is known, length of perpendicular,

$$AP = \sqrt{[(\ell r + x_1 - \alpha)^2 + (mr + y_1 - \beta)^2 + (nr + z_1 - \gamma)^2]}$$

Equation of perpendicular is given by

$$\frac{x-\alpha}{\ell r + x_1 - \alpha} = \frac{y-\beta}{mr + y_1 - \beta} = \frac{z-\gamma}{nr + z_1 - \gamma}$$

## 13. EQUATIONS OF A PLANE :

The equation of every plane is of the first degree i.e. of the form ax + by + cz + d = 0, in which a, b, c are constants, where  $a^2 + b^2 + c^2 \neq 0$  (i.e. a, b,  $c \neq 0$  simultaneously).

### (a) Vector form of equation of plane :

If  $\vec{a}$  be the position vector of a point on the plane and  $\vec{n}$  be a vector normal to the plane then it's vectorial equation is given

by  $(\vec{r} - \vec{a}).\vec{n} = 0 \Rightarrow \vec{r}.\vec{n} = d$  where  $d = \vec{a}.\vec{n} = constant$ .

#### (b) Plane Parallel to the Coordinate Planes :

- (i) Equation of y-z plane is x = 0.
- (ii) Equation of z-x plane is y = 0.
- (iii) Equation of x-y plane is z = 0.

(iv) Equation of the plane parallel to x-y plane at a distance c is z = c. Similarly, planes parallel to y-z plane and z-x plane are respectively x = c and y = c.

#### (c) Equations of Planes Parallel to the Axes :

If a = 0, the plane is parallel to x-axis i.e. equation of the plane parallel to x-axis is by + cz + d = 0.

Similarly, equations of planes parallel to y-axis and parallel to z-axis are ax + cz + d = 0 and ax + by + d = 0 respectively.

#### (d) Equation of a Plane in Intercept Form :

Equation of the plane which cuts off intercepts a, b, c from the

axes is  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ .

#### (e) Equation of a Plane in Normal Form :

If the length of the perpendicular distance of the plane from the origin is p and direction cosines of this perpendicular are (l, m, n), then the equation of the plane is lx + my + nz = p.

## (f) Vectorial form of Normal equation of plane :

If  $\hat{n}$  is a unit vector normal to the plane from the origin to the plane and d be the perpendicular distance of plane from origin then its vector equation is  $\vec{r} \cdot \hat{n} = d$ .

#### (g) Equation of a Plane through three points :

The equation of the plane through three non-collinear points

$$(\mathbf{x}_{1}, \mathbf{y}_{1}, \mathbf{z}_{1}), (\mathbf{x}_{2}, \mathbf{y}_{2}, \mathbf{z}_{2}) (\mathbf{x}_{3}, \mathbf{y}_{3}, \mathbf{z}_{3}) \text{ is } \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} & \mathbf{1} \\ \mathbf{x}_{1} & \mathbf{y}_{1} & \mathbf{z}_{1} & \mathbf{1} \\ \mathbf{x}_{2} & \mathbf{y}_{2} & \mathbf{z}_{2} & \mathbf{1} \\ \mathbf{x}_{3} & \mathbf{y}_{3} & \mathbf{z}_{3} & \mathbf{1} \end{vmatrix} = \mathbf{0}$$

#### 14. ANGLE BETWEEN TWO PLANES :

Consider two planes ax + by + cz + d = 0 and a'x + b'y + c'z + d' = 0. Angle between these planes is the angle between their normals.

$$\cos \theta = \frac{aa' + bb' + cc'}{\sqrt{a^2 + b^2 + c^2}\sqrt{a'^2 + b'^2 + c'^2}}$$

 $\therefore$  Planes are perpendicular if aa' + bb' + cc' = 0 and they are parallel if a/a' = b/b' = c/c'.

#### Planes parallel to a given Plane :

Equation of a plane parallel to the plane ax + by + cz + d = 0 is ax + by + cz + d' = 0. d' is to be found by other given condition.

#### 15. ANGLE BETWEEN A LINE AND A PLANE :

Let equations of the line and plane be  $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$  and ax + by + cz + d = 0 respectively and  $\theta$  be the angle which line makes with the plane. Then  $(\pi/2 - \theta)$  is the angle between the line and the normal to the plane.

So 
$$\sin\theta = \frac{a\ell + bm + cn}{\sqrt{a^2 + b^2 + c^2}\sqrt{\ell^2 + m^2 + n^2}}$$
  
Line is parallel to plane if  $\theta = 0$   
i.e. if al + bm + cn = 0.

Line is  $\perp$  to the plane if line is parallel to the normal of the plane

i.e. if 
$$\frac{a}{\ell} = \frac{b}{m} = \frac{c}{n}$$
.

# 16. CONDITION IN ORDER THAT THE LINE MAY LIE ON THE GIVEN PLANE :

The line  $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$  will lie on the plane Ax + By + Cz + D = 0 if (a) A $\ell$  + Bm + Cn = 0 and (b) Ax<sub>1</sub> + By<sub>1</sub> + Cz<sub>1</sub> + D = 0

#### 17. POSITION OF TWO POINTS W.R.T. A PLANE :

Two points  $P(x_1, y_1, z_1) \& Q(x_2, y_2, z_2)$  are on the same or opposite sides of a plane ax + by + cz + d = 0 according to  $ax_1 + by_1 + cz_1 + d \& ax_2 + by_2 + cz_2 + d$  are of same or opposite signs.

#### 18. IMAGE OF A POINT IN THE PLANE :

Let the image of a point  $P(x_1, y_1, z_1)$ R in a plane ax + by + cz + d = 0 is  $Q(x_2, y_2, z_2)$  and foot of perpendicular of point P on plane is  $R(x_3, y_3, z_3)$ , then (a)  $\frac{x_3 - x_1}{2} = \frac{y_3 - y_1}{b} = \frac{z_3 - z_1}{c} = -\left(\frac{ax_1 + by_1 + cz_1 + d}{c^2 + b^2 + c^2}\right)$ **(b)**  $\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{z_2 - z_1}{c} = -2\left(\frac{ax_1 + by_1 + cz_1 + d}{a^2 + b^2 + c^2}\right)$ 19. **CONDITION FOR COPLANARITY OF TWO LINES :** Let the two lines be  $\frac{\mathbf{x} - \alpha_1}{\ell_1} = \frac{\mathbf{y} - \beta_1}{\mathbf{m}_1} = \frac{\mathbf{z} - \gamma_1}{\mathbf{n}_1}$ ..... (i) and  $\frac{\mathbf{x} - \alpha_2}{\ell_2} = \frac{\mathbf{y} - \beta_2}{\mathbf{m}_2} = \frac{\mathbf{z} - \gamma_2}{\mathbf{n}_2}$ ..... (ii)  $\begin{array}{c|c} \text{These lines will coplanar if} & \begin{vmatrix} \alpha_2 - \alpha_1 & \beta_2 - \beta_1 & \gamma_2 - \gamma_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0 \end{array}$  $\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ \ell_1 & m_1 & n_1 \end{vmatrix} = 0$ the plane containing the two lines is  $\ell_2 m_2 n_4$ 

## 20. PERPENDICULAR DISTANCE OF A POINT FROM THE PLANE :

Perpendicular distance p, of the point  $A(x_1, y_1, z_1)$  from the plane ax + by + cz + d = 0 is given by

$$p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{(a^2 + b^2 + c^2)}}$$

Distance between two parallel planes  $ax + by + cz + d_1 = 0$ 

& ax + by + cz + d<sub>2</sub> = 0 is - 
$$\left| \frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}} \right|$$

## 21. A PLANE THROUGH THE LINE OF INTERSECTION OF TWO GIVEN PLANES :

Consider two planes

u = ax + by + cz + d = 0 and v = a' x + b' y + c' z + d' = 0.

The equation  $u + \lambda v = 0$ ,  $\lambda$  a real parameter, represents the plane passing through the line of intersection of given planes and if planes are parallel, this represents a plane parallel to them.

### 22. BISECTORS OF ANGLES BETWEEN TWO PLANES :

Let the equations of the two planes be ax + by + cz + d = 0 and  $a_1x + b_1y + c_1z + d_1 = 0$ .

Then equations of bisectors of angles between them are given by

$$\frac{ax + by + cz + d}{\sqrt{(a^2 + b^2 + c^2)}} = \pm \frac{a_1 x + b_1 y + c_1 z + d_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}}$$

- (a) Equation of bisector of the angle containing origin : First make both constant terms positive. Then +ve sign give the bisector of the angle which contains the origin.
- **(b) Bisector of acute/obtuse angle :** First making both constant terms positive,

 $\begin{array}{ll} aa_1 + bb_1 + cc_1 > 0 & \Rightarrow & \text{origin lies in obtuse angle} \\ aa_1 + bb_1 + cc_1 < 0 & \Rightarrow & \text{origin lies in acute angle} \end{array}$