Session 2

Transformation of Quadratic Equations, Condition for Common Roots

Transformation of Quadratic Equations

Let α, β be the roots of the equation $ax^2 + bx + c = 0$, then the equation

(i) whose roots are $\alpha + k, \beta + k$, is
$a(x-k)^2 + b(x-k) + c = 0$ [replace x by $(x-k)$]

(ii) whose roots are $\alpha - k, \beta - k$, is
$a(x+k)^2 + b(x+k) + c = 0$ [replace x by $(x+k)$]

(iii) whose roots are $\alpha k, \beta k$, is
$ax^2 + kbx + k^2c = 0$ $\left[\text{replace } x \text{ by} \left(\frac{x}{k}\right)\right]$

(iv) whose roots are $\frac{\alpha}{k}, \frac{\beta}{k}$, is
$ak^2x^2 + bkx + c = 0$ [replace x by xk]

(v) whose roots are $-\alpha, -\beta$, is
$ax^2 - bx + c = 0$ [replace x by $(-x)$]

(vi) whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}$, is
$cx^2 + bx + a = 0$ $\left[\text{replace } x \text{ by} \left(\frac{1}{x}\right)\right]$

(vii) whose roots are $-\frac{1}{\alpha}, -\frac{1}{\beta}$, is
$cx^2 - bx + a = 0$ $\left[\text{replace } x \text{ by} \left(-\frac{1}{x}\right)\right]$

(viii) whose roots are $\frac{k}{\alpha}, \frac{k}{\beta}$, is
$cx^2 + kbx + k^2a = 0$ $\left[\text{replace } x \text{ by} \left(\frac{k}{x}\right)\right]$

(ix) whose roots are $p\alpha + q, p\beta + q$, is
$a\left(\frac{x-q}{p}\right)^2 + b\left(\frac{x-q}{p}\right) + c = 0$ $\left[\text{replace } x \text{ by} \left(\frac{x-q}{p}\right)\right]$

(x) whose roots are $\alpha^n, \beta^n, n \in N$, is
$a(x^{1/n})^2 + b(x^{1/n}) + c = 0$ [replace x by $(x^{1/n})$]

(xi) whose roots are $\alpha^{1/n}, \beta^{1/n}, n \in N$ is
$a(x^n)^2 + b(x^n) + c = 0$ [replace x by (x^n)]

Example 19. If α, β be the roots of the equation $x^2 - px + q = 0$, then find the equation whose roots are $\frac{q}{p-\alpha}$ and $\frac{q}{p-\beta}$.

Sol. Let $\frac{q}{p-\alpha} = x \Rightarrow \alpha = p - \frac{q}{x}$

So, we replacing x by $p - \frac{q}{x}$ in the given equation, we get

$$\left(p - \frac{q}{x}\right)^2 - p\left(p - \frac{q}{x}\right) + q = 0$$

$$\Rightarrow \quad p^2 + \frac{q^2}{x^2} - \frac{2pq}{x} - p^2 + \frac{pq}{x} + q = 0$$

$$\Rightarrow \quad q - \frac{pq}{x} + \frac{q^2}{x^2} = 0$$

or $qx^2 - pqx + q^2 = 0$ or $x^2 - px + q = 0$

is the required equation whose roots are $\frac{q}{p-\alpha}$ and $\frac{q}{p-\beta}$.

Example 20. If α and β are the roots of $ax^2 + bx + c = 0$, then find the roots of the equation $ax^2 - bx(x-1) + c(x-1)^2 = 0$.

Sol. $\because$ $ax^2 - bx(x-1) + c(x-1)^2 = 0$ ...(i)

$$\Rightarrow \quad a\left(\frac{x}{x-1}\right)^2 - b\left(\frac{x}{x-1}\right) + c = 0$$

or $$a\left(\frac{x}{1-x}\right)^2 + b\left(\frac{x}{1-x}\right) + c = 0$$

Now, α, β are the roots of $ax^2 + bx + c = 0$.

Then, $\alpha = \frac{x}{1-x}$ and $\beta = \frac{x}{1-x}$

$$\Rightarrow \quad x = \frac{\alpha}{\alpha+1} \text{ and } x = \frac{\beta}{\beta+1}$$

Hence, $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$ are the roots of the Eq. (i).

Example 21. If α, β be the roots of the equation $3x^2 + 2x + 1 = 0$, then find value of $\left(\frac{1-\alpha}{1+\alpha}\right)^3 + \left(\frac{1-\beta}{1+\beta}\right)^3$.

Sol. Let $\frac{1-\alpha}{1+\alpha} = x \Rightarrow \alpha = \frac{1-x}{1+x}$

So, replacing x by $\frac{1-x}{1+x}$ in the given equation, we get

$$3\left(\frac{1-x}{1+x}\right)^2+2\left(\frac{1-x}{1+x}\right)+1=0 \Rightarrow x^2-2x+3=0 \quad \text{...(i)}$$

It is clear that $\frac{1-\alpha}{1+\alpha}$ and $\frac{1-\beta}{1+\beta}$ are the roots of Eq. (i).

$$\therefore \quad \left(\frac{1-\alpha}{1+\alpha}\right)+\left(\frac{1-\beta}{1+\beta}\right)=2 \quad \text{...(ii)}$$

and

$$\left(\frac{1-\alpha}{1+\alpha}\right)\left(\frac{1-\beta}{1+\beta}\right)=3 \quad \text{...(iii)}$$

$$\therefore \quad \left(\frac{1-\alpha}{1+\alpha}\right)^3+\left(\frac{1-\beta}{1+\beta}\right)^3=\left(\frac{1-\alpha}{1+\alpha}+\frac{1-\beta}{1+\beta}\right)^3-3$$

$$\left(\frac{1-\alpha}{1+\alpha}\right)\left(\frac{1-\beta}{1+\beta}\right)\left(\frac{1-\alpha}{1+\alpha}+\frac{1-\beta}{1+\beta}\right)=2^3-3\cdot3\cdot2=8-18=-10$$

Roots Under Special Cases

Consider the quadratic equation $ax^2+bx+c=0$ ...(i)

where $a, b, c \in R$ and $a \neq 0$. Then, the following hold good :

(i) If roots of Eq. (i) are equal in magnitude but opposite in sign, then sum of roots is zero as well as $D>0$, i.e. $b=0$ and $D>0$.

(ii) If roots of Eq. (i) are reciprocal to each other, then product of roots is 1 as well as $D \geq 0$ i.e., $a=c$ and $D \geq 0$.

(iii) If roots of Eq. (i) are of opposite signs, then product of roots <0 as well as $D>0$ i.e., $a>0, c<0$ and $D>0$ or $a<0, c>0$ and $D>0$.

(iv) If both roots of Eq. (i) are positive, then sum and product of roots >0 as well as $D \geq 0$ i.e., $a>0, b<0, c>0$ and $D \geq 0$ or $a<0, b>0, c<0$ and $D \geq 0$.

(v) If both roots of Eq. (i) are negative, then sum of roots <0, product of roots >0 as well as $D \geq 0$ i.e., $a>0, b>0, c>0$ and $D \geq 0$ or $a<0, b<0, c<0$ and $D \geq 0$.

(vi) If atleast one root of Eq. (i) is positive, then either one root is positive or both roots are positive i.e., point (iii) $\cup$ (iv).

(vii) If atleast one root of Eq. (i) is negative, then either one root is negative or both roots are negative i.e., point (iii) $\cup$ (v).

(viii) If greater root in magnitude of Eq. (i) is positive, then sign of $b=$ sign of $c \neq$ sign of a.

(ix) If greater root in magnitude of Eq. (i) is negative, then sign of $a=$ sign of $b \neq$ sign of c.

(x) If both roots of Eq. (i) are zero, then $b=c=0$.

(xi) If roots of Eq. (i) are 0 and $\left(-\frac{b}{a}\right)$, then $c=0$.

(xii) If roots of Eq. (i) are 1 and $\frac{c}{a}$, then $a+b+c=0$.

Example 22. ***For what values of*** m, ***the equation*** $x^2+2(m-1)x+m+5=0$ ***has*** $(m \in R)$

(i) roots are equal in magnitude but opposite in sign?
(ii) roots are reciprocals to each other?
(iii) roots are opposite in sign?
(iv) both roots are positive?
(v) both roots are negative?
(vi) atleast one root is positive?
(vii) atleast one root is negative?

Sol. Here, $a=1, b=2(m-1)$ and $c=m+5$

$\therefore \quad D=b^2-4ac=4(m-1)^2-4(m+5)$
$=4(m^2-3m-4)$

$\therefore \quad D=4(m-4)(m+1)$ and here $a=1>0$

(i) $b=0$ and $D>0$

$\Rightarrow 2(m-1)=0$ and $4(m-4)(m+1)>0$

$\Rightarrow m=1$ and $m \in(-\infty,-1) \cup(4, \infty)$

$\therefore \quad m \in \phi$ [null set]

(ii) $a=c$ and $D \geq 0$

$\Rightarrow 1=m+5$ and $4(m-4)(m+1) \geq 0$

$\Rightarrow m=-4$ and $m \in(-\infty,-1] \cup[4, \infty)$

$\therefore \quad m=-4$

(iii) $a>0, c<0$ and $D>0$

$\Rightarrow 1>0, m+5<0$ and $4(m-4)(m+1)>0$

$\Rightarrow m<-5$ and $m \in(-\infty,-1) \cup(4, \infty)$

$\therefore \quad m \in(-\infty,-5)$

(iv) $a>0, b<0, c>0$ and $D \geq 0$

$\Rightarrow 1>0, 2(m-1)<0, m+5>0$

and $4(m-4)(m+1) \geq 0$

$\Rightarrow m<1, m>-5$ and $m \in(-\infty,-1] \cup[4, \infty)$

$\Rightarrow m \in(-5,-1]$

(v) $a>0, b>0, c>0$ and $D \geq 0$

$\Rightarrow 1>0, 2(m-1)>0, m+5>0$

and $4(m-4)(m+1) \geq 0$

$\Rightarrow m>1, m>-5$ and $m \in(-\infty,-1] \cup[4, \infty)$

$\therefore \quad m \in[4, \infty)$

(vi) Either one root is positive or both roots are positive

i.e., (c) $\cup$ (d)

$\Rightarrow m \in(-\infty,-5) \cup(-5,-1]$

(vii) Either one root is negative or both roots are negative

i.e., (c) $\cup$ (e)

$\Rightarrow m \in(-\infty,-5) \cup[4, \infty)$

Condition for Common Roots

1. Only One Root is Common

Consider two quadratic equations

$$ax^2 + bx + c = 0 \text{ and } a'x^2 + b'x + c' = 0$$

[where $a, a' \neq 0$ and $ab' - a'b \neq 0$]

Let α be a common root, then

$$a\alpha^2 + b\alpha + c = 0 \text{ and } a'\alpha^2 + b'\alpha + c' = 0.$$

On solving these two equations by cross-multiplication, we have

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{ca' - c'a} = \frac{1}{ab' - a'b}$$

From first two relations, we get

$$\alpha = \frac{bc' - b'c}{ca' - c'a} \quad \text{...(i)}$$

and from last two relations, we get

$$\alpha = \frac{ca' - c'a}{ab' - a'b} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{bc' - b'c}{ca' - c'a} = \frac{ca' - c'a}{ab' - a'b}$$

$$\Rightarrow \quad (ab' - a'b)(bc' - b'c) = (ca' - c'a)^2$$

or $$\begin{vmatrix} a & b \\ a' & b' \end{vmatrix} \times \begin{vmatrix} b & c \\ b' & c' \end{vmatrix} = \begin{vmatrix} c & a \\ c' & a' \end{vmatrix}^2 \quad \text{[remember]}$$

This is the required condition for one root of two quadratic equations to be common.

2. Both Roots are Common

Let α, β be the common roots of the equations $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$, then

$$\alpha + \beta = -\frac{b}{a} = -\frac{b'}{a'} \Rightarrow \frac{a}{a'} = \frac{b}{b'} \quad \text{...(iii)}$$

and $$\alpha\beta = \frac{c}{a} = \frac{c'}{a'} \Rightarrow \frac{a}{a'} = \frac{c}{c'} \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$

This is the required condition for both roots of two quadratic equations to be identical.

Remark

To find the common root between the two equations, make the same coefficient of x^2 in both equations and then subtract of the two equations.

Example 23. Find the value of λ, so that the equations $x^2 - x - 12 = 0$ and $\lambda x^2 + 10x + 3 = 0$ may have one root in common. Also, find the common root.

Sol. $\because \quad x^2 - x - 12 = 0$

$\Rightarrow \quad (x-4)(x+3) = 0$

$\therefore \quad x = 4, -3$

If $x = 4$ is a common root, then

$$\lambda(4)^2 + 10(4) + 3 = 0$$

$\therefore \quad \lambda = -\dfrac{43}{16}$

and if $x = -3$ is a common root, then

$$\lambda(-3)^2 + 10(-3) + 3 = 0$$

$\therefore \quad \lambda = 3$

Hence, for $\lambda = -\dfrac{43}{16}$, common root is $x = 4$

and for $\lambda = 3$, common root is $x = -3$.

Example 24. ***If equations*** $ax^2 + bx + c = 0$, ***(where*** $a, b, c \in R$ ***and*** $a \neq 0$) ***and*** $x^2 + 2x + 3 = 0$ ***have a common root, then show that*** $a : b : c = 1 : 2 : 3$.

Sol. Given equations are

$$ax^2 + bx + c = 0 \quad \text{...(i)}$$

and $$x^2 + 2x + 3 = 0 \quad \text{...(ii)}$$

Clearly, roots of Eq. (ii) are imaginary, since Eqs. (i) and (ii) have a common root. Therefore, common root must be imaginary and hence both roots will be common.

Therefore, Eqs. (i) and (ii) are identical.

$\therefore \quad \dfrac{a}{1} = \dfrac{b}{2} = \dfrac{c}{3}$ or $a : b : c = 1 : 2 : 3$

Example 25. ***If*** a, b, c ***are in GP, show that the equations*** $ax^2 + 2bx + c = 0$ ***and*** $dx^2 + 2ex + f = 0$ ***have a common root, if*** $\dfrac{a}{d}, \dfrac{b}{e}, \dfrac{c}{f}$ ***are in HP.***

Sol. Given equations are

$$ax^2 + 2bx + c = 0 \quad \text{...(i)}$$

and $$dx^2 + 2ex + f = 0 \quad \text{...(ii)}$$

Since, a, b, c are in GP.

$\therefore \quad b^2 = ac$ or $b = \sqrt{ac}$

From Eq. (i), $ax^2 + 2\sqrt{ac}\,x + c = 0$

or $\quad (\sqrt{a}\,x + \sqrt{c})^2 = 0$ or $x = -\dfrac{\sqrt{c}}{\sqrt{a}}$

$\because$ Given Eqs. (i) and (ii) have a common root.

Hence, $x = -\dfrac{\sqrt{c}}{\sqrt{a}}$ also satisfied Eq. (ii), then

$$d\left(\frac{c}{a}\right) - 2e\frac{\sqrt{c}}{\sqrt{a}} + f = 0$$

$\Rightarrow$
$$\frac{d}{a} - \frac{2e}{\sqrt{ac}} + \frac{f}{c} = 0$$

or
$$\frac{d}{a} - \frac{2e}{b} + \frac{f}{c} = 0 \qquad [\because b = \sqrt{ac}]$$

or
$$\frac{d}{a} + \frac{f}{c} = \frac{2e}{b}$$

$\therefore$ $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in AP.

Hence, $\frac{a}{d}, \frac{b}{e}, \frac{c}{f}$ are in HP.

Exercise for Session 2

1. If α and β are the roots of the equation $2x^2 - 3x + 4 = 0$, then the equation whose roots are α^2 and β^2, is

(a) $4x^2 + 7x + 16 = 0$ (b) $4x^2 + 7x + 6 = 0$ (c) $4x^2 + 7x + 1 = 0$ (d) $4x^2 - 7x + 16 = 0$

2. If α, β are the roots of $x^2 - 3x + 1 = 0$, then the equation whose roots are $\left(\frac{1}{\alpha - 2}, \frac{1}{\beta - 2}\right)$, is

(a) $x^2 + x - 1 = 0$ (b) $x^2 + x + 1 = 0$ (c) $x^2 - x - 1 = 0$ (d) None of these

3. The equation formed by decreasing each root of $ax^2 + bx + c = 0$ by 1 is $2x^2 + 8x + 2 = 0$, then

(a) $a = -b$ (b) $b = -c$ (c) $c = -a$ (d) $b = a + c$

4. If the roots of equation $\frac{x^2 - bx}{ax - c} = \frac{m - 1}{m + 1}$ are equal but opposite in sign, then the value of m will be

(a) $\frac{a - b}{a + b}$ (b) $\frac{b - a}{a + b}$ (c) $\frac{a + b}{a - b}$ (d) $\frac{b + a}{b - a}$

5. If $x^2 + px + q = 0$ is the quadratic equation whose roots are $a - 2$ and $b - 2$, where a and b are the roots of $x^2 - 3x + 1 = 0$, then

(a) $p = 1, q = 5$ (b) $p = 1, q = -5$ (c) $p = -1, q = 1$ (d) None of these

6. If both roots of the equation $x^2 - (m - 3)x + m = 0$ $(m \in R)$ are positive, then

(a) $m \in (3, \infty)$ (b) $m \in (-\infty, 1]$ (c) $m \in [9, \infty)$ (d) $m \in (1, 3)$

7. If the equation $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$, where $m \in R \sim \{-1\}$, has atleast one root is negative, then

(a) $m \in (-\infty, -1)$ (b) $m \in \left(-\frac{1}{8}, \infty\right)$ (c) $m \in \left(-1, -\frac{1}{8}\right)$ (d) $m \in R$

8. If both the roots of $\lambda(6x^2 + 3) + rx + 2x^2 - 1 = 0$ and $6\lambda(2x^2 + 1) + px + 4x^2 - 2 = 0$ are common, then $2r - p$ is equal to

(a) -1 (b) 0 (c) 1 (d) 2

9. If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root $a \neq 0$, then $\frac{a^3 + b^3 + c^3}{abc}$ is equal to

(a) 1 (b) 2 (c) 3 (d) None of these

10. If $a(p + q)^2 + 2bpq + c = 0$ and $a(p + r)^2 + 2bpr + c = 0$, then qr is equal to

(a) $p^2 + \frac{c}{a}$ (b) $p^2 + \frac{a}{c}$ (c) $p^2 + \frac{a}{b}$ (d) $p^2 + \frac{b}{a}$

Answers

Exercise for Session 2

1.(a) **2.** (c) **3.** (b) **4.** (a) **5.**(d) **6.** (c)
7. (c) **8.** (b) **9.** (c) **10.** (a)