Chapter 7

Cubes and Cube Roots

Introduction to Cube and Cube Roots

Cube

The cube of a number is that number raised to the power 3.

Or

When a number is multiplied three times by itself, we can say that the number has been cubed and the product is called cube of that number.

If a is a number, then the cube of a is $a^3 = a \times a \times a$.

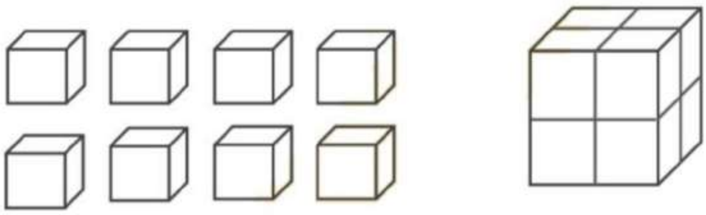
For example,

a) $2^3 = 2 \times 2 \times 2 = 8$. Thus, cube of 2 is 8.

b) $8^3 = 8 \times 8 \times 8 = 512$. Thus, cube of 8 is 512.

A cube is a three-dimensional structure that is formed when six identical squares bind to each other in an enclosed form.

$2^3 = 2 \times 2 \times 2 = 8$.



Perfect Cube

A natural number is said to be a perfect cube if it is the cube of some natural number.

A natural number n is a perfect cube if it is the cube of some natural number m. i.e., $n = m^3$.

2	216
2	108
2	54
3	27
3	9
3	3
	1

For example,

A given natural number is a perfect cube if it can be expressed as the product of triplets of equal factors.

For example,

216 is a perfect cube as there is an integer 6

such that

3	189
3	63
3	21
7	7
	1

$$216 = 6 \times 6 \times 6 = 6^3.$$

$$216 = 2 \times 2 \times 2 \times 3 \times 3 \times 3 = 2 \times 3 = 6.$$

Show that 189 is not a perfect cube.

Resolving 189 into prime factors, we get:

$$189 = 3 \times 3 \times 3 \times 7$$

Making the triplets, we find that one triplet is formed and we are left with one more factor.

Thus, 189 cannot be expressed as a product of triplets.

Hence, 189 is not a perfect cube.

Cube of first 20 numbers:

Number	Square	Number	Square
1	$1^3 = 1 \times 1 \times 1 = 1$	11	$11^3 = 11 \times 11 \times 11 = 1331$
2	$2^3 = 2 \times 2 \times 2 = 8$	12	$12^3 = 12 \times 12 \times 12 = 1728$
3	$3^3 = 3 \times 3 \times 3 = 27$	13	$13^3 = 13 \times 13 \times 13 = 2197$
4	$4^3 = 4 \times 4 \times 4 = 64$	14	$14^3 = 14 \times 14 \times 14 = 2744$
5	$5^3 = 5 \times 5 \times 5 = 125$	15	$15^3 = 15 \times 15 \times 15 = 3375$
6	$6^3 = 6 \times 6 \times 6 = 216$	16	$16^3 = 16 \times 16 \times 16 = 4096$
7	$7^3 = 7 \times 7 \times 7 = 343$	17	$17^3 = 17 \times 17 \times 17 = 4913$
8	$8^3 = 8 \times 8 \times 8 = 512$	18	$18^3 = 18 \times 18 \times 18 = 5832$
9	$9^3 = 9 \times 9 \times 9 = 729$	19	$19^3 = 19 \times 19 \times 19 = 6859$
10	$10^3 = 10 \times 10 \times 10 = 1000$	20	$20^3 = 20 \times 20 \times 20 = 8000$

Some Interesting Patterns

Some Interesting patterns

1. Adding consecutive odd numbers

Observe the following pattern of sums of odd numbers.

a) $1 = 1 = 1^3$

b) $3 + 5 = 8 = 2^3$

c) $7 + 9 + 11 = 27 = 3^3$

d) $13 + 15 + 17 + 19 = 64 = 4^3$

e) $21 + 23 + 25 + 27 + 29 = 125 = 5^3$

f) $31 + 33 + 35 + 37 + 39 + 41 = 216 = 6^3$

2. Cubes and their Prime Factors

Consider the following prime factorization of the numbers and their cubes.

Prime Factors of Number	Prime Factorization of the cube
$4 = 2 \times 2$	$4^3 = 64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^3 \times 2^3$
$6 = 2 \times 3$	$6^3 = 216 = 2 \times 2 \times 2 \times 3 \times 3 \times 3 = 2^3 \times 3^3$
$15 = 3 \times 5$	$15^3 = 3375 = 3 \times 3 \times 3 \times 5 \times 5 \times 5 = 3^3 \times 5^3$
$12 = 2 \times 2 \times 3$	$17^3 = 1728 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3 = 2^3 \times 2^3 \times 3^3$

Properties of Cubes of numbers

1. Cubes of all even numbers are even.
   For example,
   Cube of 12.
   $(12)^3 = 12 \times 12 \times 12 = 1728$ (even)
2. Cubes of all odd numbers are even.
   For example,
   Cube of 13.
   $(13)^3 = 13 \times 13 \times 13 = 2197$ (odd)

3. The sum of the cubes of first n natural numbers is equal to the square of their sum.

i.e., $1^3 + 2^3 + 3^3 + \dots\dots\dots + n^3 = (1 + 2 + 3 + \dots\dots\dots + n)^2 = (\frac{n(n+1)}{2})^2$

For example,

Find the value of $1^3 + 2^3 + \dots\dots\dots + 7^3$.

$1^3 + 2^3 + \dots\dots\dots + 7^3 = (1 + 2 + \dots\dots\dots + n)^2$

$= (7(\frac{7+1}{2}))^2 = (7 \times 4)^2 = (28)^2 = 784$

4. Cubes of the digits 1 to 9.

x	1	2	3	4	5	6	7	8	9
$x3$	1	8	27	64	125	216	343	512	729

If the number ends in 1, 4, 5, 6, 9 then its cube root also ends in 1, 4, 5, 6, 9.

Cube of 2 ends in 8 and cube of 8 ends in 2.

Cube of 3 ends in 7 and cube of 7 ends in 3.

Smallest multiple that makes any number a perfect Cube.

What is the smallest multiple by which 441 may be multiplied so that the product is a perfect cube?

3	441
3	147
7	49
7	7
	1

$441 = 3 \times 3 \times 7 \times 7$

We find that 7 and 3 occur only twice.

If we multiply 441 by 3 and 7 then,

$9261 = 3 \times 3 \times 3 \times 7 \times 7 \times 7$

To make 441 a perfect cube it must be multiplied by 7 and 3 that is 21.

Cubes of Negative Integers

The cube of a negative integer is always negative.

$(-1)^3 = (-1) \times (-1) \times (-1) = -1$

$(-2)^3 = (-2) \times (-2) \times (-2) = -8$

Cubes of Rational Numbers

$$\left(\frac{a}{b}\right)^3 = \frac{a \times a \times a}{b \times b \times b} = \frac{a^3}{b^3}$$

$$\left(\frac{3}{5}\right)^3 = \frac{3 \times 3 \times 3}{5 \times 5 \times 5} = \frac{3^3}{5^3}$$

Cube Roots

Cube Roots

The cube root of a number 'n' is that number 'm' whose cube gives 'n'.

Or

A number m is the cube root of a number n if $n = m^3$.

The cube root of a number n is denoted by $\sqrt[3]{n}$. $\sqrt[3]{n}$ is also called a radical, n is called the radicand and 3 is called the index of the radical.

For example,

We have,

$8 = 2^3 \quad \therefore \quad \sqrt[3]{8} = 2$

$343 = 7^3 \quad \therefore \quad \sqrt[3]{343} = 7$

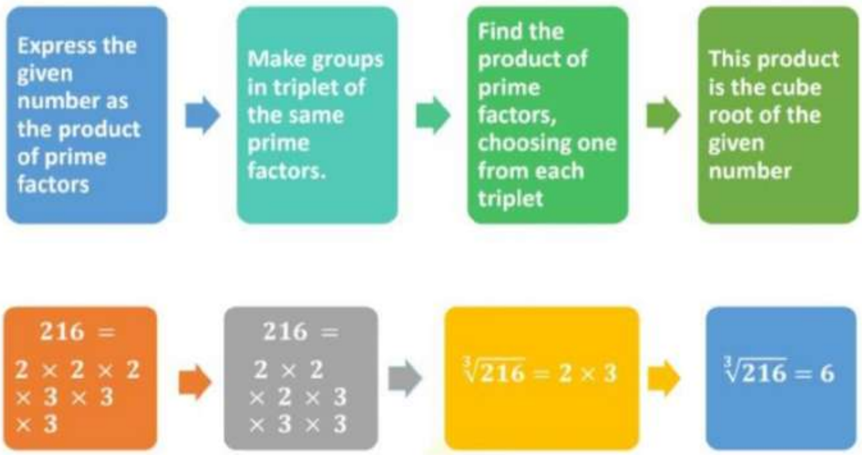
$-125 = (-5)^3 \quad \therefore \quad \sqrt[3]{-125} = -5$

$0.008 = (0.2)^3 \quad \therefore \quad \sqrt[3]{0.008} = 0.2$

$\frac{64}{125} = (\frac{4}{5})^3 \quad \therefore \quad \frac{3\sqrt{64}}{125} = \frac{4}{5}$

The symbol $\sqrt[3]{}$ for the cube root is very much similar to the symbol for square root. The only difference is that whereas in the case of square root, we use the symbol ($\sqrt{}$) for the cube root we use the same symbol ($\sqrt{}$) but with a 3 which indicates that we are taking a cube root.

Cube Root through prime factorization method



Find the cube root of 2197.

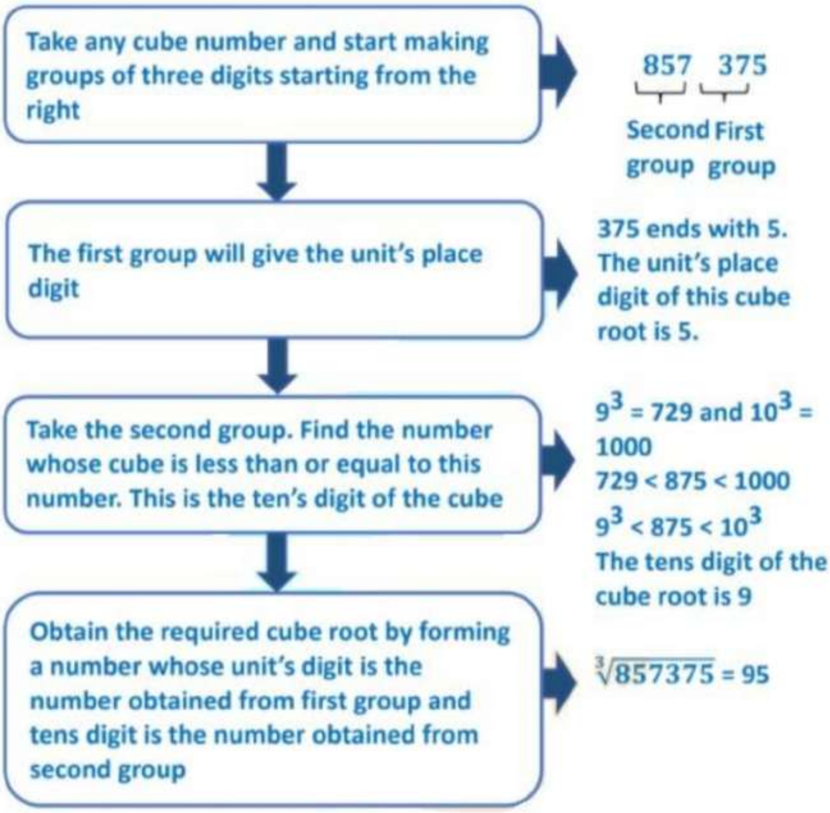
Cube root of 3375 = $\sqrt[3]{2197}$

$2197 = 13 \times 13 \times 13$

$\sqrt[3]{2197} = \sqrt[3]{13 \times 13 \times 13}$

$\sqrt[3]{2197} = 13$

13	2197
13	169
13	13
	1



Find the cube root of 4913 through Estimation.
4 913
1st group → 913
The digit in the unit place of the cube root is 7
2nd group → 4
$1^3 = 1$ and $2^3 = 8$
$1 < 4 < 8$
$1^3 < 4 < 2^3$
Therefore, the digit in the tens place will be 1

Cube root of the product of integers For any integer a and b,

For any integer a and b,

$\sqrt[3]{ab} = (\sqrt[3]{a} \times \sqrt[3]{b})$

Evaluate:

$\sqrt[3]{216 \times (-343)} = \sqrt[3]{216} \times \sqrt[3]{-343}$

$= \sqrt[3]{6 \times 6 \times 6} \times \sqrt[3]{(-7) \times (-7) \times (-7)}$

$= 6 \times (-7)$

$= -42$

Cube root of a Rational Number

For any integer a and b,

$\sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}$

Evaluate:

$\frac{\sqrt[3]{216}}{2197} = \frac{\sqrt[3]{216}}{\sqrt[3]{2197}} = \frac{\sqrt[3]{6X6X6}}{\sqrt[3]{13X13X13}}$

Cube Root of Negative Numbers
Let a be a positive integer, then $(-a)$ is a negative integer.

$\sqrt[3]{-a^3}$ = -a

Find the cube root of (-1000).

$1000 = 2 \times 2 \times 2 \times 5 \times 5 \times 5$

$\sqrt[3]{1000} = (2 \times 5) = 10$

$\sqrt[3]{-1000} = -10$