Question Set 5

VECTORS

(Marks with option : 12)

Remember :

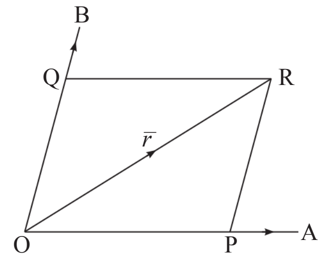
1. If $\bar{a}$ and $\bar{b}$ are collinear, then $\bar{a} = m\bar{b}$, where m is a non-zero scalar.
2. If $\bar{a}, \bar{b}, \bar{c}$ are coplanar vectors, then any one of them can be expressed as the linear combination of the other two.
3. Unit vector $\hat{a}$ along the vector $\bar{a}$ is given by $\hat{a} = \dfrac{\bar{a}}{|\bar{a}|}$.
4. If $\bar{a}$ and $\bar{b}$ are the position vectors of A and B respectively, then $\overline{AB} = \bar{b} - \bar{a}$.
5. If $\bar{a}, \bar{b}, \bar{c}$ are position vectors of the points A, B, C respectively, then the position vector of :
   (i) the point P dividing seg AB internally in the ratio $m : n$ is given by $\bar{p} = \dfrac{m\bar{b} + n\bar{a}}{m + n}$
   (ii) the point Q dividing seg AB externally in the ratio $m : n$ is given by $\bar{q} = \dfrac{m\bar{b} - n\bar{a}}{m - n}$
   (iii) midpoint M of seg AB is $\bar{m} = \dfrac{\bar{a} + \bar{b}}{2}$
   (iv) centroid G of $\triangle ABC$ is $\bar{g} = \dfrac{\bar{a} + \bar{b} + \bar{c}}{3}$.
6. If $H(\bar{h})$ is incentre of $\triangle ABC$, then $\bar{h} = \dfrac{|\overline{BC}|\,\bar{a} + |\overline{AC}|\,\bar{b} + |\overline{AB}|\,\bar{c}}{|\overline{BC}| + |\overline{AC}| + |\overline{AB}|}$.

5.1 REPRESENTATION OF VECTOR, COLLINEARITY, COPLANARITY, SECTION FORMULA

Theory Questions	3 or 4 marks each

Q. 1. If $\bar{a}$ and $\bar{b}$ any two non-zero, non-collinear vectors lying in the same plane, then prove that any vector $\bar{r}$ coplanar with them can be uniquely expressed as $\bar{r} = t_1\bar{a} + t_2\bar{b}$, where t_1 and t_2 are scalars.

Proof : Take any point O in the plane of $\bar{a}$, $\bar{b}$ and $\bar{r}$. Represents the vectors $\bar{a}$, $\bar{b}$ and $\bar{r}$ by $\overline{OA}$, $\overline{OB}$ and $\overline{OR}$. Take the point P on $\bar{a}$ and Q on $\bar{b}$ such that OPRQ is a parallelogram.



Now, $\overline{OP}$ and $\overline{OA}$ are collinear vectors.

$\therefore$ there exists a non-zero scalar t_1 such that

$\overline{OP} = t_1 \cdot \overline{OA} = t_1 \cdot \bar{a}$.

Also $\overline{OQ}$ and $\overline{OB}$ are collinear vectors.

$\therefore$ there exists a non-zero scalar t_2 such that

$\overline{OQ} = t_2 \cdot \overline{OB} = t_2 \cdot \bar{b}$.

Now, by parallelogram law of addition of vectors,

$\overline{OR} = \overline{OP} + \overline{OQ} \qquad \therefore\ \bar{r} = t_1\bar{a} + t_2\bar{b}$

Thus, $\bar{r}$ is expressed as a linear combination $t_1\bar{a} + t_2\bar{b}$.

Uniqueness :

Let, if possible, $\bar{r} = t_1'\,\bar{a} + t_2'\,\bar{b}$, where t_1', t_2' are non-zero scalars. Then

$t_1\bar{a} + t_2\bar{b} = t_1'\,\bar{a} + t_2'\,\bar{b}$

$\therefore\ (t_1 - t_1')\,\bar{a} = -(t_2 - t_2')\,\bar{b}$ ... (1)

We want to show that $t_1 = t_1'$ and $t_2 = t_2'$.

Suppose $t_1 \neq t_1'$, i.e. $t_1 - t_1' \neq 0$ and $t_2 \neq t_2'$, i.e. $t_2 - t_2' \neq 0$.

Then dividing both sides of (1) by $t_1 - t_1'$, we get

$$\bar{a} = -\left(\frac{t_2 - t_2'}{t_1 - t_1'}\right)\bar{b}$$

This shows that the vector $\bar{a}$ is a non-zero scalar multiple of $\bar{b}$.

$\therefore$ $\bar{a}$ and $\bar{b}$ are collinear vectors.

This is a contradiction, since $\bar{a}$, $\bar{b}$ are given to be non-collinear.

$\therefore\ t_1 = t_1'$

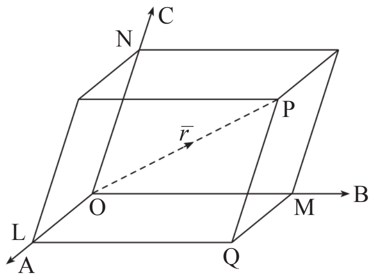
Similarly, we can show that $t_2 = t_2'$.

This shows that $\bar{r}$ is uniquely expressed as a linear combination $t_1\bar{a} + t_2\bar{b}$.

Q. 2. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are three non-zero, non-coplanar vectors, prove that any vector $\bar{r}$ in space can be uniquely expressed as a linear combination $x\bar{a} + y\bar{b} + z\bar{c}$, where x, y, z are scalars.

Proof : Let $\overline{OP}$ represent the vector $\bar{r}$ and $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ represent the three non-coplanar vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ respectively.

Through P draw planes parallel to the planes BOC, COA and AOB intersecting the lines OA, OB and OC in L, M and N respectively.



Now, $\overline{OL}$ and $\bar{a}$ are collinear vectors. Hence, there exists a non-zero scalar x such that $\overline{OL} = x\bar{a}$.

Similarly, $\overline{OM}$ and $\bar{b}$ are collinear and $\overline{ON}$ and $\bar{c}$ are collinear. Hence, there exist non-zero scalars y and z such that $\overline{OM} = y\bar{b}$ and $\overline{ON} = z\bar{c}$.

Now, $\overline{OP} = \overline{OL} + \overline{LP}$

$\therefore\ \bar{r} = \overline{OL} + \overline{LQ} + \overline{QP}$

$\quad = \overline{OL} + \overline{OM} + \overline{ON}$

$\quad = x\bar{a} + y\bar{b} + z\bar{c}.$

Thus, $\bar{r}$ is expressed as a linear combination $x\bar{a} + y\bar{b} + z\bar{c}$.

Uniqueness :

Let, if possible, $\bar{r} = x'\bar{a} + y'\bar{b} + z'\bar{c}$, where $x',\ y',\ z'$ are scalars. Then

$$x\bar{a} + y\bar{b} + z\bar{c} = x'\bar{a} + y'\bar{b} + z'\bar{c}$$

$$\therefore\ (x - x')\bar{a} + (y - y')\bar{b} = (z' - z)\bar{c} \qquad \text{... (1)}$$

We note that uniqueness of the linear combination for $\bar{r}$ will be established if we show that $x = x'$, $y = y'$ and $z = z'$.

Suppose on the contrary that $z \neq z'$, i.e. $z' - z \neq 0$.

Then dividing both sides of (1) by $z' - z\ (\neq 0)$, we get

$$\bar{c} = \left(\frac{x - x'}{z' - z}\right)\bar{a} + \left(\frac{y - y'}{z' - z}\right)\bar{b}$$

This shows that $\bar{c}$ is expressed as a linear combination of $\bar{a}$ and $\bar{b}$.

$\therefore\ \bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar. This is a contradiction, since $\bar{a}$, $\bar{b}$ and $\bar{c}$ are given to be non-coplanar.

$\therefore\ z = z'$

Similarly, we can show that $x = x'$ and $y = y'$.

This proves the uniqueness of the linear combination $x\bar{a} + y\bar{b} + z\bar{c}$.

Q. 3. Prove that three non-zero vectors $\bar{a}, \bar{b}, \bar{c}$ are coplanar if and only if there exist scalars x, y, z, not all zero simultaneously such that $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$.

Proof : Let $\bar{a}, \bar{b}, \bar{c}$ be coplanar vectors.

Case I : Suppose that any two of $\bar{a}$, $\bar{b}$ and $\bar{c}$ are collinear vectors, say $\bar{a}$ and $\bar{b}$.

$\therefore$ there exist scalars x and y at least one of which is non-zero such that $x\bar{a} + y\bar{b} = \bar{0}$.

$\therefore$ $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ is required non-zero linear combination where $z = 0$.

Case II : None of the two vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are collinear.

Then any one of them, say $\bar{a}$, will be the linear combination of $\bar{b}$ and $\bar{c}$.

$\therefore$ there exist scalars α and β such that $\bar{a} = \alpha\bar{b} + \beta\bar{c}$

$\therefore$ $(-1)\bar{a} + \alpha\bar{b} + \beta\bar{c} = \bar{0}$, i.e. $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$

where $x = -1, y = \alpha, z = \beta$ which are not all zero simultaneously.

Conversely : Let there exist scalars x, y, z not all zero such that

$x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ ... (1)

Let $x \neq 0$, then divide (1) by x, we get

i.e. $\bar{a} + \left(\frac{y}{x}\right)\bar{b} + \left(\frac{z}{x}\right)\bar{c} = \bar{0}$ $\therefore$ $\bar{a} = \left(-\frac{y}{x}\right)\bar{b} + \left(-\frac{z}{x}\right)\bar{c}$

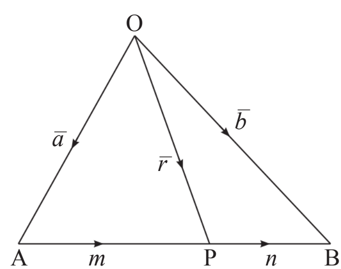
i.e. $\bar{a} = \alpha\bar{b} + \beta\bar{c}$, where $\alpha = -\frac{y}{x}$ and $\beta = -\frac{z}{x}$ are scalars.

$\therefore$ $\bar{a}$ is the linear combination of $\bar{b}$ and $\bar{c}$.

Hence, $\bar{a}, \bar{b}, \bar{c}$ are coplanar.

Q. 4. If $\bar{a}$, $\bar{b}$ and $\bar{r}$ are position vectors of the points A, B and R respectively and R divides the line segment AB internally in the ratio $m : n$, then prove that $\bar{r} = \frac{m\bar{b} + n\bar{a}}{m + n}$. ***(March '22)***

Proof : Let $\bar{a}, \bar{b}$ and $\bar{r}$ be the position vectors of the points A, B and R respectively w.r.t. some origin O. Then,



$\overline{AR}$ = (p.v. of R) − (p.v. of A) = $\bar{r} - \bar{a}$

$\overline{RB}$ = (p.v. of B) − (p.v. of R) = $\bar{b} - \bar{r}$

R divides seg AB internally in the ratio $m : n$

$\therefore$ $\frac{AR}{RB} = \frac{m}{n}$

$\therefore$ $n \cdot AR = m \cdot RB$ and A–R–B

Now $\overline{AR}$ and $\overline{RB}$ are in the same direction

$\therefore\ n\cdot\overline{AR} = m\cdot\overline{RB}$

$\therefore\ n\cdot(\bar{r}-\bar{a}) = m\cdot(\bar{b}-\bar{r})$

$\therefore\ n\cdot\bar{r} - n\cdot\bar{a} = m\cdot\bar{b} - m\cdot\bar{r}$

$\therefore\ m\cdot\bar{r} + n\cdot\bar{r} = m\cdot\bar{b} + n\cdot\bar{a}$

$\therefore\ (m+n)\bar{r} = m\bar{b} + n\bar{a}$

$\therefore\ \bar{r} = \dfrac{m\bar{b}+n\bar{a}}{m+n}.$

Solved Examples	2 marks each

Ex. 1. Find a vector in the direction of $\bar{a} = \hat{i} - 2\hat{j}$ that has magnitude 7 units.

Solution : $\bar{a} = \hat{i} - 2\hat{j}$

$\therefore\ |\bar{a}| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$

Unit vector in direction of $\bar{a} = \hat{a} = \dfrac{\bar{a}}{|\bar{a}|} = \dfrac{\hat{i}-2\hat{j}}{\sqrt{5}}$

$\therefore$ vector of magnitude 7 in the direction of $\bar{a}$ is

$7\hat{a} = 7\left(\dfrac{\hat{i}-2\hat{j}}{\sqrt{5}}\right) = \dfrac{7}{\sqrt{5}}\hat{i} - \dfrac{14}{\sqrt{5}}\hat{j}.$

Ex. 2. If $\bar{a}$, $\bar{b}$, $\bar{c}$ are the position vectors of the points A, B, C respectively and $5\bar{a} - 3\bar{b} - 2\bar{c} = \bar{0}$, then find the ratio in which the point C divides line segment BA.

Solution : $5\bar{a} - 3\bar{b} - 2\bar{c} = \bar{0}$

$\therefore\ 2\bar{c} = 5\bar{a} - 3\bar{b}$

$\therefore\ \bar{c} = \dfrac{5\bar{a}-3\bar{b}}{2} = \dfrac{5\bar{a}-3\bar{b}}{5-3}$

$\therefore$ C divides line segment BA externally in the ratio 5 : 3.

Ex. 3. If the vectors $2\hat{i} - q\hat{j} + 3\hat{k}$ and $4\hat{i} - 5\hat{j} + 6\hat{k}$ are collinear, find q.

Solution : The vectors $2\hat{i} - q\hat{j} + 3\hat{k}$ and $4\hat{i} - 5\hat{j} + 6\hat{k}$ are collinear.

$\therefore$ the coefficients of $\hat{i}, \hat{j}, \hat{k}$ are proportional.

$\therefore\ \dfrac{2}{4} = \dfrac{-q}{-5} = \dfrac{3}{6}$

$\therefore\ \dfrac{q}{5} = \dfrac{1}{2}\quad \therefore\ q = \dfrac{5}{2}.$

Ex. 4. If $\bar{a} = \hat{i} + 2\hat{j}$, $\bar{b} = -2\hat{i} + \hat{j}$, $\bar{c} = 4\hat{i} + 3\hat{j}$, find x and y such that $\bar{c} = x\bar{a} + y\bar{b}$.

Solution :

$\bar{c} = x\bar{a} + y\bar{b}$

$\therefore\ 4\hat{i} + 3\hat{j} = x(\hat{i} + 2\hat{j}) + y(-2\hat{i} + \hat{j})$

$\therefore\ 4\hat{i} + 3\hat{j} = (x - 2y)\hat{i} + (2x + y)\hat{j}$

By equality of vectors, we get

$x - 2y = 4$ ... (1)

and $2x + y = 3$ . .. (2)

Multiplying equation (2) by 2, we get

$4x + 2y = 6$ ... (3)

Adding (1) and (3), we get

$5x = 10 \qquad \therefore\ x = 2$

Substituting $x = 2$ in equation (2), we get

$2(2) + y = 3$

$\therefore\ y = 3 - 4 = -1$

Hence, $x = 2$, $y = -1$.

Ex. 5. Find the position vector of point R which divides the line joining the points P and Q whose position vectors are $2\hat{i} - \hat{j} + 3\hat{k}$ and $-5\hat{i} + 2\hat{j} - 5\hat{k}$ in the ratio 3 : 2 externally.

Solution : It is given that the points P and Q have position vectors

$\bar{p} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\bar{q} = -5\hat{i} + 2\hat{j} - 5\hat{k}$ respectively.

If R$(\bar{r})$ divides the line segment joining P and Q externally in the ratio 3 : 2, by section formula for external division

$$\bar{r} = \frac{3\bar{q} - 2\bar{p}}{3 - 2} = \frac{3(-5\hat{i} + 2\hat{j} - 5\hat{k}) - 2(2\hat{i} - \hat{j} + 3\hat{k})}{3 - 2}$$

$$= -19\hat{i} + 8\hat{j} - 21\hat{k}$$

Hence, the position vector of R is $-19\hat{i} + 8\hat{j} - 21\hat{k}$.

Ex. 6. Show that the points A = (3, 2, − 4), B = (9, 8, − 10) and C = (− 2, − 3, 1) are collinear. ***(Sept. '21)***

Solution : Let $\bar{a}$, $\bar{b}$, $\bar{c}$ be the position vectors of the points.

A = (3, 2, − 4), B = (9, 8, − 10) and C = (− 2, − 3, 1) respectively.

Then $\bar{a} = 3\hat{i} + 2\hat{j} - 4\hat{k}$, $\bar{b} = 9\hat{i} + 8\hat{j} - 10\hat{k}$, $\bar{c} = -2\hat{i} - 3\hat{j} + \hat{k}$

$\overline{AB} = \bar{b} - \bar{a}$

$= (9\hat{i} + 8\hat{j} - 10\hat{k}) - (3\hat{i} + 2\hat{j} - 4\hat{k})$

$= 6\hat{i} + 6\hat{j} - 6\hat{k}$ ... (1)

and $\overline{BC} = \bar{c} - \bar{b}$

$= (-2\hat{i} - 3\hat{j} + \hat{k}) - (9\hat{i} + 8\hat{j} - 10\hat{k})$

$= -11\hat{i} - 11\hat{j} + 11\hat{k}$

$= -11(\hat{i} + \hat{j} - \hat{k}) = -\frac{11}{6}(6\hat{i} + 6\hat{j} - 6\hat{k})$

$= -\frac{11}{6}\overline{AB}$ ... [By (1)]

$\therefore$ $\overline{BC}$ is a non-zero scalar multiple of $\overline{AB}$.

$\therefore$ they are parallel to each other.

But they have the point B in common.

$\therefore$ $\overline{BC}$ and $\overline{AB}$ are collinear vectors.

Hence, the points A, B and C are collinear.

Examples for Practice	**2 marks each**

1. If $\overline{AB} = 2\hat{i} - 4\hat{j} + 7\hat{k}$ and initial point $A = (1, 5, 0)$, find the terminal point B.

2. Find a unit vector in the direction of $\bar{u}$, where $\bar{u} = 8\hat{i} + 3\hat{j} - \hat{k}$.

3. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are the position vectors of the points A, B and C respectively, such that

(1) $3\bar{a} + 5\bar{b} = 8\bar{c}$, find the ratio in which C divides line segment AB.

(2) $3\bar{a} + 5\bar{b} - 8\bar{c} = \bar{0}$, find the ratio in which A divides BC.

(3) $2\bar{a} + 3\bar{b} - 5\bar{c} = \bar{0}$, find the ratio in which C divides AB.

4. If two of the vertices of a triangle are A (3, 1, 4) and B (−4, 5, −3) and the centroid of the triangle is at G (−1, 2, 1), then find the coordinates of the third vertex C of the triangle.

5. If the vectors $3\hat{i} - 5\hat{j} + \hat{k}$ and $9\hat{i} - 15\hat{j} + p\hat{k}$ are collinear, then find the value of p.

6. A and B are two points. The position vector of the point A is $6\hat{i} - 2\hat{j}$. A point P divides AB internally in the ratio 3 : 2. If $\hat{i} - \hat{j} + \hat{k}$ is the position vector of P, then find the position vector of the point B.

7. Find the coordinates of the point which divides the line segment joining the points $A(2, -6, 8)$ and $B(-1, 3, -4)$ externally in the ratio 1 : 3.

8. Let PQRS be a quadrilateral. If M and N are the midpoints of the sides PQ and RS respectively, then prove that $\overline{PS} + \overline{QR} = 2\overline{MN}$.

9. If G $(a, 2, -1)$ is the centroid of the triangle with vertices P (1, 3, 2), Q $(3, b, -4)$ and R $(5, 1, c)$, then find the values of a, b and c.

10. If two non-zero vectors $\overline{a}$ and $\overline{b}$ are collinear, then prove that there exist scalars m and n such that $m\overline{a} + n\overline{b} = \overline{0}$ and $(m, n) \neq (0, 0)$. ***(Sept. '21)***

ANSWERS

1. (3, 1, 7) **2.** $\dfrac{8\hat{i} + 3\hat{j} - \hat{k}}{\sqrt{74}}$

3. (1) 5 : 3 internally (2) 8 : 5 externally (3) 3 : 2 internally

4. $(-2, 0, 2)$ **5.** $p = 3$ **6.** $-\dfrac{1}{3}(7\hat{i} + \hat{j} - 5\hat{k})$

7. $\left(\dfrac{7}{2}, -\dfrac{21}{2}, 14\right)$ **9.** $a = 3, b = 2, c = -1$.

Solved Examples	**3 or 4 marks each**

Ex. 7. If the points $A(3, 0, p)$, $B(-1, q, 3)$ and $C(-3, 3, 0)$ are collinear, then find

(1) the ratio in which the point C divides the line segment AB

(2) the values of p and q.

Solution : Let $\overline{a}$, $\overline{b}$ and $\overline{c}$ be the position vectors of A, B and C respectively.

Then $\overline{a} = 3\hat{i} + 0\cdot\hat{j} + p\hat{k}$, $\overline{b} = -\hat{i} + q\hat{j} + 3\hat{k}$ and $\overline{c} = -3\hat{i} + 3\hat{j} + 0\cdot\hat{k}$.

(1) As the points A, B, C are collinear, suppose the point C divides line segment AB in the ratio $\lambda : 1$.

$\therefore$ by the section formula

$$\overline{c} = \frac{\lambda\cdot\overline{b} + 1\cdot\overline{a}}{\lambda + 1}$$

$$\therefore \ -3\hat{i} + 3\hat{j} + 0\cdot\hat{k} = \frac{\lambda(-\hat{i} + q\hat{j} + 3\hat{k}) + (3\hat{i} + 0\cdot\hat{j} + p\hat{k})}{\lambda + 1}$$

$$\therefore \ (\lambda + 1)(-3\hat{i} + 3\hat{j} + 0\cdot\hat{k}) = (-\lambda\hat{i} + \lambda q\hat{j} + 3\lambda\hat{k}) + (3\hat{i} + 0\cdot\hat{j} + p\hat{k})$$

$$\therefore \ -3(\lambda + 1)\hat{i} + 3(\lambda + 1)\hat{j} + 0\cdot\hat{k} = (-\lambda + 3)\hat{i} + \lambda q\hat{j} + (3\lambda + p)\hat{k}$$

By equality of vectors, we have

$$-3(\lambda+1) = -\lambda+3 \quad \text{... (1)}$$

$$3(\lambda+1) = \lambda q \quad \text{... (2)}$$

$$0 = 3\lambda + p \quad \text{... (3)}$$

From equation (1), $-3\lambda - 3 = -\lambda + 3$

$\therefore \quad -2\lambda = 6 \qquad \therefore \quad \lambda = -3$

$\therefore$ C divides segment AB externally in the ratio 3 : 1.

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(2) Putting $\lambda = -3$ in equation (2), we get

$$3(-3+1) = -3q$$

$\therefore \quad -6 = -3q \qquad \therefore \quad q = 2$

Also, putting $\lambda = -3$ in equation (3), we get

$$0 = -9 + p \qquad \therefore \quad p = 9$$

Hence $p = 9$ and $q = 2$.

Ex. 8. Express $-\hat{i} - 3\hat{j} + 4\hat{k}$ as the linear combination of the vectors $2\hat{i} + \hat{j} - 4\hat{k}$, $2\hat{i} - \hat{j} + 3\hat{k}$ and $3\hat{i} + \hat{j} - 2\hat{k}$.

Solution : Let $\bar{a} = 2\hat{i} + \hat{j} - 4\hat{k}$, $\bar{b} = 2\hat{i} - \hat{j} + 3\hat{k}$, $\bar{c} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\bar{p} = -\hat{i} - 3\hat{j} + 4\hat{k}$.

Let $\bar{p} = x\bar{a} + y\bar{b} + z\bar{c}$.

Then $-\hat{i} - 3\hat{j} + 4\hat{k} = x(2\hat{i} + \hat{j} - 4\hat{k}) + y(2\hat{i} - \hat{j} + 3\hat{k}) + z(3\hat{i} + \hat{j} - 2\hat{k})$

$\therefore \quad -\hat{i} - 3\hat{j} + 4\hat{k} = (2x + 2y + 3z)\hat{i} + (x - y + z)\hat{j} + (-4x + 3y - 2z)\hat{k}$

By equality of vectors

$$2x + 2y + 3z = -1$$

$$x - y + z = -3$$

$$-4x + 3y - 2z = 4$$

We have to solve these equations by using Cramer's Rule.

$$D = \begin{vmatrix} 2 & 2 & 3 \\ 1 & -1 & 1 \\ -4 & 3 & -2 \end{vmatrix}$$

$$= 2(2-3) - 2(-2+4) + 3(3-4)$$

$$= -2 - 4 - 3 = -9 \neq 0$$

$$D_x = \begin{vmatrix} -1 & 2 & 3 \\ -3 & -1 & 1 \\ 4 & 3 & -2 \end{vmatrix}$$

$$= -1(2-3) - 2(6-4) + 3(-9+4)$$

$$= 1 - 4 - 15 = -18$$

$$D_y = \begin{vmatrix} 2 & -1 & 3 \\ 1 & -3 & 1 \\ -4 & 4 & -2 \end{vmatrix}$$

$$= 2(6-4) + 1(-2+4) + 3(4-12)$$

$$= 4 + 2 - 24 = -18$$

$$D_z = \begin{vmatrix} 2 & 2 & -1 \\ 1 & -1 & -3 \\ -4 & 3 & 4 \end{vmatrix}$$

$$= 2(-4+9) - 2(4-12) - 1(3-4)$$

$$= 10 + 16 + 1 = 27$$

$$\therefore\ x = \frac{D_x}{D} = \frac{-18}{-9} = 2,\quad y = \frac{D_y}{D} = \frac{-18}{-9} = 2,\quad z = \frac{D_z}{D} = \frac{27}{-9} = -3$$

$$\therefore\ \bar{p} = 2\bar{a} + 2\bar{b} - 3\bar{c}.$$

Ex. 9. If $\bar{a}$, $\bar{b}$, $\bar{c}$ are non-coplanar vectors, then show that the vectors $-\bar{a} + 3\bar{b} - 5\bar{c}$, $-\bar{a} + \bar{b} + \bar{c}$ and $2\bar{a} - 3\bar{b} + \bar{c}$ are coplanar.

Solution : Let $\bar{p} = -\bar{a} + 3\bar{b} - 5\bar{c}$, $\bar{q} = -\bar{a} + \bar{b} + \bar{c}$ and $\bar{r} = 2\bar{a} - 3\bar{b} + \bar{c}$.

Then in order to prove that these vectors are coplanar, we should be able to find scalars x and y such that $\bar{r} = x\bar{p} + y\bar{q}$.

Now, $x\bar{p} + y\bar{q} = x(-\bar{a} + 3\bar{b} - 5\bar{c}) + y(-\bar{a} + \bar{b} + \bar{c})$

$$= (-x - y)\bar{a} + (3x + y)\bar{b} + (-5x + y)\bar{c}$$

and $\bar{r} = 2\bar{a} - 3\bar{b} + \bar{c}$

$\therefore\ \bar{r} = x\bar{p} + y\bar{q}$ gives

$$2\bar{a} - 3\bar{b} + \bar{c} = (-x - y)\bar{a} + (3x + y)\bar{b} + (-5x + y)\bar{c}$$

By equality of vectors

$$-x - y = 2 \qquad \text{... (1)}$$

$$3x + y = -3 \qquad \text{... (2)}$$

and $-5x + y = 1$ ... (3)

Adding (1) and (2), we get

$$2x = -1 \qquad \therefore\ x = -\frac{1}{2}$$

$\therefore\ -x - y = 2$ gives $\frac{1}{2} - y = 2$

$$\therefore\ y = \frac{1}{2} - 2 = -\frac{3}{2}$$

For these values of x and y,

$$-5x + y = -5\left(-\frac{1}{2}\right) + \left(-\frac{3}{2}\right) = \frac{5}{2} - \frac{3}{2} = 1$$

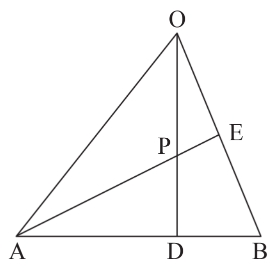
Thus, these values satisfy the third equation also.

$$\therefore\ \bar{r} = \left(-\frac{1}{2}\right)\bar{p} + \left(-\frac{3}{2}\right)\bar{q}$$

Hence, the given vectors are coplanar.

Ex. 10. In $\triangle$OAB, E is the midpoint of OB and D is the point on AB such that AD : DB = 2 : 1. If OD and AE intersect at P, then determine the ratio OP : PD using vector methods.

Solution : Let A, B, D, E, P have position vectors $\bar{a}$, $\bar{b}$, $\bar{d}$, $\bar{e}$, $\bar{p}$ respectively w.r.t. O.



$\because$ AD : DB = 2 : 1.

$\therefore$ D divides AB internally in the ratio 2 : 1.

Using section formula for internal division, we get

$$\bar{d} = \frac{2\bar{b} + \bar{a}}{2+1}$$

$$\therefore\ 3\bar{d} = 2\bar{b} + \bar{a} \qquad \text{... (1)}$$

Since E is the midpoint of OB, $\bar{e} = \overline{OE} = \frac{1}{2}\,\overline{OB} = \frac{\bar{b}}{2}$

$$\therefore\ \bar{b} = 2\bar{e} \qquad \text{... (2)}$$

$\therefore$ from (1)

$$3\bar{d} = 2(2\bar{e}) + \bar{a} \qquad \text{... [By (2)]}$$

$$= 4\bar{e} + \bar{a}$$

$$\therefore\ \frac{3\bar{d} + 2\cdot\bar{0}}{3+2} = \frac{4\bar{e} + \bar{a}}{4+1}$$

LHS is the position vector of the point which divides OD internally in the ratio 3 : 2.

RHS is the position vector of the point which divides AE internally in the ratio 4 : 1.

But OD and AE intersect at P.

$\therefore$ P divides OD internally in the ratio 3 : 2.

Hence, OP : PD = 3 : 2.

Ex. 11. If A (5, 1, p), B (1, q, p) and C (1, -2, 3) are vertices of a triangle and G$\left(r, -\frac{4}{3}, \frac{1}{3}\right)$ is its centroid, then find the values of p, q, r by vector method. ***(March '22)***

Solution : Let $\bar{a}, \bar{b}, \bar{c}$ and $\bar{g}$ be the position vectors of A, B, C and G respectively.

Then $\bar{a} = 5\hat{i} + \hat{j} + p\hat{k}$, $\bar{b} = \hat{i} + q\hat{j} + p\hat{k}$, $\bar{c} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\bar{g} = r\hat{i} - \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}$.

Since G is the centroid of the $\triangle$ABC, by the centroid formula

$$\bar{g} = \frac{\bar{a} + \bar{b} + \bar{c}}{3} \qquad \therefore\ 3\bar{g} = \bar{a} + \bar{b} + \bar{c}$$

$$\therefore\ 3\left(r\hat{i} - \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}\right) = \left(5\hat{i} + \hat{j} + p\hat{k}\right) + \left(\hat{i} + q\hat{j} + p\hat{k}\right) + \left(\hat{i} - 2\hat{j} + 3\hat{k}\right)$$

$$\therefore\ 3r\hat{i} - 4\hat{j} + \hat{k} = (5 + 1 + 1)\ \hat{i} + (1 + q - 2)\ \hat{j} + (p + p + 3)\ \hat{k}$$

$$\therefore\ 3r\hat{i} - 4\hat{j} + \hat{k} = 7\hat{i} + (q - 1)\ \hat{j} + (2p + 3)\ \hat{k}$$

By equality of vectors

$3r = 7$, $-4 = q - 1$ and $1 = 2p + 3$

$\therefore\ r = \frac{7}{3}$, $q = -3$, and $p = -1$

Hence, $p = -1$, $q = -3$, and $r = \frac{7}{3}$.

Ex. 12. Using vector method, find the incentre of the triangle whose vertices are P(0, 4, 0), Q(0, 0, 3), R(0, 4, 3).

Solution : The position vectors $\bar{p}, \bar{q}, \bar{r}$ of the vertices P, Q, R are $\bar{p} = 4\hat{j}$, $\bar{q} = 3\hat{k}$ and $\bar{r} = 4\hat{j} + 3\hat{k}$

$$\therefore\ \overline{PQ} = \bar{q} - \bar{p} = 3\hat{k} - 4\hat{j}$$
$$= -4\hat{j} + 3\hat{k}$$
$$\overline{QR} = \bar{r} - \bar{q} = (4\hat{j} + 3\hat{k}) - (3\hat{k})$$
$$= 4\hat{j}$$

and $\overline{PR} = \bar{r} - \bar{p} = (4\hat{j} + 3\hat{k}) - (4\hat{j})$

$$= 3\hat{k}$$

Let $x = |\overline{QR}|$, $y = |\overline{PR}|$ and $z = |\overline{PQ}|$

$$\therefore\ x = \sqrt{0 + 4^2 + 0} = 4$$
$$y = \sqrt{0 + 0 + 3^2} = 3$$

and $z = \sqrt{0 + (-4)^2 + 3^2} = 5$

If H($\bar{h}$) is the incentre of the triangle PQR, then

$$\bar{h} = \frac{x\bar{p} + y\bar{q} + z\bar{r}}{x + y + z}$$

$$= \frac{4(4\hat{j}) + 3(3\hat{k}) + 5(4\hat{j} + 3\hat{k})}{4 + 3 + 5}$$

$$= \frac{1}{12}(16\hat{j} + 9\hat{k} + 20\hat{j} + 15\hat{k})$$

$$= \frac{1}{12}(36\hat{j} + 24\hat{k})$$

$\therefore\ \bar{h} = 3\hat{j} + 2\hat{k}$

$\therefore\ \text{H} = (0, 3, 2)$.

Examples for Practice	**3 or 4 marks each**

1. If the points A(3, 2, p), B(q, 8, -10) and C(-2, -3, 1) are collinear, then find

(1) the ratio in which the point C divides the line segment AB

(2) the values of p and q.

2. Express $\bar{p}$ as a linear combination of $\bar{a}$, $\bar{b}$ and $\bar{c}$, where $\bar{p} = \hat{i} + 4\hat{j} - 4\hat{k}$, $\bar{a} = 2\hat{i} - \hat{j} + 3\hat{k}$, $\bar{b} = \hat{i} - 2\hat{j} + 4\hat{k}$, $\bar{c} = -\hat{i} + 3\hat{j} - 5\hat{k}$.

3. If $\bar{a}$, $\bar{b}$, $\bar{c}$ are non-zero, non-coplanar vectors, then show that the vectors $2\bar{a} - 5\bar{b} + 2\bar{c}$, $\bar{a} + 5\bar{b} - 6\bar{c}$ and $3\bar{a} - 4\bar{c}$ are coplanar.

4. If $\bar{a} + \lambda\bar{b} + 3\bar{c}$, $-2\bar{a} + 3\bar{b} - 4\bar{c}$, $\bar{a} - 3\bar{b} + 5\bar{c}$ are coplanar, then find the value of λ.

5. In a $\triangle$ABC, D and E are the points on BC and AC respectively such that BD $= 2 \cdot$ DC and AE $= 3 \cdot$ EC. Let P be the point of intersection of AD and BE. Find BP/PE using vector methods.

6. Using vector method, find the incentre of the triangle whose vertices are A(0, 3, 0), B(0, 0, 4) and C(0, 3, 4).

ANSWERS

1. (1) 5 : 11 externally (2) $p = -4$, $q = 9$

2. $\bar{p} = \bar{a} + 2\bar{b} + 3\bar{c}$

4. $\lambda = -2$

5. 8 : 3

6. (0, 2, 3)

Solved Examples	2 or 3 marks each

Ex. 13. If G and G′ are the centroids of △ ABC and △ A′B′C′ respectively, show that $\overline{AA'} + \overline{BB'} + \overline{CC'} = 3\overline{GG'}$.

Solution : Let $\bar{a}, \bar{b}, \bar{c}, \bar{a}', \bar{b}', \bar{c}', \bar{g}$ and $\bar{g}'$ be the position vectors of the points A, B, C, A′, B′, C′, G and G′ respectively.

G and G′ are the centroids of △ABC and △A′B′C′ respectively.

∴ by the centroid formula

$$\bar{g} = \frac{\bar{a}+\bar{b}+\bar{c}}{3} \quad \text{and} \quad \bar{g}' = \frac{\bar{a}'+\bar{b}'+\bar{c}'}{3}$$

$$\therefore \ \bar{a}+\bar{b}+\bar{c} = 3\bar{g} \quad \text{and} \quad \bar{a}'+\bar{b}'+\bar{c}' = 3\bar{g}' \qquad \text{... (1)}$$

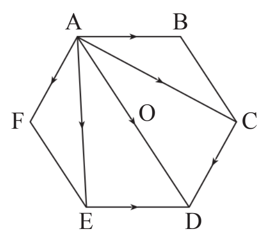
Now $\overline{AA'} = \bar{a}' - \bar{a}$, $\overline{BB'} = \bar{b}' - \bar{b}$, $\overline{CC'} = \bar{c}' - \bar{c}$ and $\overline{GG'} = \bar{g}' - \bar{g}$

$$\begin{aligned} \therefore \ \overline{AA'} + \overline{BB'} + \overline{CC'} &= (\bar{a}' - \bar{a}) + (\bar{b}' - \bar{b}) + (\bar{c}' - \bar{c}) \\ &= (\bar{a}' + \bar{b}' + \bar{c}') - (\bar{a} + \bar{b} + \bar{c}) \\ &= 3\bar{g}' - 3\bar{g} \\ &= 3(\bar{g}' - \bar{g}) \qquad \text{... [By (1)]} \\ &= 3\,\overline{GG'}. \end{aligned}$$

Ex. 14. ABCDEF is a regular hexagon. Show that

$\overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF} = 6\overline{AO}$.

Solution : ABCDEF is a regular hexagon.



$\therefore \ \overline{AB} = \overline{ED}$ and $\overline{AF} = \overline{CD}$

By the triangle law of addition of vectors,

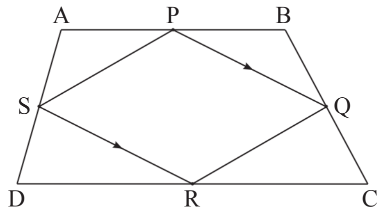
$\overline{AC} + \overline{AF} = \overline{AC} + \overline{CD} = \overline{AD}$

$\overline{AE} + \overline{AB} = \overline{AE} + \overline{ED} = \overline{AD}$

$$\begin{aligned} \therefore \ \text{LHS} &= \overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF} \\ &= \overline{AD} + (\overline{AC} + \overline{AF}) + (\overline{AE} + \overline{AB}) \\ &= \overline{AD} + \overline{AD} + \overline{AD} \\ &= 3\overline{AD} = 3\,(2\overline{AO}) \qquad \text{... [} \because \text{ O is the midlloint of AD]} \\ &= 6\overline{AO} = \text{RHS}. \end{aligned}$$

Ex. 15. Prove that the line segments joining the midpoints of the adjacent sides of a quadrilateral form a parallelogram.

Solution : Let ABCD be a quadrilateral and P, Q, R, S be the midpoints of the sides AB, BC, CD and DA respectively.



Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{p}, \bar{q}, \bar{r}$ and $\bar{s}$ be the position vectors of the points A, B, C, D, P, Q, R and S respectively.

Since P, Q, R and S are the midpoints of the sides AB, BC, CD and DA respectively,

$$\bar{p} = \frac{\bar{a}+\bar{b}}{2}, \bar{q} = \frac{\bar{b}+\bar{c}}{2}, \bar{r} = \frac{\bar{c}+\bar{d}}{2} \text{ and } \bar{s} = \frac{\bar{d}+\bar{a}}{2}$$

$$\therefore \overline{PQ} = \bar{q} - \bar{p}$$

$$= \left(\frac{\bar{b}+\bar{c}}{2}\right) - \left(\frac{\bar{a}+\bar{b}}{2}\right)$$

$$= \frac{1}{2}(\bar{b}+\bar{c}-\bar{a}-\bar{b}) = \frac{1}{2}(\bar{c}-\bar{a})$$

$$\overline{SR} = \bar{r} - \bar{s}$$

$$= \left(\frac{\bar{c}+\bar{d}}{2}\right) - \left(\frac{\bar{d}+\bar{a}}{2}\right)$$

$$= \frac{1}{2}(\bar{c}+\bar{d}-\bar{d}-\bar{a}) = \frac{1}{2}(\bar{c}-\bar{a})$$

$\therefore$ $\overline{PQ} = \overline{SR}$ $\quad\therefore$ $\overline{PQ} \parallel \overline{SR}$

Similarly, $\overline{QR} \parallel \overline{PS}$

$\therefore$ □ PQRS is a parallelogram.

Examples for Practice	**2 or 3 marks each**

1. D, E, F are the midpoints of the sides BC, CA and AB respectively of $\triangle$ABC and O is any point in the plane of $\triangle$ABC. Show that

   (1) $\overline{AD} + \overline{BE} + \overline{CF} = \bar{0}$

   (2) $\overline{AD} + \frac{2}{3}\overline{BE} + \frac{1}{3}\overline{CF} = \frac{1}{2}\overline{AC}$.

2. ABCDE is a pentagon, show that $\overline{AB} + \overline{AE} + \overline{BC} + \overline{CD} + \overline{ED} = 2\overline{AD}$.

3. If G is the centroid of $\triangle$ABC and O is any point in the plane of $\triangle$ABC, show that

   (1) $\overline{GA} + \overline{GB} + \overline{GC} = \bar{0}$ (2) $\overline{OA} + \overline{OB} + \overline{OC} = 3\overline{OG}$.

4. E and F are the midpoints of the diagonals AC and BD of the quadrilateral ABCD and G is the midpoint of seg EF. Show that

(1) $\overline{AB} + \overline{AD} + \overline{CB} + \overline{CD} = 4\overline{EF}$ (2) $\overline{GA} + \overline{GB} + \overline{GC} + \overline{GD} = \overline{0}$.

5.2 PRODUCT OF VECTORS

Remember :

1. If $\overline{a}$ and $\overline{b}$ are two vectors inclined at an angle θ, then their scalar product (denoted by $\overline{a} \cdot \overline{b}$) is defined by $\overline{a} \cdot \overline{b} = ab \cos \theta$.

2. If $\overline{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\overline{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\overline{a} \cdot \overline{b}$

$= a_1b_1 + a_2b_2 + a_3b_3$.

3. $\overline{a} \cdot \overline{b} = \overline{b} \cdot \overline{a}$.

4. $\overline{a} \cdot \overline{a} = a^2 \qquad \therefore \ a = \sqrt{\overline{a} \cdot \overline{a}}$.

5. If $\overline{a}$ is the unit vector, then $\overline{a} \cdot \overline{a} = 1$.

6. Angle θ between $\overline{a}$ and $\overline{b}$ is given by $\cos \theta = \dfrac{\overline{a} \cdot \overline{b}}{ab}$.

7. $\overline{a}$ is perpendicular to $\overline{b}$, if $\overline{a} \cdot \overline{b} = a_1b_1 + a_2b_2 + a_3b_3 = 0$.

In particular, $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} = 0, \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{j} = 0, \hat{k} \cdot \hat{i} = \hat{i} \cdot \hat{k} = 0$.

8. (i) Scalar projection of $\overline{a}$ on $\overline{b} = \dfrac{\overline{a} \cdot \overline{b}}{b}$.

Vector projection of $\overline{a}$ on $\overline{b} = \left(\dfrac{\overline{a} \cdot \overline{b}}{|\overline{b}|}\right)(\hat{b})$.

(ii) Scalar projection of $\overline{b}$ on $\overline{a} = \dfrac{\overline{a} \cdot \overline{b}}{a}$.

Vector projection of $\overline{b}$ on $\overline{a} = \left(\dfrac{\overline{a} \cdot \overline{b}}{|\overline{a}|}\right)(\hat{a})$.

9. Vector product of two vectors $\overline{a}$ and $\overline{b}$ which are inclined at an angle θ is denoted by $\overline{a} \times \overline{b}$ and is defined as $\overline{a} \times \overline{b} = ab \sin \theta \cdot \hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\overline{a}$ and $\overline{b}$ such that $\overline{a}$, $\overline{b}$ and $\hat{n}$ form a right-handed set.

10. $\overline{a} \times \overline{b} = -\overline{b} \times \overline{a}$ but $|\overline{a} \times \overline{b}| = |\overline{b} \times \overline{a}|$.

11. If $\overline{a}$ and $\overline{b}$ are non-zero vectors and $\overline{a} \times \overline{b} = \overline{0}$, then $\overline{a}$ is collinear (or parallel) to $\overline{b}$. In particular, $\overline{a} \times \overline{a} = \overline{0}$.

12. $\sin\theta = \dfrac{|\overline{a} \times \overline{b}|}{ab}$.

13. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \overline{0}$, $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$.

14. Unit vectors perpendicular to both $\overline{a}$ and $\overline{b} = \dfrac{\pm(\overline{a} \times \overline{b})}{|\overline{a} \times \overline{b}|}$.

15. If $\overline{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\overline{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\overline{a} \times \overline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

16. If $\overline{a}$ and $\overline{b}$ represent the adjacent sides of a parallelogram, then the vector area of the parallelogram is $\overline{a} \times \overline{b}$ and its area is $|\overline{a} \times \overline{b}|$.

17. If $\overline{a}$ and $\overline{b}$ represent the diagonals of a parallelogram, then its area is $\dfrac{1}{2}|\overline{a} \times \overline{b}|$.

18. Vector area of $\triangle$ABC $= \dfrac{1}{2}(\overline{AB} \times \overline{AC})$

$\therefore$ area of $\triangle$ABC $= \dfrac{1}{2}|\overline{AB} \times \overline{AC}|$.

19. If $\overline{a}, \overline{b}, \overline{c}$ are the position vectors of the vertices of $\triangle$ABC, then the vector area of $\triangle$ABC $= \dfrac{1}{2}(\overline{a} \times \overline{b} + \overline{b} \times \overline{c} + \overline{c} \times \overline{a})$.

20. $(\overline{a} \times \overline{b})^2 = a^2b^2 - (\overline{a} \cdot \overline{b})^2$

This is called **Lagrange's Identity.**

Solved Examples	2 marks each

Ex. 16. Find the value of a for which the vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} + a\hat{j} + 3\hat{k}$ are perpendicular.

Solution : Let $\overline{p} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\overline{q} = \hat{i} + a\hat{j} + 3\hat{k}$

Since $\overline{p}$ is perpendicular to $\overline{q}$, $\overline{p} \cdot \overline{q} = 0$

$\therefore\ (3\hat{i} + 2\hat{j} + 9\hat{k}) \cdot (\hat{i} + 9\hat{j} + 3\hat{k}) = 0$

$\therefore\ 3(1) + 2a + 9(3) = 0$

$\therefore\ 3 + 2a + 27 = 0$

$\therefore\ 2a = -30 \qquad \therefore\ a = -15.$

Ex. 17. Find the cosine of the angle between the vectors $\bar{a}$ and $\bar{b}$, if $\bar{a} = \hat{i} - 2\hat{j} + \hat{k}$ and $\bar{b} = 2\hat{i} - 2\hat{j} + 2\hat{k}$.

Solution : $\bar{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\bar{b} = 2\hat{i} - 2\hat{j} + 2\hat{k}$

$\therefore\ a = |\bar{a}| = \sqrt{1^2 + (-2)^2 + 1^2}$

$= \sqrt{1 + 4 + 1} = \sqrt{6}$

$b = |\bar{b}| = \sqrt{2^2 + (-2)^2 + 2^2}$

$= \sqrt{4 + 4 + 4} = \sqrt{12} = 2\sqrt{3}$

$\bar{a} \cdot \bar{b} = (\hat{i} - 2\hat{j} + \hat{k}) \cdot (2\hat{i} - 2\hat{j} + 2\hat{k})$

$= 1(2) + (-2)(-2) + 1(2)$

$= 2 + 4 + 2 = 8$

If θ is the angle between $\bar{a}$ and $\bar{b}$, then

$$\cos\theta = \frac{\bar{a} \cdot \bar{b}}{ab} = \frac{8}{\sqrt{6} \times 2\sqrt{3}} = \frac{4}{\sqrt{18}} = \frac{4}{3\sqrt{2}}.$$

Ex. 18. Find the projection of $\bar{a}$ on $\bar{b}$, where $\bar{a} = \hat{i} - \hat{j} + \hat{k}$ and $\bar{b} = \hat{i} + \hat{j} - 2\hat{k}$.

Solution : $\bar{a} = \hat{i} - \hat{j} + \hat{k}$, $\bar{b} = \hat{i} + \hat{j} - 2\hat{k}$

$\therefore\ \bar{a} \cdot \bar{b} = (\hat{i} - \hat{j} + \hat{k}) \cdot (\hat{i} + \hat{j} - 2\hat{k})$

$= 1(1) + (-1)(1) + 1(-2)$

$= 1 - 1 - 2 = -2$

and $b = |\bar{b}| = \sqrt{1^2 + 1^2 + (-2)^2}$

$= \sqrt{1 + 1 + 4} = \sqrt{6}$

$$\therefore\ \text{projection of } \bar{a} \text{ on } \bar{b} = \frac{\bar{a} \cdot \bar{b}}{b} = -\frac{2}{\sqrt{6}}.$$

Ex. 19. Find a unit vector perpendicular to the vectors $\hat{j} + 2\hat{k}$ and $\hat{i} + \hat{j}$.

Solution : Let $\bar{a} = \hat{j} + 2\hat{k}$, $\bar{b} = \hat{i} + \hat{j}$

$$\text{Then } \bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{vmatrix}$$

$= (0 - 2)\hat{i} - (0 - 2)\hat{j} + (0 - 1)\hat{k}$

$= -2\hat{i} + 2\hat{j} - \hat{k}$

$\therefore\ |\bar{a} \times \bar{b}| = \sqrt{(-2)^2 + 2^2 + (-1)^2}$

$= \sqrt{4 + 4 + 1} = \sqrt{9} = 3$

Unit vector perpendicular to both $\bar{a}$ and $\bar{b}$

$$= \pm\frac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|} = \pm\left(\frac{-2\hat{i}+2\hat{j}-\hat{k}}{3}\right)$$

$$= \pm\left(-\frac{2}{3}\hat{i}+\frac{2}{3}\hat{j}-\frac{1}{3}\hat{k}\right).$$

Ex. 20. Find the area of the parallelogram whose adjacent sides are the vectors $2\hat{i}-2\hat{j}+\hat{k}$ and $\hat{i}-3\hat{j}-3\hat{k}$. *(Sept. '21)*

Solution : Let : $\bar{a}=2\hat{i}-2\hat{j}+\hat{k}$, $\bar{b}=\hat{i}-3\hat{j}-3\hat{k}$

$$\therefore\ \bar{a}\times\bar{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -2 & 1\\ 1 & -3 & -3\end{vmatrix}$$

$$= (6+3)\hat{i}-(-6-1)\hat{j}+(-6+2)\hat{k}$$

$$= 9\hat{i}+7\hat{j}-4\hat{k}$$

$$|\bar{a}\times\bar{b}| = \sqrt{9^2+7^2+(-4)^2} = \sqrt{81+49+16} = \sqrt{146}$$

Area of the parallelogram whose adjacent sides are $\bar{a}$ and $\bar{b}$ is

$|\bar{a}\times\bar{b}| = \sqrt{146}$ sq units.

Examples for Practice	2 marks each

1. If $\bar{a}=3\hat{i}+5\hat{j}-7\hat{k}$, $\bar{b}=7\hat{i}+3\hat{k}$, find $\bar{a}\cdot\bar{b}$. Interpret the result.
2. If $\bar{a}=(3, 4, x)$ and $\bar{b}=(2, -1, 4)$, find x, so that $\bar{a}$ is perpendicular to $\bar{b}$.
3. The diagonals of a parallelogram are given by $\bar{a}=2\hat{i}-3\hat{j}+5\hat{k}$, $\bar{b}=-2\hat{i}+2\hat{j}+2\hat{k}$. Prove that parallelogram is a rhombus.
4. Find the projection of $\bar{a}=2\hat{i}+\hat{j}+3\hat{k}$ on $\bar{b}=\hat{i}-\hat{j}+2\hat{k}$.
5. If $|\bar{a}\cdot\bar{b}|=|\bar{a}\times\bar{b}|$ and $\bar{a}\cdot\bar{b}<0$, then find the angle between $\bar{a}$ and $\bar{b}$.
6. If $\bar{a}=2\hat{i}+3\hat{j}-\hat{k}$, $\bar{b}=\hat{i}-4\hat{j}+2\hat{k}$, find $(\bar{a}+\bar{b})\times(\bar{a}-\bar{b})$.
7. If $\bar{a}=2\hat{i}+\hat{j}-3\hat{k}$ and $\bar{b}=\hat{i}-2\hat{j}+\hat{k}$, find a vector of magnitude 5 perpendicular to both $\bar{a}$ and $\bar{b}$.
8. Find the area of parallelogram whose diagonals are $2\hat{i}+\hat{j}-3\hat{k}$ and $\hat{i}-3\hat{j}+4\hat{k}$.
9. Find the area of the parallelogram whose adjacent sides are $\bar{a}=3\hat{i}+\hat{j}-2\hat{k}$ and $\bar{b}=\hat{i}-3\hat{j}+4\hat{k}$.
10. Find the sine of the angle between the vectors $\bar{a}$ and $\bar{b}$, if $\bar{a}=\hat{i}+\hat{j}+\hat{k}$ and $\bar{b}=2\hat{i}-\hat{j}+2\hat{k}$.
11. If $|\bar{a}|=2$, $|\bar{b}|=5$ and $|\bar{a}\times\bar{b}|=8$, find $\bar{a}\cdot\bar{b}$.
12. If $\bar{a}\cdot\bar{b}=\sqrt{3}$ and $\bar{a}\times\bar{b}=2\hat{i}+\hat{j}+2\hat{k}$, find the angle between $\bar{a}$ and $\bar{b}$.

ANSWERS

1. 0, $\bar{a}$ is perpendicular to $\bar{b}$

2. $x = -\frac{1}{2}$

4. $\frac{7}{\sqrt{6}}$

5. $\frac{3\pi}{4}$

6. $-4\hat{i} + 10\hat{j} + 22\hat{k}$

7. $\pm \frac{5}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$

8. $\frac{1}{2}\sqrt{195}$ sq units.

9. $2\sqrt{75}$ sq units.

10. $\sqrt{\frac{2}{3}}$

11. ± 6

12. 60°.

Solved Examples	3 marks each

Ex. 21. If $\bar{a} + \bar{b} + \bar{c} = \bar{0}$, $|\bar{a}| = 3$, $|\bar{b}| = 5$ and $|\bar{c}| = 7$, find the angle between $\bar{a}$ and $\bar{b}$.

Solution : Let θ be the angle between $\bar{a}$ and $\bar{b}$.

$\because \ \bar{a} + \bar{b} + \bar{c} = \bar{0}$

$\therefore \ \bar{a} + \bar{b} = -\bar{c}$

$\therefore \ (\bar{a} + \bar{b}) \cdot (\bar{a} + \bar{b}) = (-\bar{c}) \cdot (-\bar{c})$

$\therefore \ \bar{a} \cdot (\bar{a} + \bar{b}) + \bar{b} \cdot (\bar{a} + \bar{b}) = \bar{c} \cdot \bar{c}$

$\therefore \ \bar{a} \cdot \bar{a} + \bar{a} \cdot \bar{b} + \bar{b} \cdot \bar{a} + \bar{b} \cdot \bar{b} = \bar{c} \cdot \bar{c}$

$\therefore \ |\bar{a}|^2 + \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{b} + |\bar{b}|^2 = |\bar{c}|^2$ ... $[\because \ \bar{b} \cdot \bar{a} = \bar{a} \cdot \bar{b}]$

$\therefore \ |\bar{a}|^2 + 2\bar{a} \cdot \bar{b} + |\bar{b}|^2 = |\bar{c}|^2$

$\therefore \ |\bar{a}|^2 + 2|\bar{a}| \cdot |\bar{b}| \cos\theta + |\bar{b}|^2 = |\bar{c}|^2$

$\therefore \ (3)^2 + 2(3)(5)\cos\theta + (5)^2 = (7)^2$

$\therefore \ 9 + 30\cos\theta + 25 = 49$

$\therefore \ 30\cos\theta = 15$

$\therefore \ \cos\theta = \frac{1}{2} = \cos 60°$

$\therefore \ \theta = 60°$.

Ex. 22. If θ is the angle between the unit vectors $\hat{e}_1$ and $\hat{e}_2$, prove that $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{e}_1 - \hat{e}_2|$.

Solution : Since $\hat{e}_1$ and $\hat{e}_2$ are unit vectors, $|\hat{e}_1| = 1$, $|\hat{e}_2| = 1$

Also, $\hat{e}_1 \cdot \hat{e}_1 = 1$, $\hat{e}_2 \cdot \hat{e}_2 = 1$

and $\hat{e}_1 \cdot \hat{e}_2 = |\hat{e}_1|\,|\hat{e}_2| \cos\theta = 1 \cdot 1 \cdot \cos\theta = \cos\theta = \hat{e}_2 \cdot \hat{e}_1$

Consider, $|\hat{e}_1 - \hat{e}_2|^2 = (\hat{e}_1 - \hat{e}_2)\cdot(\hat{e}_1 - \hat{e}_2) = \hat{e}_1 \cdot \hat{e}_1 - \hat{e}_1 \cdot \hat{e}_2 - \hat{e}_2 \cdot \hat{e}_1 + \hat{e}_2 \cdot \hat{e}_2$

$$= 1 - \cos\theta - \cos\theta + 1 = 2 - 2\cos\theta$$

$$= 2(1 - \cos\theta) = 2 \cdot 2\sin^2(\theta/2)$$

$\therefore\ \sin^2(\theta/2) = \frac{1}{4}|\hat{e}_1 - \hat{e}_2|^2$

$\therefore\ \sin(\theta/2) = \frac{1}{2}|\hat{e}_1 - \hat{e}_2|.$

Ex. 23. Find the projection of $\overline{AB}$ on $\overline{CD}$, where $A \equiv (2, -3, 0)$, $B \equiv (1, -4, -2)$, $C \equiv (4, 6, 8)$, $D \equiv (7, 0, 10)$.

Solution : Let $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ be the position vectors of A, B, C and D respectively with respect to the origin O.

$\therefore\ \bar{a} = 2\hat{i} - 3\hat{j}$, $\bar{b} = \hat{i} - 4\hat{j} - 2\hat{k}$, $\bar{c} = 4\hat{i} + 6\hat{j} + 8\hat{k}$, $\bar{d} = 7\hat{i} + 10\hat{k}$

$\therefore\ \overline{AB} = \bar{b} - \bar{a} = (\hat{i} - 4\hat{j} - 2\hat{k}) - (2\hat{i} - 3\hat{j})$

$$= -\hat{i} - \hat{j} - 2\hat{k}$$

and $\overline{CD} = \bar{d} - \bar{c} = (7\hat{i} + 10\hat{k}) - (4\hat{i} + 6\hat{j} + 8\hat{k})$

$$= 3\hat{i} - 6\hat{j} + 2\hat{k}$$

$\therefore\ \overline{AB} \cdot \overline{CD} = (-\hat{i} - \hat{j} - 2\hat{k}) \cdot (3\hat{i} - 6\hat{j} + 2\hat{k})$

$$= (-1)(3) + (-1)(-6) + (-2)(2)$$

$$= -3 + 6 - 4 = -1$$

$$|\overline{CD}| = \sqrt{3^2 + (-6)^2 + 2^2}$$

$$= \sqrt{9 + 36 + 4} = 49 = 7$$

$\therefore$ projection of $\overline{AB}$ on $\overline{CD} = \dfrac{\overline{AB} \cdot \overline{CD}}{|\overline{CD}|} = -\dfrac{1}{7}.$

Ex. 24. If $\bar{a} = \hat{i} + \hat{j} + \hat{k}$ and $\bar{c} = \hat{j} - \hat{k}$, find a vector $\bar{b}$ satisfying $\bar{a} \times \bar{b} = \bar{c}$ and $\bar{a} \cdot \bar{b} = 3$.

Solution : Given : $\bar{a} = \hat{i} + \hat{j} + \hat{k}$, $\bar{c} = \hat{j} - \hat{k}$

Let $\bar{b} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $\bar{a} \cdot \bar{b} = 3$ gives $(\hat{i} + \hat{j} + \hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = 3$

$\therefore\ (1)(x) + (1)(y) + (1)(z) = 3$

$\therefore\ x + y + z = 3$ ... (1)

Also, $\bar{c} = \bar{a} \times \bar{b}$

$$\therefore\ \hat{j} - \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix}$$

$$= (z - y)\hat{i} - (z - x)\hat{j} + (y - x)\hat{k}$$

$$= (z - y)\hat{i} + (x - z)\hat{j} + (y - x)\hat{k}$$

By equality of vectors

$z - y = 0$ ... (2)

$x - z = 1$ ... (3)

$y - x = -1$ ... (4)

From (2), $y = z$.

From (3), $x = 1 + z$

Substituting these values of x and y in (1), we get

$1 + z + z + z = 3 \qquad \therefore\ z = \frac{2}{3}$

$\therefore\ y = z = \frac{2}{3}$

$\therefore\ x = 1 + z = 1 + \frac{2}{3} = \frac{5}{3}$

$\therefore\ \bar{b} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

i.e. $\bar{b} = \frac{1}{3}(5\hat{i} + 2\hat{j} + 2\hat{k})$.

Ex. 25. If A (1, 1, 1), B (−2, 4, 3), C (−1, 5, 5) and D (2, 2, 6) are four points, find the vectors of magnitude 4 units perpendicular to both $\overline{\mathbf{AB}}$ and $\overline{\mathbf{CD}}$.

Solution : Let $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ be the position vectors of A, B, C and D respectively with respect to the origin O.

$\therefore\ \bar{a} = \hat{i} + \hat{j} + \hat{k},\ \bar{b} = -2\hat{i} + 4\hat{j} + 3\hat{k},\ \bar{c} = -\hat{i} + 5\hat{j} + 5\hat{k},\ \bar{d} = 2\hat{i} + 2\hat{j} + 6\hat{k}$

$\therefore\ \overline{AB} = \bar{b} - \bar{a} = (-2\hat{i} + 4\hat{j} + 3\hat{k}) - (\hat{i} + \hat{j} + \hat{k})$

$= -3\hat{i} + 3\hat{j} + 2\hat{k}$

and $\overline{CD} = \bar{d} - \bar{c} = (2\hat{i} + 2\hat{j} + 6\hat{k}) - (-\hat{i} + 5\hat{j} + 5\hat{k})$

$= 3\hat{i} - 3\hat{j} + \hat{k}$

$$\therefore\ \overline{AB}\times\overline{CD}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -3 & 3 & 2\\ 3 & -3 & 1\end{vmatrix}$$

$$=(3+6)\hat{i}-(-3-6)\hat{j}+(9-9)\hat{k}$$

$$=9\hat{i}+9\hat{j}=9(\hat{i}+\hat{j})$$

$\therefore\ |\overline{AB}\times\overline{CD}|=9\sqrt{1^2+1^2}=9\sqrt{2}$

$\therefore$ unit vectors perpendicular to $\overline{AB}$ and $\overline{CD}$

$$=\frac{\pm(\overline{AB}\times\overline{CD})}{|\overline{AB}\times\overline{CD}|}=\pm\frac{9(\hat{i}+\hat{j})}{9\sqrt{2}}=\pm\frac{\hat{i}+\hat{j}}{\sqrt{2}}$$

$\therefore$ the vectors of magnitude 4 perpendicular to both $\overline{AB}$ and $\overline{CD}=\pm\dfrac{4(\hat{i}+\hat{j})}{\sqrt{2}}$.

Ex. 26. Find the area of the triangle with vertices (1, 2, 0), (1, 0, 2) and (0, 3, 1).

Solution : Let ABC be the triangle where A (1, 2, 0), B (1, 0, 2) and C (0, 3, 1).
Let $\bar{a}$, $\bar{b}$, $\bar{c}$ be the position vectors of A, B, C respectively w.r.t. the origin.
Then $\bar{a}=\hat{i}+2\hat{j}$, $\bar{b}=\hat{i}+2\hat{k}$, $\bar{c}=3\hat{j}+\hat{k}$

$$\therefore\ \overline{AB}=\bar{b}-\bar{a}=(\hat{i}+2\hat{k})-(\hat{i}+2\hat{j})$$

$$=-2\hat{j}+2\hat{k}$$

$$\overline{AC}=\bar{c}-\bar{a}=(3\hat{j}+\hat{k})-(\hat{i}+2\hat{j})$$

$$=-\hat{i}+\hat{j}+\hat{k}$$

$$\therefore\ \overline{AB}\times\overline{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & -2 & 2\\ -1 & 1 & 1\end{vmatrix}$$

$$=(-2-2)\hat{i}-(0+2)\hat{j}+(0-2)\hat{k}$$

$$=-4\hat{i}-2\hat{j}-2\hat{k}$$

$\therefore$ area of $\triangle ABC=\dfrac{1}{2}|\overline{AB}\times\overline{AC}|$

$$=\frac{1}{2}\sqrt{(-4)^2+(-2)^2+(-2)^2}$$

$$=\frac{1}{2}\sqrt{16+4+4}=\frac{1}{2}\sqrt{24}$$

$$=\sqrt{6}\text{ sq units.}$$

Examples for Practice	3 marks each

1. Find the projection of $\overline{PQ}$ on $\overline{RS}$, where $2\hat{i}+3\hat{j}-3\hat{k}$, $\hat{i}+3\hat{j}+5\hat{k}$, $2\hat{i}-\hat{j}+3\hat{k}$ and $4\hat{i}+\hat{j}+6\hat{k}$ are the position vectors of the points P, Q, R and S respectively.
2. If $\overline{a}+\overline{b}=\overline{c}$ and $|\overline{a}|=1$, $|\overline{b}|=1$ and $|\overline{c}|=\sqrt{2}$, find the angle between $\overline{b}$ and $\overline{c}$.
3. If $\overline{a}=\hat{i}-2\hat{j}+3\hat{k}$, $\overline{b}=4\hat{i}-3\hat{j}+\hat{k}$, $\overline{c}=\hat{i}-\hat{j}+2\hat{k}$, verify that $\overline{a}\times(\overline{b}+\overline{c})=\overline{a}\times\overline{b}+\overline{a}\times\overline{c}$.
4. Find a unit vector perpendicular to $\overline{PQ}$ and $\overline{PR}$ where $P=(2,2,0)$, $Q=(0,3,5)$ and $R=(5,0,3)$. Also, find the sine of angle between $\overline{PQ}$ and $\overline{PR}$.
5. If $\overline{u}+\overline{v}+\overline{w}=\overline{0}$, show that $\overline{u}\times\overline{v}=\overline{v}\times\overline{w}=\overline{w}\times\overline{u}$.
6. Find the area of the triangle whose vertices are $(1,-2,1)$, $(2,-1,2)$ and $(-1,-1,-1)$.
7. If $\overline{a}$ and $\overline{b}$ are two vectors, show that $\begin{vmatrix} \overline{a}\cdot\overline{a} & \overline{a}\cdot\overline{b} \\ \overline{a}\cdot\overline{b} & \overline{b}\cdot\overline{b} \end{vmatrix}=(\overline{a}\times\overline{b})^2$.
8. Find the vector area of the triangle, whose vertices are $(2,3,1)$, $(1,-2,4)$ and $(3,-1,2)$.

ANSWERS

1. $\dfrac{22}{\sqrt{17}}$
2. $\dfrac{\pi}{4}$
4. $\dfrac{13\hat{i}+21\hat{j}+\hat{k}}{\sqrt{611}}$, $\sin\theta=\dfrac{\sqrt{611}}{\sqrt{30}\cdot\sqrt{22}}$
6. $\dfrac{3}{\sqrt{2}}$ sq units
8. $7\hat{i}+4\hat{j}+9\hat{k}$.

5.3 SCALAR TRIPLE PRODUCT

Remember :

1. If $\overline{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\overline{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\overline{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, then
$$[\overline{a}\ \overline{b}\ \overline{c}]=\overline{a}\cdot(\overline{b}\times\overline{c})=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$
2. $[\overline{a}\ \overline{b}\ \overline{c}]=[\overline{b}\ \overline{c}\ \overline{a}]=[\overline{c}\ \overline{a}\ \overline{b}]$.
3. If any one of $\overline{a}$, $\overline{b}$, $\overline{c}$ is a zero vector, then $[\overline{a}\ \overline{b}\ \overline{c}]=0$.
4. If any two of $\overline{a}$, $\overline{b}$, $\overline{c}$ are collinear or equal, then $[\overline{a}\ \overline{b}\ \overline{c}]=0$.
5. If $\overline{a}$, $\overline{b}$, $\overline{c}$ are the coterminus edges of a parallelopiped, then its volume $=[\overline{a}\ \overline{b}\ \overline{c}]$.

6. The vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar if $[\bar{a}\ \bar{b}\ \bar{c}] = 0$.

7. Volume of tetrahedron A–BCD $= \frac{1}{6}[\overline{AB}\ \overline{AC}\ \overline{AD}]$.

8. The points A, B, C, D are coplanar if $[\overline{AB}\ \overline{AC}\ \overline{AD}] = 0$.

9. If $\bar{a}, \bar{b}, \bar{c}$ are vectors in space, then vector triple product is defined as $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}\cdot\bar{c})\ \bar{b} - (\bar{a}\cdot\bar{b})\ \bar{c}$

10. $(\bar{a} \times \bar{b}) \times \bar{c} = (\bar{a}\cdot\bar{c})\ \bar{b} - (\bar{b}\cdot\bar{c})\ \bar{a}$.

Solved Examples	2 marks each

Ex. 27. If $\bar{a} = 3\hat{i} - 2\hat{j} + 7\hat{k}$, $\bar{b} = 5\hat{i} + \hat{j} - 2\hat{k}$ and $\bar{c} = \hat{i} + \hat{j} - \hat{k}$, then find $\bar{a}\cdot(\bar{b} \times \bar{c})$.

Solution : $\bar{a}\cdot(\bar{b} \times \bar{c}) = \begin{vmatrix} 3 & -2 & 7 \\ 5 & 1 & -2 \\ 1 & 1 & -1 \end{vmatrix}$

$= 3(-1+2) + 2(-5+2) + 7(5-1)$

$= 3 - 6 + 28 = 25.$

Ex. 28. If the vectors $3\hat{i} + 5\hat{k}$, $4\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} + \hat{j} + 4\hat{k}$ are the coterminus edges of the parallelopiped, then find the volume of the parallelopiped.

(Sept. '21) (3 marks)

Solution : Let $\bar{a} = 3\hat{i} + 5\hat{k}$, $\bar{b} = 4\hat{i} + 2\hat{j} - 3\hat{k}$, $\bar{c} = 3\hat{i} + \hat{j} + 4\hat{k}$

$\therefore\ [\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} 3 & 0 & 5 \\ 4 & 2 & -3 \\ 3 & 1 & 4 \end{vmatrix}$

$= 3(8+3) - 0(16+9) + 5(4-6)$

$= 33 - 0 - 10 = 23$

$\therefore$ volume of the parallelopiped $= [\bar{a}\ \bar{b}\ \bar{c}] = 23$ cu units.

Ex. 29. If the vectors $-3\hat{i} + 4\hat{j} - 2\hat{k}$, $\hat{i} + 2k$ and $\hat{i} - p\hat{j}$ are coplanar, then find the value of p.

Solution : Let $\bar{a} = -3\hat{i} + 4\hat{j} - 2\hat{k}$, $\bar{b} = \hat{i} + 2\hat{k}$, $\bar{c} = \hat{i} - p\hat{j}$.

Then $[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} -3 & 4 & -2 \\ 1 & 0 & 2 \\ 1 & -p & 0 \end{vmatrix}$

$= -3(0+2p) - 4(0-2) - 2(-p-0)$

$= -6p + 8 + 2p = -4p + 8$

Now, $\bar{a}, \bar{b}, \bar{c}$ are coplanar.

$\therefore\ [\bar{a}\ \bar{b}\ \bar{c}] = 0$

$\therefore\ -4p + 8 = 0$

$\therefore\ -4p = -8$

$\therefore\ p = 2.$

Ex. 30. If $\bar{c} = 3\bar{a} - 2\bar{b}$, then prove that $[\bar{a}\ \bar{b}\ \bar{c}] = 0$.

Solution : We use the results : $\bar{b} \times \bar{b} = \bar{0}$ and if in a scalar triple product, two vectors are equal, then the scalar triple product is zero.

$$\begin{aligned}[\bar{a}\ \ \bar{b}\ \ \bar{c}] &= \bar{a}\cdot(\bar{b}\times\bar{c})\\ &= \bar{a}\cdot[\bar{b}\times(3\bar{a}-2\bar{b})]\\ &= \bar{a}\cdot(3\bar{b}\times\bar{a}-2\bar{b}\times\bar{b})\\ &= \bar{a}\cdot(3\bar{b}\times\bar{a}-\bar{0}) = 3\bar{a}\cdot(\bar{b}\times\bar{a}) = 3\times 0 = 0.\end{aligned}$$

Ex. 31. If $\bar{a}, \bar{b}$ and $\bar{c}$ are any three vectors, prove that

(1) $[\bar{a}+\bar{b}\ \ \bar{b}+\bar{c}\ \ \bar{c}+\bar{a}] = 2[\bar{a}\ \bar{b}\ \bar{c}]$

(2) $[\bar{a}\ \ \bar{b}+\bar{c}\ \ \bar{a}+\bar{b}+\bar{c}] = 0$.

Solution : We use the following results :

(i) If any vector in a scalar triple product $[\bar{a}\ \bar{b}\ \bar{c}]$ is repeated, then $[\bar{a}\ \bar{b}\ \bar{c}] = 0$.

(ii) $\bar{a}\cdot(\bar{b}\times\bar{c}) = \bar{b}\cdot(\bar{c}\times\bar{a}) = \bar{c}\cdot(\bar{a}\times\bar{b})$

(1) $[\bar{a}+\bar{b}\ \ \bar{b}+\bar{c}\ \ \bar{c}+\bar{a}] = (\bar{a}+\bar{b})\cdot[(\bar{b}+\bar{c})\times(\bar{c}+\bar{a})]$

$= (\bar{a}+\bar{b})\cdot[\bar{b}\times\bar{c}+\bar{b}\times\bar{a}+\bar{c}\times\bar{c}+\bar{c}\times\bar{a}]$

$= \bar{a}\cdot(\bar{b}\times\bar{c})+\bar{a}\cdot(\bar{b}\times\bar{a})+\bar{a}\cdot(\bar{c}\times\bar{c})+\bar{a}\cdot(\bar{c}\times\bar{a})+\bar{b}\cdot(\bar{b}\times\bar{c})+\bar{b}\cdot(\bar{b}\times\bar{a})$
$\quad + \bar{b}\cdot(\bar{c}\times\bar{c})+\bar{b}\cdot(\bar{c}\times\bar{a})$

$= \bar{a}\cdot(\bar{b}\times\bar{c})+0+0+0+0+0+0+\bar{a}\cdot(\bar{b}\times\bar{c})$

$= [\bar{a}\ \ \bar{b}\ \ \bar{c}]+[\bar{a}\ \ \bar{b}\ \ \bar{c}] = 2[\bar{a}\ \ \bar{b}\ \ \bar{c}].$

- -

(2) $[\bar{a}\ \ \bar{b}+\bar{c}\ \ \bar{a}+\bar{b}+\bar{c}] = \bar{a}\cdot[(\bar{b}+\bar{c})\times(\bar{a}+\bar{b}+\bar{c})]$

$= \bar{a}\cdot(\bar{b}\times\bar{a}+\bar{b}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a}+\bar{c}\times\bar{b}+\bar{c}\times\bar{c})$

$= \bar{a}\cdot(\bar{b}\times\bar{a})+\bar{a}\cdot(\bar{b}\times\bar{b})+\bar{a}\cdot(\bar{b}\times\bar{c})+\bar{a}\cdot(\bar{c}\times\bar{a})+\bar{a}\cdot(\bar{c}\times\bar{b})+\bar{a}\cdot(\bar{c}\times\bar{c})$

$= 0+0+\bar{a}\cdot(\bar{b}\times\bar{c})+0-\bar{a}\cdot(\bar{b}\times\bar{c})+0 = 0.$

Ex. 32. $\bar{a}, \bar{b}, \bar{c}$ **are three non-coplanar vectors.**

If $\bar{p} = \dfrac{\bar{b} \times \bar{c}}{\bar{a} \cdot (\bar{b} \times \bar{c})}$, $\bar{q} = \dfrac{\bar{c} \times \bar{a}}{\bar{a} \cdot (\bar{b} \times \bar{c})}$, $\bar{r} = \dfrac{\bar{a} \times \bar{b}}{\bar{a} \cdot (\bar{b} \times \bar{c})}$,

show that $\bar{a} \cdot \bar{p} + \bar{b} \cdot \bar{q} + \bar{c} \cdot \bar{r} = 3$.

Solution : We use the results : If in a scalar triple product, two vectors are equal, then the scalar triple product is zero and

$\bar{a} \cdot (\bar{b} \times \bar{c}) = \bar{b} \cdot (\bar{c} \times \bar{a}) = \bar{c} \cdot (\bar{a} \times \bar{b})$.

$$\therefore\ \bar{a} \cdot \bar{p} + \bar{b} \cdot \bar{q} + \bar{c} \cdot \bar{r} = \bar{a} \cdot \left(\frac{\bar{b} \times \bar{c}}{\bar{a} \cdot (\bar{b} \times \bar{c})} \right) + \bar{b} \cdot \left(\frac{\bar{c} \times \bar{a}}{\bar{a} \cdot (\bar{b} \times \bar{c})} \right) + \bar{c} \cdot \left(\frac{\bar{a} \times \bar{b}}{\bar{a} \cdot (\bar{b} \times \bar{c})} \right)$$

$$= \frac{\bar{a} . (\bar{b} \times \bar{c})}{\bar{a} . (\bar{b} \times \bar{c})} + \frac{\bar{b} . (\bar{c} \times \bar{a})}{\bar{a} . (\bar{b} \times \bar{c})} + \frac{\bar{c} . (\bar{a} \times \bar{b})}{\bar{a} . (\bar{b} \times \bar{c})}$$

$$= \frac{[\bar{a}\ \bar{b}\ \bar{c}]}{[\bar{a}\ \bar{b}\ \bar{c}]} + \frac{[\bar{a}\ \bar{b}\ \bar{c}]}{[\bar{a}\ \bar{b}\ \bar{c}]} + \frac{[\bar{a}\ \bar{b}\ \bar{c}]}{[\bar{a}\ \bar{b}\ \bar{c}]} = 3.$$

Ex. 33. Show that $\bar{a} \times (\bar{b} \times \bar{c}) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times (\bar{a} \times \bar{b}) = \bar{0}$.

Solution :

$$\text{LHS} = \bar{a} \times (\bar{b} \times \bar{c}) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times (\bar{a} \times \bar{b})$$

$$= (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c} + (\bar{b} \cdot \bar{a})\bar{c} - (\bar{b} \cdot \bar{c})\bar{a} + (\bar{c} \cdot \bar{b})\bar{a} - (\bar{c} \cdot \bar{a})\bar{b}$$

$$= (\bar{c} \cdot \bar{a})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c} + (\bar{a} \cdot \bar{b})\bar{c} - (\bar{b} \cdot \bar{c})\bar{a} + (\bar{b} \cdot \bar{c})\bar{a} - (\bar{c} \cdot \bar{a})\bar{b} \quad \ldots [\because \bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}]$$

$$= \bar{0} = \text{RHS}.$$

Examples for Practice	2 marks each

1. Find $[\bar{a}\ \bar{b}\ \bar{c}]$, where :

(1) $\bar{a} = 2\hat{i} + \hat{j} - \hat{k}$, $\bar{b} = 3\hat{i} - \hat{j} - \hat{k}$, $\bar{c} = \hat{j} + 3\hat{k}$

(2) $\bar{a} = \hat{i} - \hat{j} + 4\hat{k}$, $\bar{b} = \hat{i} + \hat{j} - 4\hat{k}$, $\bar{c} = \hat{i} + \hat{j} + \hat{k}$.

2. If $\bar{a} = 3\hat{i} + p\hat{j} + 6\hat{k}$, $\bar{b} = 2\hat{i} - 5\hat{j} - 13\hat{k}$, $\bar{c} = \hat{i} + 2\hat{j} + 7\hat{k}$, find p, if $[\bar{a}\ \bar{b}\ \bar{c}] = 0$.

3. Show that the vectors $\hat{i} + 5\hat{j} - 2\hat{k}$, $3\hat{i} - \hat{j}$ and $5\hat{i} + 9\hat{j} - 4\hat{k}$ are coplanar.

4. Find the volume of the parallelopiped whose coterminus edges are :

(1) $\hat{i} + 2\hat{j} + 3\hat{k}$, $-\hat{i} + \hat{j} + 2\hat{k}$, $2\hat{i} + \hat{j} + 4\hat{k}$

(2) $\hat{i} + \hat{j}$, $\hat{j} + \hat{k}$, $\hat{i} + \hat{j} + \hat{k}$

(3) $\hat{i} + \hat{j} + 2\hat{k}$, $\hat{i} + 2\hat{j} + 4\hat{k}$, $2\hat{i} + \hat{j} + 3\hat{k}$.

5. Find λ, if the vectors $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + \lambda\hat{j} + 3\hat{k}$ and $5\hat{i} + 9\hat{j} + 7\hat{k}$ are coplanar.

6. If $\bar{a}, \bar{b}, \bar{c}$ are any three vectors, prove that

(1) $[\bar{a} + \bar{b}\ \ \bar{a} + \bar{c}\ \ \bar{b}] = [\bar{a}\ \ \bar{c}\ \ \bar{b}]$

(2) $[\bar{a} - \bar{b}\ \ \bar{b} - \bar{c}\ \ \bar{c} - \bar{a}] = 0$.

7. If $\bar{a}, \bar{b}, \bar{c}$ are three non-coplanar vectors and $\bar{p} = \dfrac{\bar{b} \times \bar{c}}{\bar{a} \cdot (\bar{b} \times \bar{c})}$, $\bar{q} = \dfrac{\bar{c} \times \bar{a}}{\bar{a} \cdot (\bar{b} \times \bar{c})}$,

$\bar{r} = \dfrac{\bar{a} \times \bar{b}}{\bar{a} \cdot (\bar{b} \times \bar{c})}$, then show that $(\bar{a} + \bar{b}) \cdot \bar{p} + (\bar{b} + \bar{c}) \cdot \bar{q} + (\bar{c} + \bar{a}) \cdot \bar{r} = 3$.

ANSWERS

1. (1) -16 (2) 10

2. $p = 1$ **4.** (1) 9 cu units (2) 1 cu unit (3) 1 cu unit.

5. $\lambda = 4$.

Solved Examples	**3 marks each**

Ex. 34. If A, B, C, D are (1, 1, 1), (2, 1, 3), (3, 2, 2), (3, 3, 4) respectively, then find the volume of the parallelopiped with AB, AC and AD as the concurrent edges.

Solution : The position vectors $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ of the points A, B, C and D w.r.t. the origin are $\bar{a} = \hat{i} + \hat{j} + \hat{k}$, $\bar{b} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\bar{c} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\bar{d} = 3\hat{i} + 3\hat{j} + 4\hat{k}$.

$\therefore \overline{AB} = \bar{b} - \bar{a} = (2\hat{i} + \hat{j} + 3\hat{k}) - (\hat{i} + \hat{j} + \hat{k})$

$= \hat{i} + 2\hat{k}$

$\overline{AC} = \bar{c} - \bar{a} = (3\hat{i} + 2\hat{j} + 2\hat{k}) - (\hat{i} + \hat{j} + \hat{k})$

$= 2\hat{i} + \hat{j} + \hat{k}$

and $\overline{AD} = \bar{d} - \bar{a} = (3\hat{i} + 3\hat{j} + 4\hat{k}) - (\hat{i} + \hat{j} + \hat{k})$

$= 2\hat{i} + 2\hat{j} + 3\hat{k}$

$$\therefore [\overline{AB}\ \overline{AC}\ \overline{AD}] = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{vmatrix}$$

$= 1(3-2) - 0(6-2) + 2(4-2)$

$= 1 - 0 + 4 = 5$

$\therefore$ volume of the parallelopiped $= |[\overline{AB}\ \overline{AC}\ \overline{AD}]|$

$= 5$ cu units.

Ex. 35. Show that the points (5, 3, 4), (4, 1, 6), (4, 5, 1) and (3, 3, 3) are coplanar.

Solution : Let $P \equiv (5, 3, 4)$, $Q \equiv (4, 1, 6)$, $R \equiv (4, 5, 1)$ and $S \equiv (3, 3, 3)$

Let $\bar{p}, \bar{q}, \bar{r}, \bar{s}$, be the position vectors of the points P, Q, R, S respectively w.r.t. the origin.

Then $\bar{p} = 5\hat{i} + 3\hat{j} + 4\hat{k}$, $\bar{q} = 4\hat{i} + \hat{j} + 6\hat{k}$, $\bar{r} = 4\hat{i} + 5\hat{j} + \hat{k}$, $\bar{s} = 3\hat{i} + 3\hat{j} + 3\hat{k}$.

$\therefore\ \overline{PQ} = \bar{q} - \bar{p} = (4\hat{i} + \hat{j} + 6\hat{k}) - (5\hat{i} + 3\hat{j} + 4\hat{k}) = -\hat{i} - 2\hat{j} + 2\hat{k}$

$\overline{PR} = \bar{r} - \bar{p} = (4\hat{i} + 5\hat{j} + \hat{k}) - (5\hat{i} + 3\hat{j} + 4\hat{k}) = -\hat{i} + 2\hat{j} - 3\hat{k}$

and $\overline{PS} = \bar{s} - \bar{p} = (3\hat{i} + 3\hat{j} + 3\hat{k}) - (5\hat{i} + 3\hat{j} + 4\hat{k}) = -2\hat{i} - \hat{k}$

$$\therefore\ [\overline{PQ}\ \overline{PR}\ \overline{PS}] = \begin{vmatrix} -1 & -2 & 2 \\ -1 & 2 & -3 \\ -2 & 0 & -1 \end{vmatrix}$$

$$= -1\,(-2+0) + 2\,(1-6) + 2\,(0+4)$$

$$= 2 - 10 + 8 = 0$$

$\therefore\ \overline{PQ}, \overline{PR}, \overline{PS}$ are coplanar.

Hence, the points P, Q, R, S are coplanar.

Ex. 36. Find the value of p if the points A(2, – 1, 1), B(4, 0, p), C (1, 1, 1) and D (2, 4, 3) are coplanar.

Solution : The position vectors $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ of the points A, B, C and D are

$\bar{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\bar{b} = 4\hat{i} + p\hat{k}$, $\bar{c} = \hat{i} + \hat{j} + \hat{k}$ and $\bar{d} = 2\hat{i} + 4\hat{j} + 3\hat{k}$.

$\therefore\ \overline{AB} = \bar{b} - \bar{a} = (4\hat{i} + p\hat{k}) - (2\hat{i} - \hat{j} + \hat{k})$

$= 2\hat{i} + \hat{j} + (p-1)\hat{k}$

$\overline{AC} = \bar{c} - \bar{a} = (\hat{i} + \hat{j} + \hat{k}) - (2\hat{i} - \hat{j} + \hat{k})$

$= -\hat{i} + 2\hat{j}$

$\overline{AD} = \bar{d} - \bar{a} = (2\hat{i} + 4\hat{j} + 3\hat{k}) - (2\hat{i} - \hat{j} + \hat{k})$

$= 5\hat{j} + 2\hat{k}$

$$\therefore\ [\overline{AB}\ \overline{AC}\ \overline{AD}] = \begin{vmatrix} 2 & 1 & p-1 \\ -1 & 2 & 0 \\ 0 & 5 & 2 \end{vmatrix}$$

$$= 2(4-0) - 1(-2-0) + (p-1)(-5-0)$$

$$= 8 + 2 - 5p + 5$$

$$= -5p + 15$$

Now, points A, B, C, D are coplanar.

$\therefore$ vectors $\overline{AB}, \overline{AC}, \overline{AD}$ are coplanar.

$\therefore\ [\overline{AB}\ \overline{AC}\ \overline{AD}] = 0$

$\therefore\ -5p + 15 = 0$

$\therefore\ p = 3.$

Ex. 37. Find the volume of a tetrahedron whose vertices are $A(-1,2,3)$, $B(3,-2,1)$, $C(2,1,3)$ and $D(-1,-2,4)$.

Solution : The position vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ of the points A, B, C and D w.r.t. the origin are

$\bar{a} = -\hat{i} + 2\hat{j} + 3\hat{k}$, $\bar{b} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\bar{c} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\bar{d} = -\hat{i} - 2\hat{j} + 4\hat{k}$.

$$\therefore\ \overline{AB} = \bar{b} - \bar{a} = (3\hat{i} - 2\hat{j} + \hat{k}) - (-\hat{i} + 2\hat{j} + 3\hat{k})$$
$$= 4\hat{i} - 4\hat{j} - 2\hat{k}$$
$$\overline{AC} = \bar{c} - \bar{a} = (2\hat{i} + \hat{j} + 3\hat{k}) - (-\hat{i} + 2\hat{j} + 3\hat{k})$$
$$= 3\hat{i} - \hat{j}$$
and
$$\overline{AD} = \bar{d} - \bar{a} = (-\hat{i} - 2\hat{j} + 4\hat{k}) - (-\hat{i} + 2\hat{j} + 3\hat{k})$$
$$= -4\hat{j} + \hat{k}$$

$$\therefore\ [\overline{AB}\ \overline{AC}\ \overline{AD}] = \begin{vmatrix} 4 & -4 & -2 \\ 3 & -1 & 0 \\ 0 & -4 & 1 \end{vmatrix}$$
$$= 4(-1+0) + 4(3-0) - 2(-12+0)$$
$$= -4 + 12 + 24 = 32$$

$$\therefore\ \text{volume of the tetrahedron} = \frac{1}{6}|[\overline{AB}\ \overline{AC}\ \overline{AD}]|$$
$$= \frac{1}{6}(32) = \frac{16}{3}\ \text{cu units.}$$

Ex. 38. If four points $A(\bar{a})$, $B(\bar{b})$, $C(\bar{c})$ and $D(\bar{d})$ are coplanar, then show that $[\bar{a}\ \bar{b}\ \bar{d}] + [\bar{b}\ \bar{c}\ \bar{d}] + [\bar{c}\ \bar{a}\ \bar{d}] = [\bar{a}\ \bar{b}\ \bar{c}]$.

Solution : $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ are the position vectors of the points A, B, C and D respectively.

$\therefore\ \overline{AB} = \bar{b} - \bar{a}$, $\overline{AC} = \bar{c} - \bar{a}$, $\overline{AD} = \bar{d} - \bar{a}$

The points A, B, C, D are coplanar.

$\therefore$ the vectors $\overline{AB}$, $\overline{AC}$, $\overline{AD}$ are coplanar.

$$\therefore\ [\overline{AB}\ \ \overline{AC}\ \ \overline{AD}] = 0$$
$$\therefore\ [\bar{b} - \bar{a}\ \ \bar{c} - \bar{a}\ \ \bar{d} - \bar{a}] = 0$$
$$\therefore\ (\bar{b} - \bar{a})\cdot[(\bar{c} - \bar{a}) \times (\bar{d} - \bar{a})] = 0$$
$$\therefore\ (\bar{b} - \bar{a})\cdot(\bar{c} \times \bar{d} - \bar{c} \times \bar{a} - \bar{a} \times \bar{d} + \bar{a} \times \bar{a}) = 0, \text{ where } \bar{a} \times \bar{a} = \bar{0}$$
$$\therefore\ (\bar{b} - \bar{a})\cdot(\bar{c} \times \bar{d} - \bar{c} \times \bar{a} - \bar{a} \times \bar{d}) = 0$$
$$\therefore\ \bar{b}\cdot(\bar{c} \times \bar{d}) - \bar{b}\cdot(\bar{c} \times \bar{a}) - \bar{b}\cdot(\bar{a} \times \bar{d}) - \bar{a}\cdot(\bar{c} \times \bar{d}) + \bar{a}\cdot(\bar{c} \times \bar{a}) + \bar{a}\cdot(\bar{a} \times \bar{d}) = 0 \quad \text{... (1)}$$

Now, $\bar{a}\cdot(\bar{c}\times\bar{a})=0$, $\bar{a}\cdot(\bar{a}\times\bar{d})=0$,

$-\bar{b}\cdot(\bar{a}\times\bar{d})=\bar{b}\cdot(\bar{d}\times\bar{a})=[\bar{b}\ \bar{d}\ \bar{a}]=[\bar{a}\ \bar{b}\ \bar{d}]$,

$-\bar{a}\cdot(\bar{c}\times\bar{d})=\bar{a}\cdot(\bar{d}\times\bar{c})=[\bar{a}\ \bar{d}\ \bar{c}]=[\bar{c}\ \bar{a}\ \bar{d}]$

Also, $\bar{b}\cdot(\bar{c}\times\bar{d})=[\bar{b}\ \bar{c}\ \bar{d}]$,

$-\bar{b}\cdot(\bar{c}\times\bar{a})=-\bar{a}\cdot(\bar{b}\times\bar{c})=-[\bar{a}\ \bar{b}\ \bar{c}]$

$\therefore$ from (1)

$[\bar{b}\ \bar{c}\ \bar{d}]-[\bar{a}\ \bar{b}\ \bar{c}]+[\bar{a}\ \bar{b}\ \bar{d}]+[\bar{c}\ \bar{a}\ \bar{d}]+0+0=0$

$\therefore$ $[\bar{a}\ \bar{b}\ \bar{d}]+[\bar{b}\ \bar{c}\ \bar{d}]+[\bar{c}\ \bar{a}\ \bar{d}]=[\bar{a}\ \bar{b}\ \bar{c}]$.

Examples for Practice	**3 marks each**

1. Find the volume of the parallelopiped with segments AB, AC and AD as concurrent edges, where :

(1) $A\equiv(3, 7, 4)$, $B\equiv(5, -2, 3)$, $C\equiv(-4, 5, 6)$ and $D\equiv(1, 2, 3)$.

(2) the position vectors of A, B, C, D are $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}-\hat{j}+3\hat{k}$, $3\hat{i}-2\hat{j}-2\hat{k}$ and $3\hat{i}+3\hat{j}+4\hat{k}$.

2. Find the volume of the tetrahedron whose vertices are $A(3, 7, 4)$, $B(5, -2, 3)$, $C(-4, 5, 6)$ and $D(1, 2, 3)$.

3. Show that the following sets of points are coplanar :

(1) (3, 9, 4), (0, -1, -1), (-4, 4, 4) and (4, 5, 1).

(2) (1, -1, -1), (3, 1, -1), (0, 2, 1) and (-2, 0, 1).

4. Show that the points $A(2, -1, 0)$, $B(-3, 0, 4)$, $C(-1, -1, 4)$ and $D(0, -5, 2)$ are non-coplanar.

5. Find the value of x, if the points $A(3, 2, 1)$, $B(4, x, 5)$, $C(4, 2, 2)$ and $D(6, 5, -1)$ are coplanar.

6. If the origin and the points $P(2, 3, 4)$, $Q(1, 2, 3)$ and $R(x, y, z)$ are coplanar, then prove that $x-2y+z=0$.

7. If $\bar{u}=\bar{i}-2\bar{j}+\bar{k}$, $\bar{v}=3\bar{i}+\bar{k}$ and $\bar{w}=\bar{j}-\bar{k}$, are given vectors, then find

(1) $[\bar{u}\times\bar{v}\ \ \bar{u}\times\bar{w}\ \ \bar{v}\times\bar{w}]$

(2) $(\bar{u}+\bar{w})\cdot[(\bar{u}\times\bar{v})\times(\bar{v}\times\bar{w})]$.

8. If $\bar{a}=\hat{i}+2\hat{j}+3\hat{k}$, $\bar{b}=3\hat{i}+2\hat{j}$ and $\bar{c}=2\hat{i}+\hat{j}+3\hat{k}$, then verify that $\bar{a}\times(\bar{b}\times\bar{c})=(\bar{a}\cdot\bar{c})\,\bar{b}-(\bar{a}\cdot\bar{b})\,\bar{c}$.

ANSWERS

1. (1) 92 cu units (2) 41 cu units.

2. $\frac{46}{3}$ cu units. **5.** $x=\frac{1}{5}$ **7.** (1) 16 (2) -12.

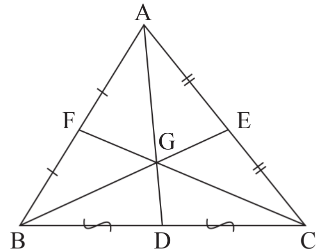
5.4 APPLICATIONS OF VECTORS

Solved Examples	3 or 4 marks each

Ex. 39. Prove that the medians of a triangle are concurrent.

Proof : Let $\bar{a}, \bar{b}, \bar{c}$ be the position vectors of the vertices A, B, C of $\triangle$ABC and $\bar{d}, \bar{e}, \bar{f}$ be the position vectors of the midpoints D, E, F of the sides BC, CA and AB respectively.

Then by the midpoint formula,



$$\bar{d} = \frac{\bar{b}+\bar{c}}{2}, \quad \bar{e} = \frac{\bar{c}+\bar{a}}{2}, \quad \bar{f} = \frac{\bar{a}+\bar{b}}{2}$$

$\therefore\ 2\bar{d} = \bar{b}+\bar{c};\ \ 2\bar{e} = \bar{c}+\bar{a};\ \ 2\bar{f} = \bar{a}+\bar{b}$

$\therefore\ 2\bar{d}+\bar{a} = \bar{a}+\bar{b}+\bar{c}$

$2\bar{e}+\bar{b} = \bar{a}+\bar{b}+\bar{c}$

$2\bar{f}+\bar{c} = \bar{a}+\bar{b}+\bar{c}$

$$\therefore\ \frac{2\bar{d}+\bar{a}}{2+1} = \frac{2\bar{e}+\bar{b}}{2+1} = \frac{2\bar{f}+\bar{c}}{2+1} = \frac{\bar{a}+\bar{b}+\bar{c}}{3} = \bar{g} \quad \text{... (Say)}$$

This shows that the point G whose position vector is $\bar{g}$, lies on the three medians AD, BE and CF dividing each of them internally in the ratio 2 : 1.

Hence, the medians are concurrent in the point *G* and its position vector is $(\bar{a}+\bar{b}+\bar{c})/3$.

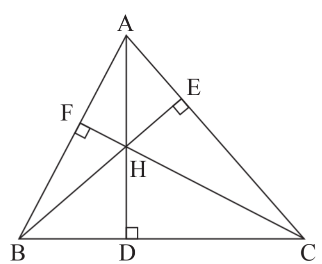
Ex. 40. Prove that the altitudes of a triangle are concurrent. ***(March '22)***

Proof : Let segments AD and CF be the altitudes of $\triangle$ABC, meeting each other in the point H.

Then it is enough to prove that $\overline{\text{HB}}$ is perpendicular to $\overline{\text{AC}}$.

Choose H as the origin and let $\bar{a}, \bar{b}, \bar{c}$ be the position vectors of the vertices A, B and C respectively w.r.t. the origin H.

Then $\overline{\text{HA}} = \bar{a}$, $\overline{\text{HB}} = \bar{b}$ and $\overline{\text{HC}} = \bar{c}$,



$\overline{\text{AB}} = \bar{b}-\bar{a}$, $\overline{\text{BC}} = \bar{c}-\bar{b}$ and $\overline{\text{AC}} = \bar{c}-\bar{a}$.

Now, $\overline{\text{HA}}$ is perpendicular to $\overline{\text{BC}}$.

$\therefore\ \overline{\text{HA}}\cdot\overline{\text{BC}} = 0 \qquad \therefore\ \bar{a}\cdot(\bar{c}-\bar{b}) = 0$

$\therefore\ \bar{a}\cdot\bar{c} - \bar{a}\cdot\bar{b} = 0 \qquad \text{... (1)}$

Also $\overline{\text{HC}}$ is perpendicular to $\overline{\text{AB}}$.

$\therefore\ \overline{HC}\cdot\overline{AB}=0 \qquad \therefore\ \bar{c}\cdot(\bar{b}-\bar{a})=0$

$\therefore\ \bar{c}\cdot\bar{b}-\bar{c}\cdot\bar{a}=0$

$\therefore\ \bar{c}\cdot\bar{b}-\bar{a}\cdot\bar{c}=0$ ... $[\because\ \bar{c}\cdot\bar{a}=\bar{a}\cdot\bar{c}]$ ... (2)

Adding (1) and (2), we get

$\bar{c}\cdot\bar{b}-\bar{a}\cdot\bar{b}=0$

$\therefore\ (\bar{c}-\bar{a})\cdot\bar{b}=0$

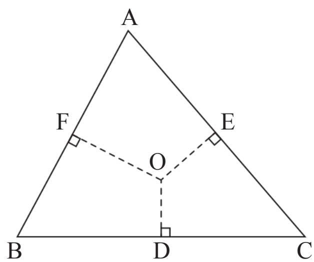
$\therefore\ \bar{b}\cdot(\bar{c}-\bar{a})=0$

$\therefore\ \overline{HB}\cdot\overline{AC}=0$

$\therefore\ \overline{HB}$ is perpendicular to $\overline{AC}$.

Hence, the altitudes of $\triangle$ABC are concurrent.

Ex. 41. Prove that the perpendicular bisectors of the sides of a triangle are concurrent.



Proof : Let D, E, F be the midpoints of the sides BC, CA and AB of $\triangle$ABC. Let the perpendicular bisectors of the sides BC and AC meet each other in the point O. Choose O as the origin and let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}$ be the position vectors of the points A, B, C, D, E, F respectively.

Here, we have to prove that $\overline{OF}=\bar{f}$ is perpendicular to $\overline{AB}=\bar{b}-\bar{a}$.

By the midpoint formula, $\bar{d}=\dfrac{\bar{b}+\bar{c}}{2},\ \ \bar{e}=\dfrac{\bar{c}+\bar{a}}{2},\ \ \bar{f}=\dfrac{\bar{a}+\bar{b}}{2}$.

Now, $\overline{OD}=\bar{d}$ is perpendicular to $\overline{BC}=\bar{c}-\bar{b}$.

$\therefore\ \bar{d}\cdot(\bar{c}-\bar{b})=0 \qquad \therefore\ \left(\dfrac{\bar{b}+\bar{c}}{2}\right)\cdot(\bar{c}-\bar{b})=0$

$\therefore\ (\bar{c}+\bar{b})\cdot(\bar{c}-\bar{b})=0$

$\therefore\ \bar{c}\cdot\bar{c}+\bar{b}\cdot\bar{c}-\bar{c}\cdot\bar{b}-\bar{b}\cdot\bar{b}=0$

$\therefore\ c^2-b^2=0 \qquad \therefore\ c^2=b^2$ ... $[\because\ \bar{c}\cdot\bar{c}=c^2,\ \bar{b}\cdot\bar{b}=b^2$ and $\bar{c}\cdot\bar{b}=\bar{b}\cdot\bar{c}]$ ... (1)

Also, $\overline{OE}=\bar{e}$ is perpendicular to $\overline{AC}=\bar{c}-\bar{a}$

$\therefore\ \bar{e}\cdot(\bar{c}-\bar{a})=0$

$\therefore\ \left(\dfrac{\bar{c}+\bar{a}}{2}\right)\cdot(\bar{c}-\bar{a})=0$

$\therefore$ as above $c^2=a^2$ ... (2)

From (1) and (2), we get

$b^2=a^2 \qquad \therefore\ b^2-a^2=0$

$\therefore\ (\bar{b}+\bar{a})\cdot(\bar{b}-\bar{a})=0$

$\therefore\ \left(\dfrac{\bar{b}+\bar{a}}{2}\right)\cdot(\bar{b}-\bar{a})=0$

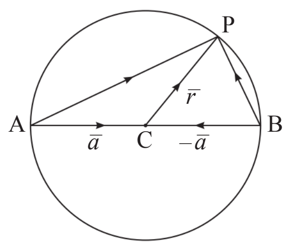
$\therefore\ \bar{f}\cdot(\bar{b}-\bar{a})=0$

$\therefore\ \bar{f}=\overline{\text{OF}}$ is perpendicular to $\bar{b}-\bar{a}=\overline{\text{AB}}$.

Hence, the perpendicular bisectors of the sides of $\triangle$ABC are concurrent.

Ex. 42. Prove by vector method, that the angle subtended on semicircle is a right angle.

Proof : Let seg AB be a diameter of a circle with centre C and P be any point on the circle other than A and B.



Then $\angle$APB is an angle subtended on a semicircle.

Let $\overline{\text{AC}}=\overline{\text{CB}}=\bar{a}$ and $\overline{\text{CP}}=\bar{r}$.

Then $|\bar{a}|=|\bar{r}|$ ... (1)

$\overline{\text{AP}}=\overline{\text{AC}}+\overline{\text{CP}}=\bar{a}+\bar{r}=\bar{r}+\bar{a}$

$\overline{\text{BP}}=\overline{\text{BC}}+\overline{\text{CP}}=-\overline{\text{CB}}+\overline{\text{CP}}=-\bar{a}+\bar{r}$

$\therefore\ \overline{\text{AP}}\cdot\overline{\text{BP}}=(\bar{r}+\bar{a})\cdot(\bar{r}-\bar{a})$

$=\bar{r}\cdot\bar{r}-\bar{r}\cdot\bar{a}+\bar{a}\cdot\bar{r}-\bar{a}\cdot\bar{a}$

$=|\bar{r}|^2-|\bar{a}|^2=0$ ... $[\because\ \bar{r}\cdot\bar{a}=\bar{a}\cdot\bar{r}]$

$\therefore\ \overline{\text{AP}}\perp\overline{\text{BP}} \qquad \therefore\ \angle$APB is a right angle.

Hence, the angle subtended on a semicircle is the right angle.

5.5 DIRECTION ANGLES AND DIRECTION COSINES

Remember :

1. If α, β, γ are the angles made by a line with positive directions of X-, Y- and Z-axes respectively, then its direction cosines are $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$. α, β, γ are called direction angles of the line.

2. If l, m, n are direction cosines of a line, then any real numbers a, b, c such that $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$ are called direction ratios or direction numbers of that line.

3. If a, b, c are the direction ratios of a line, then its direction cosines are $\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$.

4. If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, then the line PQ has direction ratios $x_2-x_1, y_2-y_1, z_2-z_1$.

5. If $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ is a vector along any line, then a_1, a_2, a_3 are the direction ratios of the line.

6. If $\hat{e}=e_1\hat{i}+e_2\hat{j}+e_3\hat{k}$ is the unit vector along any line, then e_1, e_2, e_3 are the direction cosines of the line.

7. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of the two lines, then the acute angle θ between the lines is given by $\cos\theta=|l_1l_2+m_1m_2+n_1n_2|$.

8. If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of the two lines, then the acute angle θ between the lines is given by

$$\cos\theta=\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}\right|.$$

9. Two lines with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 are
   (i) perpendicular, if $l_1l_2+m_1m_2+n_1n_2=0$
   (ii) parallel, if $l_1=l_2, m_1=m_2, n_1=n_2$.

10. Two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 are
   (i) perpendicular, if $a_1a_2+b_1b_2+c_1c_2=0$
   (ii) parallel, if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.

Theory Question	3 marks

Q. 5. If l, m, n are the direction cosines of a line, then prove that $l^2+m^2+n^2=1$. **OR**

If α, β, γ are direction angles of a line, then prove that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$.

Hence, deduce that $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$.

Ans. Let α, β, γ be the angles made by the line with X-, Y-, Z-axes respectively.

$\therefore \; l = \cos\alpha,\; m = \cos\beta$ and $n = \cos\gamma$

Let $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ be any non-zero vector along the line.

Since $\hat{i}$ is the unit vector along X-axis,

$$\bar{a}\cdot\hat{i} = |\bar{a}|\cdot|\hat{i}|\cos\alpha = a\cos\alpha$$

Also, $\bar{a}\cdot\hat{i} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot\hat{i}$

$$= a_1 \times 1 + a_2 \times 0 + a_3 \times 0 = a_1$$

$\therefore \; a\cos\alpha = a_1$ ... (1)

Since $\hat{j}$ is the unit vector along Y-axis,

$$\bar{a}\cdot\hat{j} = |\bar{a}|\cdot|\hat{j}|\cos\beta = a\cos\beta$$

Also, $\bar{a}\cdot\hat{j} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot\hat{j}$

$$= a_1 \times 0 + a_2 \times 1 + a_3 \times 0 = a_2$$

$\therefore \; a\cos\beta = a_2$ ... (2)

Similarly, $a\cos\gamma = a_3$ ... (3)

$\therefore$ from equations (1), (2) and (3)

$$a^2\cos^2\alpha + a^2\cos^2\beta + a^2\cos^2\gamma = a_1{}^2 + a_2{}^2 + a_3{}^2$$

$\therefore \; a^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = a^2$ ... $\left[\because\; a = |\bar{a}| = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}\right]$

$\therefore \; \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ ... $[\because\; a \neq 0]$

i.e. $l^2 + m^2 + n^2 = 1$.

Now, $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$

$= 1 - \cos^2\alpha + 1 - \cos^2\beta + 1 - \cos^2\gamma$

$= 3 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$

$= 3 - 1$ ... $[\because\; \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1]$

$= 2$.

Solved Examples	**2 marks each**

Ex. 43. If a vector has direction angles 45° and 60°, find the third direction angle.

Solution : Let $\alpha = 45^\circ,\; \beta = 60^\circ$

We have to find γ.

$\because \; \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore \; \cos^2 45^\circ + \cos^2 60^\circ + \cos^2\gamma = 1$

$$\therefore \; \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right) + \cos^2\gamma = 1$$

$\therefore\ \cos^2\gamma = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$\therefore\ \cos\gamma = \pm\frac{1}{2}$

$\therefore\ \cos\gamma = \frac{1}{2}$ or $\cos\gamma = -\frac{1}{2}$

$\therefore\ \cos\gamma = \cos\frac{\pi}{3}$ or $\cos\gamma = -\cos\frac{\pi}{3}$

$$= \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$$

$\therefore\ \gamma = \frac{\pi}{3}$ or $\gamma = \frac{2\pi}{3}$

Hence, the third direction angle is $\frac{\pi}{3}$ or $\frac{2\pi}{3}$.

Ex. 44. Find the direction cosines of the line which bisects the angle between positive directions of Y- and Z-axes.

Solution : Let α, β, γ be the angles made by the line with the positive directions of X-, Y- and Z-axes respectively.

Since the line bisects the angle between positive directions of Y- and Z-axes, it lies in YZ-plane.

$\therefore$ the line is perpendicular to X-axis.

$\therefore\ \alpha = 90°,\ \beta = 45°,\ \gamma = 45°$

If l, m, n are the direction cosines of the line, then

$l = \cos\alpha = \cos 90° = 0$

$m = \cos\beta = \cos 45° = \frac{1}{\sqrt{2}}$

$n = \cos\gamma = \cos 45° = \frac{1}{\sqrt{2}}$

Hence, the direction cosines of the line are $0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$.

Ex. 45. Show that no line in space can make angles $\frac{\pi}{6}$ and $\frac{\pi}{4}$ with X-axis and Y-axis.

Solution : Let, if possible, a line in space make angles $\frac{\pi}{6}$ and $\frac{\pi}{4}$ with X-axis and Y-axis.

$\therefore\ \alpha = \frac{\pi}{6},\ \beta = \frac{\pi}{4}$

Let the line make angle γ with Z-axis.

$\because\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore\ \cos^2\left(\frac{\pi}{6}\right) + \cos^2\left(\frac{\pi}{4}\right) + \cos^2\gamma = 1$

$\therefore\ \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\gamma = 1$

$\therefore\ \cos^2\gamma = 1 - \frac{3}{4} - \frac{1}{2} = -\frac{1}{4}$

This is not possible, because $\cos\gamma$ is real.

$\therefore\ \cos^2\gamma$ cannot be negative.

Hence, there is no line in space which makes angles $\frac{\pi}{6}$ and $\frac{\pi}{4}$ with X-axis and Y-axis.

Ex. 46. Find the direction ratios of a vector perpendicular to the two lines whose direction ratios are $-2, 1, -1$ and $-3, -4, 1$.

Solution : Let $\bar{a}$ and $\bar{b}$ be the vectors along the lines whose direction ratios are $-2, 1, -1$ and $-3, -4, 1$ respectively.

$\therefore\ \bar{a} = -2\hat{i} + \hat{j} - \hat{k}$ and $\bar{b} = -3\hat{i} - 4\hat{j} + \hat{k}$

A vector perpendicular to both $\bar{a}$ and $\bar{b}$ is given by

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -1 \\ -3 & -4 & 1 \end{vmatrix} = (1-4)\hat{i} - (-2-3)\hat{j} + (8+3)\hat{k}$$

$$= -3\hat{i} + 5\hat{j} + 11\hat{k}$$

$\therefore$ the direction ratios of the required vector are $-3, 5, 11$.

Ex. 47. Find the measure of acute angle between the lines whose direction ratios are 5, 12, -13 and 3, -4, 5.

Solution : Let θ be the required acute angle between the lines.

$$\therefore\ \cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1{}^2 + b_1{}^2 + c_1{}^2}\cdot\sqrt{a_2{}^2 + b_2{}^2 + c_2{}^2}}\right|$$

where $a_1 = 5,\ b_1 = 12,\ c_1 = -13$ and $a_2 = 3,\ b_2 = -4,\ c_2 = 5$

$$\therefore\ \cos\theta = \left|\frac{5(3) + 12(-4) + (-13)(5)}{\sqrt{5^2 + (12)^2 + (-13)^2}\cdot\sqrt{3^2 + (-4)^2 + 5^2}}\right|$$

$$= \left| \frac{15-48-65}{\sqrt{25+144+169} \cdot \sqrt{9+16+25}} \right| = \left| \frac{-98}{13\sqrt{2} \times 5\sqrt{2}} \right| = \frac{98}{130}$$

$\therefore \ \cos\theta = \dfrac{49}{65} \qquad \therefore \ \theta = \cos^{-1}\left(\dfrac{49}{65}\right).$

Examples for Practice	2 marks each

1. A line makes angles of measures $\frac{\pi}{6}$ and $\frac{\pi}{3}$ with X- and Z-axes respectively. Find the angle made by the line with the Y-axis.
2. A line lies in XZ-plane and makes an angle of 60° with Z-axis, find its inclination with X-axis.
3. Show that there is no line in space which makes angle of 30° with each of X- and Y-axes.
4. Find the angle between the lines whose direction ratios are 4, -3, 5 and 3, 4, 5.
5. If the direction ratios of two parallel lines are $4, -3, -1$ and $p+q$, $1+q$, 2, then find the values of p and q.
6. Direction ratios of two lines are 3, -2, k and -2, k, 4. Find k, if the lines are perpendicular to each other.
7. Find the direction ratios of a vector perpendicular to two lines whose direction ratios are 1, 3, 2 and -1, 1, 2.
8. If a line makes angles α, β, γ with the coordinate axes, prove that $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$.

ANSWERS

1. $\frac{\pi}{2}$ **2.** 30° or 150° **4.** $\frac{\pi}{3}$ **5.** $p = -13$, $q = 5$

6. $k = 3$ **7.** 1, -1, 1.

Solved Examples	3 marks each

Ex. 48. A line makes angles of measures 45° and 60° with positive directions of Y- and Z-axes respectively. Find the d.c.s. of the line and also find the vector of magnitude 5 along the direction of line.

Solution : Let α, β, γ be the direction angles of the line.

Then $\beta = 45°$ and $\gamma = 60°$

We know that, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore\ \cos^2\alpha + \cos^2 45° + \cos^2 60° = 1$

$$\therefore\ \cos^2\alpha + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 = 1$$

$$\therefore\ \cos^2\alpha + \frac{1}{2} + \frac{1}{4} = 1$$

$$\therefore\ \cos^2\alpha = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\therefore\ \cos\alpha = \pm\frac{1}{2}$$

Let l, m, n be the direction cosines of the line.

$$\text{Then}\quad l = \cos\alpha = \pm\frac{1}{2}$$

$$m = \cos\beta = \cos 45° = \frac{1}{\sqrt{2}}$$

$$n = \cos\gamma = \cos 60° = \frac{1}{2}$$

$\therefore$ direction cosines of the line are $\pm\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$

Let $\bar{e}$ be the unit vector along the line.

Then $\bar{e} = l\hat{i} + m\hat{j} + n\hat{k} = \pm\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}$

$\therefore$ the required vectors of magnitude 5 along the direction of the line are

$$5\left(\pm\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}\right)$$

i.e. $5\left(\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}\right)$ and $5\left(-\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}\right)$.

Ex. 49. Direction ratios of two lines satisfy the relation $2a - b + 2c = 0$ and $ab + bc + ca = 0$. Show that the lines are perpendicular.

Solution : Given : $2a - b + 2c = 0$ ... (1)

and $ab + bc + ca = 0$ ... (2)

From (1), $b = 2a + 2c$

Substituting the value of b in equation (2), we get

$a(2a + 2c) + (2a + 2c)\,c + ca = 0$

$\therefore\ 2a^2+2ac+2ac+2c^2+ac=0$

$\therefore\ 2a^2+5ac+2c^2=0$

$\therefore\ 2a^2+4ac+ac+2c^2=0$

$\therefore\ 2a(a+2c)+c(a+2c)=0$

$\therefore\ (a+2c)(2a+c)=0$

$\therefore\ a+2c=0 \quad \text{or} \quad 2a+c=0$

$\therefore\ a=-2c \quad \text{or} \quad c=-2a$

Now, $b=2a+2c$, therefore if $a=-2c$, then

$b=-4c+2c=-2c$

$\therefore\ a=b=-2c$

$$\therefore\ \frac{a}{-2}=\frac{b}{-2}=\frac{c}{1}$$

$\therefore$ direction ratios of the first line are $a_1=-2$, $b_1=-2$, $c_1=1$

If $c=-2a$, then

$b=2a-4a=-2a$

$\therefore\ -2a=b=c$

$$\therefore\ \frac{a}{1}=\frac{b}{-2}=\frac{c}{-2}$$

$\therefore$ direction ratios of the second line are $a_2=1$, $b_2=-2$, $c_2=-2$

Since $a_1a_2+b_1b_2+c_1c_2=-2(1)+(-2)(-2)+1(-2)$

$$=-2+4-2=0$$

Hence, the lines are perpendicular.

Ex. 50. If a line drawn from the point A (1, 2, 1) is perpendicular to the line joining P (1, 4, 6) and Q (5, 4, 4), then find the coordinates of the foot of the perpendicular.

Solution : Let AM be the perpendicular from the point A (1, 2, 1) to the line PQ, where P (1, 4, 6) and Q (5, 4, 4).

Let M divides PQ internally in the ratio $k:1$.

$$\therefore\ \text{M} \equiv \left(\frac{5k+1}{k+1}, \frac{4k+4}{k+1}, \frac{4k+6}{k+1}\right)$$

$\therefore$ the direction ratios of AM are

$$\frac{5k+1}{k+1}-1, \frac{4k+4}{k+1}-2, \frac{4k+6}{k+1}-1$$

i.e. $$\frac{5k+1-k-1}{k+1}, \frac{4k+4-2k-2}{k+1}, \frac{4k+6-k-1}{k+1}$$

i.e. $\dfrac{4k}{k+1}, \dfrac{2k+2}{k+1}, \dfrac{3k+5}{k+1}$

and the direction ratios of PQ are $5-1, 4-4, 4-6$, i.e. $4, 0, -2$

Since AM is perpendicular to PQ,

$$4\left(\frac{4k}{k+1}\right)+0\left(\frac{2k+2}{k+1}\right)-2\left(\frac{3k+5}{k+1}\right)=0$$

$\therefore\ 16k-6k-10=0$

$\therefore\ 10k=10 \qquad \therefore\ k=1$

$\therefore\ M \equiv \left(\frac{6}{2}, \frac{8}{2}, \frac{10}{2}\right), \qquad$ i.e. $M \equiv (3, 4, 5)$

Hence, the coordinates of the foot of perpendicular are $(3, 4, 5)$.

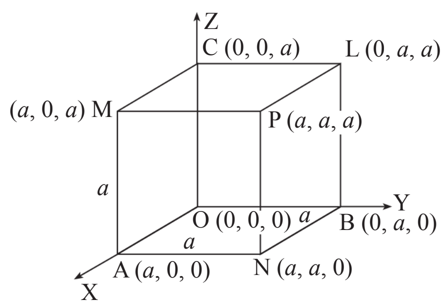
Ex. 51. Show that the angle between any two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

Solution : Take origin O as one vertex of the cube and OA, OB and OC as the positive directions of the X-axis, the Y-axis and the Z-axis respectively.

Here, the sides of the cube are

$OA = OB = OC = a$ ... (Say)

$\therefore$ the coordinates of all the vertices of the cube will be



$O \equiv (0, 0, 0)$ $\qquad A \equiv (a, 0, 0)$

$B \equiv (0, a, 0)$ $\qquad C \equiv (0, 0, a)$

$N \equiv (a, a, 0)$ $\qquad L \equiv (0, a, a)$

$M \equiv (a, 0, a)$ $\qquad P \equiv (a, a, a)$

Here the four diagonals are AL, BM, CN and OP.

Consider the diagonals OP and AL.

The direction ratios of OP are $a-0, a-0, a-0$, i.e. a, a, a

The direction ratios of AL are $0-a, a-0, a-0$, i.e. $-a, a, a$

Let θ be the angle between the diagonals OP and AL.

$$\text{Then } \cos\theta = \left|\frac{a(-a)+a(a)+a(a)}{\sqrt{a^2+a^2+a^2}\cdot\sqrt{(-a)^2+a^2+a^2}}\right| = \left|\frac{-a^2+a^2+a^2}{(\sqrt{3}a)(\sqrt{3}a)}\right| = \frac{1}{3}$$

$\therefore\ \theta = \cos^{-1}\left(\frac{1}{3}\right)$.

Examples for Practice	3 marks each

1. Find $\bar{r}$, if direction ratios of vector $\bar{r}$ are 2, -3, 6 and $|\bar{r}| = 21$ and $\bar{r}$ makes obtuse angle with the X-axis.
2. A (2, 3, 7), B (-1, 3, 2) and C (p, 5, r) are vertices of $\triangle$ABC. If the median through A is equally inclined to the coordinate axes, then find the values of p and r. Hence, find the coordinates of the vertex C.
3. Find the angle between the lines whose direction cosines l, m, n satisfy the equations $5l + m + 3n = 0$ and $5mn - 2nl + 6ln = 0$.
4. If the direction ratios of two vectors are connected by the relations $p + q + r = 0$ and $p^2 + q^2 - r^2 = 0$. Find the angle between them.
5. If M is the foot of perpendicular drawn from A (4, 3, 2) on the line joining the points B (2, 4, 1) and C (4, 5, 3), find the coordinates of M.

ANSWERS

1. $\bar{r} = -6\hat{i} + 9\hat{j} - 18\hat{k}$
2. $p = 7, r = 14$, C = (7, 5, 14)
3. $\cos^{-1}\left(\frac{1}{6}\right)$
4. $\frac{\pi}{3}$
5. $\left(\frac{28}{9}, \frac{41}{9}, \frac{19}{9}\right)$.

MULTIPLE CHOICE QUESTIONS	2 marks each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. If $\bar{a} = \hat{i} + 2\hat{j}$, $\bar{b} = -2\hat{i} + \hat{j}$ and $\bar{c} = 4\hat{i} + 3\hat{j}$, then the values of x and y such that $\bar{c} = x\bar{a} + y\bar{b}$ are

   (a) 1, 1 (b) 2, -1 (c) -1, 2 (d) 1, 0

2. If $|\bar{a}| = 3$, $|\bar{b}| = 4$, then the value of λ for which $\bar{a} + \lambda\bar{b}$ is perpendicular to $\bar{a} - \lambda\bar{b}$, is

   (a) $\frac{9}{16}$ (b) $\frac{3}{4}$ (c) $\frac{3}{2}$ (d) $\frac{4}{3}$

3. If the points A (2, 1, 1), B (0, -1, 4) and C (k, 3, -2) are collinear, then $k =$ ....

   (a) 0 (b) 1 (c) 4 (d) -4

4. If two non-zero vectors $\overline{a}$ and $\overline{b}$ are perpendicular to each other, then $(\overline{a}-\overline{b})^2 = \ldots\ldots$

(a) $a^2 - b^2$ (b) $(\overline{a}+\overline{b})^2$ (c) $\overline{a}\cdot\overline{b}$ (d) $\overline{a}\times\overline{b}$ ***(Sept. '21)***

5. If $\overline{a}, \overline{b}, \overline{c}$ are the position vectors of the points A, B, C respectively such that $3\overline{a} + 5\overline{b} = 8\overline{c}$, the ratio in which A divides BC is

(a) 8 : 5 internally (b) 8 : 5 externally

(c) 5 : 8 internally (d) 5 : 8 externally

6. The value of $\hat{i}\cdot(\hat{j}\times\hat{k})+\hat{j}\cdot(\hat{k}\times\hat{i})+\hat{k}\cdot(\hat{i}\times\hat{j})$ is

(a) 0 (b) -1 (c) 1 (d) 3 ***(March '22)***

7. If the vectors $\hat{i}-2\hat{j}+\hat{k}$, $a\hat{i}+5\hat{j}-3\hat{k}$ and $5\hat{i}-9\hat{j}+4\hat{k}$ are coplanar, then the value of a is

(a) 3 (b) -3 (c) 2 (d) -2

8. If $|\overline{a}|=3$, $|\overline{b}|=5$, $|\overline{c}|=7$ and $\overline{a}+\overline{b}+\overline{c}=\overline{0}$, then the angle between $\overline{a}$ and $\overline{b}$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

9. The volume of tetrahedron whose vertices are $(1, -6, 10)$, $(-1, -3, 7)$, $(5, -1, \lambda)$ and $(7, -4, 7)$ is 11 cu units, then the value of λ is

(a) 7 (b) 2 (c) 1 (d) 5

10. If $[\overline{a}\ \overline{b}\ \overline{c}] \neq 0$ and $\overline{p}=\dfrac{\overline{b}\times\overline{c}}{[\overline{a}\ \overline{b}\ \overline{c}]}$, $\overline{q}=\dfrac{\overline{c}\times\overline{a}}{[\overline{a}\ \overline{b}\ \overline{c}]}$, $\overline{r}=\dfrac{\overline{a}\times\overline{b}}{[\overline{a}\ \overline{b}\ \overline{c}]}$, then $\overline{a}\cdot\overline{p}+\overline{b}\cdot\overline{q}+\overline{c}\cdot\overline{r}$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

11. If α, β, γ are direction angle of a line and $\alpha = 60^\circ$, $\beta = 45^\circ$, then $\gamma = \ldots\ldots\ldots$

(a) 30° or 90° (b) 45° or 60° (c) 90° or 30° (d) 60° or 120°

12. A line lies in YZ-plane and makes angle of 30° with the Y-axis, then its inclination to the Z-axis is

(a) 30° or 60° (b) 60° or 90° (c) 60° or 120° (d) 30° or 150°

13. If $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of a line, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is

(a) 1 (b) 2 (c) 3 (d) 4

14. The values of p and q so that the line joining the points $(7, p, 2)$ and $(q, -2, 5)$ may be parallel to the line joining the points $(2, -3, 5)$ and $(-6, -15, 11)$ are

(a) $p = 4, q = -3$ (b) $p = 4, q = 3$

(c) $p = -4, q = 3$ (d) $p = -4, q = -3$

15. The measure of acute angle between the lines whose direction ratios are 3, 2, 6 and -2, 1, 2 is

(a) $\cos^{-1}\left(\frac{1}{7}\right)$ (b) $\cos^{-1}\left(\frac{8}{15}\right)$ (c) $\cos^{-1}\left(\frac{1}{3}\right)$ (d) $\cos^{-1}\left(\frac{8}{21}\right)$.

ANSWERS

1. (b) 2, -1 **2.** (b) $\frac{3}{4}$ **3.** (c) 4 **4.** (b) $(\bar{a}+\bar{b})^2$

5. (b) 8 : 5 externally **6.** (d) 3 **7.** (c) 2

8. (b) $\frac{\pi}{3}$ **9.** (a) 7 **10.** (d) 3

11. (d) 60° or 120° **12.** (c) 60° or 120°

13. (b) 2 **14.** (b) $p=4, q=3$

15. (d) $\cos^{-1}\left(\frac{8}{21}\right)$.