CHAPTER 12

Circles

STANDARD EQUATIONS OF THE CIRCLE

(*a*) **Central Form:** If (h, k) is the centre and r is the radius of the circle then its equation is $\mathbf{(x - h)^2 + (y - k)^2 = r^2}$.

(*b*) **General equation of circle:** $\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0}$, where g, f, c are constants and centre is $(-g, -f)$

i.e. $\left(-\dfrac{\text{coefficient of } x}{2}, -\dfrac{\text{coefficient of } y}{2}\right)$

and radius $r = \sqrt{g^2 + f^2 - c}$

Intercepts cut by the circle on axes

The intercepts cut by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on:

(*i*) x-axis $= 2\sqrt{g^2 - c}$

(*ii*) y-axis $= 2\sqrt{f^2 - c}$

Diameter form of circle

$$\mathbf{(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0}$$

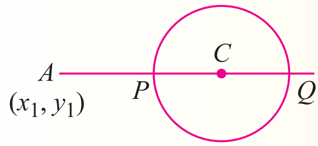
The parametric forms of the circle

(*i*) The parametric equation of the circle $x^2 + y^2 = r^2$ are $x = r\cos\theta$, $y = r\sin\theta$; $\theta \in [0, 2\pi]$.

(*ii*) The parametric equation of the circle $(x - h)^2 + (y - k)^2 = r^2$ is $x = h + r\cos\theta$, $y = k + r\sin\theta$ where θ is parameter.

POSITION OF A POINT W.R.T. CIRCLE

(*a*) Let the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and the point is (x_1, y_1) then:



Point (x_1, y_1) lies outside the circle or on the circle or inside the circle according as

$\Rightarrow \quad x_1^{\,2} + y_1^{\,2} + 2gx_1 + 2fy_1 + c >, =, < 0$ or $S_1 >, =, < 0$

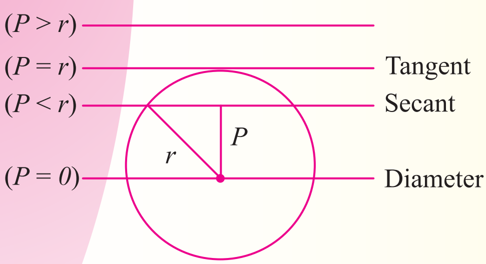
(*b*) The greatest & the least distance of a point A from a circle with centre C & radius r is $AC + r$ & $AC - r$ respectively.

(*c*) The power of point is given by S_1.

TANGENT LINE OF CIRCLE

When a straight line meet a circle on two coincident points then it is called the tangent of the circle.

(*a*) Condition of Tangency: The line $L = 0$ touches the circle $S = 0$ if P the length of the perpendicular from the centre to that line and radius of the circle r are equal i.e. $P = r$.



(*b*) Equation of the tangent (T = 0)

(*i*) Tangent at the point (x_1, y_1) on the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

(*ii*) (1) The tangent at the point $(a \cos t, a \sin t)$ on the circle $x^2 + y^2 = a^2$ is $x \cos t + y \sin t = a$.

(2) The point of intersection of the tangents at the points

$P(\alpha)$ and $Q(\beta)$ is $\left(\dfrac{a \cos \dfrac{\alpha+\beta}{2}}{\cos \dfrac{\alpha-\beta}{2}}, \dfrac{a \sin \dfrac{\alpha+\beta}{2}}{\cos \dfrac{\alpha-\beta}{2}}\right)$.

(*iii*) The equation of tangent at the point (x_1, y_1) on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is}$$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

(*iv*) If line $y = mx + c$ is a straight line touching the circle $x^2 + y^2 = a^2$, then $c = \pm a\sqrt{1+m^2}$ and contact points are $\left(\pm \frac{am}{\sqrt{1+m^2}}, \pm \frac{1}{\sqrt{1+m^2}}\right)$ or $\left(\pm \frac{a^2 m}{c}, \pm \frac{a^2}{c}\right)$ and equation of tangent is $y = mx \pm a\sqrt{1+m^2}$

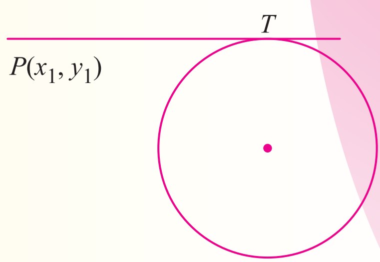
(*v*) The equation of tangent with slope m of the circle $(x - h)^2 + (y - k)^2 = a^2$ is $(y - k) = m(x - h) \pm a\sqrt{1+m^2}$

Note: To get the equation of tangent at the point (x_1, y_1) on any curve we replace xx_1 in place x^2, yy_1 in place of y^2, $\frac{x + x_1}{2}$ in place of x, $\frac{y + y_1}{2}$ in place of y, $\frac{xy_1 + yx_1}{2}$ in place of xy and c in place of c.

(*c*) Length of tangent ($\sqrt{S_1}$): The length of tangent drawn from point (x_1, y_1) out side the circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is,}$$

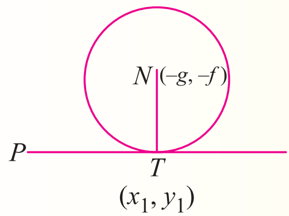
$$PT = \sqrt{S_1} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$



(*d*) Equation of Pair of tangents ($SS_1 = T^2$): $(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$ or $SS_1 = T^2$.

NORMAL OF CIRCLE

(*a*) Equation of normal at point (x_1, y_1) of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is



$$y - y_1 = \left(\frac{y_1 + f}{x_1 + g}\right)(x - x_1)$$

(b) The equation of normal on any point (x_1, y_1) of circle $x^2 + y^2 = a^2$ is $\left(\frac{y}{x} = \frac{y_1}{x_1}\right)$.

CHORD OF CONTACT

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ (i.e. $T = 0$ same as equation of tangent).

EQUATION OF THE CHORD WITH A GIVEN MIDDLE POINT ($T = S_1$)

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $T = S_1$.

DIRECTOR CIRCLE

Equation of director circle is $x^2 + y^2 = 2a^2$.

$\therefore$ director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the circle.

Note: The director circle of

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is } x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0.$$

POLE AND POLAR

The equation of the polar is the $T = 0$, so the polar of point (x_1, y_1) w.r.t. circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

Pole of a given line with respect to a circle

Similar terms we can get the coordinates of the pole. The pole of

$$lx + my + n = 0$$

w.r.t. circle $x^2 + y^2 = a^2$ will be $\left(\frac{-la^2}{n}, \frac{-ma^2}{n}\right)$.

FAMILY OF CIRCLES

(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is : $S_1 + K\,S_2 = 0$ $(K \neq -1)$.

(b) The equation of the family of cirlces passing through the point of intersection of a circle $S = 0$ & a line $L = 0$ is given by $S + KL = 0$.

(c) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form:

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$ where K is a parameter.

(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m\,(x - x_1)] = 0$, where K is a parameter.

(e) Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ & $L_3 = 0$ is given by ; $L_1L_2 + \lambda\,L_2L_3 + \mu\,L_3L_1 = 0$ provided coefficient of $xy = 0$ & coefficient of x^2 = coefficient of y^2.

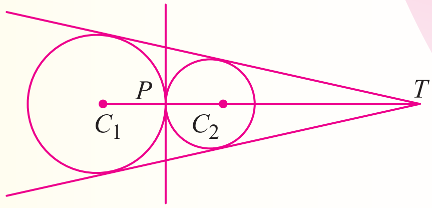
(f) Equation of circle circumscribing a quadrilateral whose side in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ are $L_1L_3 + \lambda\,L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.

DIRECT AND TRANSVERSE COMMON TANGENTS

Let two circles having centre C_1 and C_2 and radii, r_1 and r_2 and C_1C_2 is the distance between their centres then :

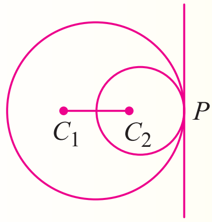
(a) Both circles will touch

(i) Externally if $C_1C_2 = r_1 + r_2$, point P divides C_1C_2 in the ratio $r_1 : r_2$ (internally).



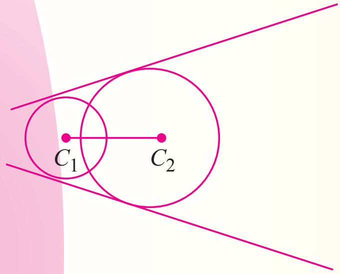
In this case there are **three common tangents**.

(*ii*) Internally if $C_1C_2 = |r_1 - r_2|$, point P divides C_1C_2 in the ratio $r_1 : r_2$ **externally** and in this case there will be only **one common tangent**.

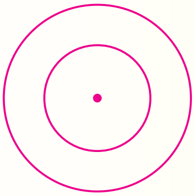


(*b*) The circles will intersect

when $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ in this case there are **two common tangents**.

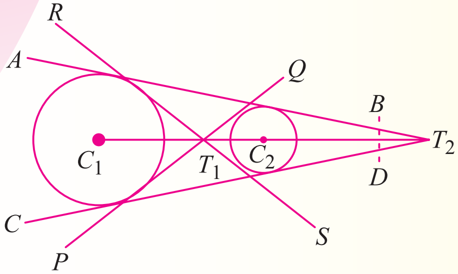


(*c*) The circles will not intersect



(***i***) One circle will lie inside the other circle if $C_1C_2 < |r_1 - r_2|$. In this case there will be no common tangent.

(***ii***) $C_1C_2 > (r_1 + r_2)$



Note: Length of direct common tangent = $\sqrt{(C_1C_2)^2 - (r_1 - r_2)^2}$

Length of transverse common tangent = $\sqrt{(C_1C_2)^2 - (r_1 + r_2)^2}$

THE ANGLE OF INTERSECTION OF TWO CIRCLES

$$\cos\theta = \frac{2g_1g_2 + 2f_1f_2 - c_1 - c_2}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_2^2 + f_2^2 - c_2}} \quad \text{or} \quad \boxed{\cos\theta = \left(\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)}$$

the circles to be orthogonal is

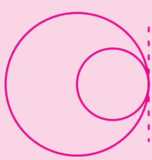
$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

RADICAL AXIS OF THE TWO CIRCLES ($S_1 - S_2 = 0$)

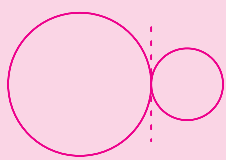
Then the equation of radical axis is given by $S_1 - S_2 = 0$.

NOTES

(i) If two circles touches each other then common tangent is radical axis.

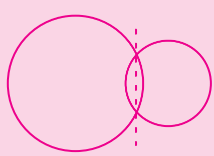


$S_1 - S_2 = 0$



$S_1 - S_2 = 0$

(ii) If two circles cuts each other then common chord is radical axis.



$S_1 - S_2 = 0$

(iii) If two circles cuts third circle orthogonally then radical axis of first two is locus of centre of third circle.

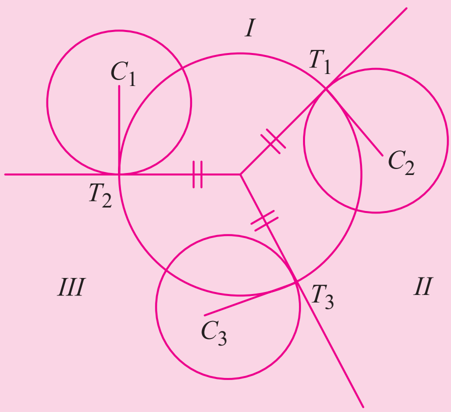
(iv) The radical axis of the two circles is perpendicular to the line joining the centre of two circles but not always pass through mid point of it.

Radical centre

The radical centre of three circles is the point from which length of tangents on three circles are equal i.e. the point of intersection of radical axis of the circles is the radical centre of the circles.

NOTES

(*i*) The circle with centre as radical centre and radius equal to the length of tangent from radical centre to any of the circle, will cut the three circles orthogonally.



(*ii*) If three circles are drawn on three sides of a triangle taking them as diameter then its orthocenter will be its radical centre.

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