8 Differential Equations and Applications

Let's Study

- Differential Equation
- Ordinary differential equation
- Order and degree of a differential equation
- Solution of a differential equation
- Formation of a differential equation
- Applications of differential equations

Let's Recall

- Independent variable
- Dependent variable
- Equation
- Derivatives
- Integration

Let's Learn

8.1 Differential Equations:

Definition: An equation involving dependent variable(s), independent variable and derivative(s)of dependent variable(s) with respect to the independent variable is called a differential equation.

For example :

1) $\frac{dy}{dx} + y = x$

2) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

3) $\frac{d^2y}{dt^2} = 2t$

4) $r\frac{dr}{d\theta} + e^{\theta} = 8$

5) $\sqrt{1 + \frac{dy}{dx}} = \frac{d^2y}{dx^2}$

6) $x\,dx + y\,dy = 0$

8.1.1 Ordinary differential equation

A differential equation in which the dependent variable, say *y*, depends only on one independent variable, say *x*, is called an ordinary differential equation.

8.1.2 Order of a differential equation

It is the order of the highest order derivative occurring in the differential equation.

$\frac{dy}{dx} + y = x$ is of order 1

$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ is of order 2

$\left(\frac{d^2y}{dx^2}\right)^2 + x\frac{dy}{dx} = 2y$ is of order 2

$\frac{2dy}{dx} = e^x$ is of order 1

8.1.3 Degree of a differential equation

It is the power of the highest order derivative when all the derivatives are made free from fractional indices and negative sign,if any.

For example -

1) $x^2\left(\frac{d^2y}{dx^2}\right)^1 + x\frac{dy}{dx} + y = 0$

In this equation, the highest order derivative is $\frac{d^2y}{dx^2}$ and its power is one. Therefore this equation has degree one.

2) $\frac{d^2y}{dx^2} = \sqrt[3]{1+\left(\frac{dy}{dx}\right)^2}$

In this equation, the highest order derivative is $\frac{d^2y}{dx^2}$ but to determine the degree of this equation, first we have to remove the cube root by raising both sides to the power 3.

$\left(\frac{d^2y}{dx^2}\right)^3 = 1+\left(\frac{dy}{dx}\right)^2$ $\therefore$ the degree of this equation is 3.

3) $\frac{dy}{dx} = \frac{2x+7}{\frac{dy}{dx}}$

The equation can be written as $\left(\frac{dy}{dx}\right)^2 = 2x+7$. Now highest order derivative is $\frac{dy}{dx}$ and its power is two.Hence the equation has degree two.

We have learnt:

To find the degree of the differential equation, make all the derivatives free from fractional indices and negative sign, if any.

8.1.4 Solution of a Differential Equation:

A function of the form $y = f(x) + c$ which satisfies the given differential equation is called the solution of the differential equation.

Every differential equation has two types of solutions: 1) General and 2) Particular

1) General Solution:

A solution of the differential equation in which the number of arbitrary constants is equal to order of differential equation is called a general solution.

2) Particular Solution:

A solution of the differential equation which can be obtained from the general solution by giving particular values to the arbitrary constants is called a particular solution.

SOLVED EXAMPLES

1) Verify that the function $y = ae^x + be^{-2x}$, $a,b \in$ R is a solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2y$.

Solution: Consider the function

$y = ae^x + be^{-2x}$ .... (I)

Differentiating both sides of equation I with respect to x, we get

$\frac{dy}{dx} = ae^x - 2be^{-2x}$ .... II and

Differentiating both sides of equation II with respect to x, we get

$\frac{d^2y}{dx^2} = ae^x + 4be^{-2x}$ III

Now, L.H.S $= \frac{d^2y}{dx^2} + \frac{dy}{dx}$

$= (ae^x + 4\,\text{b}e^{-2x}) + (ae^x - 2\text{b}e^{-2x})$

(from II and III)

$= 2\,ae^x + 2\text{b}e^{-2x}$

$= 2(ae^x + \text{b}e^{-2x}) = 2\,y$ (from I)

$=$ R.H.S.

Therefore, the given function is a general solution of the given differential equation.

2) Verify that the function $y = e^{-x} + ax + b$, where a, $b \in$ R is a solution of the differential equation $e^x\left(\frac{d^2y}{dx^2}\right) = 1$

Solution: $y = e^{-x} + ax + b$ I

Differentiating both sides of equation I with respect to x, we get

$\therefore \quad \frac{dy}{dx} = -e^{-x} + a$ II

Differentiating both sides of equation II with respect to x, we get

$$\frac{d^2y}{dx^2} = e^{-x}$$

Consider L.H.S. $= e^x \frac{d^2y}{dx^2} = e^x (e^{-x})$

$= e^0 = 1 =$ R.H.S.

Therefore, the given function is a general solution of the given differential equation.

EXERCISE 8.1

1. Determine the order and degree of the following differential equations.

i) $\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 8 = 0$

ii) $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = a^x$

iii) $\frac{d^4y}{dx^4} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$

iv) $(y''')^2 + 2(y'')^2 + 6y' + 7y = 0$

v) $\sqrt{1 + \frac{1}{\left(\frac{dy}{dx}\right)^2}} = \left(\frac{dy}{dx}\right)^{3/2}$

vi) $\frac{dy}{dx} = 7\frac{d^2y}{dx^2}$

vii) $\left(\frac{d^3y}{dx^3}\right)^{1/6} = 9$

2. In each of the following examples, verify that the given function is a solution of the corresponding differential equation.

	Solution	D.E.
i)	$xy = \log y +$ k	$y'(1 - xy) = y^2$
ii)	$y = x^n$	$x^2\frac{d^2y}{dx^2} - n \times \frac{xdy}{dx} + n\,y = 0$
iii)	$y = e^x$,	$\frac{dy}{dx} = y$
iv)	$y = 1 - \log x$	$x^2\frac{d^2y}{dx^2} = 1$
v)	$y = ae^x + be^{-x}$	$\frac{d^2y}{dx^2} = y$
vi)	$ax^2 + by^2 = 5$	$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = y.\frac{dy}{dx}$

8.1.5 Formation of a differential equation:

By eliminating arbitary constants

If the order of a differential equation is n, differentiate the equation n times to eliminate arbitrary constants.

SOLVED EXAMPLES

1. Form the differential equation of the line having x-intercept 'a' and y-intercept 'b'.

Solution: The equation of a line is given by,

$\frac{x}{a} + \frac{y}{b} = 1$ I

Differeentiating equation I with r. t. x we get,

$\frac{1}{a} + \frac{1}{b}\frac{dy}{dx} = 0, \quad \therefore \frac{1}{b}\frac{dy}{dx} = -\frac{1}{a}$

$\therefore \frac{dy}{dx} = \frac{-b}{a}$ II

Differentiating equation II with r. t. x we get, $\frac{d^2y}{dx^2} = 0$ is the required differential equation.

2. Obtain the differential equation from the relation $Ax^2 + By^2 = 1$, where A and B are constant.

Solution: The given equation is

$Ax^2 + By^2 = 1$ I

Differentiating equation I twice with respect to x, we get,

$$2Ax + 2\,By\,\frac{dy}{dx} = 0$$

$$Ax + By\,\frac{dy}{dx} = 0 \quad \text{.................... II and}$$

$$A + B\left(y\frac{d^2y}{ax^2} + \left(\frac{dy}{dx}\right)^2\right) = 0 \text{ .... III}$$

since the equations I, II & III are consistent in A and B,

$$\therefore \begin{vmatrix} x^2 & y^2 & 1 \\ x & y\dfrac{dy}{dx} & 0 \\ 1 & y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 & 0 \end{vmatrix} = 0$$

$$\therefore \left\{x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] - 1 \times y.\frac{dy}{dx}\right\} = 0$$

$$\therefore xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

is the required differential equation.

3. Form the differential equation whose general solution is $x^3 + y^3 = 4ax$

Solution: Given equation is

$$x^3 + y^3 = 4ax \quad \text{.............I}$$

Since the given equation contains only one arbitrary constant, the required differential equation will be of order one.

Differentiating equation I with respect to x, we get,

$$3x^2 + 3y^2\,\frac{dy}{dx} = 4a \quad \text{.................. II}$$

To eliminate a from the equations I & II,

substitute the value of 4a from equation II in I

$$x^3 + y^3 = x\,(3x^2 + 3y^2\,\frac{dy}{dx})$$

$$x^3 + y^3 = 3x^3 + 3xy^2\,\frac{dy}{dx}$$

that is $2x^3 - y^3 + 3xy^2\,\dfrac{dy}{dx} = 0.$

is the required differential equation.

We have learnt:

To form a differential equation by eliminating arbitrary constants, if 'n' arbitrary constants are present in the given equation then differentiate the given equation 'n' times.

EXERCISE 8.2

1. Obtain the differential equation by eliminating arbitrary constants from the following equations.

i) $y = Ae^{3x} + B.e^{-3x}$

ii) $y = c_2 + \dfrac{c_1}{x}$

iii) $y = (c_1 + c_2x)\,e^x$

iv) $y = c_1e^{3x} + c_2e^{2x}$

v) $y^2 = (x + c)^3$

2. Find the differential equation by eliminating arbitrary constants from the relation $x^2 + y^2 = 2ax$

3. Form the differential equation by eliminating arbitrary constants from the relation $bx + ay = ab$.

4. Find the differential equation whose general solution is $x^3 + y^3 = 35ax$.

5. Form the differential equation from the relation $x^2 + 4y^2 = 4b^2$.

8.2.1 Solution of a Differential Equation:

Variable Separable Method.

Sometimes, a differential equation of first order and first degree can be written in the form

$f(x)\,dx + g(y)\,dy = 0$ I

where $f(x)$ and $g(y)$ are functions of x and y respectively.

This is said to be Variable Separable form, whose solution is obtained by integrating equation I and is given by

$\int f(x)dx + \int g(y)\,dy = c$, where c is the constant of integration.

Now, we solve some examples using variable separable method.

SOLVED EXAMPLES

1. Solve the differential equation $\frac{dy}{dx} = \frac{1+y}{1+x}$

Solution : Separating the variables,the given equation can be written as,

$$\frac{dy}{1+y} = \frac{dx}{1+x}$$

Integrating both sides we get,

$$\int \frac{dy}{1+y} = \int \frac{dx}{1+x} + c$$

$$\log(1+y) = \log(1+x) + \log c$$

$$\log\left(\frac{1+y}{1+x}\right) = \log c$$

$$\therefore \frac{1+y}{1+x} = c$$

2. Solve the differential equation $3e^x dx + (1 + e^x)\,dy = 0$

Solution : Given equation is

$3e^x dx + (1 + e^x)\,dy = 0$

This equation can be written as

$$\frac{3e^x}{1+e^x}\,dx + dy = 0.$$

Integrating both sides we get,

$$\int \frac{3e^x}{1+e^x}dx + \int 1 dy = 0$$

$$\therefore 3\log(1 + e^x) + y = c$$

3. Solve $y - x\frac{dy}{dx} = 0$

Solution: Given equation is $y - x\frac{dy}{dx} = 0$.

Separating the variables we get, Integrating $\frac{dx}{x} = \frac{dy}{y}$ both sides we get,

$$\int \frac{dx}{x} = \int \frac{dy}{y} + c$$

$$\log x = \log y + \log c$$

$$\log x - \log y = \log c$$

$$\log(x/y) = \log c \quad \therefore \frac{x}{y} = c$$

Note - When variables are not separated we use the method of substitution

SOLVED EXAMPLES

1. Solve

$(2x - 2y + 3)\,dx - (x - y + 1)\,dy = 0$, hence find the particular solution if $x = 0, y = 1$.

Solution : The given equation is

$$(2x - 2y + 3)\,dx - (x - y + 1)\,dy = 0$$

$$(x - y + 1)\,dy = (2x - 2y + 3)\,dx$$

$$\frac{dy}{dx} = \frac{2x-2y+3}{x-y+1} = \frac{2(x-y)+3}{x-y+1}$$

This equation cannot be written in variable separable form.

Use the method of substitution.

Put $x - y = t$

$$\therefore \frac{dt}{dx} = 1 - \frac{dy}{dx} \quad \therefore \frac{dy}{dx} = 1 - \frac{dt}{dx}$$

Using these in given equation we get,

$$1 - \frac{dt}{dx} = \frac{2t+3}{t+1}$$

$$1 - \frac{2t+3}{t+1} = \frac{dt}{dx}$$

$$\frac{dt}{dx}=\frac{t+1-2t-3}{t+1}=\frac{-t-2}{t+1}$$

$$\frac{t+1}{t+2}.dt=-dx$$

Integrating we get, $\int\frac{t+1}{t+2}\,dt=-\int 1dx+c$

$$=\int\frac{(t+2)-1}{t+2}\,dt=\int -1.dx+c$$

$$=\int\left(1-\frac{1}{t+2}\right)dt=\int -1.dx+c$$

$t-\log(t+2)=-x+c$,

Resubstituting the value of t, we get,

$x-y-\log(x-y+2)=-x+c$

$2x-y-\log(x-y+2)=c$ I

which is the required general solution.

To determine the particular solution

we have $x=0$ and $y=1$, Substitute in I

$2(0)-1-\log(0-1+2)=c$,

$c=-1$

$2x-y-\log(x-y+2)+1=0$ is the particular solution.

We have leant:

To solve a differential equation of first order and first degree, separate the variables and integrate the equation.

EXERCISE 8.3

1. Solve the following differential equations

i) $\frac{dy}{dx}=x^2y+y$

ii) $\frac{d\theta}{dt}=-k(\theta-\theta_0)$

iii) $(x^2-yx^2)\,dy+(y^2+xy^2)\,dx=0$

iv) $y^3-\frac{dy}{dx}=x\frac{dy}{dx}$

2. For each of the following differential equations find the particular solution.

i) $(x-y^2x)dx-(y+x^2y)\,dy=0$,

when $x=2, y=0$

ii) $(x+1)\frac{dy}{dx}-1=2e^{-y}$,

when $y=0, x=1$

iii) $y(1+\log x)\,dx/dy-x\log x=0$,

when $x=e, y=e^2$.

iv) $\frac{dy}{dx}=(4x+y+1)$, when $y=1, x=0$

8.3.1 Homogeneous Differential Equation:

Definition : A differential equation

$f(x,y)\,dx+g(x,y)\,dy=0$ is said to be Homogeneous Differential Equation

if $f(x,y)$ and $g(x,y)$ are homogeneous functions of the same degree.

For example:

1) $x^3dx+y^3dy=0$ is homogeneous differential equation because x^3 and y^3 are homogeneous functions of the same degree.

2) $x^2ydx+8y^3dy=0$ is homogeneous differential equation because x^2y and y^3 are homogeneous functions of the same degree.

8.3.2 Solution of Homogeneous Differential Equation:

Method to solve Homogeneous Differential Equation:

To solve homogeneous differential equation

$f(x,y)dx+g(x,y)\,dy=0$, ...

we write it as

$\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$ II

To solve this equation we substitute

$y=tx$

$$\frac{dy}{dx}=t+x\frac{dt}{dx}$$

Then equation II is converted into variable separable form and hence it can be solved.

Let's note : After solving equation II, resubstitution t = $\frac{y}{x}$ will give the required solution of the given equation.

SOLVED EXAMPLES

1. Solve : $\left(1+2e^{\frac{x}{y}}\right)dx+2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$

Solution: The given equation can be written as

$$\frac{dy}{dx}=\frac{1+2e^{\frac{x}{y}}}{-2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}$$

This is Homogeneous Differential Equation

To solve it, substitute y = t x

differentiating with respect to x we get,

$$\frac{dy}{dx}=t+x\frac{dt}{dx}$$

equation I can be written as

$$t+x\frac{dt}{dx}=-\frac{1+2e^{\frac{1}{t}}}{2e^{1/t}\left(1-\frac{1}{t}\right)}$$

$$x\frac{dt}{dx}=-\frac{1+2e^{1/t}}{2e^{1/t}\left(1-\frac{1}{t}\right)}-t$$

$$=\frac{-\left(1+2e^{1/t}+(t-1)2e^{1/t}\right)}{2e^{1/t}\left(1-\frac{1}{t}\right)}$$

$$=-\frac{\left(1+2te^{1/t}\right)}{2e^{1/t}\left(1-\frac{1}{t}\right)}$$

$$\therefore\ \frac{2e^{1/t}\left(1-\frac{1}{t}\right)}{1+2te^{1/t}}dt=-\frac{dx}{x}$$

Integrating both sides, we get

$$\int\frac{2e^{1/t}\left(1-1/t\right)}{1+2te^{1/t}}\,dt=-\int\frac{dx}{x}$$

log [1 + 2t ($e^{1/t}$)] + $log\ x = log\ c$

log [1 + 2t $e^{1/t}$] $x = log\ c$

$x(1 + 2t\, e^{1/t}) = c$

Resubstitute the value of t = $\frac{y}{x}$ We get

$$x\left(1+2\left(\frac{y}{x}\right)e^{\frac{x}{y}}\right)=c,$$

$x + 2ye^{x/y} = c$

which is the required general solution.

2. Solve : $(x^2 + y^2)$ dx – $2xy$ dy = 0

Solution: The given equation can be written as

$(x^2 + y^2)$ dx = $2xy$ dy

$$\frac{dy}{dx}=\frac{\left(x^2+y^2\right)}{2xy}\ \text{..................}\ \text{I}$$

To solve it, substitute y = t x.

Differentiating with respect to x we get,

$$\frac{dy}{dx}=t+x\frac{dt}{dx}$$

Equation I can be written as

$$t+x\frac{dt}{dx}=\frac{(x^2+t^2x^2)}{2xtx}=\frac{x^2(1+t^2)}{2x^2t}=\frac{1+t^2}{2t}$$

$$x\frac{dt}{dx}=\frac{1+t^2}{2t}-t=\frac{1-t^2}{2t}$$

$$\therefore\ \frac{2t}{1-t^2}\,dt=\frac{dx}{x}$$

which is variable separable form.

Integrating both sides, we get,

$$\int\frac{2t}{1-t^2}\,dt=\int\frac{dx}{x}$$

$-log\ (1 - t^2) = log\ x$ + log$c$,

$log\ x + log\ (1 - t^2) = log\ c$

$\log x(1-t^2) = \log c$,

$x(1-t^2) = c$. Resubstitute the value of $t = \frac{y}{x}$, we get

$$x\left(1-\frac{y^2}{x^2}\right) = c\ ,\quad \frac{x(x^2-y^2)}{x^2} = c$$

$(x^2 - y^2) = cx$

which is the required general solution.

We have learnt :

To solve a homogeneous differential equation, separate the variables using substitution $\frac{y}{x} = t$ and integrate it.

EXERCISE 8.4

Solve the following differential equations.

1. $x dx + 2y\, dx = 0$
2. $y^2 dx + (xy + x^2)\, dy = 0$
3. $x^2 y\, dx - (x^3 + y^3)\, dy = 0$
4. $\frac{dy}{dx} + \frac{x-2y}{2x-y} = 0$
5. $(x^2 - y^2)\, dx + 2xy\, dy = 0$
6. $xy\, dy/dx = x^2 + 2y^2$
7. $x^2\, dy/dx = x^2 + xy - y^2$

8.4.1 Linear Differential Equation :

General Form

The general form of a linear differential equation of first degree is

$$\frac{dy}{dx} + P\,y = Q \text{ ....I},$$

where P and Q are functions of x only or constants.

8.4.2 Solution of Linear Differential Equation:

To solve $\frac{dy}{dx} + Py = Q$ I

The solution of equation I is given by

$y.\ (\text{I.F.}) = \int Q.\ (\text{I.F})\, dx + c$

where I.F. (Integrating factor) $= e^{\int pdx}$

Let's Note: If given equation is linear in x, that is $\frac{dx}{dy} + P.x = Q$, where P and Q are functions of y only then its solution is given by

$x.(\text{I.F.}) = \int Q.\ (\text{I.F.})\, dy + c$,

where I.F. $= e^{\int pdy}$

Working rule to solve first order Linear Differential Equation.

i. Write the equation in the form $\frac{dy}{dx} + P\,y = Q$.

ii. Find I.F $= e^{\int pdx}$

iii. The solution of the given differential equation is $y.\ (\text{I.F.}) = \int Q.\ (\text{I.F.})\, dx + c$

SOLVED EXAMPLES

1. Solve $\frac{dy}{dx} = x + y$

Solution : Given equation can be written as $\frac{dy}{dx} - y = x$

Here $P = -1$ and $Q = x$

$$\text{I.F.} = e^{\int pdx} = e^{\int -1dx} = e^{-x}$$

Hence the solution of the given equation is given by

$$y.e^{-x} = \int x.e^{-x}dx + c$$

$$y.e^{-x} = \frac{xe^{-x}}{-1} - \int \frac{e^{-x}}{-1}dx + c$$

$$y.e^{-x} = -e^{-x}\ (x+1) + c$$

$$x + y + 1 = ce^x$$

2. Solve $\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$

Solution : Given equation is of the type.

$$\frac{dy}{dx} + Py = Q,$$

where $P = \frac{1}{x}$ and $Q = x^3 - 3$

$\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

The solution of the above equation is given by

$y.(\text{I.F.}) = \int Q.(\text{I.F.})\,dx + c$

$y.\,x = \int (x^3 - 3).\,x\,dx + c$

$xy = \int (x^4 - 3x)\,dx + c$

$xy = \frac{x^5}{5} - \frac{3x^2}{2} + c$, which is the solution of the given differential equation.

We have learnt:

To solve first order Linear Differential Equation

i. wirte the equation in the form $\frac{dy}{dx} + P\,y = Q$.

ii. Find $\text{I.F.} = e^{\int p dx}$

iii) The solution of the given differential equation is $y.(\text{I.F.}) = \int Q.(\text{I.F.})\,dx + c$

EXERCISE 8.5

Solve the following differential equations

i. $\frac{dy}{dx} + y = e^{-x}$

ii. $\frac{dy}{dx} + y = 3$

iii. $x\frac{dy}{dx} + 2y = x^2 \log x$

iv. $(x + y)\frac{dy}{dx} = 1$

v. $ydx + (x - y^2)dy = 0$

vi $\frac{dy}{dx} + 2xy = x$

vii. $(x + a)\frac{dy}{dx} = -y + (a)$

viii. $dr + (2r)d\theta = 8d\theta$

8.5.1 Applications of Differential Equations

1) Population Growth and Growth of Bacteria.

If the population (P) increases at time t then the rate of change of P is proportional to the population present at that time.

This is $\frac{dp}{dt} \propto p$, $\frac{dp}{dt} = kp$

where k is the constant of proportionality.

Integrating $\int \frac{dp}{p} = \int k dt + c$

$\log P = kt + c$

$\therefore\ P = e^{kt + c} = a.e^{kt}$ where $e^c = a$,

which gives the population at any time t.

SOLVED EXAMPLES

1. Bacteria increase at the rate proportional to the number of bacteria present. If the original number N doubles in 3 hours, find in how many hours the number of bacteria will be 4N.

Solution: Let x be the number of bacteria at time t. Since the rate of increase of x is proportinal to x, the differential equation can be written as $\frac{dx}{dt} \propto x$

$\frac{dx}{dt} = kx$, where k is constant of proportionality. Integrating we get,

$\int \frac{dx}{x} = k\int 1.dt + c$

Solving this differential equation we get,

$\log x = kt + c$,

$x = a\,e^{kt}$, where $a = e^c$ I

given that when $t = 0$, $x = N$.

From equation I we get $N = a.1$,

$a = N$, $x = N\,e^{kt}$ II

Also when $t = 3$, $x = 2N$,

From equation II, we have $2N = N\,e^{3k}$

$e^{3k} = 2$ i.e. $e^{K} = 2^{1/3}$ III

Now we have to find out t when $x = 4N$

From equation II, we get $4N = N e^{kt}$

$4 = e^{kt}$, $2^2 = 2^{t/3}$, $t/3 = 2$, $t = 6$.

Hence number of bacteria will be 4N in 6 hours.

2. The population of a country doubles in 60 years, in how many years will it be triple when the rate of increase is proportional to the number of inhabitants

(given $\log_2 3 = 1.5894$)

Solution:

Let P be the population at time t.

Since the rate of increase of P is proportional to P itself, the differential equation can be written as $\frac{dp}{dt} \propto p$

$$\frac{dp}{dt} = kp$$

where k is the constant of proportionality

Integrating $\left[\int \frac{dp}{p} = \int k dt\right]$,

$\therefore$ $\log P = kt + c$ ....(1)

i) $t = 0$, $P = N$ from I

we get $\log N = 0 + c$, $c = \log N$

ii) when $t = 60$, $P = 2N$,

$\log 2N = 60\,k + \log N$

$\log 2N - \log N = 60\,k$

$\log 2N/N = 60k$, $k = \frac{\log 2}{60}$

iii) Now $P = 3N$, $t = ?$

from (I) $\log P = \frac{\log 2}{60} t + \log N$

$\log 3N - \log N = t\,\frac{\log 2}{60}$

$\frac{\log 3}{\log 2} \times 60 = t$,

$\therefore$ $t = 1.5894 \times 60 = 95.36$ years.

We have learnt :

For growth

If the population (P) increases at time t then the rate of change in P is proportional to the population present at that time $\frac{dp}{dt} \propto p$. $\frac{dp}{dt} = kP$, where k is the constant of proportionality. Integrating $\int \frac{dp}{p} = \int k dt$,

we get $\log P = kt + c$, $P = e^{kt+c} = a.e^{kt}$ (where $e^{c} = a$)

Radio Active Decay:

We know that the radio active substances like radium, uranium etc. disintegrate with time. It means the mass of the substance decreases with time.

The rate of disintegration of such elements is proportional to the amount present at that time.

If x is the amount of radioactive material present at time t then

$\frac{dx}{dt} = -kx$, where k is the constant of proportionality and $k \neq 0$. The negative sign appears because x decreases as t increases.

Integrating we get,

$$\int \frac{dx}{x} = -\int k\,dt + c$$

$\log x = -kt + c$, $x = e^{-kt+c} = e^{-kt}.e^{c}$

$x = a.e^{-kt}$, (where $a = e^{c}$) I

If x_o is the initial amount of radio active substance at time $t = 0$, then from equation I

$x_o = a.1$, $a = x_o$,

$x = x_o.e^{-kt}$ II

SOLVED EXAMPLES

1. The rate of disintegration of a radio active element at time t is proportional to its mass at that time. The original mass of 800 gm will disintegrate into its mass of 400 gm after 5 days. Find the mass remaining after 30 days.

Solution:

If x is the amount of material present at time t then

$\frac{dx}{dt} = -kt$, where k is constant of proportionality

$$\int \frac{dx}{x} = -\int k\,dt + c$$

$$\log x = -kt + c$$

$$x = e^{-kt+c} = e^{-kt}.e^{c}$$

$x = a.e^{-kt}$, where $a = e^{c}$. I

Given when $t = 0$, $x = 800$

From I we get, $800 = a.1 = a$

$x = 800\, e^{-kt}$ II

when $t = 5$, $x = 400$ from II

$$400 = 800\, e^{-5k}$$

$$e^{-5k} = \frac{1}{2}$$

Now we have to find x, when $t = 30$

From II we have

$$x = 800\, e^{-30k} = 800\, (e^{-5k})^6$$

$$= 800\left(\frac{1}{2}\right)^6 = \frac{800}{64} = 12.5$$

The mass remaining after 30 days will be 12.5 mg.

We have learnt :

For decay

If x is the amount of any dacaying material present at time t then

$\frac{dx}{dt} = -kx$, where k is constant of proportionality and $k \neq 0$. The negative sign appears because x decreases as t increases, Interating we get

$\int \frac{dx}{x} = \int -k\, dt$ that is $\log x = -kt + c$

$\therefore \log x = -kt + c$, $x = e^{-kt+c} = e^{-kt}.e^{c}$

$\therefore x = a.e^{-kt}$, where $a = e^{c}$.

EXERCISE 8.6

1. In a certain culture of bacteria, the rate of increase is proportional to the number present. If it is found that the number doubles in 4 hours, find the number of times the bacteria are increased in 12 hours.

2. If the population of a town increases at a rate proportional to the population at that time. If the population increases from 40 thousands to 60 thousands in 40 years, what will be the population in another 20 years?

   (Given : $\sqrt{\frac{3}{2}} = 1.2247$)

3. The rate of growth of bacteria is proportional to the number present. If initially, there were 1000 bacteria and the number doubles in 1 hour, find the number of bacteria after 5/2 hours.

   (Given : $\sqrt{2} = 1.414$)

4. Find the population of a city at any time t given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years the population increased from 30000 to 40000.

5. The rate of depreciation $\frac{dv}{dt}$ of a machine is inversely proportional to the square of t + 1, where V is the value of the machine t years after it was purchased. The initial value of the machine was Rs. 8,00,000 and its value decreased Rs. 1,00,000 in the first year.

   Find its value after 6 years.

Let's Remember

1. An equation which involves polynomials of differentials of dependent variables with respect to the independent variable is called a differential equation.

2. A differential equation in which the dependent variable, say y, depends only on one independent variable, say x, is called an ordinary differential equation.

3. Order of a differential equation : It is the order of highest-order derivative occuring in the differential equation.

4. Degree of a differential equation : It is the power of the highest-order derivative when all the derivatives are made free from negative and / or fractional indices, if any.

5. **(i) General Solution :** A solution of differential equation in which the number of arbitrary constants is equal to the order of differential equation is called a general solution.

   (ii) Particular Solution : A solution of a differential equation which can be obtained from the general solution by giving particular values to the arbitrary constants.

6. Order and degree of a differential equation are always positive integers.

7. Homogeneous Differential Equation: Definition : A differential equation $f(x,y)dx + g(x,y)dy = 0$ is said to be Homogeneous Differential Equation if $f(x,y)$ and $g(x,y)$ are homogeneous functions of the same degree.

8. The general form of a linear differential equation of the first order is $\frac{dy}{dx} = PY + Q$ (I), Where P and Q are functions of x only or constants.

   The solution of the above equation (I) is given by y. (I.F.) = ∫Q.(I.F.) dx + c where I.F. (Integrating factor) = $e^{\int pdx}$

9. If given equation is not linear in y that is $\frac{dy}{dx} + P.x = Q$ then its solution is given by x. (I.F.) = ∫Q. (I.F.) dy +c, where I.F. = $e^{\int pdy}$.

MISCELLANEOUS EXERCISE - 8

I) Choose the correct alternative.

1. The order and degree of $\left(\frac{dy}{dx}\right)^3 - \frac{d^3y}{dx^3} + ye^x$ are respectively.

   a) 3,1   b) 1,3

   c) 3,3   d) 1,1

2) The order and degree of $\left[1+\left(\frac{dy}{dx}\right)^3\right]^{\frac{2}{3}} = 8\frac{d^3y}{dx^3}$ are respectively

   a) 3,1   b) 1,3

   c) 3,3   d) 1,1

3) The differential equation of $y = k_1 + \frac{k_2}{x}$ is

   a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$ b) $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$

   c) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$ d) $x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$

4. The differential equation of $y = k_1e^x + k_2e^{-x}$ is

   a) $\frac{d^2y}{dx^2} - y = e^x$ b) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

   c) $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$ d) $\frac{d^2y}{dx^2} + y = 0$

5. The solution of $\frac{dy}{dx} = 1$ is

   a) $x + y = c$

   b) $xy = c$

   c) $x^2 + y^2 = c$

   d) $y - x = c$

6) The solution of $\frac{dy}{dx} + \frac{x^2}{y^2} = 0$ is

   a) $x^3 + y^3 = 7$

   b)) $x^2 + y^2 = c$

   c) $x^3 + y^3 = c$

   d) $x + y = c$

7) The solution of $x\frac{dy}{dx} = y\log y$ is

a) $y = ae^x$

b) $y = be^{2x}$

c) $y = be^{-2x}$

d) $y = e^{ax}$

8) Bacterial increases at the rate proportional to the number present. If the original number M doubles in 3 hours, then the number of bacteria will be 4M in

a) 4 hours b) 6 hours

c) 8 hours d) 10 hours

9) The integrating factor of $\frac{dy}{dx} + y = e^{-x}$ is

a) x b) $-x$

c) e^x d) e^{-x}

10) The integrating factor of $\frac{d^2y}{dx^2} - y = e^x$ is e^{-x} then its solution is

a) $ye^{-x} = x + c$ b) $ye^x = x + c$

c) $ye^x = 2x + c$ d) $ye^{-x} = 2x + c$

II. Fill in the blanks.

1) The order of highest derivative occurring in the differential equation is called .... of the differential equation.

2) The power of the highest ordered derivative when all the derivatives are made free from negative and / or fractional indices if any is called .... of the differential equation.

3) A solution of differential equation which can be obtained from the general solution by giving particular values to the arbitrary constants is called .... solution.

4) Order and degree of a differential equation are always .... integers.

5) The integrating factor of the differential equation $\frac{dy}{dx} - y = x$ is ....

6) The differential equation by eliminating arbitrary constants from bx + ay = ab is ....

III State whether each of the following is True or False.

1) The integrating factor of the differential equation $\frac{dy}{dx} - y = x$ is e^{-x}.

2) Order and degree of a differential equation are always positive integers

3) The degree of a differential equation is the power of the highest ordered derivative when all the derivatives are made free form negative and / or fractional indices if any.

4) The order of highest derivative occuring in the differential equation is called degree of the differential equation.

5) The power of the highest ordered derivative when all the derivatives are made free from negative and / or fractional indices if any is called order of the differential equation.

6) The degree of the differential equation $e^{\frac{dy}{dx}} = \frac{dy}{dx} + c$ is not defined.

IV. Solve the following.

1. Find the order and degree of the following differential equations:

i) $\left[\frac{d^3y}{dx^3} + x\right]^{3/2} = \frac{d^2y}{dx^2}$

ii) $x + \frac{dy}{dx} = 1 + \left(\frac{dy}{dx}\right)^2$

2. Verify y = log$x$ + c is a solution of the differential equation $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$.

3) Solve the differential equations

i) $\frac{dy}{dx} = 1 + x + y + xy$

ii) $e^{dy/dx} = x$

iii) $dr = a\,r\,d\theta - \theta\,dr$

iv) Find the differential equation of family of curves $y = e^x(ax + bx^2)$

where A and B are arbitrary constant.

4) Solve $\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$

when $x = \frac{2}{3}$ and $y = \frac{1}{3}$

5) Solve $ydx - xdy = -\log x\, dx$

6) Solve $y\log y \frac{dx}{dy} + x - \log y = 0$

7) Solve $(x+y)dy = a^2 dx$

8) Solve $\frac{dy}{dx} + \frac{2}{x} y = x^2$

9) The rate of growth of population is proporational to the number present.

If the population doubled in the last 25 years and the present population is 1 lac, when will the city have population 4,000,000?

10) The resale value of a machine decreases over a 10 year period at a rate that depends on the age of the machine. When the machine is x years old, the rate at which its value is changing is ₹ 2200 (x – 10) per year. Express the value of the machine as a function of its age and initial value. If the machine was originally worth ₹ 1,20,000 how much will it be worth when it is 10 years old?

11) $y^2 dx + (xy + x^2)dy = 0$

12) $x^2 y dx - (x^3 + y^3)dy = 0$

13) $yx \frac{dy}{dx} = x^2 + 2y^2$

14) $(x + 2y^3)\frac{dy}{dx} = y$

15) $ydx - xdy + \log x dx = 0$

16) $\frac{dy}{dx} = \log x$

17) $y\log y \frac{dy}{dx} = \log y - x$

Activities

1) Complete the following activity.

The equation $\frac{dy}{dx} - y = 2x$ is of the form

$\square$

Where P = $\square$ and, Q = $\square$

$\therefore$ I.F. = $e^{\int dx}$ = $\square$

$\therefore$ the solution of the linear differential equation is

y $\square$ $= \int 2x$ (I.F.) dx + c.

$\therefore$ $ye^{-x} = \int 2x$ $\square$ dx + c

$ye^{-x} = 2\int x$ $\square$ dx

$= 2\{x\int e^{-x}\, dx - \int$ $\square$ $dx \frac{d}{dx}$ $\square$ $dx\} + c$

$= 2\ \{x \frac{e^{-x}}{\square} - \int \frac{e^{-x}}{\square}.1\, dx$

$\therefore$ $e^{\frac{-x}{y}} = -2xe^{-x} + 2\int$ $\square$ $dx + c_1$

$e^{-x}y = -2xe^{-x} + 2$ $\square$ $+ c_2$

$y +$ $\square$ $+$ $\square$ $= ce^{x}$ is the required general solution of the given differential equation.

2) Verify $y = a + \frac{b}{x}$ is a solution of

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

$y = a + \frac{b}{x}$

$\frac{dy}{dx} =$ $\square$

$\frac{d^2y}{dx^2} =$ $\square$

Consider $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx}$

$= x$ $\square$ $+ 2$ $\square$

$=$ $\square$

Hence $y = a + \frac{b}{x}$ is a solution of

$\square$

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