DAY TEN

Real Function

Learning & Revision for the Day

- Real Valued Function and Real Function
- Domain and Range of real Function
- Algebra of Real Functions
- Inverse Function
- Basic Functions
- Nature of a Functions

Real Valued Function and Real Function

A Function $f: A \rightarrow B$ is said to be a **real valued function** if $B \subseteq R$ (the set of real numbers), if both A and B are subset of R (the set of real numbers) then f is called a **real function**.

NOTE Every real function is a real valued function but converse need not be true.

Domain and Range of Real Function

The **domain** of $y = f(x)$ is the set of all real x for which $f(x)$ is defined (real).

Range of $y = f(x)$ is collection of all distinct images corresponding to each real number in the domain.

NOTE If $f: A \rightarrow B$, then A will be domain of f and B will be codomain of f.

To find range

(i) First of all find the domain of $y = f(x)$.

(ii) If domain has finite number of points, then range is the set of f – images of these points.

(iii) If domain is R or $R - \{\text{some finite points}\}$, express x in terms of y and find the values of y for which the values of x lie in the domain.

(iv) If domain is a finite interval, find the least and the greatest values for range using monotonicity.

Algebra of Real Functions

Let $f : X \rightarrow R$ and $g : X \longrightarrow R$ be two real functions. Then,

- The sum $f + g : X \longrightarrow R$ defined as $(f + g)(x) = f(x) + g(x)$.
- The difference $f - g$: $X - R$, defined as
$$(f - g)(x) = f(x) - g(x)$$

- The **product** $fg : X \longrightarrow R$, defined as $(fg)(x) = f(x)\, g(x)$
- $f + g$ and fg are defined only, if f and g have the same domain. In case the domain of f and g are different, domain of $f + g$ or fg = Domain of $f \cap$ Domain of g.
- The product $cf : X \longrightarrow R$, defined as $(cf)(x) = cf(x)$, where c is a real number.
- The quotient $\frac{f}{g}$ is a function defined as $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0, x \in X$
- If domain of $y = f(x)$ and $y = g(x)$ are D_1 and D_2 respectively, then the domain of $f(x) \pm g(x)$ or $f(x) \cdot g(x)$ is $D_1 \cap D_2$, while domain of $\frac{f(x)}{g(x)}$ is

$$D_1 \cap D_2 - \{x : g(x) = 0\}.$$

Equal or Identical Functions

Two functions f and g are said to be equal, if

(i) the domain of f = the domain of g

(ii) the range of f = the range of g

(iii) $f(x) = g(x), \forall\, x \in$ domain

Inverse Functions

- If $f : A \to B$ is a bijective function, then the mapping $f^{-1} : B \to A$ which associate each element $b \in B$ to a unique element $a \in A$ such that $f(a) = b$, is called the **inverse function** of f.

$$f^{-1}(b) = a \Leftrightarrow f(a) = b$$

- The curves $y = f(x)$ and $y = f^{-1}(x)$ are mirror images of each other in the line mirror $y = x$.
- f is invertible iff f is one-one and onto.
- Inverse of bijective function is unique and bijective.
- The solution of $f(x) = f^{-1}(x)$ are same as the solution of $f(x) = x$.
- If $fo\,g = I = gof$, then f and g are inverse of each other.
- $fof^{-1} = I_B, f^{-1}of = I_A$ and $(f^{-1})^{-1} = f$.
- If f and g are two bijections such that (gof) exists, then gof is also bijective function and $(gof)^{-1} = f^{-1}og^{-1}$.

Basic Functions

Basic functions can be categorised into the following categories.

1. Algebraic Functions

A function, say $f(x)$, is called an algebraic function, if it consists finite number of terms involving powers and roots of the independent variable x and the four algebraic operations $+, -, \times$ and $\div$.

Some algebraic functions are given below

(i) **Polynomial Function**

(a) The function

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$$

where, $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $n \in N$ is known as **polynomial function**. If $a_0 \neq 0$, then n is the degree of polynomial function.

(b) Domain of polynomial function is R.

(c) A polynomial of odd degree has its range $(-\infty, \infty)$ but a polynomial of even degree has a range which is always subset of R.

(ii) **Constant Function** The function $f(x) = k$, where k is constant, is known as constant function. Its domain is R and range is $\{k\}$,

(iii) **Identity Function** The function $f(x) = x$, is known as **identity function**. Its domain is R and range is R.

(iv) **Rational Function** The function $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$, is called rational function.

Its domain is $R - \{x \mid q(x) = 0\}$.

(v) **Irrational Function** The algebraic functions containing one or more terms having non-integral rational power of x are called irrational functions.

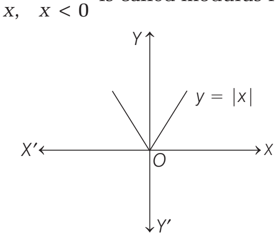
e.g., $y = f(x) = 2\sqrt{x} - \sqrt[3]{x} + 6$

(vi) **Reciprocal Function** The function $f(x) = \frac{1}{x}$ is called the reciprocal function of x. Its domain is $R - \{0\}$ and range is $R - \{0\}$.

2. Piecewise Functions

Piecewise functions are special type of algebraic functions.

(i) **Absolute Valued Function** (Modulus Function) The function $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ is called modulus function.



Its domain is R and range is $[0, \infty)$.

Properties of Modulus Function

(a) $|x| \leq a \Rightarrow -a \leq x \leq a \,(a > 0)$

(b) $|x| \geq a \Rightarrow x \leq -a$ or $x \geq a \,(a > 0)$

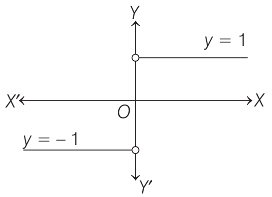
(c) $|x \pm y| \leq |x| + |y|$

(d) $|x \pm y| \geq ||x| - |y||$

(ii) **Signum Function** The function $f(x)$

$$= \operatorname{sgn}(x) = \begin{cases} \dfrac{|x|}{x} \text{ or } \dfrac{x}{|x|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$$

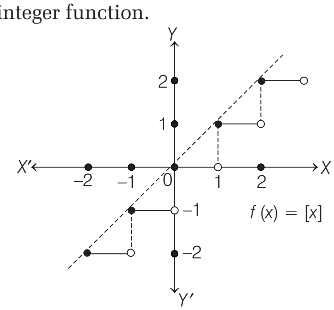
called signum function.



Its domain is R and range is $\{-1, 0, 1\}$.

(iii) **Greatest Integer Function** The symbol $[x]$ indicates the integral part of x which is nearest and smaller than to x. It is also known as floor of x.

The function $f(x) = [x] = \begin{cases} x \ \forall \ x \in I \\ n, \quad n \le x < n+1, \ n \in I \end{cases}$ is called greatest integer function.



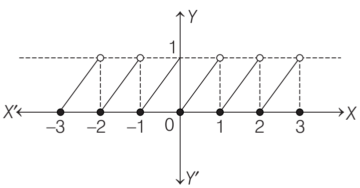
Its domain is R and range is I.

(iv) **Fractional Part Function** The symbol $\{x\}$ indicates the fractional part of x. i.e. $\{x\} = x - [x], x \in R$

$\therefore \qquad y = \{x\} = x - [x]$

The function $f(x) = \{x\} = \begin{cases} 0, \quad \forall \quad x \in I \\ x - n, \quad n \le x < n+1, \ n \in I \end{cases}$

is called the fractional part function.

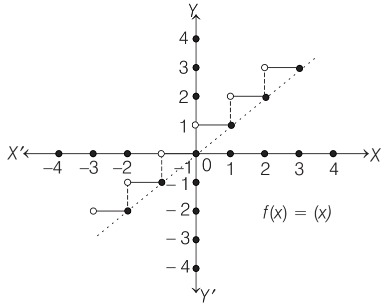


Its domain is R and range is $(0, 1)$.

(v) **Least Integer Function** The symbol (x) indicates the integer part of x which is nearest and greater than x.

The function $f(x) = (x) = \begin{cases} x, \ \forall \ x \in I \\ n+1, \ n < x \le n+1 \quad n \in I \end{cases}$

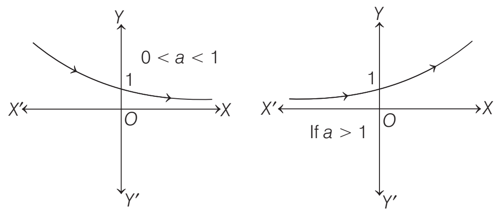
is called least integer function.



Its domain $\in R$ and range $\in I$.

(vi) **Transcendental function** The function which is not algebraic is called transcendental function.

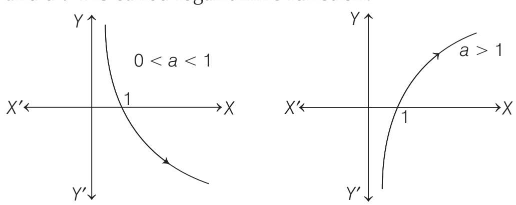
(vii) **Exponential Function** The function $f(x) = a^x$, $a > 0, a \neq 1$, is called an exponential function.



Its domain is R and range is $(0, \infty)$.

It is a one-one into function.

(viii) **Logarithmic Function** The function $f(x) = \log_a x, (x, a > 0)$ and $a \neq 1$ is called logarithmic function.

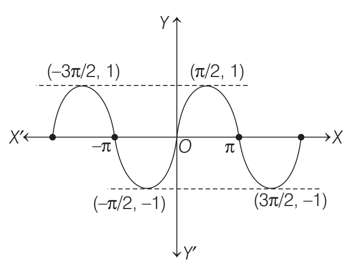


Its domain is $(0, \infty)$ and range is R.

It is a one-one into function.

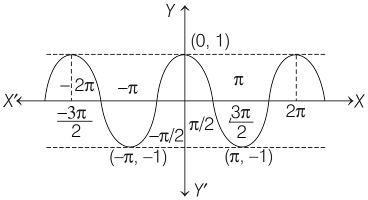
(ix) **Trigonometric Functions** Some standard trigonometric functions with their domain and range, are given below

(a) **Sine Function** $f(x) = \sin x$,



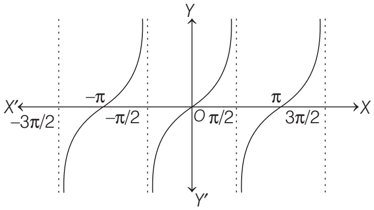
Its domain is R and the range is $[-1, 1]$.

(b) **Cosine Function** $f(x) = \cos x$,



Its domain is R and the range is $[-1, 1]$.

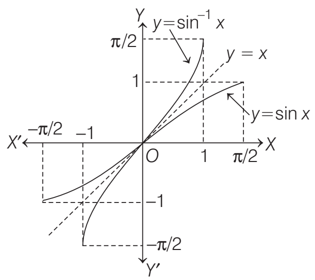
(c) **Tangent Function** $f(x) = \tan x$,



Its domain is $R - \left\{\frac{(2n+1)\pi}{2}, n \in I\right\}$ and range is R.

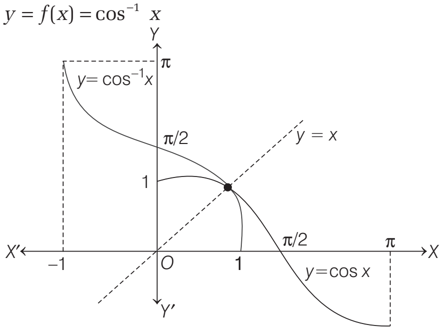
(x) **Inverse Trigonometric Function** Some standard inverse trigonometric functions with their domain and range, are given below.

(a) $y = f(x) = \sin^{-1} x$



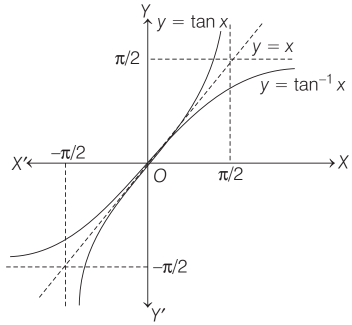
Its domain is $[-1, 1]$ and range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) $y = f(x) = \cos^{-1} x$



Its, domain is $[-1, 1]$ and range is $[0, \pi]$.

(c) $y = f(x) = \tan^{-1} x$



Its domain is R and range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Nature of a Function

A function $f(x)$ is said to be an **odd** function, if

$$f(-x) = -f(x), \forall x.$$

A function $f(x)$ is said to be an **even** function, if $f(-x) = f(x), \forall x$.

Different Conditions for Even and Odd Functions

$f(x)$	$g(x)$	$f(x) + g(x)$	$f(x) - g(x)$
Odd	Odd	Odd	Odd
Even	Even	Even	Even
Odd	Even	Neither odd nor even	Neither odd nor even
Even	Odd	Neither odd nor even	Neither odd nor even
$f(x)g(x)$	$f(x)/g(x)$	$(gof)(x)$	$(fog)(x)$
Even	Even	Odd	Odd
Even	Even	Even	Even
Odd	Odd	Even	Even
Odd	Odd	Even	Even

NOTE

- Every function can be expressed as the sum of an even and an odd function.
- Zero function $f(x) = 0$ is the only function which is both even and odd.
- Graph of odd function is symmetrical about origin.
- Graph of even function is always symmetrical about Y-axis.

3. Periodic Function

- A function $f(x)$ is said to be periodic function, if there exists a positive real number T, such that $f(x + T) = f(x), \forall x \in R$.
- The smallest value of T is called the Fundamental period of $f(x)$.

Properties of Periodic Function

(i) If $f(x)$ is periodic with period T, then $cf(x)$, $f(x+c)$ and $f(x) \pm c$ is periodic with period T.

(ii) If $f(x)$ is periodic with period T, then $kf(cx+d)$ has period $\frac{T}{|c|}$.

(iv) If $f(x)$ is periodic with period T_1 and $g(x)$ is periodic with period T_2, then $f(x)+g(x)$ is periodic with period equal to LCM of T_1 and T_2, provided there is no positive k, such that $f(k+x)=g(x)$ and $g(k+x)=f(x)$.

(iv) If $f(x)$ is a periodic function with period T and $g(x)$ is any function, such that range of $f \subseteq$ domain of g, then gof is also periodic with period T.

Periods of Some Important Functions

Function	Periods
$\sin x, \cos x, \sec x, \text{cosec } x, (\sin x)^{2n+1}$, $(\cos x)^{2n+1}, (\sec x)^{2n+1}, (\text{cosec } x)^{2n+1}$	2π
$\tan x, \cot x, \tan^n x, \cot^n x, (\sin x)^{2n}, (\cos x)^{2n}$, $(\sec x)^{2n}, (\text{cosec } x)^{2n}, \|\sin x\|, \|\cos x\|, \|\tan x\|$, $\|\cot x\|, \|\sec x\|, \|\text{cosec } x\|$	π
$\|\sin x+\cos x\|, \sin^4 x+\cos^4 x$, $\|\sec x\|+\|\text{cosec } x\|, \|\tan x\|+\|\cot x\|$	$\pi/2$
$x-[x]$	1
Algebraic functions like $\sqrt{x}, x^2, x^2+5, c, \ldots$ etc.	Period does not exist

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 Two sets A and B are defined as follows
$A=\{(x, y): y=e^{2x}, x \in R\}$ and
$B=\{(x, y): y=x^2, x \in R\}$, then
(a) $A \subset B$ (b) $B \subset A$
(c) $A \cup B$ (d) $A \cap B=\phi$

2 Domain of definition of the function $f(x)=\sqrt{\sin^{-1}(2x)+\frac{\pi}{6}}$ for real valued x, is
(a) $\left[-\frac{1}{4}, \frac{1}{2}\right]$ (b) $\left[-\frac{1}{2}, \frac{1}{2}\right]$
(c) $\left(-\frac{1}{2}, \frac{1}{9}\right)$ (d) $\left[-\frac{1}{4}, \frac{1}{4}\right]$

3 The domain of the function $f(x)=\frac{1}{\sqrt{|x|-x}}$ is
(a) $(0, \infty)$ (b) $(-\infty, 0)$
(c) $(-\infty, \infty)-(0)$ (d) $(-\infty, \infty)$

4 Domain of definition of the function
$f(x)=\frac{3}{4-x^2}+\log_{10}(x^3-x)$, is → **AIEEE 2003**
(a) $(1, 2)$
(b) $(-1, 0) \cup (1, 2)$
(c) $(1, 2) \cup (2, \infty)$
(d) $(-1, 0) \cup (1, 2) \cup (2, \infty)$

5. If $f: R \to R$ is a function satisfying the property $f(x+1)+f(x+3)=2$ for all $x \in R$, then f is
(a) periodic with period 3
(b) periodic with period 4
(c) non- periodic
(d) periodic with period 5

6 The period of the function $f(x)=\sin^3 x+\cos^3 x$ is
(a) 2π (b) π
(c) $\frac{2\pi}{3}$ (d) None of these

7 Let $f: R \to R$ be defined by $f(x)=x^2+1$.
Then, pre-images of 17 and –3, respectively are
(a) $\phi, \{4, -4\}$ (b) $\{3, -3\}, \phi$ → **NCERT Exemplar**
(c) $\{4, -4\}, \phi$ (d) $\{4, -4\}, \{2, -2\}$

8 Suppose $f(x)=(x+1)^2$ for $x \geq -1$. If $g(x)$ is the function, whose graph is reflection of the graph of $f(x)$ w.r.t. the line $y=x$, then $g(x)$ is equal to
(a) $-\sqrt{x}-1, x \geq 0$
(b) $\frac{1}{(x+1)^2}, x > -1$
(c) $\sqrt{x+1}, x \geq -1$
(d) $\sqrt{x}-1, x \geq 0$

9 The function $f: R \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ defined as $f(x)=\frac{x}{1+x^2}$ is → **JEE Main 2017**
(a) invertible
(b) injective but not surjective
(c) surjective but not injective
(d) neither injective nor surjective

10 The inverse of the function $f(x)=\frac{e^x-e^{-x}}{e^x+e^{-x}}+2$ is given by
(a) $\log_e\left(\frac{x-2}{x-1}\right)^{1/2}$ (b) $\log_e\left(\frac{x-1}{3-x}\right)^{1/2}$
(c) $\log_e\left(\frac{x}{2-x}\right)^{1/2}$ (d) $\log_e\left(\frac{x-1}{x+1}\right)^{-2}$

11 If $f(x)$ is an invertible function, and $g(x) = 2f(x) + 5$, then the value of g^{-1} is

(a) $2f^{-1}(x) - 5$ (b) $\frac{1}{2f^{-1}(x)+5}$
(c) $\frac{1}{2}f^{-1}(x) + 5$ (d) $f^{-1}\left(\frac{x-5}{2}\right)$

12 Let $f : (2, 3) \to (0, 1)$ be defined by $f(x) = x - [x]$, then $f^{-1}(x)$ is equal to

(a) $x - 2$ (b) $x + 1$ (c) $x - 1$ (d) $x + 2$

13 For a real number x, $[x]$ denotes the integral part of x.

The value of $\left[\frac{1}{2}\right] + \left[\frac{1}{2} + \frac{1}{100}\right] + \left[\frac{1}{2} + \frac{2}{100}\right] + \ldots + \left[\frac{1}{2} + \frac{99}{100}\right]$ is

(a) 49 (b) 50 (c) 48 (d) 51

14 If $f^2(x) \cdot f\left(\frac{1-x}{1+x}\right) = x^3$, [where, $x \neq -1, 1$ and $f(x) \neq 0$], then find $|[f(-2)]|$ (where [.] is the greatest integer function)

(a) $1/x$ (b) $1 - x$ (c) 1 (d) 2

15 If $g(x) = 1 + x - [x]$ and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}, \forall x$, then $f\{g(x)\}$ is equal to

(a) x (b) 1 (c) $f(x)$ (d) $g(x)$

16. The function $f(x) = \log(x + \sqrt{x^2 + 1})$, is

(a) an even function (b) an odd function
(c) a periodic function
(d) neither an even nor an odd function

17. **Statement I** $f(x) = |x - 2| + |x - 3| + |x - 5|$ is an odd function for all values of x lie between 3 and 5.

Statement II For odd function $f(-x) = -f(x)$

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

18. If domain of $f(x)$ and $g(x)$ are D_1 and D_2 respectively, then domain of $f(x) + g(x)$ is $D_1 \cap D_2$, then

Statement I The domain of the function

$$f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x \text{ is } [-1, 1].$$

Statement II $\sin^{-1} x$ and $\cos^{-1} x$ is defined in $|x| \leq 1$ and $\tan^{-1} x$ is defined for all x.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

19. **Statement I** The period of

$f(x) = 2\cos\frac{1}{3}(x - \pi) + 4\sin\frac{1}{3}(x - \pi)$ is 3π.

Statement II If T is the period of $f(x)$, then the period of $f(ax + b)$ is $\frac{T}{|a|}$.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

20. If the range of $f(x)$ is collection of all outputs $f(x)$ corresponding to each real number in the domain, then

Statement I The range of $\log\left(\frac{1}{1+x^2}\right)$ is $(-\infty, \infty)$.

Statement II When $0 < x \leq 1$, $\log x \in (-\infty, 0]$.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 Domain of $f(x) = \sqrt{\frac{x-1}{x - 2\{x\}}}$, where {·} denotes the fractional part of x, is

(a) $(-\infty, 0) \cup (0, 2]$ (b) $[1, 2)$
(c) $(-\infty, \infty) \sim [0, 2)$ (d) $(-\infty, 0) \cup (0, 1] \cup [2, \infty)$

2 Range of $f(x) = [|\sin x| + |\cos x|]$, where [·] denotes the greatest integer function, is

(a) {0} (b) {0, 1} (c) {1} (d) None of these

3 If $[x^2] + x - a = 0$ has a solution, where $a \in N$ and $a \leq 20$, then total number of different values of a can be

(a) 2 (b) 3 (c) 4 (d) 6

4 Total number of solutions of $[x]^2 = x + 2\{x\}$, where [·] and {·} denotes the greatest integer function and fractional part respectively, is equal to

(a) 2 (b) 4
(c) 6 (d) None of these

5 If $f(x)=\sin x+\cos x, g(x)=x^2-1$, then $g\{f(x)\}$ is invertible in the domain

(a) $\left[0, \frac{\pi}{2}\right]$ (b) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

(c) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) $[0, \pi]$

6 Let $f(x)=x^{10}+a\cdot x^8+b\cdot x^6+cx^4+dx^2$ be a polynomial function with real corfficient. If $f(1)=1$ and $f(2)=-5$, then the minimum number of distinct real zeroes of $f(x)$ is

(a) 5 (b) 6

(c) 7 (d) 8

7 If $f:R\to R, f(x)=x^3+3$, and $g:R\to R, g(x)=2x+1$, then $f^{-1}og^{-1}(23)$ equals

(a) 2 (b) 3

(c) 4 (d) 5

8 If $f(x)$ and $g(x)$ are two functions such that $f(x)=[x]+[-x]$ and $g(x)=\{x\}\forall x\in R$ and $h(x)=f(g(x))$; then which of the following is incorrect ?
([·] denotes greatest integer function and {·} denotes fractional part function).

(a) $f(x)$ and $h(x)$ are inertial functions
(b) $f(x)=g(x)$ has no solution
(c) $f(x)+h(x)>0$ has no solution
(b) $f(x)-h(x)$ is a periodic function

9 The period of the function $f(x)=[6x+7]+\cos\pi x-6x$, where [·] denotes the greatest integer function, is

(a) 3 (b) 2π (c) 2 (d) None of these

10 The number of real solutions of the equation $\log_{0.5}|x|=2|x|$ is.

(a) 1 (b) 2 (c) 0 (d) None of these

ANSWERS

SESSION 1	**1** (d)	**2** (a)	**3** (b)	**4** (d)	**5** (b)	**6** (a)	**7** (c)	**8** (d)	**9** (c)	**10** (b)
	11 (d)	**12** (d)	**13** (b)	**14** (d)	**15** (b)	**16** (b)	**17** (b)	**18** (a)	**19** (d)	**20** (d)
SESSION 2	**1** (d)	**2** (c)	**3** (c)	**4** (b)	**5** (b)	**6** (a)	**7** (a)	**8** (b)	**9** (c)	**10** (b)

Hints and Explanations

SESSION 1

1 Set A represents the set of points lying on the graph of an exponential function and set B represents the set of points lying on the graph of the polynomial.
Take $e^{2x}=x^2$, then the two curves does not intersect. Hence, there is no point common between them.

2 For $f(x)$ to be defined,

$\sin^{-1}(2x)+\frac{\pi}{6}\geq 0$

$\Rightarrow \quad -\frac{\pi}{6}\leq\sin^{-1}(2x)\leq\frac{\pi}{2}$

$\Rightarrow \quad \sin\left(-\frac{\pi}{6}\right)\leq 2x\leq\sin\left(\frac{\pi}{2}\right)$

$\Rightarrow \quad -\frac{1}{4}\leq x\leq\frac{1}{2}$

$\Rightarrow \quad x\in\left[-\frac{1}{4},\frac{1}{2}\right]$

3 $y=\frac{1}{\sqrt{|x|-x}}$

For domain, $|x|-x>0$

$\Rightarrow \quad |x|>x$

i.e. only possible, if $x<0$.

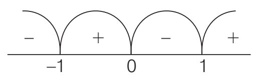
$\therefore \quad x\in(-\infty, 0)$

4 Given, $f(x)=\frac{3}{4-x^2}+\log_{10}(x^3-x)$

For domain of $f(x)$,

$x^3-x>0$

$\Rightarrow \quad x(x-1)(x+1)>0$



$\Rightarrow \quad x\in(-1,0)\cup(1,\infty)$

and $\quad 4-x^2\neq 0$

$\Rightarrow \quad x\neq\pm 2$

$\Rightarrow \quad x\in(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

So, common region is
$(-1,0)\cup(1,2)\cup(2,\infty)$.

6 We have,

$f(x+1)+f(x+3)=2 \quad$ ...(i)

On replacing x by $x+2$, we get

$f(x+3)+f(x+5)=2 \quad$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$f(x+1)-f(x+5)=0$

$\Rightarrow \quad f(x+1)=f(x+5)$

Now, on replacing x by $x-1$, we get

$f(x)=f(x+4)$

Hence, f is periodic with period 4.

6 $f(x)=\left[\frac{3\sin x-\sin 3x}{4}+\frac{3\cos x+\cos 3x}{4}\right]$

$\therefore$ Period of $f(x)$ = LCM of period of

$\{\sin x, \cos x, \sin 3x, \cos 3x\}$

$=\frac{\text{LCM of } \{2\pi, 2\pi\}}{\text{HCF of } \{1, 3\}}=2\pi$

7 Let $y=x^2+1$

$\Rightarrow \quad x=\pm\sqrt{y-1}$

$\therefore \quad f^{-1}(x)=\pm\sqrt{x-1}$

$\therefore \quad f^{-1}(17)=\pm\sqrt{17-1}=\pm 4$

and $f^{-1}(-3)=\pm\sqrt{-3-1}$

$=\pm\sqrt{-4}\notin R$

$\therefore \quad f^{-1}(-3)=\phi$

8 Let $y=(x+1)^2$ for $x\geq -1$

$\Rightarrow \quad \pm\sqrt{y}=x+1\Rightarrow \quad \sqrt{y}=x+1$

$\Rightarrow \quad y\geq 0, x+1\geq 0$

$\Rightarrow \quad x=\sqrt{y}-1$

$\Rightarrow \quad f^{-1}(y)=\sqrt{y}-1$

$\Rightarrow \quad f^{-1}(x)=\sqrt{x}-1, x\geq 0$

9 We have, $f(x) = \dfrac{x}{1+x^2}$

$$\therefore \quad f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}}{1+\frac{1}{x^2}}$$

$$= \frac{x}{1+x^2} = f(x)$$

$$\therefore \quad f\left(\frac{1}{2}\right) = f(2)$$

$$\text{or} \quad f\left(\frac{1}{3}\right) = f(3)$$

and so on.
So, $f(x)$ is many-one function.
Again, let $y = f(x)$

$$\Rightarrow \quad y = \frac{x}{1+x^2}$$

$$\Rightarrow \quad y + x^2 y = x$$

$$\Rightarrow \quad yx^2 - x + y = 0$$

As, $\quad x \in R$

$$\therefore \quad (-1)^2 - 4(y)(y) \geq 0$$

$$\Rightarrow \quad 1 - 4y^2 \geq 0$$

$$\Rightarrow \quad y \in \left[\frac{-1}{2}, \frac{1}{2}\right]$$

$$\therefore \text{Range} = \text{Codomain} = \left[\frac{-1}{2}, \frac{1}{2}\right]$$

So, $f(x)$ is surjective.
Hence, $f(x)$ is surjective but not injective.

10 Given, $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} + 2$

$$\Rightarrow \quad y = \frac{e^{2x} - 1}{e^{2x} + 1} + 2$$

$$\Rightarrow \quad e^{2x} = \frac{1-y}{y-3} = \frac{y-1}{3-y}$$

$$\Rightarrow \quad x = \frac{1}{2} \log_e \left(\frac{y-1}{3-y}\right)$$

$$\Rightarrow \quad f^{-1}(y) = \log_e \left(\frac{y-1}{3-y}\right)^{1/2}$$

$$\Rightarrow \quad f^{-1}(x) = \log_e \left(\frac{x-1}{3-x}\right)^{1/2}$$

11 We have, $g(x) = 2f(x) + 5$

Now, on replacing x by $g^{-1}(x)$, we get

$$g(g^{-1}(x)) = 2f(g^{-1}(x)) + 5$$

$$\Rightarrow \quad x = 2f(g^{-1}(x)) + 5$$

$$\Rightarrow \quad f(g^{-1}(x)) = \frac{x-5}{2}$$

$$\Rightarrow \quad g^{-1}(x) = f^{-1}\left(\frac{x-5}{2}\right)$$

12 $f: (2, 3) \to (0, 1)$ and $f(x) = x - [x]$

$$\therefore \quad f(x) = y = x - 2 \Rightarrow x = y + 2$$

$$\Rightarrow \quad f^{-1}(x) = x + 2$$

13 $\because [x]$ denotes the integral part of x.

Hence, after term $\left[\dfrac{1}{2} + \dfrac{50}{100}\right]$ each term will be one. Hence, the sum of given series will be 50.

14 $f^2(x) \cdot f\left(\dfrac{1-x}{1+x}\right) = x^3 \qquad \ldots\text{(i)}$

On replacing x by $\dfrac{1-x}{1+x}$, we get

$$f^2\left(\frac{1-x}{1+x}\right) f(x) = \left(\frac{1-x}{1+x}\right)^3 \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii),

$$f^3(x) = x^6 \left(\frac{1+x}{1-x}\right)^3$$

$$\Rightarrow \quad f(x) = x^2 \left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow \quad f(-2) = \frac{-4}{3} \Rightarrow [f(-2)] = -2$$

$$\Rightarrow \quad |[f(-2)]| = 2$$

15 $\because \; g(x) = 1 + x - [x] \quad$ [put $x = n \in Z$]

$$\therefore \quad g(x) = 1 + x - x = 1$$

and $g(x) = 1 + n + k - n = 1 + k$ [put $x = n + k$]
[where, $n \in Z, 0 < k < 1$]

$$\text{Now, } f\{g(x)\} = \begin{cases} -1, & g(x) < 0 \\ 0, & g(x) = 0 \\ 1, & g(x) > 0 \end{cases}$$

Clearly, $\quad g(x) > 0, \forall \; x$
So, $\quad f\{g(x)\} = 1, \forall \; x$

16 Given that, $f(x) = \log(x + \sqrt{x^2+1})$

Now, $\quad f(-x) = \log(-x + \sqrt{x^2+1})$

$$\therefore \quad f(x) + f(-x) = \log(x + \sqrt{x^2+1}) + \log(-x + \sqrt{x^2+1})$$

$$= \log(1) = 0$$

Hence, $f(x)$ is an odd function.

17 Here, $f(x) = \begin{cases} -3x + 10, & \forall \; x \leq 2 \\ -x + 6, & \forall \; 2 < x \leq 3 \\ x, & \forall \; 3 < x \leq 5 \\ 3x - 10, & \forall \; x > 5 \end{cases}$

$$\therefore \quad f(x) = x, \forall \; 3 < x < 5$$

$$\Rightarrow \quad f(-x) = -x = -f(x)$$

18 Since, $\sin^{-1} x$ is defined in $[-1, 1]$, $\cos^{-1} x$ is defined in $[-1, 1]$ and $\tan^{-1} x$ is defined in R.
Hence, $f(x)$ is defined in $[-1, 1]$.

19 Period of $2\cos \dfrac{1}{3}(x - \pi)$ and $4\sin \dfrac{1}{3}(x - \pi)$ are $\dfrac{2\pi}{1/3}, \dfrac{2\pi}{1/3}$ or $6\pi, 6\pi$

$\therefore$ Period of their sum $= 6\pi$

20 Range of $\dfrac{1}{1+x^2}$ is $(0, 1)$ and domain R

$$\therefore \quad \log\left(\frac{1}{1+x^2}\right) \in (-\infty, 0]$$

SESSION 2

1 We have, $\dfrac{x-1}{x-2\{x\}} \geq 0$, here two cases arise

***Case* I** $x \geq 1$ and $x > 2\{x\}$

$$\Rightarrow \quad x \geq 2$$

$$\therefore \quad x \in [2, \infty).$$

***Case* II** $x \leq 1$ and $x < 2\{x\}$

$$\Rightarrow \quad x < 1 \text{ and } x \neq 0.$$

$\therefore \; x \in (-\infty, 0) \cup (0, 1)$.
Finally, $x = 1$ is also a point of the domain.

2 $y = |\sin x| + |\cos x|$

$$\Rightarrow \quad y^2 = 1 + |\sin 2x|$$

$$\Rightarrow \quad 1 \leq y^2 \leq 2 \Rightarrow \quad y \in [1, \sqrt{2}]$$

$$\therefore \quad f(x) = [y] = 1, \forall \; x \in R$$

3 Since, $[x^2] + x - a = 0$

$\therefore x$ has to be an integer.

$$\Rightarrow \quad a = x^2 + x = x(x+1)$$

Thus, a can be 2, 6, 12, 20.

4 $[x]^2 = x + 2\{x\}$

$$\Rightarrow \quad [x]^2 = [x] + 3\{x\}$$

$$\Rightarrow \quad \{x\} = \frac{[x]^2 - [x]}{3}$$

$$\Rightarrow \quad 0 \leq \frac{[x]^2 - [x]}{3} < 1$$

$$\Rightarrow \quad [x] \in \left(\frac{1-\sqrt{13}}{2}, 0\right] \cup \left[1, \frac{1+\sqrt{13}}{2}\right)$$

$$\Rightarrow \quad [x] = -1, 0, 1, 2$$

$$\Rightarrow \quad \{x\} = \frac{2}{3}, 0, 0, \frac{2}{3}$$

$$\therefore \quad x = -\frac{1}{3}, 0, 1, \frac{8}{3}$$

5 $g\{f(x)\} = (\sin x + \cos x)^2 - 1$ is invertible.

$\Rightarrow \; g\{f(x)\} = \sin 2x$

We know that, $\sin x$ is bijective only when $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

Thus, $g\{f(x)\}$ is bijective, if

$$-\frac{\pi}{2} \leq 2x \leq \frac{\pi}{2}.$$

$$\therefore \quad -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$$

6 Since, $f(x)$ is an even function.

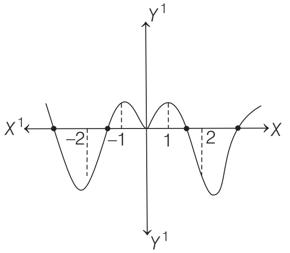
$\therefore$ Its graph is symmetrical about Y-axis

Also, we have,

$f(1) = 1$ and $f(2) = -5$

$\Rightarrow$ $f(-1) = 1$ and $f(-2) = -5$

According to these information, we have the following graph



Thus, minimum number of zeroes is 5.

7 Clearly, $f^{-1}og^{-1}(23) = (gof)^{-1}(23)$

Here, $gof(x) = 2(x^3 + 3) + 1$
$= 2x^3 + 7$

Now, let $y = (gof)^{-1}(23)$

$\Rightarrow$ $(gof)(y) = 23$

$\Rightarrow$ $2y^3 + 7 = 23$

$\Rightarrow$ $2y^3 = 16$

$\Rightarrow$ $y^3 = 8$

$\Rightarrow$ $y = 2$

Hence, $f^{-1}og^{-1}(23) = 2$

8 We have,

$$f(x) = [x] + [-x] = \begin{cases} 0, & \text{if } x \in I \\ -1, & \text{if } x \notin I \end{cases}$$

$$g(x) = \{x\} = \begin{cases} 0, & \text{if } x \in I \\ (x), & \text{if } x \notin I \end{cases}$$

and $h(x) = f(g(x))$

$$= f(\{x\}) = \begin{cases} f(0), & x \in \text{I} \\ f(\{x\}), & x \notin \text{I} \end{cases} = \begin{cases} 0, & x \in \text{I} \\ -1, & x \notin \text{I} \end{cases}$$

Clearly, option (b) is incorrect.

9 We have, $f(x) = [6x + 7] + \cos \pi x - 6x$

$= [6x] + 7 + \cos \pi x - 6x$

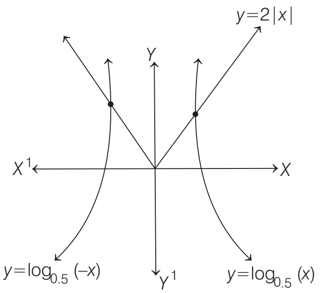
$= 7 + \cos \pi x - \{6x\}$

$[\because \{x\} = x - [x]]$

Now, as $\{6x\}$ has period $\frac{1}{6}$ and $\cos \pi x$ has the period 2, therefore the period of $f(x) = \text{LCM}\left(2, \frac{1}{6}\right)$ which is 2.

Hence, the period is 2.

10 For the solution of given equation, let us draw the graph of $y = \log_{0.5}|x|$ and $y = 2|x|$



From the graph it is clear that there are two solution.