RELATIONS AND FUNCTIONS

(TOTAL MARKS 10)

FIVE MARK

1. Show that the function $f: R \to R$ is defined by $f(x) = 4x + 3$ is invertible. Also write the inverse of $f(x)$

Solution : One-One : Consider any two elements $x_1, x_2 \in R$

Consider $f(x_1) = f(x_2)$

$4x_1 + 3 = 4x_2 + 3$

$4x_1 = 4x_2$

$x_1 = x_2$

$\therefore f$ is One-One

Onto : Let y is any elements in R

Let $y = f(x)$

$y = 4x + 3$

$4x = y - 3$

$x = \frac{y-3}{4} \in R$

$\therefore$ f is onto

$\therefore$ f is both One-One and Onto

$\therefore$ f is invertible function

To find f^{-1} : Let $y = f(x)$

$y = 4x + 3$

$4x = y - 3$

$x = \frac{y-3}{4}$

$f^{-1}(y) = \frac{y-3}{4}$

2. Prove that the function $f : N \to Y$ defined $f(x) = 4x + 3$, where $Y = [y : y = 4x + 3,\ x \in N]$ Is invertible. Also write inverse of $f(x)$.

Solution : One-One : Consider any two elements $x_1, x_2 \in N$

Consider $f(x_1) = f(x_2)$

$4x_1 + 3 = 4x_2 + 3$

$4x_1 = 4x_2$

$x_1 = x_2$

$\therefore f$ is One-One

Onto : Let y is any elements in Y

Let $y = f(x)$

$y = 4x + 3$

$4x = y - 3$

$x = \frac{y-3}{4} \in N$

$\therefore$ f is onto

$\therefore$ f is both One-One and Onto

$\therefore$ f is invertible function

To find f^{-1} : Let $y = f(x)$

$y = 4x + 3$

$4x = y - 3$

$x = \frac{y-3}{4}$

$f^{-1}(y) = \frac{y-3}{4}$

3. Prove that the function $f : N \to Y$ defined by $f(x) = x^2$ where $Y = \{y : y = x^2,\ x \in N\}$ is invertible. Also find the inverse of f.

Solution : One-One : Consider any two elements $x_{1,} x_2 \in N$

Consider $f(x_1) = f(x_2)$

${x_1}^2 = {x_2}^2$

${x_1}^2 - {x_2}^2 = 0$

$(x_1 + x_2)(x_1 - x_2) = 0$

$(x_1 + x_2) = 0$ *and* $(x_1 - x_2) = 0$

$x_1 = -x_2 \notin N$ and $x_1 = x_2 \in N$

$\therefore f$ is One-One

Onto : Let y is any elements in Y

Let $y = f(x)$

$y = x^2$

$x = \sqrt{y} \in N$

$\therefore$ f is onto

$\therefore$ f is both One-One and Onto

$\therefore$ f is invertible function

To find f^{-1}: Let $y = f(x)$

$y = x^2$

$x = \sqrt{y}$

$f^{-1}(y) = \sqrt{y}$

4. Let $f : N \longrightarrow R$ be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that $f : N \longrightarrow S$, where S is the range of f, is invertible. Find the inverse of f.

Solution : One-One : Consider any two elements $x_1, x_2 \in N$

Consider $f(x_1) = f(x_2)$

$4{x_1}^2 + 12x_1 + 15 = 4{x_2}^2 + 12x_2 + 15$

$\Longrightarrow 4{x_1}^2 + 12x_1 = 4{x_2}^2 + 12x_2$

$4{x_1}^2 - 4{x_2}^2 + 12x_1 - 12x_2 = 0$

$4({x_1}^2 - {x_2}^2) + 12(x_1 - x_2) = 0$

$4(x_1 + x_2)(x_1 - x_2) + 12(x_1 - x_2) = 0$

$(x_1 - x_2)[4(x_1 + x_2) + 12] = 0$

$x_1 - x_2 = 0$ and $4(x_1 + x_2) + 12 \neq 0$

$\therefore\ x_1 = x_2$

$\therefore f$ is One-One

Onto: Let y is any elements in S

Let $y = f(x)$

$y = 4x^2 + 12x + 15$

$4x^2 + 12x + 15 - y = 0$

$\therefore a = 4, \quad b = 12\ and\ c = 15 - y$

WKT $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-12 \pm \sqrt{144 - 4(4)(15 - y)}}{8}$

$x = \frac{-12 \pm \sqrt{144 - 16(15 - y)}}{8}$

$x = \frac{-12 \pm \sqrt{16(9 - 15 + y)}}{8}$

$x = \frac{-12 \pm 4\sqrt{y - 6}}{8}$

$x = 4\left[\frac{-3 \pm \sqrt{y - 6}}{8}\right]$

$x = \frac{-3 + \sqrt{y - 6}}{2} \in N$

$\therefore$ f is onto

$\therefore$ f is both One-One and Onto

$\therefore$ f is invertible function

TO find f^{-1}_ Let $y = f(x)$

$y = 4x^2 + 12x + 15$

$4x^2 + 12x + 15 - y = 0$

$\therefore a = 4, \quad b = 12\ and\ c = 15 - y$

WKT $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-12 \pm \sqrt{144 - 4(4)(15 - y)}}{8}$

$x = \frac{-12 \pm \sqrt{144 - 16(15 - y)}}{8}$

$x = \frac{-12 \pm \sqrt{16(9 - 15 + y)}}{8}$

$x = \frac{-12 \pm 4\sqrt{y - 6}}{8}$

$x = 4\left[\frac{-3 \pm \sqrt{y - 6}}{8}\right]$

$x = \frac{-3 + \sqrt{y - 6}}{2}$

$f^{-1}(y) = \frac{-3 + \sqrt{y - 6}}{2}$

5. Consider $f : R_+ \rightarrow [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\frac{\sqrt{y+6} - 1}{3}\right)$

Solution : One-One : Consider any two elements $x_1, x_2 \in R_+$

Consider $f(x_1) = f(x_2)$

$9{x_1}^2 + 6x_1 - 5 = 9{x_2}^2 + 6x_2 - 5$

$\Rightarrow 9{x_1}^2 + 6x_1 = 9{x_2}^2 + 6x_2$

$9{x_1}^2 - 9{x_2}^2 + 6x_1 - 6x_2 = 0$

$9({x_1}^2 - {x_2}^2) + 6(x_1 - x_2) = 0$

$9(x_1 + x_2)(x_1 - x_2) + 6(x_1 - x_2) = 0$

$(x_1 - x_2)[9(x_1 + x_2) + 6] = 0$

$x_1 - x_2 = 0$ and $9(x_1 + x_2) + 6 \neq 0$

$x_1 = x_2$

$\therefore f$ is One-One

Onto Let y is any elements in $[-5, \infty)$

Let $y = f(x)$

$y = 9x^2 + 6x - 5$

$9x^2 + 6x - 5 - y = 0$

$\therefore a = 9, \quad b = 6\ and\ c = -5 - y$

WKT $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-6 \pm \sqrt{36 - 4(9)(-5-y)}}{18}$

$x = \frac{-6 \pm \sqrt{36 - 36(-5-y)}}{18}$

$x = \frac{-6 \pm \sqrt{36(1+5+y)}}{18}$

$x = \frac{-6 \pm 6\sqrt{y+6)}}{18}$

$x = 6\left[\frac{-1 \pm \sqrt{y+6}}{18}\right]$

$x = \frac{-1 + \sqrt{y+6}}{3} \in R_+$

$\therefore$ f is onto

$\therefore f$ is both One-One and Onto

$\therefore f$ is invertible function

TO find f^{-1} : Let $y = f(x)$

$y = 9x^2 + 6x - 5$

$9x^2 + 6x - 5 - y = 0$

$\therefore a = 9, \quad b = 6\ and\ c = -5 - y$

WKT $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-6 \pm \sqrt{36 - 4(9)(-5-y)}}{18}$

$x = \frac{-6 \pm \sqrt{36 - 36(-5-y)}}{18}$

$x = \frac{-6 \pm \sqrt{36(1+5+y)}}{18}$

$x = \frac{-6 \pm 6\sqrt{y+6)}}{18}$

$x = 6\left[\frac{-1 \pm \sqrt{y+6}}{18}\right]$

$x = \frac{-1 + \sqrt{y+6}}{3}$

$f^{-1}(y) = \frac{-1+\sqrt{y+6}}{3}$

6. Let R_+ be the set of all non-negative real numbers. Show that the function $f : R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$ is invertible and write the inverse of f.

Solution : One-One : Consider any two elements $x_1, x_2 \in R_+$

Consider $f(x_1) = f(x_2)$

$${x_1}^2 + 4 = {x_2}^2 + 4$$

$${x_1}^2 = {x_2}^2$$

$$(x_1 + x_2)(x_1 - x_2) = 0$$

$$(x_1 + x_2) = 0 \; and \; (x_1 - x_2) = 0$$

$x_1 = -x_2 \notin R_+$ and $x_1 = x_2 \in R_+$

$\therefore f$ is One-One

Onto : Let y is any elements in $[4, \infty)$

Let $y = f(x)$

$$y = x^2 + 4$$

$$x^2 = y - 4$$

$$x = \sqrt{y-4} \in R_+$$

$\therefore$ f is onto

$\therefore$ f is both One-One and Onto

$\therefore$ f is invertible function

To find f^{-1} : Let $y = f(x)$

$$y = x^2 + 4$$

$$x^2 = y - 4$$

$$x = \sqrt{y-4}$$

$$f^{-1}(y) = \sqrt{y-4}$$

7. If $f : A \to B$ defined by $f(x) = \frac{4x+3}{6x-4}$ where $A = R - \left\{\frac{2}{3}\right\}$ and $B = R - \left\{\frac{2}{3}\right\}$.Show that $f : N \to S$ where S is the range of function f is invertible and $f^{-1} = f$.

Solution : One-One : Consider any two elements $x_1, x_2 \in R_+$

Consider $f(x_1) = f(x_2)$

$$\frac{4x_1+3}{6x_1-4} = \frac{4x_2+3}{6x_2-4}$$

$$(4x_1 + 3)(6x_2 - 4) = (6x_1 - 4)(4x_2 + 3)$$

$$24x_1x_2 - 16x_1 + 18x_2 - 12 = 24x_1x_2 + 18x_1 - 16x_2 - 12$$

$$-16x_1 + 18x_2 = 18x_1 - 16x_2$$

$$-16x_1 - 18x_1 = -16x_2 - 18x_2$$

$$-34x_1 = -34x_2$$

$$x_1 = x_2$$

$\therefore$ f is One-One

Onto : Let y is any elements in $[-5, \infty)$

Let $y = f(x)$

$$y = \frac{4x+3}{6x-4}$$

$$y(6x - 4) = 4x + 3$$

$$6xy - 4y = 4x + 3$$

$$6xy - 4x = 3 + 4y$$

$$x(6y - 4) = 3 + 4y$$

$x = \frac{3+4y}{6y-4} \in A$

$\therefore$ f is onto

$\therefore$ f is both One-One and Onto

$\therefore$ f is invertible function

To find f^{-1} : Let $y = f(x)$

$$y = \frac{4x+3}{6x-4}$$

$$y(6x-4) = 4x+3$$

$$6xy - 4y = 4x + 3$$

$$6xy - 4x = 3 + 4y$$

$$x(6y-4) = 3 + 4y$$

$$x = \frac{3+4y}{6y-4}$$

$$f^{-1}(y) = \frac{3+4y}{6y-4}$$

8. Let $A = R - \{3\}$ and $B = R - \{1\}$. Consider the function $f: A \rightarrow B$ defined by $f(x) = \frac{x-2}{x-3}$ Is f one-one and onto? Justify your answer.

Solution: **One-One :** Consider any two elements $x_1, x_2 \in A$

Consider $f(x_1) = f(x_2)$

$$\frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3}$$

$$(x_1-2)(x_2-3) = (x_1-3)(x_2-2)$$

$$x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$$

$$-3x_1 - 2x_2 = -2x_1 - 3x_2$$

$$-3x_1 + 2x_1 = -3x_2 + 2x_2$$

$$-x_1 = -x_2$$

$$\therefore x_1 = x_2$$

$\therefore$ f is One-One

Onto : Let y is any elements in B

Let $y = f(x)$

$$y = \frac{x-2}{x-3}$$

$$y(x-3) = x - 2$$

$$xy - 3y = x - 2$$

$$xy - x = -2 + 3$$

$$x(y-1) = -2 + 3y$$

$$\therefore x = \frac{-2+3y}{y-1} \in A$$

$\therefore$ f is onto

$\therefore$ f is both One-One and Onto