CHAPTER IV.

ARITHMETICAL PROGRESSION.

38. Definition. Quantities are said to be in **Arithmetical Progression** when they increase or decrease by a *common difference.*

Thus each of the following series forms an Arithmetical Progression:

$$3,\ 7,\ 11,\ 15, \ldots\ldots\ldots\ldots\ldots\ldots$$

$$8,\ 2,\ -4,\ -10, \ldots\ldots\ldots\ldots\ldots\ldots$$

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots\ldots\ldots$$

The common difference is found by subtracting *any* term of the series from that which *follows* it. In the first of the above examples the common difference is 4; in the second it is -6; in the third it is d.

39. If we examine the series

$$a,\ a+d,\ a+2d,\ a+3d, \ldots$$

we notice that *in any term the coefficient of* d *is always less by one than the number of the term in the series.*

Thus the 3^{rd} term is $a+2d$;

6^{th} term is $a+5d$;

20^{th} term is $a+19d$;

and, generally, the p^{th} term is $a+(p-1)\,d$.

If n be the number of terms, and if l denote the last, or n^{th} term, we have $l = a + (n-1)\,d$.

40. *To find the sum of a number of terms in Arithmetical Progression.*

Let a denote the first term, d the common difference, and n the number of terms. Also let l denote the last term, and s

the required sum; then

$$s = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l;$$

and, by writing the series in the reverse order,

$$s = l + (l - d) + (l - 2d) + \ldots + (a + 2d) + (a + d) + a.$$

Adding together these two series,

$$2s = (a + l) + (a + l) + (a + l) + \ldots \text{ to } n \text{ terms}$$
$$= n(a + l),$$

$$\therefore \; s = \frac{n}{2}(a + l) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and $$l = a + (n - 1)d \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\therefore \; s = \frac{n}{2}\{2a + (n - 1)d\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

41. In the last article we have three useful formulæ (1), (2), (3); in each of these any one of the letters may denote the unknown quantity when the three others are known. For instance, in (1) if we substitute given values for s, n, l, we obtain an equation for finding a; and similarly in the other formulæ. But it is necessary to guard against a too mechanical use of these general formulæ, and it will often be found better to solve simple questions by a mental rather than by an actual reference to the requisite formula.

Example 1. Find the sum of the series $5\frac{1}{2}$, $6\frac{3}{4}$, 8,......to 17 terms.
Here the common difference is $1\frac{1}{4}$; hence from (3),

the sum $$= \frac{17}{2}\left\{2 \times \frac{11}{2} + 16 \times 1\tfrac{1}{4}\right\}$$
$$= \frac{17}{2}(11 + 20)$$
$$= \frac{17 \times 31}{2}$$
$$= 263\tfrac{1}{2}.$$

Example 2. The first term of a series is 5, the last 45, and the sum 400: find the number of terms, and the common difference.

If n be the number of terms, then from (1)

$$400 = \frac{n}{2}(5 + 45);$$

whence $$n = 16.$$

If d be the common difference

$$45 = \text{the } 16^{\text{th}} \text{ term} = 5 + 15d;$$

whence $$d = 2\tfrac{2}{3}.$$

42. If *any two* terms of an Arithmetical Progression be given, the series can be completely determined; for the data furnish *two* simultaneous equations, the solution of which will give the first term and the common difference.

Example. The 54^{th} and 4^{th} terms of an A. P. are -61 and 64; find the 23^{rd} term.

If a be the first term, and d the common difference,

$$-61 = \text{the } 54^{\text{th}} \text{ term} = a + 53d;$$

and $$64 = \text{the } 4^{\text{th}} \text{ term} = a + 3d;$$

whence we obtain $$d = -\frac{5}{2},\ a = 71\tfrac{1}{2};$$

and the 23^{rd} term $= a + 22d = 16\tfrac{1}{2}$.

43. DEFINITION. When three quantities are in Arithmetical Progression the middle one is said to be the **arithmetic mean** of the other two.

Thus a is the arithmetic mean between $a - d$ and $a + d$.

44. *To find the arithmetic mean between two given quantities.*

Let a and b be the two quantities; A the arithmetic mean. Then since a, A, b are in A.P. we must have

$$b - A = A - a,$$

each being equal to the common difference;

whence $$A = \frac{a + b}{2}.$$

45. Between two given quantities it is always possible to insert any number of terms such that the whole series thus formed shall be in A.P.; and by an extension of the definition in Art. 43, the terms thus inserted are called the *arithmetic means.*

Example. Insert 20 arithmetic means between 4 and 67.

Including the extremes, the number of terms will be 22; so that we have to find a series of 22 terms in A.P., of which 4 is the first and 67 the last.

Let d be the common difference;

then $$67 = \text{the } 22^{\text{nd}} \text{ term} = 4 + 21d;$$

whence $d = 3$, and the series is 4, 7, 10,......61, 64, 67;
and the required means are 7, 10, 13,......58, 71, 64.

46. *To insert a given number of arithmetic means between two given quantities.*

Let a and b be the given quantities, n the number of means.

Including the extremes the number of terms will be $n+2$; so that we have to find a series of $n+2$ terms in A.P., of which a is the first, and b is the last.

Let d be the common difference;

then $$b = \text{the } (n+2)^{\text{th}} \text{ term}$$
$$= a + (n+1)\,d;$$

whence $$d = \frac{b-a}{n+1};$$

and the required means are

$$a + \frac{b-a}{n+1}, \quad a + \frac{2\,(b-a)}{n+1}, \ldots\ldots a + \frac{n\,(b-a)}{n+1}.$$

Example 1. The sum of three numbers in A.P. is 27, and the sum of their squares is 293; find them.

Let a be the *middle* number, d the common difference; then the three numbers are $a-d$, a, $a+d$.

Hence $$a-d+a+a+d=27;$$

whence $a=9$, and the three numbers are $9-d$, 9, $9+d$.

$$\therefore\ (9-d)^2+81+(9+d)^2=293;$$

whence $$d=\pm 5;$$

and the numbers are 4, 9, 14.

Example 2. Find the sum of the first p terms of the series whose n^{th} term is $3n-1$.

By putting $n=1$, and $n=p$ respectively, we obtain

$$\text{first term}=2, \quad \text{last term}=3p-1;$$

$$\therefore\ \text{sum}=\frac{p}{2}(2+3p-1)=\frac{p}{2}(3p+1).$$

EXAMPLES. IV. a.

1. Sum $2, 3\frac{1}{4}, 4\frac{1}{2}, \ldots$ to 20 terms.

2. Sum 49, 44, 39,... to 17 terms.

3. Sum $\frac{3}{4}, \frac{2}{3}, \frac{7}{12}, \ldots$ to 19 terms.

4. Sum $3, \frac{7}{3}, 1\frac{2}{3}, \ldots$ to n terms.

5. Sum $3 \cdot 75, 3 \cdot 5, 3 \cdot 25, \ldots$ to 16 terms.

6. Sum $-7\frac{1}{2}, -7, -6\frac{1}{2}, \ldots$ to 24 terms.

7. Sum $1 \cdot 3, -3 \cdot 1, -7 \cdot 5, \ldots$ to 10 terms.

8. Sum $\frac{6}{\sqrt{3}}, 3\sqrt{3}, \frac{12}{\sqrt{3}}, \ldots$ to 50 terms.

9. Sum $\frac{3}{\sqrt{5}}, \frac{4}{\sqrt{5}}, \sqrt{5}, \ldots$ to 25 terms.

10. Sum $a-3b, 2a-5b, 3a-7b, \ldots$ to 40 terms.

11. Sum $2a-b, 4a-3b, 6a-5b, \ldots$ to n terms.

12. Sum $\frac{a+b}{2}, a, \frac{3a-b}{2}, \ldots$ to 21 terms.

13. Insert 19 arithmetic means between $\frac{1}{4}$ and $-9\frac{3}{4}$.

14. Insert 17 arithmetic means between $3\frac{1}{2}$ and $-41\frac{1}{2}$.

15. Insert 18 arithmetic means between $-35x$ and $3x$.

16. Insert x arithmetic means between x^2 and 1.

17. Find the sum of the first n odd numbers.

18. In an A. P. the first term is 2, the last term 29, the sum 155; find the difference.

19. The sum of 15 terms of an A. P. is 600, and the common difference is 5; find the first term.

20. The third term of an A. P. is 18, and the seventh term is 30; find the sum of 17 terms.

21. The sum of three numbers in A. P. is 27, and their product is 504; find them.

22. The sum of three numbers in A. P. is 12, and the sum of their cubes is 408; find them.

23. Find the sum of 15 terms of the series whose n^{th} term is $4n+1$.

24. Find the sum of 35 terms of the series whose p^{th} term is $\frac{p}{7}+2$.

25. Find the sum of p terms of the series whose n^{th} term is $\frac{n}{a}+b$.

26. Find the sum of n terms of the series

$$\frac{2a^2-1}{a}, \quad 4a-\frac{3}{a}, \quad \frac{6a^2-5}{a}, \ldots$$

47. In an Arithmetical Progression when s, a, d are given, to determine the values of n we have the quadratic equation

$$s = \frac{n}{2}\left\{2a + (n-1)d\right\};$$

when both roots are positive and integral there is no difficulty in interpreting the result corresponding to each. In some cases a suitable interpretation can be given for a negative value of n.

Example. How many terms of the series $-9, -6, -3, \ldots$ must be taken that the sum may be 66?

Here $$\frac{n}{2}\{-18+(n-1)\,3\} = 66;$$

that is, $$n^2 - 7n - 44 = 0,$$

or $$(n-11)(n+4) = 0;$$

$$\therefore\ n = 11 \text{ or } -4.$$

If we take 11 terms of the series, we have

$$-9,\ -6,\ -3,\ 0,\ 3,\ 6,\ 9,\ 12,\ 15,\ 18,\ 21;$$

the sum of which is 66.

If we begin at the *last* of these terms and *count backwards* four terms, the sum is also 66; and thus, although the negative solution does not directly answer the question proposed, we are enabled to give it an intelligible meaning, and we see that it answers a question closely connected with that to which the positive solution applies.

48. We can justify this interpretation in the general case in the following way.

The equation to determine n is

$$dn^2 + (2a-d)\,n - 2s = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Since in the case under discussion the roots of this equation have opposite signs, let us denote them by n_1 and $-n_2$. The last term of the series corresponding to n_1 is

$$a + (n_1 - 1)\,d;$$

if we begin at this term and count *backwards*, the common difference must be denoted by $-d$, and the sum of n_2 terms is

$$\frac{n_2}{2}\left\{2\,(a + \overline{n_1 - 1}d) + (n_2 - 1)(-d)\right\},$$

and we shall shew that this is equal to s.

For the expression $= \frac{n_2}{2}\left\{2a + (2n_1 - n_2 - 1)\,d\right\}$

$$= \frac{1}{2}\left\{2an_2 + 2n_1n_2d - n_2(n_2 + 1)\,d\right\}$$

$$= \frac{1}{2}\left\{2n_1n_2d - (dn_2^2 - \overline{2a - d}\,.\,n_2)\right\}$$

$$= \frac{1}{2}(4s - 2s) = s,$$

since $-n_2$ satisfies $dn^2 + (2a - d)\,n - 2s = 0$, and $-n_1n_2$ is the product of the roots of this equation.

49. When the value of n is fractional there is no exact number of terms which corresponds to such a solution.

Example. How many terms of the series 26, 21, 16,...must be taken to amount to 74?

Here $\frac{n}{2}\{52 + (n-1)(-5)\} = 74;$

that is, $5n^2 - 57n + 148 = 0,$

or $(n-4)(5n-37) = 0;$

$$\therefore\ n = 4 \text{ or } 7\tfrac{2}{5}.$$

Thus the number of terms is 4. It will be found that the sum of 7 terms is greater, while the sum of 8 terms is less than 74.

50. We add some Miscellaneous Examples.

Example 1. The sums of n terms of two arithmetic series are in the ratio of $7n+1 : 4n+27$; find the ratio of their 11^{th} terms.

Let the first term and common difference of the two series be a_1, d_1 and a_2, d_2 respectively.

We have $\frac{2a_1 + (n-1)\,d_1}{2a_2 + (n-1)\,d_2} = \frac{7n+1}{4n+27}.$

Now we have to find the value of $\frac{a_1 + 10d_1}{a_2 + 10d_2}$; hence, by putting $n = 21$, we obtain

$$\frac{2a_1 + 20d_1}{2a_2 + 20d_2} = \frac{148}{111} = \frac{4}{3};$$

thus the required ratio is $4 : 3$.

Example 2. If $S_1, S_2, S_3, \ldots S_p$ are the sums of n terms of arithmetic series whose first terms are 1, 2, 3, 4,... and whose common differences are 1, 3, 5, 7,...; find the value of

$$S_1 + S_2 + S_3 + \ldots + S_p.$$

We have
$$S_1=\frac{n}{2}\{2+(n-1)\}=\frac{n(n+1)}{2},$$
$$S_2=\frac{n}{2}\{4+(n-1)3\}=\frac{n(3n+1)}{2},$$
$$S_3=\frac{n}{2}\{6+(n-1)5\}=\frac{n(5n+1)}{2},$$
$$S_p=\frac{n}{2}\{2p+(n-1)(2p-1)\}=\frac{n}{2}\{(2p-1)n+1\};$$

$$\therefore \text{ the required sum}=\frac{n}{2}\{(n+1)+(3n+1)+\ldots\ldots(\overline{2p-1}\,.\,n+1)\}$$
$$=\frac{n}{2}\{(n+3n+5n+\ldots\overline{2p-1}\,.\,n)+p\}$$
$$=\frac{n}{2}\{n(1+3+5+\ldots\overline{2p-1})+p\}$$
$$=\frac{n}{2}(np^2+p)$$
$$=\frac{np}{2}(np+1).$$

EXAMPLES. IV. b.

1. Given $a=-2$, $d=4$ and $s=160$, find n.

2. How many terms of the series 12, 16, 20,... must be taken to make 208?

3. In an A. P. the third term is four times the first term, and the sixth term is 17; find the series.

4. The 2nd, 31st, and last terms of an A. P. are $7\frac{3}{4}$, $\frac{1}{2}$ and $-6\frac{1}{2}$ respectively; find the first term and the number of terms.

5. The 4th, 42nd, and last terms of an A. P. are 0, -95 and -125 respectively; find the first term and the number of terms.

6. A man arranges to pay off a debt of £3600 by 40 annual instalments which form an arithmetic series. When 30 of the instalments are paid he dies leaving a third of the debt unpaid: find the value of the first instalment.

7. Between two numbers whose sum is $2\frac{1}{6}$ an even number of arithmetic means is inserted; the sum of these means exceeds their number by unity: how many means are there?

8. The sum of n terms of the series 2, 5, 8,... is 950: find n.