CHAPTER NINETEEN

Ellipse

Definition 1:

An ellipse is the locus of a point which moves in a plane such that the sum of its distances from two fixed points is always constant and the constant is greater than the distance between the fixed points.

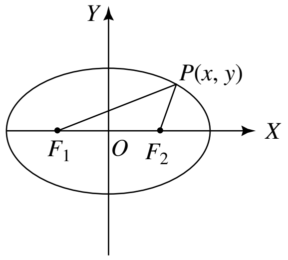

Fig. 19.1

(i) Two fixed points are called the *foci* of the ellipse.
(ii) Mid-point of the line joining the foci is called the *centre* of the ellipse.
(iii) The line segment through the foci of the ellipse is called the *major axis* of the ellipse.
(iv) The line segment through the centre and perpendicular to the major axis is called the *minor axis* of the ellipse.
(v) The end points of the major axis are called the *vertices* of the ellipse.
(vi) A line segment through a focus perpendicular to the major axis is called a *Latus rectum* of the ellipse.
(vii) Major axis and minor axis are called the *principal axis* of the ellipse.

Standard Equation of the Ellipse

Let F_1 $(-c, 0)$ and F_2 $(c, 0)$ be the foci of the ellipse. Then O (0, 0), the mid-point of $F_1 F_2$ is the centre of the ellipse.

Let P (x, y) be any point on the ellipse. Then by definition $PF_1 + PF_2 = 2a$ (constant)

$$\Rightarrow \sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2} = 2a. \ (2a > 2c)$$

Also $[(x + c)^2 + y^2] - [(x - c)^2 + y^2] = 4cx$.

$$\Rightarrow \sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2} = \frac{4cx}{2a} = \frac{2cx}{a}$$

$$\Rightarrow (x-c)^2 + y^2 = \left(a - \frac{cx}{a}\right)^2$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

Let $a^2 - c^2 = b^2$, then locus of $P(x, y)$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b > 0$.

Definition 2:

An ellipse is the locus of a point which moves in a plane in such a way that the ratio of its distance from a fixed point in the plane to its distance from a fixed line in the plane (not passing through fixed point) is a constant and equal to e, where $0 < e < 1$.

(i) Fixed point is called a *focus* of the ellipse.
(ii) Fixed line is called a *directrix* of the ellipse.
(iii) The constant ratio e is called the *eccentricity* of the ellipse.

Standard Equation of an Ellipse referred to its principle axes along the coordinate axes is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

$b^2 = a^2 (1 - e^2)$

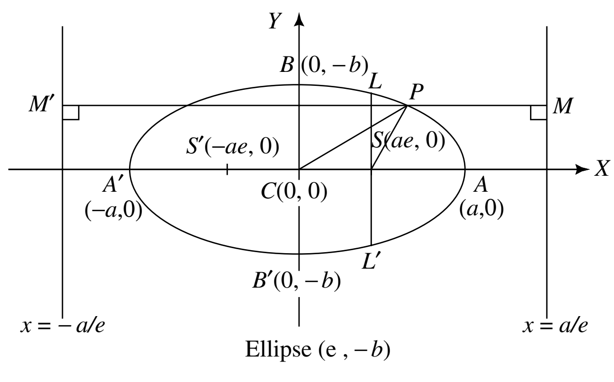

Fig. 19.2

Let $P(x, y)$ be any point on the ellipse.

$S = (ae, 0)$ be a focus and $x = \frac{a}{e}$ be a directrix.

Then according to the definition

$$(x - ae)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

Since $e < 1$, $1 - e^2 > 0$ so let $a^2(1 - e^2) = b^2$ and the required equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

In the figure

(i) $A'A$ is the *major axis* of the ellipse along x-axis of length $2a$; $A(a, 0)$, $A'(-a, 0)$ are the vertices of the ellipse;

(ii) BB' is the minor axis of the ellipse along y-axis of length $2b$.

(iii) $O(0, 0)$ is the centre of the ellipse.

(iv) $e = \sqrt{\frac{a^2 - b^2}{a^2}}$ is the eccentricity of the ellipse.

From symmetry we observe that if

$S' = (-ae, 0)$ be taken as a focus and $x = \frac{-a}{e}$ is taken as a directrix, same ellipse is described. So

(v) $S'(-ae, 0)$ and $S(ae, 0)$ are the two foci of the ellipse.

(vi) $x = \frac{-a}{e}$ and $x = \frac{a}{e}$ are the two directrices of the ellipse.

(vii) $x = \pm ae$ are the two latera recta of the ellipse.

(viii) Latus rectum $x = ae$ meets the ellipse at point $\left(ae, \pm\frac{b^2}{a}\right)$, so length of each latus rectum is $\frac{2b^2}{a}$.

Illustration 1

Find the length and equations of the latera recta of the ellipse

$$\frac{x^2}{36} + \frac{y^2}{25} = 1$$

Solution: Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

where $a^2 = 36$, $b^2 = 25 \Rightarrow e^2 = 1 - \frac{25}{36} = \frac{11}{36}$

Equations of latera recta are $x = \pm ae$

$$\Rightarrow x = \pm 6 \times \frac{\sqrt{11}}{6} = \pm\sqrt{11}$$

and length of each latus rectum $= \frac{2b^2}{a} = \frac{2 \times 25}{6}$

$= \frac{25}{3}$ units.

Illustration 2

Find the centre, foci, the length of the axis, eccentricity and the equation of the directrices of the ellipse.

$9x^2 + 16y^2 - 18x + 32y - 119 = 0$

Solution: The equation of the ellipse can be written as

$9(x - 1)^2 + 16(y + 1)^2 = 144$

$$\Rightarrow \frac{(x-1)^2}{16} + \frac{(y+1)^2}{9} = 1$$

Which can be written as $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$

where $X = x - 1$, $Y = y + 1$, $a = 4$, $b = 3$.

centre is $X = 0$, $Y = 0$ i.e. $(1, -1)$

foci are $X = \pm ae$, $Y = 0 \Rightarrow x - 1 = \pm 4\sqrt{1 - \frac{b^2}{a^2}} = \pm\sqrt{7}$

$\Rightarrow x = 1 \pm \sqrt{7}$; $y = -1$

Length of the major axis $= 2a = 8$

Length of the minor axis $= 2b = 6$.

eccentricity $= \sqrt{1 - \frac{b^2}{a^2}} = \frac{\sqrt{7}}{4}$

Equations of the directrices are $X = \pm \frac{a}{e}$.

$$\Rightarrow x - 1 = \pm \frac{4 \times 4}{\sqrt{7}} \Rightarrow x = 1 \pm \frac{16}{\sqrt{7}}$$

Illustration 3

Find the equation of the ellipse whose eccentricity is $\frac{1}{2}$, a focus is (2, 3) and a directrix is $x = 7$. Find the length of the major and minor axes of the ellipse.

Solution: Equation of the ellipse by definition is

$$(x - 2)^2 + (y - 3)^2 = \frac{1}{4}(7 - x)^2$$

$\Rightarrow 3x^2 + 4y^2 - 2x - 24y + 3 = 0$

which can be written as

$$3\left(x - \frac{1}{3}\right)^2 + 4(y - 3)^2 = 3 \times \frac{1}{9} + 4 \times 9 - 3 = \frac{100}{3}$$

$$\text{or } \frac{\left(x - \frac{1}{3}\right)^2}{\frac{100}{9}} + \frac{(y-3)^2}{\frac{100}{12}} = 1$$

So length of the major axis = $2\sqrt{\dfrac{100}{9}} = \dfrac{20}{3}$.

Length of the minor axis = $2\sqrt{\dfrac{100}{12}} = \dfrac{20}{2\sqrt{3}} = \dfrac{10}{\sqrt{3}}$.

Some Properties and Standard Results for the Ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$

1. The parametric equations of the ellipse or the coordinates of any point on the ellipse are $x = a\cos\theta$, $y = b\sin\theta$. The point is denoted by "θ". θ is called the eccentric angle of the point.
2. An equation of the tangent at the above point "θ" is $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$ and at (x', y') is $\dfrac{xx'}{a^2}+\dfrac{yy'}{b^2} = 1$.

Illustration 4

Find an equation of the tangent to the ellipse $\dfrac{x^2}{81}+\dfrac{y^2}{49} = 1$ at the point P whose eccentric angle is $\pi/6$. Also find the coordinates of P.

Solution: Coordinates of P are $\left(9\cos\dfrac{\pi}{6}, 7\sin\dfrac{\pi}{6}\right) = \left(\dfrac{9\sqrt{3}}{2}, \dfrac{7}{2}\right)$

An equation of the tangent at P to the ellipse is $\dfrac{x}{9}\cos\dfrac{\pi}{6}+\dfrac{y}{7}\sin\dfrac{\pi}{6} = 1$

$\Rightarrow \dfrac{\sqrt{3}x}{9\times 2}+\dfrac{y}{7\times 2} = 1 \Rightarrow 7\sqrt{3}x + 9y = 126.$

3. An equation of the normal at $P(\theta)$ is
$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta} = a^2-b^2$$
and at (x', y') is
$$\frac{x-x'}{x'/a^2} = \frac{y-y'}{y'/b^2}$$

Illustration 5

$5\sqrt{3}x + 7y = 70$ is a tangent to the ellipse $\dfrac{x^2}{49}+\dfrac{y^2}{25} = 1$ at the point P. Find the coordinates of P and the equation of the normal at P.

Solution: Let the coordinates of P be $(7\cos\theta, 5\sin\theta)$

Equation of the tangent at P is

$$\frac{x}{7}\cos\theta+\frac{y}{5}\sin\theta = 1 \qquad (1)$$

Given equation can be written as

$$\frac{x\sqrt{3}}{14}+\frac{y}{10} = 1 \qquad (2)$$

Comparing (1) and (2) we get

$\cos\theta = \dfrac{\sqrt{3}}{2}$, $\sin\theta = \dfrac{1}{2} \Rightarrow \theta = 30°$

So the coordinates of P are $\left(\dfrac{7\sqrt{3}}{2}, \dfrac{5}{2}\right)$.

Equation of the normal at P is

$\dfrac{7x}{\cos\theta}-\dfrac{5y}{\sin\theta} = 1$

$\Rightarrow 14x - 10\sqrt{3}\,y = \sqrt{3}$

4. The condition that the line $y = mx + c$ is a tangent to the ellipse is $c^2 = a^2m^2 + b^2$. So equation of any tangent to the ellipse (not parallel to y-axis) can be written as $y = mx \pm \sqrt{a^2m^2+b^2}$

Illustration 6

If $y = mx + 5$ is a tangent to the ellipse $4x^2 + 25y^2 = 100$, then find the value of $100m^2$.

Solution: Equation of the ellipse is

$\dfrac{x^2}{25}+\dfrac{y^2}{4} = 1$

If $y = mx + 5$ touches the ellipse.

Then $(5)^2 = 25m^2 + 4 \Rightarrow 25m^2 = 21$

$\Rightarrow 100m^2 = 84.$

5. An equation of the *chord of contact* of the point (x', y') joining the points of contact of the tangents drawn from (x', y') to the ellipse is
$$\frac{xx'}{a^2}+\frac{yy'}{b^2} = 1$$

Note

- This equation is same as the tangent at (x', y') to the ellipse. When $P(x', y')$ lies on the ellipse the two points of contact coincide with P and the tangent at P and the chord of contact of P are same).

6. An equation of the chord of the ellipse whose mid-point is (x', y') is $T = S'$, where
$$T \equiv \frac{xx'}{a^2}+\frac{yy'}{b^2} - 1 \text{ and } S' \equiv \frac{x'^2}{a^2}+\frac{y'^2}{b^2} = 1.$$

7. An equation of pair of tangents from a point $P(x', y')$ outside the ellipse to the ellipse is $SS' = T^2$, $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$.

Illustration 7

Find the equation of the chord of contact of the point (3, 1) to the ellipse $x^2 + 9y^2 = 9$. Also find the mid-point of this chord of contact.

Solution: Equation of the ellipse is

$$\frac{x^2}{9} + \frac{y^2}{1} = 1$$

Equation of the chord of contact of (3, 1) is

$$\frac{x(3)}{9} + \frac{y(1)}{1} = 1 \Rightarrow x + 3y = 3 \qquad (1)$$

If (h, k) is the mid-point of this chord, then its equation is

$$\frac{hx}{9} + \frac{ky}{1} - 1 = \frac{h^2}{9} + \frac{k^2}{1} - 1 \qquad (2)$$

From (1) and (2) we get

$$\frac{h}{1} = \frac{9k}{3} = \frac{h^2 + 9k^2}{3}$$

$$\Rightarrow h = \frac{3}{2}, k = \frac{1}{2}.$$

So the mid point of the chord of contact is $\left(\frac{3}{2}, \frac{1}{2}\right)$.

8. *Director circle* of the ellipse is the locus of the point of intersection of the tangents to the ellipse which intersect at right angles and its equation is $x^2 + y^2 = a^2 + b^2$.

   So a pair of tangent draw from any point on the director circle to the ellipse are at right angles.

9. *Auxiliary circle* of the ellipse is the circle on the major-axis as a diameter and its equation is $x^2 + y^2 = a^2$. If P is a point on the ellipse and Q is a point on the auxiliary circle such that Q lies on the ordinate produced of the point P. If C is the centre of the ellipse and CA is the semi-major axis of the ellipse then $ACQ = \theta$ is called the *eccentric angle* of the point P and the coordinate of P are $(a \cos \theta, b \sin \theta)$. Also $PQ = (a - b) \sin \theta$.

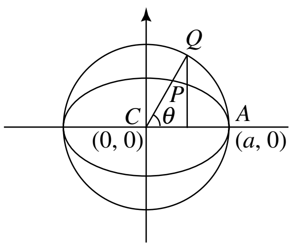

Fig. 19.3

Illustration 8

Find the equation of the ellipse whose auxiliary circle is the director circle of the ellipse $\frac{x^2}{36} + \frac{y^2}{13} = 1$ and the length of a latus rectum is 2 units.

Solution: Equation of the director circle of the ellipse is $x^2 + y^2 = 36 + 13 = 49$.

So if the required equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then auxiliary circle is $x^2 + y^2 = a^2 = 49 \Rightarrow a^2 = 49 \Rightarrow a = 7$.

Also length of a latus rectum $= \frac{2b^2}{a} = 2$

$\Rightarrow b^2 = a = 7$

and the required equation is

$$\frac{x^2}{49} + \frac{y^2}{7} = 1.$$

10. *A diameter* of an ellipse is the locus of the mid points of a system of parallel chords of the ellipse and its equation is

$$y = -\frac{b^2}{a^2 m} x,$$

   where m is the slope of the parallel chords of the ellipse which are bisected by it. This is a line through the centre of the ellipse. Two diameters of an ellipse are said to be *conjugate* when each bisects that chords parallel to the others. Thus two diameters $y = mx$ and $y = m'x$ of the ellipse are conjugate if $mm' = -\frac{b^2}{a^2}$.

Illustration 9

Find the equation of the diameters of the ellipse $\frac{x^2}{36} + \frac{y^2}{16} = 1$ bisecting the chords parallel to the line $y = x - 1$. Also find the equation of the conjugate diameter bisecting the chords parallel of this diameter.

Solution: Slope of the given line is $m = 1$.

So equation of the diameter bisecting the chords with slope 1 is $y = -\frac{16}{36} x \Rightarrow 4x + 9y = 0$. Slope of the conjugate diameter is m' such that $-\frac{16}{36} m' = -\frac{16}{36} \Rightarrow m' = 1$ and the required equation of the conjugate diameter is $y = x$.

11. The tangent and normal at any point of an ellipse bisect the angle between the focal radii to that point.

12. The locus of the feet of the perpendiculars from the foci on any tangent to an ellipse is the auxiliary circle.

13. If W, W' are the feet of the perpendiculars from the foci S and S' respectively on the tangent at any point P of an ellipse with centre C, then CW is parallel to $S'P$ and CW' is parallel to SP.

14. If the normal at any point P of the ellipse $x^2/a^2 + y^2/b^2 = 1$ meet the major and minor axes in G and g respectively and CF is perpendicular from the centre C on the normal then
$PF \,.\, PG = b^2$ and $PF.\, Pg = a^2$

SOLVED EXAMPLES

Concept-based
Straight Objective Type Questions

Example 1: Equation of a directrix of the ellipse $\frac{x^2}{36} + \frac{y^2}{4} = 1$ is

(a) $9x - 8 = 0$ (b) $8x - 9 = 0$
(c) $\sqrt{2}\,x + 9 = 0$ (d) $x + 9\sqrt{2} = 0$

Ans. (c)

Solution: Let $a^2 = 36$, $b^2 = 4$ then if e is the eccentricity of the ellipse

$$e^2 = \frac{36-4}{36} = \frac{32}{36} = \frac{8}{9} \Rightarrow e = \frac{2\sqrt{2}}{3}$$

Equation of the directrices are

$$x = \pm\frac{a}{e} = \pm\frac{6\times 3}{2\sqrt{2}} = \pm\frac{9}{\sqrt{2}}$$

$$\sqrt{2}x \pm 9 = 0$$

Example 2: The locus of a point whose distance from the point (3, 0) is 3/5 times its distance from the line $x = p$ is an ellipse with centre at the origin. The value of p is

(a) 5 (b) 7
(c) $\frac{25}{3}$ (d) $\frac{25}{9}$

Ans. (c)

Solution: Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $e = \frac{3}{5}$ is the eccentricity Focus is $(3, 0) = (ae, 0)$ $\Rightarrow a = 5$

Equation of the directrix is $x = \frac{a}{e} = \frac{5\times 5}{3} = \frac{25}{3}$ which is same as $x = p \Rightarrow p = \frac{25}{3}$.

Example 3: An equation of the ellipse whose length of the major axis is 10 and foci are $(\pm 2, 0)$ is

(a) $\frac{x^2}{25} + \frac{y^2}{21} = 1$ (b) $\frac{x^2}{25} + \frac{y^2}{4} = 1$
(c) $\frac{x^2}{25} + \frac{y^2}{16} = 1$ (d) $\frac{x^2}{29} + \frac{y^2}{25} = 1$

Ans. (a)

Solution: Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

where $2a = 10 \Rightarrow a = 5$

$$ae = 2 \Rightarrow e = \frac{2}{5}$$

$$b^2 = a^2(1 - e^2) = 25\left(1 - \frac{4}{25}\right) = 21$$

Required equation of the ellipse is

$$\frac{x^2}{25} + \frac{y^2}{21} = 1$$

Example 4: If $F_1 = (3, 0)$, $F_2 = (-3, 0)$ and P is any point on the curve $16x^2 + 25y^2 = 400$, then $PF_1 + PF_2$ equals

(a) 8 (b) 6
(c) 10 (d) 12

Ans. (c)

Solution: Equation of the curve is $\frac{x^2}{25} + \frac{y^2}{16} = 1$

Here $a^2 = 25$, $b^2 = 16$, $e^2 = \frac{25-16}{25} = \frac{9}{25}$

Foci are $\left(\pm 5\times\frac{3}{5}, 0\right) = (\pm 3, 0)$

$\Rightarrow$ F_1, F_2 are the foci of the ellipse.

So $PF_1 + PF_2 = e \times$ distance between the directrices

$$= e\times\frac{2a}{e} = 2a = 10$$

Example 5: Equation of a common tangent to the circle $x^2 + y^2 = 16$, parabola $x^2 = y - 4$ and the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 0$ is

(a) $x = 4$ (b) $x = -4$
(c) $y = 4$ (d) $y = 5$

Ans. (c)

Solution: From the figure we find $y = 4$ is a common tangent to the three curves.

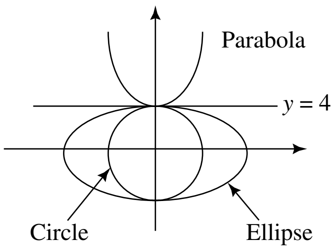

Fig. 19.4

Example 6: If $y = x + c$ is a normal to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, then c^2 is equal to

(a) $\frac{13}{25}$ (b) $\frac{25}{13}$
(c) $\frac{25}{9}$ (d) $\frac{13}{4}$

Ans. (b)

Solution: Any point on the ellipse is $P(3\cos\theta, 2\sin\theta)$ equation of the normal at P is

$3.x \sec\theta - 2.y \operatorname{cosec}\theta = 9 - 4$

Comparing it with $y = x + c$, we get

$$\frac{1}{3\sec\theta} = \frac{1}{2\operatorname{cosec}\theta} = \frac{-c}{5}$$

$\Rightarrow$ $\cos\theta = \frac{-3c}{5}$, $\sin\theta = \frac{-2c}{5}$.

$\Rightarrow$ $9c^2 + 4c^2 = 25$ as $\cos^2\theta + \sin^2\theta = 1$

$\Rightarrow$ $c^2 = \frac{25}{13}$

Example 7: If $F_1(-3, 4)$ and $F_2(2, 5)$ are the foci of an ellipse passing through the origin, then the eccentricity of the ellipse is

(a) $\frac{\sqrt{29}}{5+\sqrt{26}}$ (b) $\frac{\sqrt{21}}{5+\sqrt{29}}$
(c) $\frac{\sqrt{26}}{5+\sqrt{29}}$ (d) $\frac{\sqrt{29}}{5+\sqrt{21}}$

Ans. (c)

Solution: If $2a$ is the length of the major axis and e is the eccentricity of the ellipse

then $F_1F_2 = 2ae$

$\Rightarrow$ $\sqrt{26} = 2ae$ (1)

Also $OF_1 + OF_2 = 2a$, O being the origin.
(As O lies on the ellipse)

$\Rightarrow$ $\sqrt{25} + \sqrt{29} = 2a$ (2)

From (1) and (2)

$$e = \frac{\sqrt{26}}{5+\sqrt{29}}$$

Example 8: E_1: $\frac{x^2}{81} + \frac{y^2}{b^2} = 1$ and E_2: $\frac{x^2}{b^2} + \frac{y^2}{49} = 1$ are two ellipses having the same eccentricity if b^2 is equal to

(a) 32 (b) 63
(c) 65 (d) 64

Ans. (b)

Solution: We have $\frac{81-b^2}{81} = \frac{b^2-49}{b^2}$

$\Rightarrow$ $b^4 = 49 \times 81 \Rightarrow b^2 = 63$.

Example 9: If the normal at any point P on the ellipse $\frac{x^2}{64} + \frac{y^2}{36} = 1$ meets the major axis at G_1 and the minor axis at G_2 then the ratio of PG_1 and PG_2 is equal to

(a) 6 : 8 (b) 8 : 6
(c) 9 : 16 (d) 16 : 9

Ans. (c)

Solution: Equation of the normal at $P(8\cos\theta, 6\sin\theta)$ is $8x\sec\theta - 6y\operatorname{cosec}\theta = 64 - 36 = 28$.

So G_1 is $\left(\frac{28}{8}\cos\theta, 0\right)$ and G_2 is $\left(0, \frac{-28}{6}\sin\theta\right)$

$$(PG_1)^2 = \left(8 - \frac{28}{8}\right)^2 \cos^2\theta + 36\sin^2\theta$$

$$= \frac{36}{64}(36\cos^2\theta + 64\sin^2\theta)$$

$$(PG_2)^2 = 64\cos^2\theta + \left(6 + \frac{28}{6}\right)^2 \sin^2\theta$$

$$= \frac{64}{36}(36\cos^2\theta + 64\sin^2\theta)$$

$$\left(\frac{PG_1}{PG_2}\right)^2 = \frac{36}{64} \times \frac{36}{64} \Rightarrow \frac{PG_1}{PG_2} = \frac{36}{64} = \frac{9}{16}.$$

Example 10: The tangent at the point $P\left(\frac{4}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ to the ellipse $9x^2 + 16y^2 = 144$ meets the axis of x at A and the axis of y at B. If C is the centre of the ellipse, then area of the $\Delta\, ABC$ is (in sq. units)

(a) 12 (b) 16
(c) 9 (d) 24

Ans. (a)

Solution: Ellipse is $\frac{x^2}{16} + \frac{y^2}{9} = 1.$

Equation of the tangent at $P\left(\frac{4}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ is

$$\frac{x \cdot 4}{16\sqrt{2}} + \frac{y \cdot 3}{9\sqrt{2}} = 1$$

coordinate of A are $(4\sqrt{2}, 0)$

and of B are $(0, 3\sqrt{2})$

C is (0, 0)

Area of the triangle $ABC = \frac{1}{2} \times 4\sqrt{2} \times 3\sqrt{2} = 12$ sq. units

Example 11: A point on the ellipse $4x^2 + 9y^2 = 36$. Where the normal is parallel to the line $4x - 2y - 5 = 0$ is:

(a) $\left(\frac{9}{5}, \frac{8}{5}\right)$ (b) $\left(\frac{8}{5}, -\frac{9}{5}\right)$
(c) $\left(-\frac{9}{5}, \frac{8}{5}\right)$ (d) $\left(\frac{8}{5}, \frac{9}{5}\right)$

Ans. (a)

Solution: Slope of the normal is 2, so slope of the tangent is $-\frac{1}{2}$.

Equation of the tangent is $y = -\frac{1}{2}x \pm \sqrt{9 \times \frac{1}{4} + 4}$

$$\Rightarrow \quad x + 2y = \pm 5 \qquad (1)$$

If it touches at (x_1, y_1), then equation of the tangent at (x_1, y_1) to the ellipse is

$$\frac{xx_1}{9} + \frac{yy_1}{4} = 1 \qquad (2)$$

Comparing (1) and (2) we get

$$\frac{x_1}{9} = \frac{y_1}{8} = \pm\frac{1}{5}$$

$$\Rightarrow \quad (x_1, y_1) = \left(\frac{9}{5}, \frac{8}{5}\right)$$

Example 12: The equation of the parabola whose vertex is at the centre of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and the focus coincide with the focus of the ellipse on the positive side of the major axis of the ellipse is

(a) $y^2 = 3x$ (b) $y^2 = 4x$
(c) $y^2 = 5x$ (d) $y^2 = 12\,x$

Ans. (d)

Solution: Centre of the ellipse is the origin and focus is $(ae, 0) = \left(\sqrt{a^2 - b^2}, 0\right) = (3, 0)$. Required equation of the parabola is

$$y^2 = 4 \times 3x \Rightarrow y^2 = 12x.$$

Example 13: Consider the ellipses E_1: $\frac{x^2}{9} + \frac{y^2}{5} = 1$ and E_2: $\frac{x^2}{5} + \frac{y^2}{9} = 1$. Both the ellipses have

(a) the same foci (b) same major axis
(c) the same minor axis (d) the same eccentricity

Ans. (d)

Solution: Major axis and minor axis of E_1 are respectively the minor axis and major axis of E_2

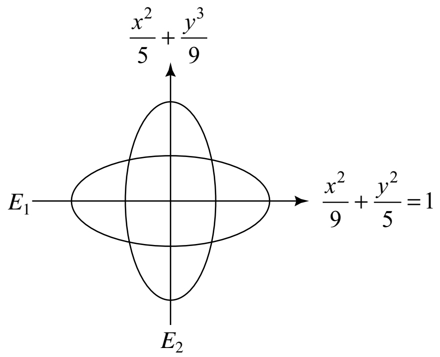

Fig. 19.5

Foci of E_1 are $(\pm 2, 0)$
Foci of E_2 are $(0, \pm 2)$

Eccentricity of both $= \sqrt{\frac{9-5}{9}} = \frac{2}{3}$

Example 14: If the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is 4π, then the maximum area of a rectangle inscribed in the ellipse in sq. units is

(a) 4 (b) 8
(c) 16 (d) 32

Ans. (b)

Solution: Area of the ellipse

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{b}{a}\left|\int_0^a 4\sqrt{a^2 - x^2}\,dx\right|$$

$$= \frac{b}{a}\ \text{(Area of the circle of radius } a\text{)}$$

$$= \frac{b}{a} \times \pi a^2 = \pi ab = 4\pi\ \text{(given)}$$

$\Rightarrow ab = 4$

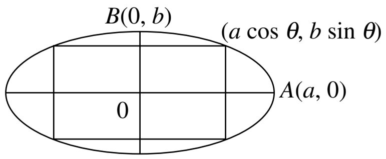

Fig. 19.6

Area of a rectangle inscribed in the ellipse is $4a \cos \theta \times b \sin \theta = 2ab \sin 2\theta$ which is maximum when $\sin 2\theta = 1$

Hence the maximum area of the rectangle is $2ab = 8$ (sq. units)

Example 15: Equation of a circle described on the Latus rectum of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ as a diameter with centre on the + ve x-axis is

(a) $25x^2 + 25y^2 - 150y - 31 = 0$
(b) $25x^2 + 25y^2 - 150x + 31 = 0$
(c) $25x^2 + 25y^2 - 150x - 31 = 0$
(d) $25x^2 + 25y^2 - 150y + 31 = 0$

Ans. (c)

Solution: Coordinates of the centre which is the focus of the ellipse on the +ve x-axis is $(\sqrt{25-16}, 0)$ i.e (3, 0) and radius of the circle is equal to the length of the semi-latus rectum i.e. $\frac{16}{5}$. So the required equation of the circle is

$$(x - 3)^2 + y^2 = \left(\frac{16}{5}\right)^2$$

or $25x^2 + 25y^2 - 150x - 31 = 0$

LEVEL 1

Straight Objective Type Questions

Example 16: The eccentricity of an ellipse with its centre at the origin is ½. If one of the directrix is $x = 4$, then the equation of the ellipse is

(a) $4x^2 + 3y^2 = 12$ (b) $3x^2 + 4y^2 = 12$
(c) $3x^2 + 4y^2 = 1$ (d) $4x^2 + 3y^2 = 1$

Ans. (b)

Solution: Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ then } \frac{a^2 - b^2}{a^2} = \frac{1}{4} = e^2$$

and directrix $x = a/e = 4 \Rightarrow a = 2, b^2 = 3$
and the ellipse is $x^2/4 + y^2/3 = 1$

Example 17: In an ellipse, the distance between the foci is 6 and minor axis is 8, then the eccentricity is

(a) $1/\sqrt{5}$ (b) 3/5
(c) 1/2 (d) 4/5

Ans. (b)

Solution: Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $b = 4$.

If e is the eccentricity then $2ae = 6$
$\Rightarrow a^2e^2 = 9$
$\Rightarrow a^2 - b^2 = 9 \Rightarrow a^2 = 25$
$\Rightarrow a = 5$ and $e = 3/5$

Example 18: The locus of the point of intersection of the tangents to the ellipse $x^2/a^2 + y^2/b^2 = 1$ which are at right angles is

(a) a circle (b) a parabola
(c) an ellipse (d) a hyperbola

Ans. (a)

Solution: Equation of the tangent to the given ellipse with slope m is

$$y = mx + \sqrt{a^2m^2 + b^2} \qquad (1)$$

and the equation of tangent perpendicular to (1) is

$$my + x = \sqrt{a^2 + b^2m^2} \qquad (2)$$

Squaring and adding (1) and (2) to eliminate m, we get

$$(y - mx)^2 + (my + x)^2 = a^2m^2 + b^2 + a^2 + b^2m^2$$

$\Rightarrow$ $(x^2 + y^2)(1 + m^2) = (a^2 + b^2)(1 + m^2)$

$\Rightarrow$ $x^2 + y^2 = a^2 + b^2$

which is a circle.

Note

- The circle $x^2 + y^2 = a^2 + b^2$ is called the *Director Circle* of the ellipse $x^2/a^2 + y^2/b^2 = 1$, whose centre is at the centre of the ellipse and whose radius is equal to the length of the line joining the end points of the major axis minor axis of the ellipse.

Example 19: The normal at an end of a latus rectum of the ellipse $x^2/a^2 + y^2/b^2 = 1$ passes through an end of the minor axis if

(a) $e^4 + e^2 = 1$ (b) $e^3 + e^2 = 1$
(c) $e^2 + e = 1$ (d) $e^3 + e = 1$

Ans. (a)

Solution: Let an end of a latus rectum be $\left(ae, b\sqrt{1-e^2}\right)$, then the equation of the normal at this end is

$$\frac{x-ae}{ae/a^2} = \frac{y-b\sqrt{1-e^2}}{b\sqrt{1-e^2}/b^2}$$

It will pass through the end $(0, -b)$ if

$$-a^2 = \frac{-b^2\left(1+\sqrt{1-e^2}\right)}{\sqrt{1-e^2}} \text{ or } \frac{b^2}{a^2} = \frac{\sqrt{1-e^2}}{1+\sqrt{1-e^2}}$$

or $(1 - e^2)\left[1+\sqrt{1-e^2}\right] = \sqrt{1-e^2}$

or $\sqrt{1-e^2} + 1 - e^2 = 1$ or $e^4 + e^2 = 1$.

Example 20: The locus of the middle points of the portions of the tangents of the ellipse $x^2/a^2 + y^2/b^2 = 1$ included between the axis is the curve.

(a) $x^2/a^2 + y^2/b^2 = 4$ (b) $a^2/x^2 + b^2/y^2 = 4$
(c) $a^2 x^2 + b^2 y^2 = 4$ (d) $b^2 x^2 + a^2 y^2 = 4$

Ans. (b)

Solution: Equation of a tangent to the ellipse can be written as $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ which meets the axes at A $(a/\cos\theta, 0)$ and B $(0, b/\sin\theta)$. If (h, k) is the middle point of AB, then

$$h = a/2\cos\theta,\ k = b/2\sin\theta$$

Eliminating θ we get $(a/2h)^2 + (b/2k)^2 = 1$

$\Rightarrow$ locus of P (h, k) is $a^2/x^2 + b^2/y^2 = 4$.

Example 21: In a model, it is shown that an arc of a bridge in semi-elliptical with major axis horizontal. If the length of the base is 9 m and the highest part of the bridge is 3 m from the horizontal; the best approximation of the height of the arch at 2 m from the centre of the base is

(a) 11/4 *m* (b) 8/3 *m*
(c) 7/2 *m* (d) 2 *m*

Ans. (b)

Solution: Let the equation of the semi elliptical arc be $x^2/a^2 + y^2/b^2 = 1$ $(y > 0)$.
Length of the major axis $= 2a = 9 \Rightarrow a = 9/2$
Length of the semi minor axis $= b = 3$.

So the equation of the arc becomes $\frac{4x^2}{81} + \frac{y^2}{9} = 1$

If $x = 2$ then $y^2 = \frac{65}{9} \Rightarrow y = \frac{1}{3}\sqrt{65} = \frac{8}{3}$ approximately.

Example 22: The locus of the foot of the perpendicular drawn from the centre to any tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is

(a) a circle (b) an ellipse
(c) a hyperbola (d) none of these

Ans. (d)

Solution: Equation of any tangent to the ellipse is

$$y = mx + \sqrt{a^2m^2 + b^2} \quad (1)$$

Equation of the line through the centre (0, 0) perpendicular to (1) is

$$y = (-1/m)\,x \quad (2)$$

Eliminating m from (1) and (2) we get the required locus of the foot of the perpendicular

as $$y = -\frac{x^2}{y} + \sqrt{a^2\frac{x^2}{y^2} + b^2}$$

or $$(x^2 + y^2)^2 = a^2x^2 + b^2y^2.$$

which does not represent a circle, an ellipses or a hyperbola.

Example 23: Sum of the focal distance of any point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ is equal to the length of the

(a) major axis (b) minor axis
(c) latus rectum (d) none of these

Ans. (a)

Solution: If P be any point on the ellipse, then distance of P from a focus is e times its distance from the corresponding directrix (by definition of ellipse). So that the sum of the focal distance is equal to e times the distance between the two directrices of the ellipse, $x = \pm a/e$. Hence the required distance $= e \times 2a/e = 2a$ = the length of the major axis of the ellipse.

Example 24: The line passing through the extremity A of major axis and extremity B of the minor axes of the ellipse $9x^2 + 16y^2 = 144$ meets the circle $x^2 + y^2 = 16$ at the point P. Then the area of the triangle OAP, O being the origin (in square units) is

(a) 96/25 (b) 192/25
(c) 48/25 (d) 96/50

Ans. (b)

◎ **Solution:** Coordinates of A are (4, 0) and of B are (0, 3). So the equation of AB is $\frac{x}{4}+\frac{y}{3}=1$. Which meets the circle $x^2+y^2=16$ at points whose y coordinate is given by

$$y^2+\left(\frac{12-4y}{3}\right)^2=16$$

$\Rightarrow$ $y=0$ or $y=96/25$

$y=0$ corresponds to the point A.

So the y-coordinate of P is 96/25

Area of the triangle OAP

$= (1/2)\, OA \times (y\text{-coordinate of } P)$

$= (1/2) \times 4 \times (96/25)$

$= 192/25.$

Example 25: The ellipse $x^2+4y^2=4$ is inscribed in a rectangle aligned with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point (4, 0). Then equation of the ellipse is

(a) $4x^2+48y^2=48$ (b) $4x^2+6y^2=48$

(c) $x^2+16y^2=16$ (d) $x^2+12y^2=16$

Ans. (d)

◎ **Solution:** Let the equation of the required ellipse by $x^2/a^2+y^2/b^2=1$

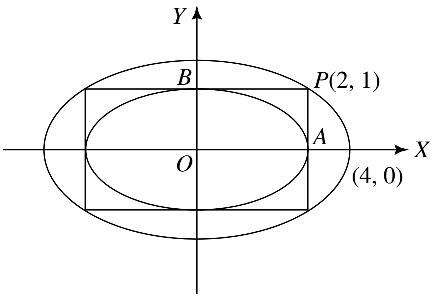

Fig. 19.7

It passes through (4, 0) and (2, 1) as $OB=1$ and $OA=2$ are respectively the lengths of the semi-minor axis and semi-major axis of the given ellipse, coordinates of P are (2, 1)

So, $\frac{16}{a^2}=1$ and $\frac{4}{a^2}+\frac{1}{b^2}=1$

$\Rightarrow$ $a^2=16$ and $b^2=4/3$.

and the required equation is

$$\frac{x^2}{16}+\frac{3y^2}{4}=1$$

$\Rightarrow$ $x^2+12y^2=16.$

Example 26: A parabola has its latus rectum along PQ, where $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1>0$, $y_2>0$ are the end points of the latus rectums of the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$. Coordinates of the focus of the parabola are

(a) (0, – 1/2) (b) (0, 0)

(c) (0, 1/2) (d) (0, 1)

Ans. (c)

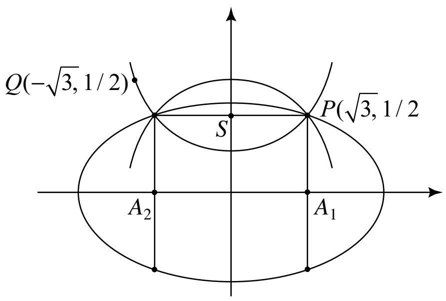

Fig. 19.8

◎ **Solution:** Eccentricity of the ellipse is

$$e^2=\frac{a^2-b^2}{a^2}=\frac{4-1}{4}=\frac{3}{4}$$

$\Rightarrow$ $e=\sqrt{3}/2$

$\Rightarrow$ coordinates of the foci of the ellipse are

$(\pm 2\times\sqrt{3}/2, 0) = (\pm\sqrt{3}, 0)$

Length of the latus rectum of the ellipse is $\frac{2\times 1}{2}=1$. So the coordinates of P are $(\sqrt{3}, 1/2)$ and of Q are $(-\sqrt{3}, 1/2)$. Focus of the parabola is the mid point of PQ i.e., (0, 1/2)

Note

❑ There are two such parabola.

Example 27: Tangents are drawn from the point $P(3, 4)$ to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ touching the ellipse at point A and B. The equation of the locus of the point whose distances from the point P and the line AB are equal is

(a) $9x^2+y^2-6xy-54x-62y+241=0$

(b) $x^2+9y^2+6xy-54x+62y-241=0$

(c) $9x^2+9y^2-6xy-54x-62y-241=0$

(d) $x^2+y^2-2xy+27x+31y-120=0$

Ans. (a)

◎ **Solution:** AB being the chord of contact of the ellipse from $P(3, 4)$ has its equation.

$$\frac{3x}{9}+\frac{4y}{4}=1 \Rightarrow x+3y=3$$

If (h, k) is any point on the locus, then

$$\sqrt{(h-3)^2+(k-4)^2}=\left|\frac{h+3k-3}{\sqrt{1+9}}\right|$$

$\Rightarrow \qquad 10(h^2 + k^2 - 6h - 8k + 25) = (h + 3k - 3)^2$
Locus of (h, k) is
$9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

Example 28: Equation of the ellipse whose axes are the axes of coordinates, which passes through the point (– 3, 1) and has eccentricity $\sqrt{2/5}$ is

(a) $5x^2 + 3y^2 - 32 = 0$ (b) $3x^2 + 5y^2 - 32 = 0$
(c) $5x^2 + 3y^2 - 48 = 0$ (d) $3x^2 + 5y^2 - 15 = 0$

Ans. (b)

Solution: Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We have $b^2 = a^2(1 - e^2) = a^2(1 - 2/5) = 3a^2/5$
Since it passes through the point (– 3, 1)

$$\frac{9}{a^2} + \frac{1}{b^2} = 1 \Rightarrow \frac{9}{a^2} + \frac{5}{3a^2} = 1$$

$$\Rightarrow \qquad a^2 = 9 + 5/3 = 32/3. \Rightarrow b^2 = 32/5$$

So the required equation is

$$\frac{x^2}{32/3} + \frac{y^2}{32/5} = 1 \text{ or } 3x^2 + 5y^2 = 32$$

Example 29: An ellipse is drawn by taking a diameter of the circle $(x - 1)^2 + y^2 = 1$ as its semi-minor axis and a diameter of circle $x^2 + (y - 2)^2 = 4$ as its semi-major axis. If the centre of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is

(a) $4x^2 + y^2 = 8$ (b) $x^2 + 4y^2 = 16$
(c) $4x^2 + y^2 = 4$ (d) $x^2 + 4y^2 = 8$

Ans. (b)

Solution: Diameter of $(x - 1)^2 + y^2 = 1$ is $2 \times 1 = 2 = b$ and of $x^2 + (y - 2)^2 = 4$ is $2 \times 2 = 4 = a$
So the required equation of the ellipse is

$$\frac{x^2}{(4)^2} + \frac{y^2}{(2)^2} = 1 \Rightarrow x^2 + 4y^2 = 16$$

Example 30: The ellipse E_1: $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0, 4) circumscribes the rectangle R. The eccentricity of the ellipse E_2 is

(a) $\sqrt{2}/2$ (b) $\sqrt{3}/2$
(c) 1/2 (d) 3/4

Ans. (c)

Solution: Let the required ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ since it passes through (0, 4) $\Rightarrow b^2 = 16$.

It also passes through the point (3, 2)

$$\Rightarrow \frac{9}{a^2} + \frac{4}{16} = 1$$

$\Rightarrow a^2 = 12$

Equation of the ellipse is $\frac{x^2}{12} + \frac{y^2}{16} = 1$

So eccentricity of the ellipse is $\sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$

Note Fig ia Similar to the Fig of Example 25.

Example 31: A rod of length 12 cm moves with its ends always touching the coordinate axes. The locus of a point P on the rod, which is 3 cm from the end in contact with x-axis is an ellipse whose eccentricity is:

(a) $\frac{2}{3}$ (b) $\frac{\sqrt{2}}{3}$
(c) $\frac{2\sqrt{2}}{3}$ (d) $\frac{2\sqrt{3}}{2}$

Ans. (c)

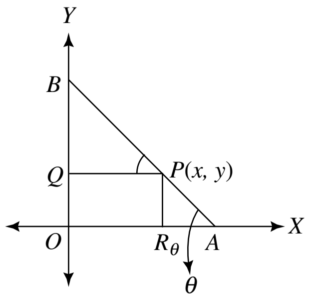

Fig. 19.9

Solution: Let AB be the rod making an angle θ with x-axis and $P(x, y)$ be any point on the rod such that $AP = 3$ cm.
$\Rightarrow PB = AB - AP = 12 - 3 = 9$ cm.

$$\cos\theta = \frac{PQ}{PB} = \frac{x}{9}$$

$$\sin\theta = \frac{PR}{PA} = \frac{y}{3}$$

so eliminating θ we get the required locus as $\frac{x^2}{81} + \frac{y^2}{9} = 1$

which is an ellipse with eccentricity $\sqrt{\frac{81 - 9}{81}} = \frac{2\sqrt{2}}{3}$.

Example 32: Eccentricity of the ellipse whose latus rectum is half of its major axis is:

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$
(c) $\frac{2}{3}$ (d) $\frac{1}{3}$

Ans. (b)

Solution: Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Length of the latus rectum $= \frac{2b^2}{a} = a$, half of major axis

$\Rightarrow \quad 2b^2 = a^2 \Rightarrow e^2 = \frac{a^2-b^2}{a^2} = \frac{1}{2}$

$\Rightarrow \quad e = \frac{1}{\sqrt{2}}$.

Example 33: If the equation $5[(x-2)^2 + cy - 3)^2] = (\lambda^2 - 2\lambda + 1)(2x + y - 1)^2$ represents an ellipse then

(a) $\lambda \in (0, 2)$ (b) $\lambda \in (-1, 1)$
(c) $\lambda \in (0, 2) - \{1\}$ (d) $\lambda \in (-1, 1) - \{0\}$

Ans. (c)

Solution: Given equation represents an ellipse with eccentricity $\sqrt{\lambda^2 - 2\lambda + 1} = e$

Since $0 < e < 1$

$\Rightarrow \quad 0 < |\lambda - 1| < 1$

$\Rightarrow \quad \lambda \in (0, 2) \sim \{1\}$

Example 34: If the line $lx + my + n = 0$ cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ at points whose eccentric angles differ by $\frac{\pi}{2}$, then the value of $\frac{a^2l^2+b^2m^2}{n^2}$ is equal to

(a) 1 (b) 2
(c) $\frac{1}{2}$ (d) $\frac{3}{2}$.

Ans. (b)

Solution: Let the line meet the ellipse at $P(a\cos\theta, b\sin\theta)$ and $Q\left(a\cos\left(\frac{\pi}{2}+\theta\right), b\sin\left(\frac{\pi}{2}+\theta\right)\right)$. Since they lie on the line $lx + my + n = 0$

$la\cos\theta + m\,b\sin\theta + n = 0$

and $-l\,a\sin\theta + m\,b\cos\theta + n = 0$

Eliminating θ by squaring and adding we get

$a^2l^2 + b^2m^2 = 2n^2$

$\Rightarrow \frac{a^2l^2+b^2m^2}{n^2} = 2.$

Example 35: An ellipse has OB as semi minor axis F and F' its foci and the angle FBF' is a right angle. The eccentricity of the ellipse is

(a) $\frac{1}{4}$ (b) $\frac{1}{\sqrt{3}}$
(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{2}$

Ans. (c)

Solution: Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$F(ae, 0)$, $F'(-ae, 0)$

BF is perpendicular to BF'

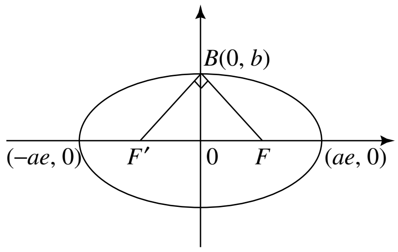

Fig. 19.10

$\Rightarrow \frac{b}{-ae}\times\frac{b}{ae} = -1 \Rightarrow b^2 = a^2e^2$

$\Rightarrow a^2(1-e^2) = a^2e^2 \Rightarrow 2e^2 = 1 \Rightarrow e = \frac{1}{\sqrt{2}}$

Example 36: The locus of the foot of the perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent is

(a) $(x^2 - y^2)^2 = 6x^2 + 2y^2$
(b) $(x^2 - y^2)^2 = 6x^2 - 2y^2$
(c) $(x^2 + y^2)^2 = 6x^2 + 2y^2$
(d) $(x^2 + y^2)^2 = 6x^2 - 2y^2$

Ans. (c)

Solution: Equation of any tangent to the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$ is $y = mx \pm \sqrt{6m^2+2}$. If (h, k) is the foot of the perpendicular from (0, 0), the centre of the ellipse on this tangent then $k = mh \pm \sqrt{6m^2+2}$ and $\frac{k}{h}\times m = -1$

Eliminating m, we get $(h^2 + k^2)^2 = 6h^2 + 2k^2$

So, required locus is $(x^2 + y^2)^2 = 6x^2 + 2y^2$

Example 37: The area (in sq. units) of the quadrilateral formed by the tangents at the end points of the latera recta to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ is:

(a) $\frac{27}{4}$ (b) 18
(c) $\frac{27}{2}$ (d) 27

Ans. (d)

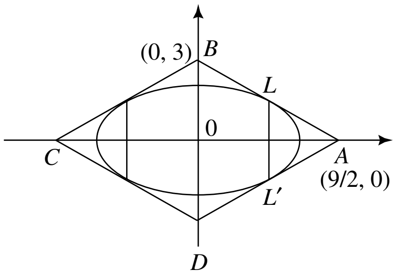

Fig. 19.11

◎ **Solution:** we have $a^2 = 9$, $b^2 = 5$

$$e^2 = 1 - \frac{5}{9} = \frac{4}{9}$$

$$\Rightarrow \qquad e = \frac{2}{3}$$

Coordinates of L are $\left(ae, \frac{b^2}{a}\right) = \left(2, \frac{5}{3}\right)$

Equation of the tangent at L is

$$\frac{2x}{9} + \frac{5}{3}\left(\frac{y}{5}\right) = 1 \Rightarrow 2x + 3y = 9.$$

It meets the axes at $A\left(\frac{9}{2}, 0\right)$ and $B(0, 3)$

From symmetry, we observe that the quadrilateral is a rhombus whose area is $4 \times$ area of the $\Delta\, AOB$.

Hence the required area is $4 \times \frac{9}{2} \times 3 \times \frac{1}{2}$

$= 27$ sq. units.

◉ **Example 38:** If the distance between the foci of an ellipse is half the length of its latus rectum, then eccentricity of the ellipse is:

(a) $\frac{1}{2}$ (b) $\frac{2\sqrt{2}-1}{2}$

(c) $\sqrt{2}-1$ (d) $\frac{\sqrt{2}-1}{2}$

Ans. (c)

◎ **Solution:** Let the equation of the ellipse be

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then

$$2ae = \frac{1}{2}\left(\frac{2b^2}{a}\right)$$

$$\Rightarrow \qquad b^2 = 2a^2e$$

$$\Rightarrow \qquad a^2(1 - e^2) = 2a^2e$$

$$\Rightarrow \qquad e^2 + 2e = 1 \Rightarrow (e + 1)^2 = 2$$

$$\Rightarrow \qquad e = \sqrt{2} - 1.$$

◉ **Example 39:** The tangent to the ellipse $3x^2 + 16y^2 = 12$, at the point $\left(1, \frac{3}{4}\right)$ intersects the curves $y^2 + x = 0$ at:

(a) no point (b) exactly one point

(c) two distinct point (d) more than two points.

Ans. (b)

◎ **Solution:** Tangent at $\left(1, \frac{3}{4}\right)$ to the ellipse is $\frac{x \cdot 1}{4} + \frac{y\left(\frac{3}{4}\right)}{\frac{3}{4}}$

$= 1 \Rightarrow x + 4y = 4$

which intersects the curve $y^2 + x = 0$ at points for which $y^2 + (4 - 4y) = 0$

$\Rightarrow (y - 2)^2 = 0 \Rightarrow y = 2$

and the point of intersection is $(-4, 2)$ i.e. exactly one point.

◉ **Example 40:** The line $2x + y = 3$ intersects the ellipse $4x^2 + y^2 = 5$ at two points. The tangents to the ellipse at these two points intersect at the point:

(a) $\left(\frac{5}{6}, \frac{5}{3}\right)$ (b) $\left(\frac{5}{6}, \frac{5}{6}\right)$

(c) $\left(\frac{5}{3}, \frac{5}{6}\right)$ (d) $\left(\frac{5}{3}, \frac{5}{3}\right)$

Ans. (a)

◎ **Solution:** Let the line intersect the ellipse at P and Q and the tangents at P and Q to the ellipse intersect at $R(h, k)$, then PQ is the chord of contact of R w.r.t. the ellipse; so its equation is $4hk + ky = 5$

But the equation of the chord of contact is given as $2x + 3y = 3$

$$\Rightarrow \qquad \frac{4h}{2} = \frac{k}{1} = \frac{5}{3}$$

$$\Rightarrow \qquad h = \frac{5}{6},\ k = \frac{5}{3}$$

and the required point is $\left(\frac{5}{6}, \frac{5}{3}\right)$

Assertion-Reason Type Questions

Example 41: $E: 2x^2 + 3y^2 = 6$, $e: x^2 + y^2 = 2x + 4y + 4 = 0$

Statement 1: The area of the ellipse E is more than the area of the circle C.

Statement 2: The length of the semi-major axis of E is more than the radius of C.
Ans. (b)

Solution: Equation of the ellipse is $\frac{x^2}{3} + \frac{y^2}{2} = 1$

whose area is $\pi\sqrt{3}\sqrt{2} = \pi\sqrt{6}$. sq. units.

Equation of the circle is
$(x - 1)^2 + (y + 2)^2 = 1$
whose area is π sq. units
So area of $E >$ area of C and thus Statement 1 is true.

Length of the semi major axis of E is $\sqrt{3} > 1$, the radius of C so statement 2 is also true but does not justify statement 1.

Example 42: Statement 1: The line $2x + 3y = 1$ intersects the ellipse $\frac{x^2}{4} + \frac{y^2}{2} = 1$ at P and Q. The tangents to the ellipse at P and Q. intersect at the point (8, 6).

Statement 2: Equation of the chord of contact to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ from an external point (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.
Ans. (a)

Solution: Statement 2 is true (see theory).
In statement 1, let the coordinates of the required point be (h, k), then $2x + 3y = 1$ is the chord of contact of the ellipse and by statement 2, its equation is $\frac{hx}{4} + \frac{ky}{2} = 1$. Comparing the two equations, we get $h = 8$, $k = 6$ and the statement 1 is also true.

Example 43: Statement 1: The locus of the centre of a variable circle touching two circles C_1: $(x - 2)^2 + (y - 1)^2 = 16$ and C_2: $(x - 4)^2 + (y - 1)^2 = 1$ is an ellipse.

Statement 2: If a circle $S_2 = 0$ lies completely inside the circle $S_1 = 0$, then the locus of the centre of a variable circle touching both the circles is an ellipse.

Ans. (a)

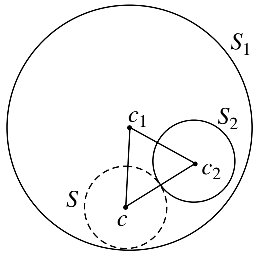

Fig. 19.12

Solution: Let C_1, C_2 be the centres and r_1, r_2 the radii of the circles S_1 and S_2 respectively. C be the centre and r the radius of the circles touching S_1 and S_2.
Then $CC_1 = r_1 - r$ and $CC_2 = r_2 + r$
$\Rightarrow CC_1 + CC_2 = r_1 + r_2$ (constant)
So distance of C from two fixed points C_1 and C is constant and hence by definition of the ellipse locus of C is an ellipse.
In statement 1, the circle C_2 lies inside the circle C_1 and hence by statement 2, the required locus is an ellipse and thus the statement 1 is also true.

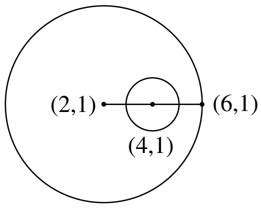

Fig. 19.13

Example 44: Statement 1: If the line $\frac{x}{a} + \frac{y}{b} = \sqrt{2}$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and θ is the eccentric angle of the point of contact, then $\sin^2\theta = \frac{1}{2}$.

Statement 2: If the length of the semi major axis of an ellipse is $\sqrt{2}$ times the length of the semi-minor axis, then the eccentricity e of the ellipse satisfies $2e^2 - 1 = 0$.
Ans. (b)

Solution: Let the coordinates of the point of contact in statement 1 be $(a \cos \theta, b \sin \theta)$. Equation of the tangent at this point to the ellipse is

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

Comparing with the given line

$\cos\theta = \sin\theta = \dfrac{1}{\sqrt{2}} \Rightarrow \sin^2\theta = \dfrac{1}{2}$

and the statement 1 is true.

In statement 2, let the equation of the ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ s.t $a=\sqrt{2}\ b$

$\Rightarrow a^2 = 2b^2$ and $e^2 = \dfrac{a^2-b^2}{a^2} = \dfrac{1}{2}$ thus statement 2 is also true but does not justify statement 1.

Example 45: **Statement 1:** If L and M are the feet of the perpendiculars draw from the foci of the ellipse $\dfrac{x^2}{100}+\dfrac{y^2}{50}=1$ on any tangent to it, then L and M lie on the circle $x^2+y^2=150$.

Statement 2: The locus of feet of the perpendiculars from the foci upon any tangent is the auxiliary circle of the ellipse.
Ans. (d)

Solution: Let the equation of the ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Equation of any tangent is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta = 1 \qquad (1)$$

Equation of the perpendiculars from $(\pm ae, 0)$ to the tangent (1) is

$$\frac{x\sin\theta}{b}-\frac{y\cos\theta}{a} = \pm\frac{ae\sin\theta}{b} \qquad (2)$$

Eliminating θ from (1) and (2) we get the required locus by squaring and adding

$$x^2\left[\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right]+y^2\left[\frac{\sin^2\theta}{b^2}+\frac{\cos^2\theta}{a^2}\right]=1+\frac{a^2e^2\sin^2\theta}{b^2}$$

$$\Rightarrow \quad (x^2+y^2)\left(\frac{b^2\cos^2\theta+a^2\sin^2\theta}{a^2b^2}\right)$$

$$=\left[\frac{b^2+(a^2-b^2)\sin^2\theta}{a^2b^2}\right]a^2$$

$$=\left[\frac{b^2\cos^2\theta+a^2\sin^2\theta}{a^2b^2}\right]a^2$$

$$\Rightarrow \quad x^2+y^2=a^2$$

which is the auxiliary circle of the ellipse. Hence the statement 2 is true. Using it, we find statement 1 is false.

(Note $x^2+y^2=150$ is the director circle of the ellipse $\dfrac{x^2}{100}+\dfrac{y^2}{50}=1$)

Example 46: The tangent at a point $P(a\cos\alpha, b\sin\alpha)$ to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ meets the auxiliary circle at A and B such that $\underline{|AOB} = \dfrac{\pi}{2}$, O being the origin.

Statement 1: Equation of the directrices of the ellipse are $x = \pm\, a\sqrt{1+\sin^2\alpha}$.

Statement 2: Eccentricity e of the ellipse satisfies $e^2\sin^2\alpha + e^2 - 1 = 0$.
Ans. (a)

Solution: Equation of the tangent at P to the ellipse is

$$\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha = 1 \qquad (1)$$

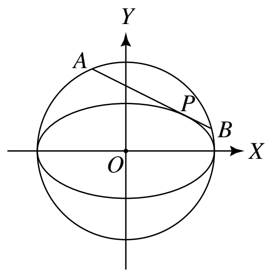

Fig. 19.14

Equation of the auxiliary circle is $x^2+y^2=a^2$ (2)

Equation of the pair of lines OA and OB is $x^2+y^2=a^2\left(\dfrac{x}{a}\cos\alpha+\dfrac{y}{b}\sin\alpha\right)^2$

(Making (2) homogeneous with the help of (1))

Since $OA \perp OB$

$$(1-\cos^2\alpha)+\left(1-\frac{a^2}{b^2}\sin^2\alpha\right)=0$$

$\Rightarrow b^2 = (a^2-b^2)\sin^2\alpha \Rightarrow 1-e^2 = e^2\sin^2\alpha$
$\Rightarrow e^2\sin^2\alpha + e^2 - 1 = 0$
So statement 2 is true.

Using it we find the directrices of the ellipse are $x = \pm\dfrac{a}{e} = \pm a\sqrt{1+\sin^2\alpha}$ and the statement 1 is also true.

Example 47: The tangent and normal at a point $P(x_1, y_1)$, $x_1>0$, $y_1>0$ on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ meets the major axis at T and N respectively.

Statement 1: *PN* is the internal bisector of $\angle F_1PF_2$; F_1, F_2 being the foci of the ellipse.

Statement 2: *PT* is constant for all positions of the point *P*.

Ans. (c)

◎ **Solution:** Equation of the tangent at $P(x_1, y_1)$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = 1$ which meets the major axis at $\left(\frac{a^2}{x_1}, 0\right)$

So $$(PT)^2 = \left(x_1 - \frac{a^2}{x_1}\right)^2 + y_1^2$$

$$= \frac{(x_1^2 - a^2)^2}{x_1^2} + y_1^2$$

Which shows that *PT* depends on the position of *P* and thus the statement 2 is false.

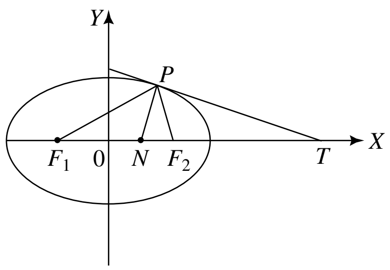

Fig. 19.15

For statement 1, equation of the normal at $P(x_1, y_1)$ is

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2 = a^2e^2$$

So coordinates of *N* are $(e^2x_1, 0)$

F_1 are $(-ae, 0)$, F_2 are $(ae, 0)$

Also $$PF_1 = e\left|\frac{a}{e} + x_1\right|$$
$$= a + ex_1$$
$$PF_2 = a - ex_1$$
$$NF_1 = e^2x_1 + ae = ePF_1$$
$$NF_2 = ae - e^2x_1 = e(PF_2)$$

$\Rightarrow \frac{PF_1}{PF_2} = \frac{NF_1}{NF_2}$ which shows that *PN* bisects the $\angle F_1PF_2$ and the statement 1 is true.

◉ **Example 48: Statement 1:** Equations of the tangents drawn at the ends of the major axis of the ellipse $9x^2 + 5y^2 - 30y = 0$ is $y = 0$, $y = 6$.

Statement 2. The tangents drawn at the ends of the major axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ are always parallel to the *y*-axis.

Ans. (c)

◎ **Solution:** Equation of the ellipse in statement 1 is: $\frac{x^2}{5}+\frac{(y-3)^2}{9} = 1$ whose centre is (0, 3) and the ends of the major axis are (0, 0) and (6, 0) as the length of semi major axis is 3.

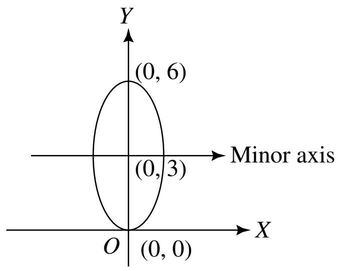

Fig. 19.16

The tangents at these ends are $y = 6$ and $y = 0$.

Statement 1: is true.

Statement 2: is false as the tangents at the ends of the major axis are parallel to *y*-axis only when $a > b$.

◉ **Example 49: Statement 1:** If the length of the latus rectum of an ellipse is $\frac{1}{3}$ of the major axis, then the eccentricity of the ellipse is $\sqrt{\frac{2}{3}}$.

Statement 2: If a focus of an ellipse is at the origin, directrix is the line $x = 4$ and the eccentricity is $\sqrt{\frac{2}{3}}$, then the length of the semi major axis is $4\sqrt{6}$.

Ans. (b)

◎ **Solution:** In statement 1, if the equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, then $\frac{2b^2}{a} = \frac{1}{3}(2a)$

$\Rightarrow \frac{b^2}{a^2} = \frac{1}{3} = 1 - e^2$, e being the eccentricity

$\Rightarrow e = \sqrt{\frac{2}{3}}$ and the statement 1 is true

In statement 2, the equation of the ellipse is

$$x^2 + y^2 = \frac{2}{3}(4 - x)^2$$

$$\Rightarrow 3(x^2 + y^2) = 2(16 - 8x + x^2)$$

$$\Rightarrow x^2 + 16x + 3y^2 = 32$$

$$\Rightarrow \frac{(x+8)^2}{96} + \frac{y^2}{32} = 1$$

Length of the semi major axis = $\sqrt{96} = 4\sqrt{6}$.

Thus statement 2 is also true but does not justify statement 1.

Example 50: Statement 1: An equation of a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$ is $y = 2x + 2\sqrt{3}$.

Statement 2: If the line $y = mx + 4\sqrt{3}/m$, $(m \neq 0)$ is a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$

Ans. (a)

Solution: Equation of a tangent to $y^2 = 16\sqrt{3}x$ is $y = mx + 4\sqrt{3}/m$.

And to $\frac{x^2}{2} + \frac{y^2}{4} = 1$ is $x = m_1 y + \sqrt{4m_1^2 + 2}$

or $\quad y = \frac{1}{m_1}x - \sqrt{4 + \frac{2}{m_1^2}},\ m = \frac{1}{m_1}$

and $\left(\frac{4\sqrt{3}}{m}\right)^2 = \left[-\sqrt{4 + \frac{2}{m_1^2}}\right]^2$

$\Rightarrow \quad \frac{48}{m^2} = 4 + 2m^2$

$\Rightarrow \quad m^4 + 2m^2 - 24 = 0$

$\Rightarrow \quad m^2 = 4 \Rightarrow m = \pm 2.$

Showing that both the statement are true and statement 2 is a correct explanation for statement 1.

LEVEL 2

Straight Objective Type Questions

Example 51: If p is the length of the perpendicular from a focus upon the tangent at any point P of the ellipse $x^2/a^2 + y^2/b^2 = 1$ and r is the distance of P from the focus, then $\frac{2a}{r} - \frac{b^2}{p^2} =$

(a) -1 (b) 0
(c) 1 (d) 2

Ans. (c)

Solution: The equation of the tangent at $P(a\cos\theta, b\sin\theta)$ to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

length of the perpendicular from the focus $(ae, 0)$ on the line is

$$p = \left|\frac{e\cos\theta - 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right| = \left|\frac{ab(e\cos\theta - 1)}{\sqrt{b^2\cos^2\theta + a^2(1 - \cos^2\theta)}}\right|$$

$$= \left|\frac{ab(e\cos\theta - 1)}{\sqrt{a^2 - a^2e^2\cos^2\theta}}\right| = b\sqrt{\frac{1 - e\cos\theta}{1 + e\cos\theta}}$$

$$\Rightarrow \quad \frac{b^2}{p^2} = \frac{1 + e\cos\theta}{1 - e\cos\theta}$$

Now $\quad r^2 = (ae - a\cos\theta)^2 + b^2\sin^2\theta$

$= a^2[(e - \cos\theta)^2 + (1 - e^2)\sin^2\theta]$

$= a^2[e^2\cos^2\theta - 2e\cos\theta + 1]$

$= a^2(1 - e\cos\theta)^2$

$\Rightarrow \quad r = a(1 - e\cos\theta)$

Now $\frac{2a}{r} - \frac{b^2}{p^2} = \frac{2}{1 - e\cos\theta} - \frac{1 + e\cos\theta}{1 - e\cos\theta} = 1$

Example 52: If $y = x$ and $3y + 2x = 0$ are the equations of a pair of conjugate diameters of an ellipse, then the eccentricity of the ellipse is

(a) $\sqrt{2/3}$ (b) $1/\sqrt{3}$
(c) $1/\sqrt{2}$ (d) $2/\sqrt{5}$

Ans. (b)

Solution: Let the equation of the ellipse be $x^2/a^2 + y^2/b^2 = 1$

Slope of the given diameters are $m_1 = 1$, $m_2 = -2/3$.

$\Rightarrow m_1m_2 = -2/3 = -b^2/a^2$

[using the condition of conjugacy of two diameters]

$3b^2 = 2a^2 \Rightarrow 3a^2(1 - e^2) = 2a^2$,

$1 - e^2 = 2/3 \Rightarrow e^2 = 1/3 \Rightarrow e = 1/\sqrt{3}$

Example 53: If α, β are the eccentric angles of the extremities of a focal chord of the ellipse $x^2/16 + y^2/9 = 1$, then $\tan(\alpha/2)\tan(\beta/2) =$

(a) $\dfrac{\sqrt{7}+4}{\sqrt{7}-4}$ (b) $-\dfrac{9}{23}$

(c) $\dfrac{\sqrt{5}-4}{\sqrt{5}+4}$ (d) $\dfrac{8\sqrt{7}-23}{9}$

Ans. (d)

◎ **Solution:** The eccentricity $e = \sqrt{1-\dfrac{9}{16}} = \dfrac{\sqrt{7}}{4}$.

Let P ($4\cos\alpha$, $3\sin\alpha$) and Q ($4\cos\beta$, $3\sin\beta$) be a focal chord of the ellipse passing through the focus at $(\sqrt{7}, 0)$.

Then $$\frac{3\sin\beta}{4\cos\beta-\sqrt{7}} = \frac{3\sin\alpha}{4\cos\alpha-\sqrt{7}}$$

$$\Rightarrow \quad \frac{\sin(\alpha-\beta)}{\sin\alpha-\sin\beta} = \frac{\sqrt{7}}{4}$$

$$\Rightarrow \quad \frac{\cos[(\alpha-\beta)/2]}{\cos[(\alpha+\beta)/2]} = \frac{\sqrt{7}}{4}$$

$$\Rightarrow \quad \tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right) = \frac{\sqrt{7}-4}{\sqrt{7}+4} = \frac{23-8\sqrt{7}}{-9}$$

◉ **Example 54:** If an ellipse slides between two perpendicular straight lines, then the locus of its centre is

(a) a parabola (b) an ellipse
(c) a hyperbola (d) a circle

Ans. (d)

◎ **Solution:** Let $2a$, $2b$ be the length of the major and minor axes respectively of the ellipse. If the ellipse slides between two perpendicular lines, the point of intersection P of these lines being the point of intersection of perpendicular tangents lies on the Director circle of the ellipse. This means that the centre C of the ellipse is always at a constant distance $\sqrt{a^2+b^2}$ from P. Hence the locus of C is a circle.

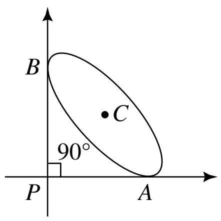

Fig. 19.17

◉ **Example 55.** The locus of the points of intersection of the tangents at the extremities of the chords of the ellipse $x^2 + 2y^2 = 6$ which touch the ellipse $x^2 + 4y^2 = 4$ is

(a) $x^2 + y^2 = 4$ (b) $x^2 + y^2 = 6$
(c) $x^2 + y^2 = 9$ (d) none of these

Ans. (c)

◎ **Solution:** We can write $x^2 + 4y^2 = 4$ as $\dfrac{x^2}{4}+\dfrac{y^2}{1} = 1$ (i)

Equation of a tangent to the ellipse (i) is

$$\frac{x}{2}\cos\theta + y\sin\theta = 1 \quad \text{(ii)}$$

Equation of the ellipse $x^2 + 2y^2 = 6$ can be written as

$$\frac{x^2}{6}+\frac{y^2}{3} = 1 \quad \text{(iii)}$$

Suppose (ii) meets the ellipse (iii) at P and Q and the tangents at P and Q to the ellipse (iii) intersect at (h, k), then (ii) is the chord of contact of (h, k) with respect to the ellipse (iii) and thus its equation is $\dfrac{hx}{6}+\dfrac{ky}{3} = 1$ (iv)

Since (ii) and (iv) represent the same line

$$\frac{h/6}{(\cos\theta)/2} = \frac{k/3}{\sin\theta} = 1$$

$\Rightarrow$ $h = 3\cos\theta$, $k = 3\sin\theta$.

and the locus of (h, k) is $x^2 + y^2 = 9$

◉ **Example 56:** The sum of the squares of the perpendiculars on any tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ from two points on the minor axis each at a distance $\sqrt{a^2-b^2}$ from the centre is

(a) $2a^2$ (b) $2b^2$
(c) $a^2 + b^2$ (d) $a^2 - b^2$

Ans. (a)

◎ **Solution:** The eccentricity e of the given ellipse is given by $e^2 = 1 - b^2/a^2 \Rightarrow a^2 - b^2 = a^2e^2$. So the point on the minor axis, i.e. y-axis at a distance $\sqrt{a^2-b^2}$ from the centre (0, 0) of the ellipse are $(0, \pm ae)$.

The equation of the tangent at any point ($a\cos\theta$, $b\sin\theta$) on the ellipse is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

So the required sum is

$$\left[\frac{\dfrac{ae\sin\theta}{b}-1}{\sqrt{\dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta}{b^2}}}\right]^2 + \left[\frac{\dfrac{-ae\sin\theta}{b}-1}{\sqrt{\dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta}{b^2}}}\right]^2$$

$$= \frac{(ae\sin\theta-b)^2+(ae\sin\theta+b)^2}{(b^2\cos^2\theta+a^2\sin^2\theta)}\times a^2$$

$$= \frac{2a^2(a^2e^2\sin^2\theta+b^2)}{b^2\cos^2\theta+a^2\sin^2\theta}$$

$$= \frac{2a^2[(a^2-b^2)\sin^2\theta + b^2)}{b^2\cos^2\theta + a^2\sin^2\theta} = 2a^2$$

Example 57. $y = mx + c$ is a normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ if c^2 is equal to

(a) $\dfrac{(a^2-b^2)^2}{a^2m^2+b^2}$ (b) $\dfrac{(a^2-b^2)^2}{a^2m^2}$

(c) $\dfrac{(a^2-b^2)^2m^2}{a^2+b^2m^2}$ (d) $\dfrac{(a^2-b^2)^2m^2}{a^2m^2+b^2}$

Ans. (c)

Solution: Equation of a normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is

$\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$, if it represents, $mx - y = -c$, then

$$\frac{a}{m\cos\theta} = \frac{b}{\sin\theta} = \frac{-(a^2-b^2)}{c}$$

$$\Rightarrow \quad \cos\theta = \frac{ac}{m(a^2-b^2)}, \sin\theta = -\frac{bc}{a^2-b^2}$$

$$\Rightarrow \quad \frac{c^2}{(a^2-b^2)^2}\left[\frac{a^2}{m^2} + b^2\right] = 1$$

$$\Rightarrow \quad c^2 = \frac{(a^2-b^2)^2m^2}{a^2+b^2m^2}$$

Example 58: A tangent at any point to the ellipse $4x^2 + 9y^2 = 36$ is cut by the tangent at the extremities of the major axis at T and T'. The circle on TT' as diameter passes through the point.

(a) $(0, \sqrt{5})$ (b) $(\sqrt{5}, 0)$

(c) (2, 1) (d) $(0, -\sqrt{5})$

Ans. (b)

Solution: Any point on the ellipse is $P(3\cos\theta, 2\sin\theta)$.

Equation of the tangent at P is $\dfrac{x}{3}\cos\theta + \dfrac{y}{2}\sin\theta = 1$ which meets the tangents $x = 3$ and $x = -3$ at the extremities of the major axis at

$T\left(3, \dfrac{2(1-\cos\theta)}{\sin\theta}\right)$ and $T'\left(-3, \dfrac{2(1+\cos\theta)}{\sin\theta}\right)$

Equation of the circle on TT' as diameter is

$$(x-3)(x+3) + \left(y - \frac{2(1-\cos\theta)}{\sin\theta}\right)\left(y - \frac{2(1+\cos\theta)}{\sin\theta}\right) = 0$$

$\Rightarrow x^2 + y^2 - \dfrac{4}{\sin\theta}y - 5 = 0$, which passes through $(\sqrt{5}, 0)$

Example 59: P is a point on the ellipse E: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and P' be the corresponding point on the auxiliary circle C: $x^2 + y^2 = a^2$. The normal at P to E and at P' to C intersect on circle whose radius is

(a) $a + b$ (b) $a - b$

(c) $2a$ (d) $2b$.

Ans. (a)

Solution: $P(a\cos\theta, b\sin\theta)$, $P'(a\cos\theta, a\sin\theta)$

Equation of the normal at P is

$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$ (1)

Equation of the normal at P' is

$x\sec\theta - y\operatorname{cosec}\theta = 0$ (2)

From (1) and (2) we get

$(a-b)y\operatorname{cosec}\theta = a^2 - b^2$

$\Rightarrow y = (a+b)\sin\theta$.

From (2) we get $x = (a+b)\cos\theta$.

So point of intersection of (1) and (2) lie on a circle of radius $a + b$.

Example 60: If the common tangent in the first quadrant of the circle $x^2 + y^2 = 16$ and the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$ meets the coordinate axes at A and B, the length of AB is

(a) $7/\sqrt{3}$ units (b) $14/\sqrt{3}$ units

(c) $4\sqrt{7}$ units (d) 14 units

Ans. (b)

Solution: An equation of the tangent to the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$ is

$$y = mn + \sqrt{25m^2+4} \quad (1)$$

As the tangent is in the first quadrant $m < 0$.

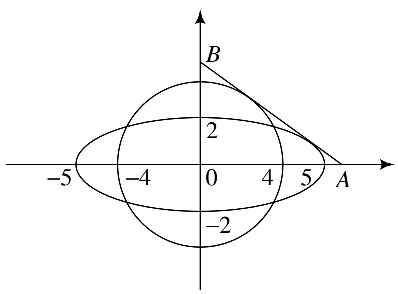

Fig. 19.18

The line will touch the circle $x^2 + y^2 = 16$ if

$$\frac{m(0) - 0 + \sqrt{25m^2+4}}{\sqrt{m^2+1}} = 4$$

$$\Rightarrow \quad 25m^2 + 4 = 16\,(m^2 + 1)$$

$$\Rightarrow \quad m^2 = \frac{4}{3} \Rightarrow m = \frac{-2}{\sqrt{3}}$$

So equation of the tangent is

$$y = \frac{-2}{\sqrt{3}}x + \sqrt{25\left(\frac{4}{3}\right)+4}$$

$$\Rightarrow \quad 2x + \sqrt{3}y = 4\sqrt{7}$$

which meets the coordinate axes at $A\left(\frac{4\sqrt{7}}{2}, 0\right)$ and $B\left(0, \frac{4\sqrt{7}}{\sqrt{3}}\right)$

so that $(AB)^2 = \left(\frac{4\sqrt{7}}{2}\right)^2 + \left(\frac{4\sqrt{7}}{\sqrt{3}}\right)^2$

$$= (4\sqrt{7})^2 \times \frac{7}{12}$$

$$\Rightarrow \quad AB = \frac{14}{\sqrt{3}} \text{ units}$$

EXERCISE

Concept-based Straight Objective Type Questions

1. Equation of an ellipse with centre at the origin passing through (5, 0) and having eccentricity 2/3 is:
   (a) $4x^2 + 9y^2 = 100$ (b) $9x^2 + 5y^2 = 225$
   (c) $5x^2 + 9y^2 = 125$ (d) $6x^2 + 4y^2 = 150$

2. Equation of a tangent to the ellipse $\frac{x^2}{36} + \frac{y^2}{25} = 1$, passing through the point where a directrix of the ellipse meets the + ve x-axis is
   (a) $5y + \sqrt{11}x = 36$ (b) $6y + \sqrt{11}x = 25$
   (c) $6y + \sqrt{11}x = 36$ (d) $5y - \sqrt{11}x = 36$

3. Equation of a line joining a foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ to a foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{25} = 1$ is,
   (a) $x + y = 4$ (b) $x + y = 5$
   (c) $x + y = 3$ (d) $5x + 3y = 1$

4. If the eccentricity of the ellipse $\frac{x^2}{25} + \frac{y^2}{a^2} = 1$ and $\frac{x^2}{a^2} + \frac{y^2}{16} = 1$ is same, then the value of a^2 is:
   (a) 9 (b) 41
   (c) 15 (d) 20

5. If $y = mx + 4$ is a tangent to the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$, then $625m^4 + 25m^2 - 156$ is equal to
   (a) – 1 (b) 0
   (c) 1 (d) 4

6. Length of a latera recta of the ellipse $\frac{x^2}{81} + \frac{y^2}{63} = 1$ is (in units)
   (a) 9 (b) 7
   (c) 14 (d) 18

7. The normal at the point $(3, 2\sqrt{3})$ on the ellipse $\frac{x^2}{36} + \frac{y^2}{16} = 1$ meets the major axis of the ellipse at $(\alpha, 0)$, the value of α is:
   (a) $\frac{5}{3}$ (b) $\frac{3}{5}$
   (c) $\frac{2}{3}$ (d) $\frac{3}{2}$

8. In an ellipse, the distance between the foci is 6 and length of semi minor axis is 4, then eccentricity of the ellipse is
   (a) $\frac{2}{5}$ (b) $\frac{3}{5}$
   (c) $\frac{2}{3}$ (d) $\frac{1}{\sqrt{2}}$

9. If F_1 and F_2 are the foci of the ellipse $4x^2 + 9y^2 = 36$, P is a point on the ellipse such that $PF_1 : PF_2 = 2 : 1$; then area of $\Delta\, PF_1F_2$ is:
   (a) 1 sq. unit (b) 2 sq. unit
   (c) 3 sq. unit (d) 4 sq. unit

10. If the eccentric angles of two points P and Q on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ are α, β such that $\alpha + \beta = \frac{\pi}{2}$, then the locus of the point of intersection of the normals at P and Q is
(a) $ax + by = 0$ (b) $ax - by = 0$
(c) $x + y = 0$ (d) $x + y = a + b$

11. $P_1(\theta_1)$ and $P_2(\theta_2)$ are two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ such that $\tan\theta_1 \tan\theta_2 = \frac{-a^2}{b^2}$. The chord joining P_1 and P_2 of the ellipse subtends a right angle at the
(a) focus $(ae, 0)$ (b) focus $(-ae, 0)$
(c) centre $(0, 0)$ (d) vertex $(a, 0)$

12. Coordinates of the foci of the ellipse $2x^2 + 3y^2 - 4x - 12y + 13 = 0$ are
(a) $\left(1\pm\frac{1}{\sqrt{6}}, 2\right)$ (b) $\left(2, 1\pm\frac{1}{\sqrt{6}}\right)$
(c) $(1, 2)$ (d) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right)$

13. C_1: $x^2 + y^2 = a^2$ and C_2: $x^2 + y^2 = b^2$ are two con-tentric circles. A line through the centre of these circles intersects them at P and Q respectively. If the lines through P and Q parallel to y-axis and x-axis respectively meet at the point R, then the locus of R is
(a) $x^2 + y^2 = a^2 + b^2$ (b) $x^2 - y^2 = a^2 - b^2$
(c) $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ (d) $\frac{x^2}{b^2}+\frac{y^2}{a^2} = 1$

14. P is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ $(a > b)$ and Q is the point corresponding to P on the auxiliary circle $x^2 + y^2 = a^2$. N is the foot of the perpendicular from P on the major axis of the ellipse. $\frac{PN}{PQ}$ is equal to:
(a) $\frac{b}{a-b}$ (b) $\frac{a}{a-b}$
(c) $\frac{b}{a+b}$ (d) $\frac{a}{a+b}$

15. If the normal at any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ meets the axis of x at G and the axis of y at g, then $PG : pg$ is:
(a) $a^2 : b^2$ (b) $a : b$
(c) $b : a$ (d) $b^2 : a^2$

LEVEL 1

Straight Objective Type Questions

16. On the ellipse $4x^2 + 9y^2 = 1$, the point at which the tangent is parallel to the line $8x = 9y$ is
(a) $(2/5, 1/5)$ (b) $(-2/5, 1/5)$
(c) $(-2/5, -1/5)$ (d) none of these

17. The coordinates of the foci of the ellipse $4x^2 + 9y^2 = 1$ are
(a) $(\pm\sqrt{5}/3, 0)$ (b) $(\pm\sqrt{5}/6, 0)$
(c) $(0, \pm\sqrt{5}/3)$ (d) $(0, \pm\sqrt{5}/6)$

18. The distance of the point $(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$ on the ellipse from the centre of the ellipse is 2 if $\theta =$
(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/3$ (d) none of these

19. An equation of the normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ with eccentricity e at the positive end of the latus rectum is
(a) $x + ey + e^3 a = 0$ (b) $x - ey - ae^3 = 0$
(c) $x - ey + e^3 a = 0$ (d) $x + ey - e^3 a = 0$

20. The circle $x^2 + y^2 = c^2$ contains the ellipse $x^2/a^2 + y^2/b^2 = 1$ if
(a) $c < a$ (b) $c < b$
(c) $c > \max\{a, b\}$ (d) $c > b$

21. In an ellipse, if the lines joining a focus to the extre-milites of the minor axis make an equilateral triangle with the minor axis, the eccentricity of the ellipse is
(a) 3/4 (b) $\sqrt{3}/2$
(c) 1/2 (d) 2/3

22. An equilateral triangle is inscribed in the ellipse $x^2 + 3y^2 = 3$ such that one vertex of the triangle is (0, 1) and one altitude of the triangle is along the y-axis. The length of its side is

(a) $4\sqrt{3}/5$ (b) $3\sqrt{3}/5$
(c) $6\sqrt{3}/5$ (d) $2\sqrt{3}$

23. If the equation $\dfrac{x^2}{10-2a}+\dfrac{y^2}{4-2a} = 1$ represents an ellipse, then a lies in the interval
(a) $(-\infty, 5)$ (b) $(2, 5)$
(c) $(-\infty, 2)$ (d) $(5, \infty)$

24. The curve represented by $x = 3(\cos t + \sin t)$, $y = 4(\cos t - \sin t)$ is an ellipse whose eccentricity is e, such that $16e^2 + 7$ is equal to:
(a) 14 (b) 12
(c) 11 (d) 21

25. The eccentricity of an ellipse, with centre at the origin is 2/3. If one of directrices is $x = 6$, then equation of the ellipse is
(a) $\dfrac{x^2}{9}+\dfrac{y^2}{4} = 1$ (b) $\dfrac{x^2}{9}+\dfrac{y^2}{5} = 1$
(c) $5x^2 + 9y^2 = 80$ (d) $3x^2 + 2y^2 = 6$

26. The locus of the foot of the perpendicular from a focus of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$ on any tangent is
(a) $x^2 + y^2 = a^2 + b^2$ (b) $x^2 + y^2 = a^2$
(c) $x^2 + y^2 = b^2$ (d) $x^2 + y^2 = 2a^2$

27. If the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} =1$ and the circle $x^2 + y^2 = r^2$ where $b < r < a$ intersect in four points and the slope of a common tangent to the ellipse and the circle is $\dfrac{b}{a}$, then r^2 is equal to
(a) $\dfrac{2b^2}{a^2+b^2}$ (b) $\dfrac{2a^2b^2}{a^2+b^2}$
(c) $\dfrac{2a^2}{a^2+b^2}$ (d) $\dfrac{2a^2b^2}{a^2-b^2}$

28. The product of the perpendiculars from the foci of the ellipse $\dfrac{x^2}{144}+\dfrac{y^2}{100} = 1$ on any tangent is:
(a) 144 (b) 100
(c) 122 (d) 200

29. The locus of the foot of the perpendicular drawn from the centre on any tangent to the ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{16} = 1$ is:
(a) $25x^2 + 16y^2 = (x^2 + y^2)^2$
(b) $25x^2 + 16y^2 = 9$
(c) $16x^2 + 25y^2 = 1$
(d) $x^2 + y^2 = 41$

30. If the normal at $P\left(2, \dfrac{3\sqrt{3}}{2}\right)$ meets the major axis of the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{9} = 1$ at Q, S_1 and S_2 are the foci of the ellipse, then $S_1Q : S_2Q$ is equal to:
(a) $\dfrac{8}{\sqrt{7}}$ (b) $\dfrac{7+\sqrt{8}}{7-\sqrt{8}}$
(c) $\dfrac{8+\sqrt{7}}{8-\sqrt{7}}$ (d) $\dfrac{\sqrt{7}}{4}$

31. Chord of the ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{16} = 1$ whose middle point is $\left(\dfrac{1}{2}, \dfrac{2}{5}\right)$ meets the minor axis at A and major axis at B, length of AB (in units) is:
(a) $\dfrac{\sqrt{41}}{5}$ (b) $\dfrac{2\sqrt{41}}{5}$
(c) $\dfrac{3\sqrt{41}}{5}$ (d) $\dfrac{7\sqrt{41}}{5}$

32. If B is an end of the minor axis of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} =1$, F_1 and F_2 are the foci such that the triang BF_1F_2 is equilateral, then eccentricity of the ellipse is:
(a) $\dfrac{1}{2}$ (b) $\dfrac{1}{\sqrt{2}}$
(c) $\dfrac{1}{3}$ (d) $\dfrac{\sqrt{3}}{2}$

33. Length of the major axis of the ellipse $(5x - 10)^2 + (5y + 15)^2 = \dfrac{(3x-4y+7)}{4}$ is:
(a) $\dfrac{10}{3}$ units (b) 5 units
(c) $\dfrac{20}{3}$ units (d) $\dfrac{5}{3}$ units

34. A tangent to the ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{16} = 1$ at any point meets the line $x = 0$ at a point Q. R is the image of Q in the line $y = x$. The circle on QR as a diameter

passes through a fixed point whose coordinate are:

(a) (5, 5) (b) (4, 5)
(c) (1, 1) (d) (0, 0)

35. P is a point on a directrix of an ellipse S is the corresponding focus and C is the centre of the ellipse. The line PC meets the ellipse at A. The angle which the tangent at A makes with PS is α. If the eccentricity of the ellipse is e, then α is equal to

(a) $\tan^{-1} e$ (b) $\tan^{-1} (1/e)$
(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

36. If the tangent with slope –2 of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is a normal to the circle $x^2 + y^2 - 4x + 1 = 0$, then the maximum value of ab is

(a) 1 (b) 2
(c) 3 (d) 4

37. If a line $3px + 2\sqrt{1-p^2}\, y = 1$ touches a fixed ellipse E for all $p \in [-1, 1]$, then equation of a directrix of the ellipse is:

(a) $x = \frac{3\sqrt{5}}{10}$ (b) $y = \frac{3\sqrt{5}}{10}$
(c) $x = \frac{\sqrt{5}}{3}$ (d) $y = \frac{\sqrt{5}}{3}$

38. If a and b are the natural numbers such that $a + b = ab$, then equation of the chord of the ellipse $x^2 + 4y^2 = 4$ with (a, b) as the mid point is:

(a) $x + 4y = 2$ (b) $x + 4y = 10$
(c) $4x + y = 2$ (d) $4x + y = 10$

39. If $\frac{x^2}{\sec^2\theta}+\frac{y^2}{\tan^2\theta} = 1$ represents an ellipse with eccentricity e and length of the major axis l then

(a) e is independent of θ
(b) l is independent of θ
(c) el is independent of θ
(d) $l^2 - e^2$ is independent of θ

40. $3x^2 + 4y^2 - 6x + 8y + k = 0$ represents an ellipse with eccentricity 1/2,

(a) for all value of k (b) for $k > 7$
(c) $k < 7$ (d) $k = 7$

Assertion-Reason Type Questions

41. The tangent at a point P on the ellipse, which is not an extremity of major axis meets a directrix at T.

Statement 1: The circle on PT as diameter passes through the focus of the ellipse corresponding to the directix on which T lies.

Statement 2: PT subtends a right angle at the focus of the ellipse corresponding to the directix on which T lies.

42. **Statement 1:** If the normal at an end of a latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ meets the major axis at G, O is the centre of the ellipse, then $OG = ae^3$, e being the eccentricity of the ellipse.

Statement 2: Equation of the normal at a point $(a \cos \theta, b \sin \theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is $\frac{ax}{\cos\theta}+\frac{by}{\sin\theta} = a^2 + b^2$

43. **Statement 1:** In a triangle PQR, if the base QR is fixed and perimeter of the triangle is constant, then vertex P traces an ellipse.

Statement 2: If the sum of the distances of a point A from two fixed points B and C is constant $> BC$, then A traces an ellipse.

44. **Statement 1:** The locus of a moving point (x, y) satisfying $\sqrt{(x-2)^2+y^2}+\sqrt{(x+2)^2+y^2} = 4$ is an ellipse.

Statement 2: The distance between (–2, 0) and (2, 0) in 4 units.

45. **Statement 1:** Product of the perpendiculars drawn from the foci on any tangent to the ellipse $\frac{x^2}{15}+\frac{y^2}{7} = 1$ is 7.

Statement 2: Foot of the perpendiculars drawn from the foci on any tangent lies on the circle $x^2 + y^2 = 15$.

46. **Statement 1:** Origin is the centre of the conic $x^2 + y^2 + xy = 1$.

    Statement 2: A point is the centre of a conic if all chords of the conic through this point are bisected at this point.

47. For all real p, the line $2\,px + y\sqrt{1-p^2} = 1$ touches a fixed ellipse whose axes are the coordinate axes.

    Statement 1: Equation of the director circle of the ellipse is $4x^2 + 4y^2 = 5$.

    Statement 2: Length of the major and minor axes of the ellipse are 2 and 1 units respectively.

48. **Statement 1:** If the extremities of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ $(a > b)$, having positive ordinates lies on the parabola $x^2 = -2(y - 2)$, then $a = 2$.

    Statement 2: If the length of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is equal to the distance between the foci, then the eccentricity e of the ellipse satisfies $e^2 + e - 1 = 0$.

49. **Statement 1:** Two diameters $y = mx$ and $y = m'x$ $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ are conjugate if $mm' = -\frac{a^2}{b^2}$.

    Statement 2: Two diameters of an ellipse are said to be conjugate when each bisects the chords parallel to the other.

50. E: $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ and $P : y^2 = 4bx$, $a > b$.

    Statement 1: The tangent at the positive end of the minor axis of the ellipse. E passes through the positive end of the latus rectum of the parabola P.

    Statement 2: If the latus rectum of the parabola P is same as that of the ellipse E, then eccentricity of E is $1/\sqrt{2}$.

LEVEL 2

Straight Objective Type Questions

51. If chords of contact of the tangent from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ are at right angles, then $\frac{x_1x_2}{a^2}\times\frac{y_1y_2}{b^2}$ is equal to

    (a) $\frac{a^2}{b^2}$ (b) $\frac{-b^2}{a^2}$

    (c) $\frac{-a^4}{b^4}$ (d) $\frac{b^4}{a^4}$

52. If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14}+\frac{y^2}{5}=1$ intersect it again at the point $Q(2\theta)$, then $\cos\theta =$

    (a) $\frac{-2}{3}$ (b) $\frac{2}{3}$

    (c) $\frac{-6}{7}$ (d) $\frac{6}{7}$

53. Let E be the ellipse $x^2/9 + y^2/4 = 1$ and C be the circle $x^2 + y^2 = 9$. Let P (1, 2) and Q (2, 1) be two points, then

    (a) Q lies inside C but outside E

    (b) Q lies outside both C and E

    (c) P lies inside both C and E

    (d) P lies inside C but out side E

54. If CF is perpendicular from the centre of the ellipse $\frac{x^2}{25}+\frac{y^2}{9} = 1$ to the tangent at P, G is the point where the normal at P meets the major axis, then $CF \cdot PG =$

    (a) 9 (b) 18

    (c) 25 (d) 34

55. If the tangent at $P(\theta)$ on the ellipse $16x^2 + 11y^2 = 256$ touches the circle $x^2 + y^2 + 2x - 15 = 0$, then $\theta =$

    (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$

    (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

56. If $lx + my + n = 0$ is an equation of the line joining the extremities of a pair of semi-conjugate diameters of the ellipse $\frac{x^2}{9}+\frac{y^2}{4} = 1$, then $\frac{9l^2+4m^2}{n^2}$ is equal to

(a) -1 (b) 0
(c) 1 (d) 2

57. Let f be a strictly decreasing function defined on **R** such that $f(x) > 0 \ \forall \ x \in$ **R**, and $\dfrac{x^2}{f(a^2+5a+3)} + \dfrac{y^2}{f(3a+15)} = 1$ represents an ellipse with major axis along the y-axis, then
(a) $a \notin (-\infty, -6)$ (b) $a \notin (2, \infty)$
(c) $a \notin (-6, 2)$ (d) $a > 0$

58. Number of points from which two perpendicular tangents can be drawn to both the ellipses E_1: $\dfrac{x^2}{a^2+2} + \dfrac{y^2}{b^2} = 1$ and E_2: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2+1} = 1$ is
(a) 0 (b) 1
(c) 2 (d) infinitely many

59. Let d be the perpendicular distance from the centre of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ to the tangent at a point P on the ellipse. If F_1 and F_2 are two foci of the ellipse then $(PF_1 - PF_2)^2 =$
(a) $4b^2\left(1-\dfrac{a^2}{d^2}\right)$ (b) $4d^2\left(1-\dfrac{b^2}{a^2}\right)$
(c) $4a^2\left(1-\dfrac{b^2}{d^2}\right)$ (d) $4a^2$

60. If from a point $(a, 0)$ three distinct chords of the ellipse $x^2 + 2y^2 = 1$ are drawn, which are bisected by the parabola $y^2 = 4x$, then
(a) $4 < a < 8$ (b) $8 < a < 4+\sqrt{17}$
(c) $0 < a < \sqrt{17}-4$ (d) $0 < a < 4$

Previous Years' AIEEE/JEE Main Questions

1. The eccentricity of an ellipse, with its centre at the origin is 1/2. If one of the directrix is $x = 4$, then the equation of the ellipse is
(a) $4x^2 + 3y^2 = 12$ (b) $3x^2 + 4y^2 = 12$
(c) $3x^2 + 4y^2 = 1$ (d) $4x^2 + 3y^2 = 1$ **[2004]**

2. Area of the greaterst rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$ is
(a) $\sqrt{ab}$ (b) a/b
(c) $2ab$ (d) ab **[2005]**

3. An ellipse has OB as semi minor axis. F and F' its foci and the angle FBF' is a right angle. The eccentricity of the ellipse is
(a) 1/4 (b) $1/\sqrt{3}$
(c) $1/\sqrt{2}$ (d) 1/2 **[2005]**

4. In an ellipse, the distance between foci is 6 and minor axis is 8. Then its eccentricity is
(a) $1/\sqrt{5}$ (b) 3/5
(c) $\dfrac{1}{2}$ (d) $\dfrac{4}{5}$ **[2006]**

5. A focus of an ellipse is at origin. The directrix is the line $x = 4$ and the eccentricity is 1/2. Then the length of the semi-major axis is
(a) 5/3 (b) 8/3
(c) 2/3 (d) 4/3 **[2008]**

6. The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle aligned with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point (4, 0). Then the equation of the ellipse is
(a) $4x^2 + 48y^2 = 48$ (b) $4x^2 + 6y^2 = 48$
(c) $x^2 + 16y^2 = 16$ (d) $x^2 + 12y^2 = 16$ **[2009]**

7. Equation of the ellipse whose axes are the axes of coordinate, which passes through the point $(-3, 1)$ and has eccentricity $\sqrt{2/5}$ is
(a) $5x^2 + 3y^2 - 32 = 0$ (b) $3x^2 + 5y^2 - 32 = 0$
(c) $5x^2 + 3y^2 - 48 = 0$ (d) $3x^2 + 5y^2 - 15 = 0$
[2011]

8. An ellipse is drawn by taking a diameter of the circle $(x-1)^2 + y^2 = 1$ as its semi-minor axis and a diameter of the circle $x^2 + (y-2)^2 = 4$ as its semi-major axis. If the centre of the ellipse is at the origin and

its axes are the coordinate axes, then the equation of the ellipse is

(a) $4x^2 + y^2 = 8$ (b) $x^2 + 4y^2 = 16$
(c) $4x^2 + y^2 = 4$ (d) $x^2 + 4y^2 = 8$ **[2012]**

9. If a and c are positive real numbers and the ellipse $\frac{x^2}{4c^2}+\frac{y^2}{c^2} = 1$ has four distinct points common with circle $x^2 + y^2 = 9a^2$, then

(a) $9ac - 9a^2 - 2c^2 < 0$ (b) $6ac + 9a^2 - 2c^2 < 0$
(c) $9ac - 9a^2 - 2c^2 > 0$ (d) $6ac + 9a^2 - 2c^2 > 0$
[2013, online]

10. Let the equation of two ellipses be E_1: $\frac{x^2}{3}+\frac{y^2}{2} = 1$ and E_2: $\frac{x^2}{16}+\frac{y^2}{b^2} = 1$. If the product of their eccentricities is 1/2, then the length of the minor axis of E_2 is:

(a) 8 (b) 9
(c) 4 (d) 2 **[2013, online]**

11. If the curves $\frac{x^2}{\alpha}+\frac{y^2}{4} = 1$ and $y^2 = 16x$ intersect at right angles, then a value of α is

(a) 2 (b) 4/3
(c) 1/2 (d) 3/4 **[2013, online]**

12. A point on the ellipse $4x^2 + 9y^2 = 36$ where the normal is parallel to the $4x - 2y - 5 = 0$, is:

(a) $\left(\frac{9}{5},\frac{8}{5}\right)$ (b) $\left(\frac{8}{5},\frac{-9}{5}\right)$
(c) $\left(-\frac{9}{5},\frac{8}{5}\right)$ (d) $\left(\frac{8}{5},\frac{9}{5}\right)$ **[2013, online]**

13. The focus of the foot of perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent is:

(a) $(x^2 - y^2)^2 = 6x^2 + 2y^2$ (b) $(x^2 - y^2)^2 = 6x^2 - 2y^2$
(c) $(x^2 + y^2)^2 = 6x^2 + 2y^2$ (d) $(x^2 + y^2)^2 = 6x^2 - 2y^2$
[2014]

14. If OB is the semi-minor axis of an ellipse, F_1 and F_2 are its foci and the angle between F_1B and F_2B is a right angle, then the square of the eccentricity of the ellipse is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$
(c) $\frac{1}{2\sqrt{2}}$ (d) $\frac{1}{4}$ **[2014, online]**

15. The minimum area of the triangle formed by any tangent to the ellipse $\frac{x^2}{16}+\frac{y^2}{81} = 1$ and the coordinate - axes is:

(a) 12 (b) 18
(c) 26 (d) 36 **[2014, online]**

16. The area (in sq. units) of the quadrilateral formed by the tangents at the end points of the latera recta to the ellipse $\frac{x^2}{9}+\frac{y^2}{5} = 1$ is:

(a) $\frac{27}{4}$ (b) 18
(c) $\frac{27}{2}$ (d) 27 **[2015]**

17. If the distance between the foci of an ellipse is half the length of its latus rectum, then the eccentricity of the ellipse is:

(a) $\frac{1}{2}$ (b) $\frac{2\sqrt{2}-1}{2}$
(c) $\sqrt{2}-1$ (d) $\frac{\sqrt{2}-1}{2}$ **[2015, online]**

18. If the tangent at a point on the ellipse $\frac{x^2}{27}+\frac{y^2}{3} = 1$ meets the coordinate axes at A and B, and O is the origin, then the minimum area (in sq. units) of the triangle OAB is

(a) $3\sqrt{3}$ (b) $\frac{9}{2}$
(c) 9 (d) $9\sqrt{3}$ **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. In an ellipse, the distance between its directrices is four times the distance between its foci. If (– 2, 0) is one of its vertices, then the equation of the ellipse is

(a) $3x^2 + 4y^2 = 1$ (b) $4x^2 + 3y^2 = 12$
(c) $3x^2 + 4y^2 = 12$ (d) $4x^2 + 3y^2 = 1$ **[2007]**

2. The tangent to the ellipse $3x^2 + 16y^2 = 12$, at the point (1, 3/4), intersects the curve $y^2 + x = 0$ at:
   (a) no point
   (b) exactly one point
   (c) two distinct points
   (d) more than two points **[2012]**
3. Let P be a point in the first quadrant lying on the ellipse $9x^2 + 16y^2 = 144$, such that the tangent at P to the ellipse is inclined at an angle 135° to the positive direction of x-axis. Then the coordinates of P are:
   (a) $\left(\frac{\sqrt{143}}{3}, \frac{1}{4}\right)$ (b) $\left(\frac{8}{9}, \frac{\sqrt{77}}{3}\right)$
   (c) $\left(\frac{4}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ (d) $\left(\frac{16}{5}, \frac{9}{5}\right)$ **[2014]**
4. The line $2x + y = 3$ intersects the ellipse $4x^2 + y^2 = 5$ at two points. The tangents to the ellipse at these two points intersect at the point:
   (a) $\left(\frac{5}{6}, \frac{5}{3}\right)$ (b) $\left(\frac{5}{6}, \frac{5}{6}\right)$
   (c) $\left(\frac{5}{3}, \frac{5}{6}\right)$ (d) $\left(\frac{5}{3}, \frac{5}{3}\right)$ **[2015]**

Answers

Concept-based

1. (c)	**2.** (c)	**3.** (a)	**4.** (d)
5. (b)	**6.** (c)	**7.** (a)	**8.** (b)
9. (d)	**10.** (a)	**11.** (c)	**12.** (a)
13. (c)	**14.** (a)	**15.** (d)	

Level 1

16. (b)	**17.** (b)	**18.** (b)	**19.** (b)
20. (c)	**21.** (b)	**22.** (c)	**23.** (c)
24. (a)	**25.** (c)	**26.** (b)	**27.** (b)
28. (b)	**29.** (a)	**30.** (c)	**31.** (a)
32. (a)	**33.** (b)	**34.** (d)	**35.** (c)
36. (d)	**37.** (b)	**38.** (b)	**39.** (c)
40. (c)	**41.** (a)	**42.** (c)	**43.** (a)
44. (c)	**45.** (b)	**46.** (a)	**47.** (a)
48. (b)	**49.** (d)	**50.** (d)	

Level 2

51. (c)	**52.** (a)	**53.** (d)	**54.** (a)
55. (c)	**56.** (d)	**57.** (c)	**58.** (a)
59. (c)	**60.** (b)		

Previous Years' AIEEE/JEE Main Questions

1. (b)	**2.** (c)	**3.** (c)	**4.** (b)
5. (b)	**6.** (d)	**7.** (b)	**8.** (c)
9. (a)	**10.** (c)	**11.** (a)	**12.** (a)
13. (c)	**14.** (a)	**15.** (d)	**16.** (d)
17. (c)	**18.** (c)		

Previous Years' B-Architecture Entrance Examination Questions

1. (c)	**2.** (b)	**3.** (d)	**4.** (a)

Hints and Solutions

Concept-based

1. Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It passes through (5, 0) $\Rightarrow a^2 = 25$

   $b^2 = a^2(1 - e^2) \Rightarrow b^2 = 25\left(1 - \frac{4}{9}\right) = \frac{125}{9}$.

   Hence the required equation of the ellipse is $5x^2 + 9y^2 = 125$
2. Equation of a tangent to the ellipse is

   $y = mx + \sqrt{36m^2 + 25}$ (1)

   Equation of the directrix is

   $x = \frac{6 \times 6}{\sqrt{36 - 25}} = \frac{36}{\sqrt{11}}$

   which meets the +ve x axis at $\left(\frac{36}{\sqrt{11}}, 0\right)$.

   The tangent (1) passes through this point if $\left[m\left(\frac{36}{\sqrt{11}}\right)\right]^2 = 36m^2 + 25$

   $\Rightarrow m^2 = \frac{11}{36} \Rightarrow m = \pm\frac{\sqrt{11}}{6}$.

   Since the tangent meets the directrix on positive x-axis, $m < 0$, and the required equation is $y = -\frac{\sqrt{11}}{6}x + \sqrt{36 \times \frac{11}{36} + 25}$

   $6y + \sqrt{11}x = 36$.
3. Foci of $\frac{x^2}{25} + \frac{y^2}{9} = 1$ is $(\pm 4, 0)$ and that of $\frac{x^2}{9} + \frac{y^2}{25} = 1$

is $(0, \pm 4)$. So the equation joining two foci $(4, 0)$ and $(0, 4)$ is $x + y = 4$.

4. $\dfrac{a^2 - 16}{a^2} = \dfrac{25 - a^2}{25} \Rightarrow a^4 = 16 \times 25$

   $= a^2 = 20$.

5. $(4)^2 = 25m^2 + 4 \Rightarrow 25m^2 = 12$

   So $625m^4 + 25m^2 - 156$

   $= 144 + 12 - 156 = 0$

6. Length of the latera recta of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $\dfrac{2b^2}{a}$ so the required length is $\dfrac{2 \times 63}{9} = 14$ units

7. Equation of the normal is

   $$\frac{\frac{x-3}{3}}{36} = \frac{\frac{y-2\sqrt{3}}{2\sqrt{3}}}{16}$$

   $\Rightarrow 12\sqrt{3}\,(x - 3) = 8\,(y - 2\sqrt{3})$

   which meets the major axis $y = 0$ at point for which

   $12\sqrt{3}\,(x - 3) = -16\sqrt{3} \Rightarrow x = \dfrac{5}{3}$.

   $\Rightarrow \alpha = \dfrac{5}{3}$.

8. $2ae = 6$ and $b = 4$. So $a^2e^2 = a^2 - b^2$

   $\Rightarrow a^2 = 25$ and $e = \dfrac{3}{5}$.

9. Equation of the ellipse is

   $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

   $PF_1 + PF_2 = 2(3) = 6$

   $\Rightarrow PF_1 = 4, PF_2 = 2$

   $F_1F_2 = 2ae = 2\sqrt{9-4} = 2\sqrt{5}$

   $(PF_1)^2 + (PF_2)^2 = 16 + 4 = (2\sqrt{5})^2 = (F_1F_2)^2$

   So $\Delta\, PF_1F_2$ is right angled and hence its area is $\dfrac{1}{2}(PF_1)(PF_2) = 4$ sq. units.

10. Equations of the normal at P and Q are

    $ax \sec \alpha - by \operatorname{cosec} \alpha = a^2 - b^2$

    $ax \sec \beta - by \operatorname{cosec} \beta = a^2 - b^2$

    If they intersect at (h, k).

    $ah \sec \alpha - bk \operatorname{cosec} \alpha = ah \operatorname{cosec} \alpha - bk \sec \alpha$ $(\because \beta = \pi/2 - \alpha)$

    $(ah + bk)(\sec \alpha - \operatorname{cosec} \alpha) = 0 \Rightarrow ah + bk = 0$

    Locus of (h, k) is $ax + by = 0$

11. $P_1\,(a \cos \theta_1, b \sin \theta_1)$, $P_2\,(a \cos \theta_2, b \sin \theta_2)$

    Slope of $OP_1 = \dfrac{b \sin\theta_1}{a \cos\theta_1} = \dfrac{b}{a}\tan\theta_1$

    Slope of $OP_2 = \dfrac{b \sin\theta_2}{a \cos\theta_2} = \dfrac{b}{a}\tan\theta_2$

    Product of the slope $= \dfrac{b^2}{a^2}\tan\theta_1 \tan\theta_2$

    $= \dfrac{b^2}{a^2}\left(\dfrac{-a^2}{b^2}\right) = -1$

    So PQ subtends a right angle at $(0, 0)$, the centre.

12. Equation of the ellipse is

    $$\frac{(x-1)^2}{\left(\frac{1}{\sqrt{2}}\right)^2} + \frac{(y-2)^2}{\left(\frac{1}{\sqrt{3}}\right)^2} = 1$$

    i.e. $\dfrac{X^2}{a^2} + \dfrac{Y^2}{b^2} = 1$

    where $X = x - 1$, $Y = y - 2$

    $a = \dfrac{1}{\sqrt{2}}$, $b = \dfrac{1}{\sqrt{3}}$

    If e is the eccentricity, then

    $e = \sqrt{1 - \dfrac{2}{3}} = \dfrac{1}{\sqrt{3}}$.

    Coordinates of the foci are

    $X = \pm ae$, $Y = 0$

    $\Rightarrow x - 1 = \pm\dfrac{1}{\sqrt{2}} \times \dfrac{1}{\sqrt{3}}$, $y - 2 = 0$

    $\Rightarrow x = 1 \pm \dfrac{1}{\sqrt{6}}$, $y = 2$

    Hence the required coordinate are $\left(1 \pm \dfrac{1}{\sqrt{6}}, 2\right)$

13. Coordinates of P are $(a \cos \theta, a \sin \theta)$ and of Q are $(b \cos \theta, b \sin \theta)$. Equation of the line through P parallel to y-axis is $x = a \cos \theta$ and of the line through Q parallel to x-axis is $y = b \sin \theta$. So the locus of the point of intersection is $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^2 = \cos^2\theta + \sin^2\theta = 1$

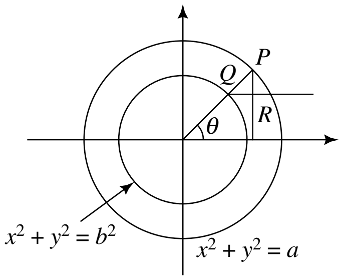

Fig. 19.19

14. Let the coordinates of P be $(a \cos \theta, b \sin \theta)$ then coordinate of Q are $(a \cos \theta, a \sin \theta)$
$PN = b \sin \theta$, $QN = a \sin \theta$
$\Rightarrow PQ = (a - b) \sin \theta$
and $\dfrac{PN}{PQ} = \dfrac{b}{a-b}$.

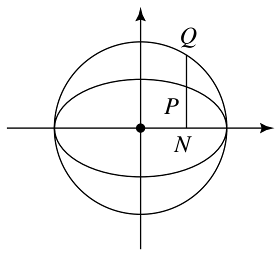

Fig. 19.20

15. Equation of the normal at P $(a \cos \theta, b \sin \theta)$ on the ellipse is $ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2$.
So coordinates of G are $\left(\dfrac{a^2-b^2}{a\sec\theta}, 0\right)$ and of g are $\left(0, -\dfrac{a^2-b^2}{a \operatorname{cosec} \theta}\right)$

$$(PG)^2 = \left(a\cos\theta - \frac{a^2-b^2}{a}\cos\theta\right)^2 + b^2 \sin^2 \theta$$

$$= \frac{b^2}{a^2} (b^2 \cos^2 \theta + a^2 \sin^2 \theta)$$

and $(Pg)^2 = \dfrac{a^2}{b^2} (b^2 \cos^2 \theta + a^2 \sin^2 \theta)$

So $PG : Pg = b^2 : a^2$

Level 1

16. Let the point be $((1/2) \cos \theta, (1/3) \sin \theta)$, then the slope of the tangent is $-\dfrac{1/3}{1/2} \cot \theta = \dfrac{8}{9}$
$\Rightarrow \tan \theta = \dfrac{3}{4} \Rightarrow \sin \theta = \pm \dfrac{3}{5}$ and $\cos \theta = \pm \dfrac{4}{5}$ and the required point can be $\left(-\dfrac{4}{5}\times\dfrac{1}{2}, \dfrac{3}{5}\times\dfrac{1}{3}\right) = \left(-\dfrac{2}{5}, \dfrac{1}{5}\right)$
17. Coordinates of the focii are $(\pm ae, 0)$ where $a^2 = 1/4$, $b^2 = 1/9$ and $a^2e^2 = a^2 - b^2 = 5/36$.
18. $6 \cos^2 \theta + 2 \sin^2 \theta = 4 \Rightarrow \cos^2 \theta = 1/2$ so $\theta = \pi/4$.
19. Positive end of the latus rectum is $(ae, b^2/a)$ equation of the normal is
$y - b^2/a = (1/e)(x - ae)$
$\Rightarrow ae\, y - ax + (a^2 - b^2)e = 0$
$\Rightarrow ae\, y - ax + a^2e^3 = 0 \Rightarrow x - ey - ae^3 = 0$
20. Radius of the circle must be greater than the major axis of the ellipse.
21. $a^2e^2 + b^2 = (2b)^2 \Rightarrow a^2e^2 = 3a^2 (1 - e^2)$
$\Rightarrow e^2 = 3/4$.
22. Point C on the ellipse is $(\sqrt{3}\cos\theta, -\sin\theta)$,
$\underline{|OBC} = \pi/6$
$\Rightarrow \dfrac{1+\sin\theta}{-\sqrt{3}\cos\theta} = \tan(\pi/2 + \pi/6) = -\sqrt{3}$
$\Rightarrow (1 + \sin \theta)^2 = 9 \cos^2 \theta$.
$\Rightarrow \sin \theta = 4/5, \cos \theta = 3/5, BC = 6\sqrt{3}/5$.

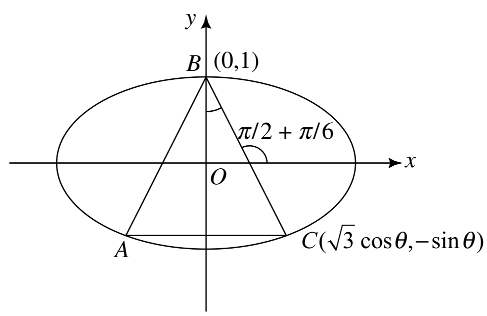

Fig. 19.21

23. $10 - 2a > 0, 4 - 2a > 0$
$\Rightarrow a < 5, a < 2$.
$\Rightarrow a \in (-\infty, 2)$
24. $\left(\dfrac{x}{3}\right)^2 + \left(\dfrac{y}{4}\right)^2 = 2(\cos^2 t + \sin^2 t) = 2$
$\Rightarrow \dfrac{x^2}{18} + \dfrac{y^2}{32} = 1$
which is an ellipse with $e^2 = \dfrac{32-18}{32} = \dfrac{7}{16}$
$\Rightarrow 16e^2 - 7 = 0$
$\Rightarrow 16e^2 + 7 = 14$

25. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ then

$e = \frac{2}{3}$, $\frac{a}{e} = 6 \Rightarrow a = 4$ $e^2 = 1-\frac{b^2}{a^2} \Rightarrow b^2 = \frac{80}{9}$.

Hence the required equation is

$\frac{x^2}{16}+\frac{9y^2}{80} = 1 \Rightarrow 5x^2 + 9y^2 = 80$

26. Equation of a tangent is $y = mx + \sqrt{a^2m^2+b^2}$ (1)

Equation of the perpendicular from the focus $(ae, 0)$ on it is

$y = -\frac{1}{m}(x - ae)$ or $my + x = ae$ (2)

Squaring and adding (1) and (2), eliminating m,

$(y - mx)^2 + (my + x)^2 = a^2m^2 + b^2 + a^2e^2$

$\Rightarrow (1 + m^2)(x^2 + y^2) = (1 + m^2)a^2$

$\Rightarrow x^2 + y^2 = a^2$ is the required locus

27. An equation of the tangent to the ellipse with slope $\frac{b}{a}$ is

$y = \frac{b}{a}x \pm \sqrt{a^2 \times \frac{b^2}{a^2}+b^2}$

As it touches the circle $x^2 + y^2 = r^2$

$2b^2 = r^2\left(1+\frac{b^2}{a^2}\right)$

$\Rightarrow r^2 = \frac{2a^2b^2}{a^2+b^2}$.

28. The product of perpendiculars from foci upon any tangent to the ellipse is equal to square of the semi-minor axis (see theory). So the required product is 100.

29. Let $P(h, k)$ be the foot of the perpendicular, O the centre then slope of $OP = \frac{k}{h}$ and the slope of the tangent is $-\frac{h}{k}$. Equation of the tangent is

$y - k = -\frac{h}{k}(x - h)$

$\Rightarrow y = -\frac{h}{k}x + \frac{h^2+k^2}{k}$

which touches the given ellipse

if $\left(\frac{h^2+k^2}{k}\right)^2 = 25\left(\frac{h}{k}\right)^2 + 16$

$\Rightarrow 25h^2 + 16k^2 = (h^2 + x^2)^2$

Locus of (h, k) is

$25x^2 + 16y^2 = (x^2 + y^2)^2$

30. $e^2 = \frac{7}{16} \Rightarrow e = \frac{\sqrt{7}}{4}$

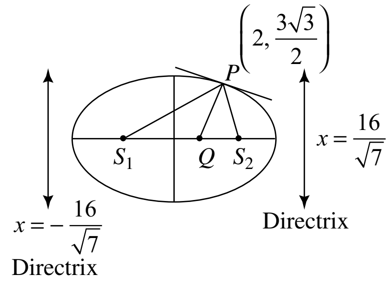

Fig. 19.22

Coordinates of S_1 are $(-\sqrt{7}, 0)$ and S_2 are $(\sqrt{7}, 0)$ PQ is the internal bisector of $\angle S_1PS_2$

So $\frac{S_1Q}{S_2Q} = \frac{PS_1}{PS_2} = \frac{\frac{16}{\sqrt{7}}+2}{\frac{16}{\sqrt{7}}-2}$

$= \frac{8+\sqrt{7}}{8-\sqrt{7}}$

31. Equation of the chord is

$\frac{1/4}{25} + \frac{4/25}{16} - 1 = \frac{(1/2)x}{25} + \frac{(2/5)y}{16} - 1$ $(T = S_1)$

$\Rightarrow 4x + 5y = 4$

which meets minor axis at $A\left(0, \frac{4}{5}\right)$ and major axis at $B(1, 0)$.

So the length of $AB = \sqrt{1+\frac{16}{25}} = \frac{\sqrt{41}}{5}$ units.

32. $F_1(-ae, 0)$, $F_2(ae, 0)$, $B(0, b)$

$(F_1F_2)^2 = (BF_1)^2$

$\Rightarrow 4a^2e^2 = a^2e^2 + b^2$

$\Rightarrow 3a^2e^2 = a^2(1 - e^2)$

$\Rightarrow e^2 = \frac{1}{4} \Rightarrow e = \frac{1}{2}$

33. Equation of the ellipse is

$(x - 2)^2 + (y + 3)^2 = \left(\frac{1}{2}\right)^2\left[\frac{3x-4y+7}{5}\right]^2$

Centre is $(2, -3)$, eccentricity $e = \frac{1}{2}$ and a directrix is $3x - 4y + 7 = 0$.

Length of the perpendicular from the centre on the directrix is

$$\left|\frac{3\times 2-4(-3)+7}{5}\right| = 5$$

$$\Rightarrow \frac{a}{e} = 5 \Rightarrow 2a = 5$$

Hence the length of the major axis is 5 units.

34. The equation of the tangent to the ellipse at $P(5\cos\theta, 4\sin\theta)$ is

$$\frac{x\cos\theta}{5}+\frac{y\sin\theta}{4} = 1$$

which meets $x = 0$ at $Q\ (0, 4\ \text{cosec}\ \theta)$

Image of Q in $y = x$ is $R\ (4\ \text{cosec}\ \theta, 0)$

Equation of the circle on QR as a diameter is

$x(x-4\ \text{cosec}\ \theta)+y(y-4\ \text{cosec}\ \theta) = 0$

$\Rightarrow x^2+y^2-4(x+y)\ \text{cosec}\ \theta = 0$

which passes through the point of intersection of $x^2+y^2 = 0$ and $x+y = 0$ i.e the point (0, 0).

35. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

A be the point $(a\cos\theta, b\sin\theta)$

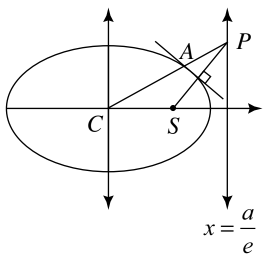

Fig. 19.23

Equation of AC is

$$y = \frac{b}{a}\tan\theta\, x.$$

which meets the directrix

$$x = \frac{a}{e} \text{ at } P\left(\frac{a}{e}, \frac{b}{e}\tan\theta\right)$$

slope of the tangent at $A = \frac{-b}{a\tan\theta}$.

S is $(ae, 0)$

$$\text{So slope of } PS = \frac{\frac{b}{e}\tan\theta}{\frac{a}{e}-ae} = \frac{b\tan\theta}{a(1-e^2)} = \frac{b\tan\theta}{a\times\frac{b^2}{a^2}}$$

$$= \frac{a}{b}\tan\theta.$$

So PS is perpendicular to the tangent at A. $\Rightarrow \alpha = \frac{\pi}{2}$

36. Equation of a tangent with slope-2 is

$$y = -2x \pm \sqrt{4a^2+b^2}$$

This is a normal to the circle $x^2+y^2-4x+1 = 0$ if it passes through the centre (2, 0) of the circle.

$0 = -4 \pm \sqrt{4a^2+b^2}$

$\Rightarrow 4a^2+b^2 = 16$

Using A.M. $\geq$ G. M.

$$\frac{4a^2+b^2}{2} \geq \sqrt{4a^2b^2}$$

$8 \geq 2ab.$

$\Rightarrow ab \leq 4.$

37. Let $p = \cos\theta$, $0 \leq \theta \leq \pi$. The equation of the line becomes $(3\cos\theta)x+(2\sin\theta)y = 1$

$$\text{or } \frac{x}{1/3}\cos\theta+\frac{y}{1/2}\sin\theta = 1$$

which touches the ellipse

$$\frac{x^2}{\frac{1}{9}}+\frac{y^2}{\frac{1}{4}} = 1$$

eccentricity of the ellipse is

$$\sqrt{\frac{\frac{1}{4}-\frac{1}{9}}{\frac{1}{4}}} = \frac{\sqrt{5}}{3}$$

and the equation of a directrix is

$$y = \frac{1/2}{\sqrt{5}/3} = \frac{3\sqrt{5}}{10}.$$

38. Since a and b are natural numbers such that $a+b = ab$, $a = b = 2$

Equation of the chord of the ellipse $\frac{x^2}{4}+\frac{y^2}{1} = 1$ having (a, b) as the mid point is $\frac{x(2)}{4}+\frac{y(2)}{1}-1 = \frac{2^2}{4}+\frac{2^2}{1}-1\ (T = S_1)$

$\Rightarrow x+4y = 10.$

39. $l = 2\sec\theta$, $e = \sqrt{\frac{\sec^2\theta-\tan^2\theta}{\sec^2\theta}} = \cos\theta$

$\Rightarrow el = 2$ which is independent of θ.

40. Given equation can be written as $3(x-1)^2 + 4(y+1)^2 = 7 - k$ which represent an ellipse if $k < 7$

$$\Rightarrow \frac{(x-1)^2}{\frac{7-k}{3}} + \frac{(y+1)^2}{\frac{7-k}{4}} = 1$$

So $\dfrac{\frac{7-k}{3} - \frac{7-k}{4}}{\frac{7-k}{3}} = e^2 = \dfrac{1}{4}$ for all $k < 7$

41. Tangent at $P(a\cos\theta, b\sin\theta)$ be

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

It meets the directrix $x = \dfrac{a}{e}$ at the point T

$$\left(\frac{a}{e}, \frac{b(e-\cos\theta)}{e\sin\theta}\right).$$

Focus $S(ae, 0)$

Slope of $SP = \dfrac{b\sin\theta}{a(\cos\theta - e)}$,

Slope of $ST = \dfrac{b(e-\cos\theta)}{ae\sin\theta(1-e^2)}$

Product of the slope $= -\dfrac{b^2}{a^2(1-e^2)} = -1$

$\Rightarrow$ Statement-2 is true using which statement-1 is also true.

42. **Statement-2** is false. Equation of the normal is $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$

In statement-1 $L(ae, b^2/a) = (a\cos\theta, b\sin\theta)$

$\Rightarrow \cos\theta = e$.

So normal at L, $\dfrac{ax}{e} - \dfrac{by}{\sqrt{1-e^2}} = a^2e^2$

which meets the major axis $y = 0$ at $x = ae^3$ and the statement-1 is true.

43. By definition of the ellipse statement-2 is true and using it statement-1 is also true.

44. Statement-2 is true but statement-1 is false as the locus represents a straight line.

45. Statement-1 is true as $p_1p_2 = b^2 = 7$. Statement-2 is also true as the foot of the perpendicular lie on the auxiliary circle which is $x^2 + y^2 = 15$ but does not justify statement-1.

46. Let $y = mx$ be a chord of the conic $x^2 + y^2 + xy = 1$ through (0, 0), which meets the conic at points for which $x^2 + m^2x^2 + mx^2 - 1 = 0$.

$\Rightarrow (1 + m + m^2)\, x^2 - 1 = 0$

$\Rightarrow x_1 + x_2 = 0 \Rightarrow y_1 + y_2 = 0$

Showing that (0, 0) is the mid point of the chord and hence statement-1 is true using statement-2 which is true.

47. Let the equation of the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Equation of any tangent is $y = mx \pm \sqrt{a^2m^2 + b^2}$.

Comparing with given equation we get

$m = -\dfrac{2p}{\sqrt{1-p^2}}$ and $a^2m^2 + b^2 = \dfrac{1}{1-p^2}$

$$\Rightarrow a^2 \times \frac{4p^2}{1-p^2} + b^2 = \frac{1}{1-p^2}$$

$\Rightarrow p^2(4a^2 - b^2) + b^2 - 1 = 0$

which is true for all p if $b^2 = 1$, $4a^2 = b^2 = 1$

so the equation of the ellipse is

$$\frac{x^2}{\frac{1}{4}} + \frac{y^2}{1} = 1$$

Length of the major axis $= 2b = 2$ and of minor axis $= 2a = 1$.

Thus statement-2 is true. Using it in statement-1 equation of the director circle is $x^2 + y^2 = a^2 + b^2$ or $x^2 + y^2 = 1 + \dfrac{1}{4} = \dfrac{5}{4} \Rightarrow 4x^2 + 4y^2 = 5$ and thus statement-1 is also true.

48. Extremities of the latus rectum are $\left(\pm ae, \dfrac{b^2}{a}\right)$ since they lie on the parabola $x^2 = -2(y-2)$

We have $a^2e^2 = -2\left(\dfrac{b^2}{a} - 2\right)$, $b^2 = a^2(1 - e^2)$

$\Rightarrow a^2e^2 - 2a\,e^2 + 2a - 4 = 0$

$\Rightarrow (ae^2 + 2)(a - 2) = 0 \Rightarrow a = 2$

Thus statement-1 is true.

In statement-2 $\dfrac{2b^2}{a} = 2ae \Rightarrow b^2 = a^2e$

$\Rightarrow a^2(1 - e^2) = a^2e \Rightarrow e^2 + e - 1 = 0$

and the statement-2 is also true but does not justify statement-1.

49. Statement-2 is true by definition of conjugate diameters.

Let $y = mx$ and $y = m'x$ be two conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Let (h, k) be the mid point of chord whose step is m then

$$\frac{hx}{a^2} + \frac{ky}{b^2} - 1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$$

$$\Rightarrow m = -\frac{b^2}{a^2}\frac{h}{k} \Rightarrow \text{Locus of } (h, k) \text{ is } y = \frac{-b^2}{a^2 m}x$$

$$\Rightarrow \frac{-b^2}{a^2 m} = m' \Rightarrow m'm = \frac{-b^2}{a^2}. \text{ (using statement-2)}$$

and thus statement-1 is false.

50. Tangent at the positive end of the minor axis of E is $y = b$ which meets the parabola $y^2 = 4bx$ at the point $\left(\frac{b}{4}, b\right)$ which is not an end of the latus rectum of P.

so statement-1 is false.

In statement-2 if $x = b$ and $x = ae$ represent the same line then $b = ae$

$$\Rightarrow \frac{b}{a} = e \Rightarrow a^2(1 - e^2) = a^2e^2$$

$$\Rightarrow e^2 = \frac{1}{2} \Rightarrow e = \frac{1}{\sqrt{2}}.$$

Thus statement-2 is true.

Level 2

51. Chords of contacts are

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \text{ and } \frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1$$

Since they are at right angles

$$\frac{-b^2}{a^2} \times \frac{x_2}{y_2} \times \frac{-b^2}{a^2} \times \frac{x_1}{y_1} = -1$$

$$\Rightarrow \frac{x_1x_2}{y_1y_2} = \frac{-a^4}{b^4}$$

52. Normal at $P\ (\sqrt{14}\cos\theta, \sqrt{5}\sin\theta)$ to the ellipse is

$$\frac{\sqrt{14}x}{\cos\theta} - \frac{\sqrt{5}y}{\sin\theta} = 14 - 5 = 9.$$

Since it passes through $Q\ (2\theta)$

$$\frac{\sqrt{14} \times \sqrt{14}\cos 2\theta}{\cos\theta} - \frac{\sqrt{5} \times \sqrt{5}\sin 2\theta}{\sin\theta} = 9$$

$$\Rightarrow \frac{14(2\cos^2\theta - 1)}{\cos\theta} - \frac{5 \times 2\sin\theta\cos\theta}{\sin\theta} = 9$$

$\Rightarrow 18\cos^2\theta - 9\cos\theta - 14 = 0$

$\Rightarrow (3\cos\theta + 2)(6\cos\theta - 7) = 0$

$\Rightarrow \cos\theta = -2/3.$

53. C: $x^2 + y^2 - 9 = 0$

E: $x^2/9 + y^2/4 - 1 = 0$

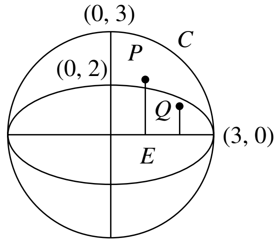

Fig. 19.24

For $P(1, 2)$, C: $1 + 4 - 9 < 0$, E: $\frac{1}{9} + \frac{4}{4} - 1 > 0$

So P lies outside E and inside C.

For $Q(2, 1)$, C: $4 + 1 - 9 < 0$, E: $\frac{4}{9} + \frac{1}{4} - 1 < 0$

So Q lies inside both C and E.

Note: We get the result from the figure.

54. Let $P(5\cos\theta, 3\sin\theta)$

Equation of the tangent at P is

$$\frac{x\cos\theta}{5} + \frac{y\sin\theta}{3} = 1$$

$$\Rightarrow CF = \frac{1}{\sqrt{\frac{\cos^2\theta}{25} + \frac{\sin^2\theta}{9}}} = \frac{15}{\sqrt{9\cos^2\theta + 25\sin^2\theta}}$$

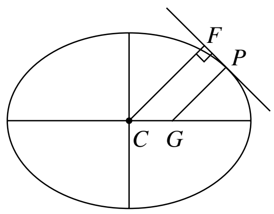

Fig. 19.25

Equation of the normal at P is

$$\frac{5x}{\cos\theta} - \frac{3y}{\sin\theta} = 25 - 9 = 16$$

Coordinates of G are $\left(\frac{16\cos\theta}{5}, 0\right)$

$$\Rightarrow PG = \frac{3}{5}\sqrt{9\cos^2\theta + 25\sin^2\theta}$$

Hence $CF \cdot PG = 9$.

55. Centre of the circle is $(-1, 0)$ and radius is 4. Equation of the tangent at $P(\theta)$ to the ellipse is

$$\frac{x}{4}\cos\theta + \frac{y}{16/\sqrt{11}}\sin\theta = 1$$

If this touches the circle, then

$$\frac{\frac{-\cos\theta}{4} - 1}{\sqrt{\frac{\cos^2\theta}{16} + \frac{\sin^2\theta}{256/11}}} = 4$$

$$\Rightarrow (\cos\theta + 4)^2 = 256\left(\frac{\cos^2\theta}{16} + \frac{11\sin^2\theta}{256}\right)$$

$\Rightarrow 4\cos^2\theta - 8\cos\theta - 5 = 0$
$\Rightarrow (2\cos\theta - 5)(2\cos\theta + 1) = 0$
$\Rightarrow \cos\theta = -1/2 \Rightarrow \theta = 2\pi/3$

56. Let the extremities of a pair of semi-conjugate diameters of the ellipse be $P(3\cos\theta, 2\sin\theta)$ and $Q(3\cos\alpha, 2\sin\alpha)$, C being the centre.

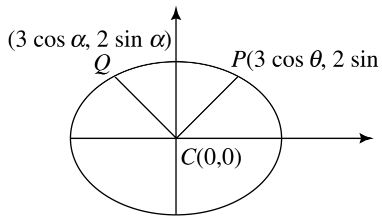

Fig. 19.26

Then slope of $CP \times$ slope of $CQ = -\frac{4}{9}$

$$\Rightarrow \frac{2}{3}\tan\theta \times \frac{2}{3}\tan\alpha = -\frac{4}{9}$$

$\Rightarrow 1 + \tan\theta\tan\alpha = 0 \Rightarrow \alpha - \theta = \pi/2$
$\Rightarrow$ coordinates of Q are $(-3\sin\theta, 2\cos\theta)$ and the equation of PQ is
$2(\sin\theta - \cos\theta)x - 3(\cos\theta + \sin\theta)y + 6 = 0$
which represents $lx + my + n = 0$, then

$$\frac{l}{2(\sin\theta - \cos\theta)} = \frac{m}{-3(\cos\theta + \sin\theta)} = \frac{n}{6}$$

So $\frac{9l^2 + 4m^2}{n^2} = (\sin\theta - \cos\theta)^2 + (\cos\theta + \sin\theta)^2 = 2.$

57. We must have $f(a^2 + 5a + 3) < f(3a + 15)$
$\Rightarrow a^2 + 5a + 3 > 3a + 15$ [$\because f$ is strictly decreasing]
$\Rightarrow a^2 + 2a - 12 > 0$
$\Rightarrow (a + 6)(a - 2) > 0$
$\Rightarrow a \notin (-6, 2)$.

58. The director circle of E_1 is $x^2 + y^2 = a^2 + 2 + b^2$ and of E_2 is $x^2 + y^2 = a^2 + b^2 + 1$.
Both are concentric circle such that one lies inside the other so that they have no point in common.

59. Equation of the tangent at $P(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

$$d = \frac{1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}} \Rightarrow \frac{1}{d^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} \quad (1)$$

$F_1(-ae, 0)$, $F_2(ae, 0)$

$$\Rightarrow PF_1 = e\left(a\cos\theta + \frac{a}{e}\right) = a(1 + e\cos\theta)$$

$PF_2 = a(1 - e\cos\theta)$
So $(PF_1 - PF_2)^2 = 4a^2e^2\cos^2\theta$

$$= 4(a^2 - b^2) \times \frac{\frac{1}{d^2} - \frac{1}{b^2}}{\frac{1}{a^2} - \frac{1}{b^2}}$$

$$= 4a^2\left(1 - \frac{b^2}{d^2}\right).$$

60. Let the middle point of the chord be $(t^2, 2t)$
It must lie inside the ellipse
So $t^4 + 8t^2 - 1 < 0$
$\Rightarrow t^2 \in (0, -4 + \sqrt{17})$
Equation of the chord of the ellipse with $(t^2, 2t)$ as the mid-point is
$t^2x + 4ty = t^4 + 8t^2$ $(T = S_1)$
Since it passes through $(a, 0)$
$at^2 = t^4 + 8t^2$
$\Rightarrow t^4 + (8 - a)t^2 = 0$
$\Rightarrow t^2 = 0$ or $t^2 = a - 8$
$\Rightarrow a = t^2 + 8$
$\Rightarrow a \in (8, 4 + \sqrt{17})$

Previous Years' AIEEE/JEE Main Questions

1. Equation one of directrix of ellipse with centre at the origin is $x = a/e$

$\therefore \frac{a}{e} = 4 \Rightarrow a = 2$ [$\because e = 1/2$]

Also, $b^2 = a^2(1 - e^2) = 4\left(1 - \frac{1}{4}\right) = 3$

Thus, equation of required ellipse is

$$\frac{x^2}{4}+\frac{y^2}{3} = 1 \text{ or } 3x^2 + 4y^2 = 12$$

2. $AB = 2\,b \sin\theta$

$AD = 2\,a\cos\theta$

Area of rectangle $= 4ab\sin\theta\cos\theta$

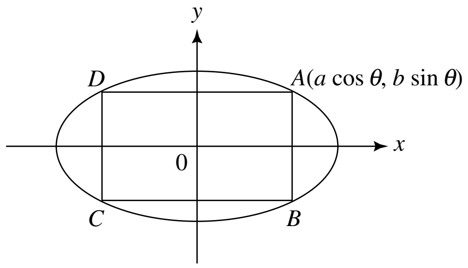

Fig. 19.27

$= 2ab \sin 2\theta$

Greatest area is obtained when $\theta = \pi/4$ and greatest area $= 2ab$.

3. Slope of BF is $m_1 = \dfrac{-b}{ae}$

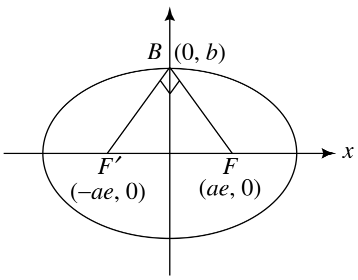

Fig. 19.28

and that of BF' is $m_2 = \dfrac{-b}{-ae} = \dfrac{b}{ae}$

As $BF \perp BF'$, $m_1 m_2 = -1$

$\Rightarrow \dfrac{b^2}{a^2e^2} = 1$

$\Rightarrow a^2(1-e^2) = a^2e^2$

$\Rightarrow 2e^2 = 1$ or $e = \dfrac{1}{\sqrt{2}}$

4. We have $2ae = 6$ and $2b = 8$

$\Rightarrow ae = 3,\ b = 4$

Now, $b^2 = a^2(1-e^2)$

$\Rightarrow 16 = a^2 - 9 \Rightarrow a = 5$

$\therefore\ e = 3/5$

5. Equation of the ellipse is

$$x^2 + y^2 = \frac{1}{4}(4-x)^2$$

$\Rightarrow 3x^2 + 8x + 4y^2 = 16$

$$\Rightarrow \left(x+\frac{4}{3}\right)^2 + \frac{4}{3}y^2 = \frac{16}{3}+\frac{16}{9} = \frac{64}{9}$$

$$\Rightarrow \frac{\left(x+\frac{4}{3}\right)^2}{\frac{64}{9}} + \frac{y^2}{\frac{3}{4}\times\frac{64}{9}} = 1$$

Length of the semi-major axis $= \sqrt{\dfrac{64}{9}} = \dfrac{8}{3}$.

6. The required ellipse passes through (4, 0) and (2, 1). Let equation of ellipse be

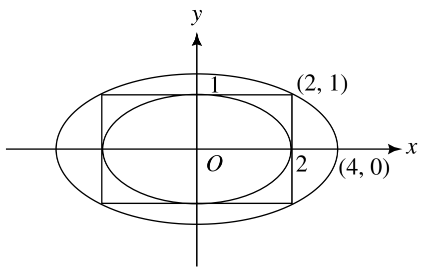

Fig. 19.29

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

As it pass through (4, 0) and (2, 1)

$\dfrac{16}{a^2} = 1$ and $\dfrac{4}{a^2}+\dfrac{1}{b^2} = 1$

$\Rightarrow a^2 = 16$ and $b^2 = 4/3$

Thus, required ellipse is

$$\frac{x^2}{16}+\frac{y^2}{4/3} = 1 \Rightarrow x^2 + 12y^2 = 16$$

7. Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

Since it passes through $(-3, 1)$, we have

$\dfrac{9}{a^2}+\dfrac{1}{b^2} = 1$, also $b^2 = a^2\left(1-\dfrac{2}{5}\right) = \dfrac{3}{5}a^2$

$\Rightarrow 9 + \dfrac{5}{3} = a^2 \Rightarrow a^2 = \dfrac{32}{3},\ b^2 = \dfrac{32}{5}$.

and the required equation is

$$\frac{x^2}{\frac{32}{3}}+\frac{y^2}{\frac{32}{5}} = 1 \Rightarrow 3x^2 + 5y^2 = 32$$

8. Minor axis is along the x-axis and its length is $2(1) = 2$. Major axis is along the y-axis and its length is $2(2) = 4$.

Equation of the ellipse is

$$\frac{x^2}{1^2} + \frac{y^2}{2^2} = 1 \Rightarrow 4x^2 + y^2 = 4$$

9. We have $x^2 + 4y^2 = 4c^2$ as the ellipse.
Points of intersection with the circle $x^2 + y^2 = 9a^2$ are given by $9a^2 - y^2 + 4y^2 = 4c^2$
$\Rightarrow 3y^2 = 4c^2 - 9a^2$
$\Rightarrow 4c^2 > 9a^2 \Rightarrow 3a < 2c$
$\Rightarrow 9a^2 + 2c^2 < 6c^2,\ 9ac < 6c^2$
$\Rightarrow 9ac - (9a^2 + 2c^2) < 0$

10. $e_1^2 = \dfrac{3-2}{3} = \dfrac{1}{3},\ e_2^2 = \dfrac{16-b^2}{16}$

$$(e_1e_2)^2 = \frac{16-b^2}{48} = \left(\frac{1}{2}\right)^2$$

$\Rightarrow 16 - b^2 = 12 \Rightarrow b^2 = 4 \Rightarrow b = 2$
and the required length is 4.

11. Let $P(x_1, y_1)$ be a common point.
Tangents to the curves at Pare

$$yy_1 = 8(x + x_1) \text{ and } \frac{xx_1}{\alpha} + \frac{yy_1}{4} = 1$$

Product of the slopes $= \dfrac{8}{y_1} \times \left(\dfrac{-4x_1}{\alpha y_1}\right) = -1$

$$\Rightarrow y_1^2 = \frac{32}{\alpha} x_1 \qquad (1)$$

As P lies on the parabola $y^2 = 16x$.

$$\Rightarrow y_1^2 = 16x_1 \qquad (2)$$

From (1) and (2) we get $\alpha = 2$

12. Slope of the tangent is $-\dfrac{1}{2}$, so its equation is

$$y = -\frac{1}{2}x \pm \sqrt{9 \times \frac{1}{4} + 4} \Rightarrow x + 2y = \pm 5$$

If it meets the parabola $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ at (x_1, y_1)

its equation is $\dfrac{xx_1}{9} + \dfrac{yy_1}{4} = 1$

So $\dfrac{x_1}{9} = \dfrac{y_1}{8} = \pm\dfrac{1}{5}$

$\Rightarrow (x_1, y_1) = \left(\dfrac{9}{5}, \dfrac{8}{5}\right)$ taking +ve sign.

13. Equation of any tangent to the ellipse
$x^2 + 3y^2 = 6$ is $y = mx \pm\sqrt{6m^2 + 2}$
If (h, k) is the foot of the perpendicular from (0, 0), the centre of the ellipse on this tangent then

$$k = mh \pm\sqrt{6m^2 + 2} \text{ and } \left(\frac{k}{h}\right) m = -1$$

Eliminating m we get
$(h^2 + k^2)^2 = 6h^2 + 2k^2$
Required locus is $(x^2 + y^2)^2 = 6x^2 + 2y^2$

14. Coordinates of F_1, F_2 and B are $(ae, 0)$, $(-ae, 0)$ and $(0, b)$ respectively. Slope of $B F_1 = -b/ae$ and slope of $B F_2$ is b/ae.
As $F_1 B \perp F_2 B$,

$$\left(\frac{-b}{ae}\right)\left(\frac{b}{ae}\right) = -1 \Rightarrow b^2 = a^2e^2$$

$\Rightarrow a^2(1 - e^2) = a^2e^2 \Rightarrow e^2 = 1/2$

15. An equation of tangent to the ellipse

$\dfrac{x^2}{16} + \dfrac{y^2}{81} = 1$ is

$$\frac{x}{4}\cos\theta + \frac{y}{9}\sin\theta = 1$$

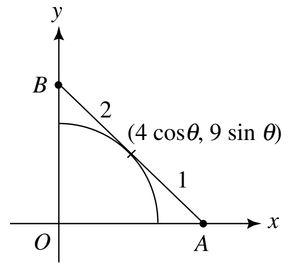

Fig. 19.30

Coordinates of A and B are $(4\sec\theta, 0)$ and $(0, 9\operatorname{cosec}\theta)$ respectively.

Area of $\Delta OAB = \dfrac{1}{2}\dfrac{36}{|\sin\theta\cos\theta|} = \dfrac{36}{\sin 2\theta}$

So minimum value of area 36.

16. We have $a^2 = 9$, $b^2 = 5$, $e^2 = 1 - \dfrac{5}{9} = \dfrac{4}{9} \Rightarrow e = \dfrac{2}{3}$.

Coordinates of L are $(ae, b^2/a) = (2, 5/3)$

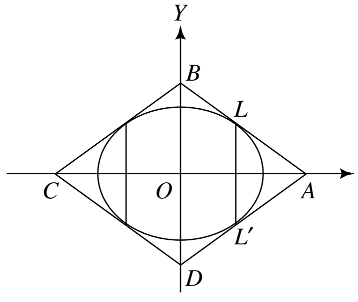

Fig. 19.31

Equation of tangent at L is

$$\frac{2x}{9}+\frac{5}{3}\left(\frac{y}{5}\right)=1$$

or $2x + 3y = 9$

It meets the axes at $A(9/2, 0)$ and $B(0, 3)$.

Area of quadrilateral

$= 4[\text{area of } \Delta OAB]$

$$= 4\left[\frac{1}{2}\left(\frac{9}{2}\right)(3)\right] = 27\ (\text{unit})^2$$

17. We have distance between the foci $= 2ae$

length of the latus rectum $= \dfrac{2b^2}{a}$

So $2\,ae = \dfrac{1}{2}\left(2\dfrac{b^2}{a}\right) \Rightarrow b^2 = 2\,a^2e$

$\Rightarrow a^2\,(1 - e^2) = 2\,a^2e$

$\Rightarrow e^2 + 2e = 1 \Rightarrow (e + 1)^2 = 2$

$\Rightarrow e = \sqrt{2}-1$

18. An equation of tangent at $(3\sqrt{3}\cos\theta, \sqrt{3}\sin\theta)$ to the ellipse $\dfrac{1}{27}x^2+\dfrac{1}{3}y^2 = 1$ is

$x\cos\theta + 3y\sin\theta = 3\sqrt{3}$.

It meets the axes in $A\,(3\sqrt{3}\sec\theta, 0)$, and

$B\ \ (0, \sqrt{3}\,\text{cosec}\,\theta)$

Let A = area of ΔOAB is $\dfrac{1}{2}|(3\sqrt{3}\sec\theta)(\sqrt{3}\text{cosec}\,\theta)|$

$$= \frac{9}{|\sin 2\theta|} \geq 9$$

Thus, least value of A is 9 which is attained when $\theta = \pi/4, 3\pi/4, 5\pi/4$, or $7\pi/4$.

Previous Years' B-Architecture Entrance Examination Questions

1. $2\,\dfrac{a}{e} = 4 \times 2ae \Rightarrow e = \dfrac{1}{2}$

$a = 2$, so $b^2 = 4\left(1-\dfrac{1}{4}\right) = 3$

Hence the required equation is

$$\frac{x^2}{4}+\frac{y^2}{3} = 1 \Rightarrow 3x^2 + 4y^2 = 12$$

2. Tangent at $\left(1, \dfrac{3}{4}\right)$ to the ellipse $3x^2 + 16y^2 = 12$ is

$$\frac{x \cdot 1}{4}+\frac{y\left(\frac{3}{4}\right)}{\frac{3}{4}} = 1$$

$\Rightarrow x + 4y = 4$

which intersects the curve $y^2 + x = 0$ at points for which

$y^2 + (4 - 4y) = 0 \Rightarrow (y - 2)^2 = 0 \Rightarrow y = 2$

and the points of intersection is $(-4, 2)$ exactly one point.

3. Equation of the tangent with slope tan 135° to the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{9} = 1$ is $y = -x + \sqrt{16(1)+9}$

$\Rightarrow x + y = 5$ which meets the given ellipse at $\left(\dfrac{16}{5}, \dfrac{9}{5}\right)$

4. Suppose the tangent intersect at (h, k), then equation of chord of contact is

$4hx + ky = 5$.

But equation of chord of contact is:

$2x + y = 3$

$\therefore\ \dfrac{4h}{2} = \dfrac{k}{1} = \dfrac{5}{3}$

$\Rightarrow h = 5/6,\ k = 5/3$.

Thus, required point is $\left(\dfrac{5}{6}, \dfrac{5}{3}\right)$.