

JEE ADVANCED

1. If $f(x) = \sqrt{\frac{x - \sin x}{x + \cos^2 x}}$, then $\lim_{x \rightarrow \infty} f(x)$ is

- a. 0 b. ∞
c. 1 d. none of these

2. If $G(x) = -\sqrt{25 - x^2}$, then $\lim_{x \rightarrow 1} \frac{G(x) - G(1)}{x - 1}$ is

- a. $\frac{1}{24}$ b. $\frac{1}{5}$
c. $-\sqrt{24}$ d. none of these

3. $\lim_{n \rightarrow \infty} \left\{ \frac{1}{1-n^2} + \frac{2}{1-n^2} + \dots + \frac{n}{1-n^2} \right\}$ is equal to

- a. 0 b. $-\frac{1}{2}$
c. $\frac{1}{2}$ d. none of these

4. If $f(x) = \begin{cases} \frac{\sin[x]}{[x]}, & \text{for } [x] \neq 0 \\ 0, & \text{for } [x] = 0 \end{cases}$, where $[x]$ denotes the

greatest integer less than or equal to x , then $\lim_{x \rightarrow 0} f(x)$ is

- a. 1
b. 0
c. -1
d. none of these

5. The value of $\lim_{x \rightarrow 0} \frac{\sqrt{\frac{1}{2}(1 - \cos 2x)}}{x}$ is

- a. 1
b. -1
c. 0
d. none of these

6. $\lim_{x \rightarrow 1} \frac{\sqrt{1 - \cos 2(x-1)}}{x-1}$

- a.** exists and it equals $\sqrt{2}$

- b.** exists and it equals $-\sqrt{2}$
c. does not exist because $x - 1 \rightarrow 0$
d. does not exist because the left-hand limit is not equal to the right-hand limit (IIT-JEE 1998)

7. Let n be an odd integer. If $\sin n\theta = \sum_{r=0}^n b_r \sin^r \theta$, for every value of θ , then

- a.** $b_0 = 1, b_1 = 3$ **b.** $b_0 = 0, b_1 = n$
c. $b_0 = -1, b_1 = n$ **d.** $b_0 = 0, b_1 = n^2 - 3n + 3$

8. $\lim_{x \rightarrow 0} \frac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2}$ is equal to

- a. 2
b. -2
c. 1/2
d. -1/2 (IIT-JEE 1999)

9. For $x \in \mathbb{R}$, $\lim_{x \rightarrow \infty} \left(\frac{x-3}{x+2} \right)^x$ is equal to

- a. e
c. e^{-5}
- b. e^{-1}
d. e^5
- (IIT-JEE 2000)

10. $\lim_{x \rightarrow 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ is equal to

- a. $-\pi$ b. π
c. $\pi/2$ d. 1 (IIT-JEE 2001)

11. The integer n for which $\lim_{x \rightarrow 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite nonzero number is

- a. 1 b. 2
c. 3 d. 4 (IIT-JEE 2002)

12. If $\lim_{x \rightarrow 0} \frac{\{(a-n)nx - \tan x\} \sin nx}{x^2} = 0$, where n is nonzero real number, then a is

- a. 0 b. $\frac{n+1}{n}$
c. n d. $n + \frac{1}{n}$ (IIT-JEE 2003)

- 13.** The value of $\lim_{x \rightarrow 0} ((\sin x)^{1/x} + (1+x)^{\sin x}) = 0$, where $x > 0$, is

- a. 0 b. -1
c. 1 d. 2 (IIT-JEE 2006)

14. Let $g(x) = \frac{(x-1)^n}{\log \cos^m(x-1)}$; $0 < x < 2$, m and n are integers, $m \neq 0$, $n > 0$ and let p be the left hand derivative of $|x-1|$ at $x=1$. If $\lim_{x \rightarrow 1^+} g(x) = p$, then

- a. $n=1, m=1$ b. $n=1, m=-1$
c. $n=2, m=2$ d. $n>2, m=n$

(IIT-JEE 2008)

15. If $\lim_{x \rightarrow 0} [1 + x \ln(1+b^2)]^{1/x} = 2b \sin^2 \theta$, $b > 0$,

smf $\theta \in (-\pi, \pi]$, then the value of θ is

- a. $\pm \frac{\pi}{4}$ b. $\pm \frac{\pi}{3}$
c. $\pm \frac{\pi}{6}$ d. $\pm \frac{\pi}{2}$ (IIT-JEE 2011)

16. If $\lim_{x \rightarrow \infty} \left(\frac{x^2 + x + 1}{x + 1} - ax - b \right) = 4$, then

- a. $a=1, b=4$ b. $a=1, b=-4$
c. $a=2, b=-3$ d. $a=2, b=3$

(IIT-JEE 2012)

17. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $(\sqrt[3]{1+a}-1)x^2 + (\sqrt{1+a}-1)x + (\sqrt[6]{1+a}-1) = 0$ where $a > -1$. Then $\lim_{a \rightarrow 0^+} \alpha(a)$ and $\lim_{a \rightarrow 0^+} \beta(a)$ are

- a. $-\frac{5}{2}$ and 1 b. $-\frac{1}{2}$ and -1
c. $-\frac{7}{2}$ and 2 d. none of these

(IIT-JEE 2012)

Multiple Correct Answers Type

1. Let $L = \lim_{x \rightarrow 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}$, $a > 0$. If L is finite, then

- a. $a=2$ b. $a=1$
c. $L = \frac{1}{64}$ d. $L = \frac{1}{32}$ (IIT-JEE 2009)

Integer Answer Type

1. The largest value of the non-negative integer a for which

$$\lim_{x \rightarrow 1} \left\{ \frac{-ax + \sin(x-1) + a}{x + \sin(x-1) - 1} \right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4} \text{ is}$$

(JEE Advanced 2014)

2. Let m and n be two positive integers greater than 1. If

$$\lim_{\alpha \rightarrow 0} \frac{e^{\cos(\alpha^n)} - e}{\alpha^m} = -\frac{e}{2}, \text{ then the value of } \frac{m}{n} \text{ is}$$

(JEE Advanced 2015)

Fill in the Blanks Type

1. $\lim_{x \rightarrow 1} (1-x) \tan \frac{\pi x}{2} = \underline{\hspace{2cm}}$. (IIT-JEE 1984)

2. If $f(x) = \begin{cases} \sin x, & x \neq n\pi, n \in I \\ 2, & \text{otherwise} \end{cases}$

$$\text{and } g(x) = \begin{cases} x^2 + 1, & x \neq 0, \\ 4, & x = 0 \\ 5, & x = 2 \end{cases}$$

then $\lim_{x \rightarrow 0} g\{f(x)\}$ is =

$\underline{\hspace{2cm}}$. (IIT-JEE 1986)

3. $\lim_{x \rightarrow \infty} \left[\frac{x^4 \sin\left(\frac{1}{x}\right) + x^2}{(1+|x|^3)} \right] = \underline{\hspace{2cm}}$. (IIT-JEE 1987)

4. ABC is an isosceles triangle inscribed in a circle of radius r . If $AB = AC$ and h is the altitude from A to BC , then

triangle ABC has perimeter $P = 2(\sqrt{2hr} - h^2 + \sqrt{2hr})$

and area $A = \underline{\hspace{2cm}}$ and also

$$\lim_{h \rightarrow 0} \frac{A}{P^3} = \underline{\hspace{2cm}}. \quad (\text{IIT-JEE 1989})$$

5. $\lim_{x \rightarrow \infty} \left(\frac{x+6}{x+1} \right)^{x+4} = \underline{\hspace{2cm}}$. (IIT-JEE 1990)

6. $\lim_{x \rightarrow 0} \left(\frac{1+5x^2}{1+3x^2} \right)^{1/x^2} = \underline{\hspace{2cm}}$. (IIT-JEE 1996)

7. $\lim_{h \rightarrow 0} \frac{\ln(1+2h) - 2\ln(1+h)}{h^2} = \underline{\hspace{2cm}}$. (IIT-JEE 1997)

True/False Type

1. If $\lim_{x \rightarrow a} [f(x)g(x)]$ exists, then both $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist. (IIT-JEE 1981)

Subjective Type

1. Evaluate $\lim_{x \rightarrow a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}$, ($a \neq 0$). (IIT-JEE 1978)

2. $f(x)$ is the integral of $\frac{2\sin x - \sin 2x}{x^3}$, $x \neq 0$. Find

$$\lim_{x \rightarrow 0} f'(x) \left[\text{where } f'(x) = \frac{df(x)}{dx} \right]. \quad (\text{IIT-JEE 1979})$$

3. Evaluate $\lim_{h \rightarrow 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$.

(IIT-JEE 1980)

4. Use the formula $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$ to find $\lim_{x \rightarrow 0} \frac{2^x - 1}{(1+x)^{1/2} - 1}$.

(IIT-JEE 1982)

5. Find $\lim_{x \rightarrow 0} \{\tan(\pi/4 + x)\}^{1/x}$.

(IIT-JEE 1993)

Answer Key

JEE Advanced

Single Correct Answer Type

- | | | | |
|--------|--------|--------|--------|
| 1. c. | 2. d. | 3. b. | 4. d. |
| 5. d. | 6. d. | 7. b. | 8. c. |
| 9. c. | 10. b. | 11. c. | 12. d. |
| 13. c. | 14. c. | 15. d. | 16. b. |
| 17. b. | | | |

Multiple Correct Answers Type

1. a., c.

Integer Answer Type

1. (2) 2. (2)

Fill in the Blanks Type

- | | | |
|--|----------|-------|
| 1. $\frac{2}{\pi}$ | 2. 1 | 3. -1 |
| 4. $A = h\sqrt{2rh - h^2}, \frac{1}{128r}$ | 5. e^5 | |
| 6. e^2 | 7. -1 | |

True/False Type

1. False

Subjective Type

- | | | |
|--------------------------|----------|-----------------------------|
| 1. $\frac{2}{3\sqrt{3}}$ | 2. 1 | 3. $a^2 \sin a + 2a \sin a$ |
| 4. $2 \ln 2$ | 5. e^2 | |

Hints and Solutions

JEE Advanced

Single Correct Answer Type

$$1. \text{ c. } \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \sqrt{\frac{1 - \frac{\sin x}{x}}{1 + \frac{\cos^2 x}{x}}} = \sqrt{\frac{1-0}{1+0}} = 1$$

$$\begin{aligned} 2. \text{ d. } \lim_{x \rightarrow 1} \frac{G(x) - G(1)}{x - 1} &= \lim_{x \rightarrow 1} \frac{-\sqrt{25-x^2} - (-\sqrt{24})}{x-1} \\ &= \lim_{x \rightarrow 1} \frac{\sqrt{24} - \sqrt{25-x^2}}{x-1} \times \frac{\sqrt{24} + \sqrt{25-x^2}}{\sqrt{24} + \sqrt{25-x^2}} \\ &= \lim_{x \rightarrow 1} \frac{x^2 - 1}{(x-1)[\sqrt{24} + \sqrt{25-x^2}]} \\ &= \lim_{x \rightarrow 1} \frac{x+1}{\sqrt{24} + \sqrt{25-x^2}} \\ &= \frac{2}{2\sqrt{24}} = \frac{1}{2\sqrt{6}} \end{aligned}$$

Alternative Method:

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{G(x) - G(1)}{x - 1} &= \lim_{x \rightarrow 1} \frac{-\sqrt{25-x^2} - (-\sqrt{24})}{x-1} \quad \left(\frac{0}{0} \text{ form}\right) \\ &= \lim_{x \rightarrow 1} \frac{x}{\sqrt{25-x^2}} \quad (\text{Applying L'Hospital's Rule}) \\ &= \frac{1}{\sqrt{24}} = \frac{1}{2\sqrt{6}} \end{aligned}$$

$$\begin{aligned} 3. \text{ b. } \lim_{n \rightarrow \infty} \left(\frac{1}{1-n^2} + \frac{2}{1-n^2} + \dots + \frac{n}{1-n^2} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1+2+3+\dots+n}{1-n^2} \\ &= \lim_{n \rightarrow \infty} \frac{\frac{n(n+1)}{2}}{1-n^2} \\ &= \lim_{n \rightarrow \infty} \frac{1+\frac{1}{n}}{2\left[\frac{1}{n^2}-1\right]} = -1/2 \end{aligned}$$

4. d. The given function is

$$f(x) = \begin{cases} \frac{\sin[x]}{[x]}, & \text{if } x \in (-\infty, 0) \cup [1, \infty) \\ 0, & \text{if } x \in [0, 1) \end{cases}$$

$$\begin{aligned} \therefore \lim_{x \rightarrow 0^-} f(x) &= \lim_{h \rightarrow 0} \frac{\sin[-h]}{[-h]} \\ &= \lim_{h \rightarrow 0} \frac{\sin(-1)}{(-1)} = \sin 1 \end{aligned}$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} 0 = 0$$

$$\therefore \lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$$

Therefore, $\lim_{x \rightarrow 0} f(x)$ does not exist.

$$\begin{aligned} 5. \text{ d. } \lim_{x \rightarrow 0} \frac{\sqrt{\frac{1}{2}(1 - \cos 2x)}}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{\frac{1}{2} \cdot 2 \sin^2 x}}{x} = \lim_{x \rightarrow 0} \frac{|\sin x|}{x} \\ \therefore \text{L.H.L.} &= \lim_{h \rightarrow 0} \frac{|\sin(0-h)|}{0-h} = \lim_{h \rightarrow 0} \frac{-\sin h}{-h} = \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \end{aligned}$$

$$\text{R.H.L.} = \lim_{h \rightarrow 0} \frac{|\sin(0+h)|}{0+h} = \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$$

As L.H.L. $\neq$ R.H.L., the given limit does not exist.

$$\begin{aligned} 6. \text{ d. } \text{L.H.L.} &= \lim_{x \rightarrow 1^-} \frac{\sqrt{1 - \cos[2(x-1)]}}{x-1} \\ &= \lim_{x \rightarrow 1^-} \frac{\sqrt{2 \sin^2(x-1)}}{x-1} \\ &= \sqrt{2} \lim_{x \rightarrow 1^-} \frac{|\sin(x-1)|}{x-1} \\ &= \sqrt{2} \lim_{h \rightarrow 0} \frac{|\sin(-h)|}{-h} = \sqrt{2} \lim_{h \rightarrow 0} \frac{\sin h}{-h} = -\sqrt{2} \\ \text{R.H.L.} &= \lim_{x \rightarrow 1^+} \sqrt{2} \frac{|\sin(x-1)|}{x-1} \\ &= \lim_{h \rightarrow 0} \sqrt{2} \frac{|\sin h|}{h} \\ &= \lim_{h \rightarrow 0} \sqrt{2} \frac{\sin h}{h} = \sqrt{2} \end{aligned}$$

L.H.L. $\neq$ R.H.L. Therefore, $\lim_{x \rightarrow 1} f(x)$ does not exist.

7. b. Putting $\theta = 0$, we get $b_0 = 0$

$$\therefore \sin n\theta = \sum_{r=1}^n b_r \sin^r \theta$$

$$\Rightarrow \frac{\sin n\theta}{\sin \theta} = \sum_{r=1}^n b_r (\sin \theta)^{r-1}$$

$$= b_1 + b_2 \sin \theta + b_3 \sin^2 \theta + \dots + b_n \sin^{n-1} \theta$$

Taking limit as $\theta \rightarrow 0$, we obtain

$$\lim_{\theta \rightarrow 0} \frac{\sin n\theta}{\sin \theta} = b_1 \Rightarrow b_1 = n.$$

$$8. \text{ c. } \lim_{x \rightarrow 0} \frac{x \tan 2x - 2x \tan x}{4 \sin^4 x} = \lim_{x \rightarrow 0} \frac{x}{4 \sin^4 x} \left[\frac{2 \tan x}{1 - \tan^2 x} - 2 \tan x \right]$$

$$\begin{aligned}
 &= \lim_{x \rightarrow 0} \frac{x \tan^3 x}{2 \sin^4 x (1 - \tan^2 x)} \\
 &= \frac{1}{2} \lim_{x \rightarrow 0} \frac{x}{\sin x} \cdot \frac{1}{\cos^3 x} \cdot \frac{1}{1 - \tan^2 x} \\
 &= \frac{1}{2} \times 1 \times \frac{1}{1^3} \times \frac{1}{1-0} = \frac{1}{2}
 \end{aligned}$$

$$9. \text{ c. } \lim_{x \rightarrow \infty} \left(\frac{x-3}{x+2} \right)^x = e^{\lim_{x \rightarrow \infty} \left[\frac{x-3}{x+2} - 1 \right] x} = e^{\lim_{x \rightarrow \infty} \left[\frac{-5x}{x+2} \right]} = e^{-5}$$

$$10. \text{ b. } \lim_{x \rightarrow 0} \frac{\sin(\pi \cos^2 x)}{x^2} = \lim_{x \rightarrow 0} \frac{\sin(\pi - \pi \cos^2 x)}{x^2} \quad [\sin(\pi - \theta) = \sin \theta]$$

$$= \lim_{x \rightarrow 0} \frac{\sin(\pi \sin^2 x)}{\pi \sin^2 x} \times \frac{(\pi \sin^2 x)}{x^2} = \pi$$

$$\begin{aligned}
 11. \text{ c. } L &= \lim_{x \rightarrow 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n} \\
 &= -\lim_{x \rightarrow 0} \frac{(1 - \cos x)(1 + \cos x)(\cos x - e^x)}{(1 + \cos x)x^n} \\
 &= \lim_{x \rightarrow 0} \frac{\left(\frac{\sin x}{x} \right)^2 \left(\frac{1 - \cos x}{x} + \frac{e^x - 1}{x} \right)}{x^{n-3}} \cdot \frac{1}{1 + \cos x}
 \end{aligned}$$

L is finite nonzero. Then $n = 3$ (as for $n = 1, 2$, $L = 0$, and for $n = 4$, $L = \infty$).

$$12. \text{ d. } \text{Given } \lim_{x \rightarrow 0} \frac{[(a-n)nx - \tan x] \sin nx}{x^2} = 0, \text{ where } a \text{ is non-zero number. Therefore,}$$

$$n \lim_{x \rightarrow 0} \frac{\sin nx}{nx} \left[(a-n)n - \frac{\tan x}{x} \right] = 0$$

$$\text{or } n[(a-n)n - 1] = 0$$

$$\text{or } a = \frac{1}{n} + n$$

$$13. \text{ c. } \lim_{x \rightarrow 0} \left[(\sin x)^{1/x} + (1/x)^{\sin x} \right]$$

$$= \lim_{x \rightarrow 0} (\sin x)^{1/x} + \lim_{x \rightarrow 0} \left(\frac{1}{x} \right)^{\sin x}$$

$$= 0 + e^{\lim_{x \rightarrow 0} \sin x \log \left(\frac{1}{x} \right)}$$

$$= e^{\lim_{x \rightarrow 0} \frac{-\log x}{\csc x}} = e^{\lim_{x \rightarrow 0} \frac{-1/x}{-\csc x \cot x}}$$

[Using L'Hospital's rule]

$$= e^{\lim_{x \rightarrow 0} \frac{\sin x}{x} \tan x} = e^0 = 1$$

$$14. \text{ c. } |x-1| = 1-x, x < 1$$

$\therefore p =$ left hand derivative of $|x-1|$ at $x=1 = -1$

$$\Rightarrow \lim_{x \rightarrow 1^-} g(x) = -1$$

$$\Rightarrow \lim_{h \rightarrow 0} g(1+h) = -1$$

$$\Rightarrow \lim_{h \rightarrow 0} \left(\frac{h^n}{\log \cos h} \right) = -1$$

$$\Rightarrow \lim_{h \rightarrow 0} \left(\frac{h^n}{m \log \cos h} \right) = -1$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{n \cdot h^{n-1}}{m \cdot (-\tan h)} = -1 \quad (\text{Applying L'Hospital's Rule})$$

$$\Rightarrow -\left(\frac{n}{m} \right) \lim_{h \rightarrow 0} \left(\frac{h^{n-1}}{\tan h} \right) = -1$$

which holds if $n = m = 2$.

$$15. \text{ d. } \lim_{x \rightarrow 0} [1 + x \ln(1+b^2)]^{1/x} = 2b \sin^2 \theta$$

$$\Rightarrow e^{\lim_{x \rightarrow 0} [1 + x \ln(1+b^2) - 1] \frac{1}{x}} = 2b \sin^2 \theta$$

$$\Rightarrow e^{\lim_{x \rightarrow 0} \ln(1+b^2)} = 2b \sin^2 \theta$$

$$\Rightarrow e^{\ln(1+b^2)} = 2b \sin^2 \theta$$

$$\Rightarrow 1 + b^2 = 2b \sin^2 \theta$$

$$\Rightarrow b + \frac{1}{b} = 2 \sin^2 \theta$$

Since $b > 0$,

$$2 \sin^2 \theta = b + \frac{1}{b} \geq 2$$

$$\Rightarrow \sin^2 \theta \geq 1$$

$$\Rightarrow \sin^2 \theta = 1$$

$$\Rightarrow \theta = \pm \pi/2$$

$$16. \text{ b. } \lim_{x \rightarrow \infty} \left(\frac{x^2 + x + 1}{x + 1} - ax - b \right) = 4$$

$$\Rightarrow \lim_{x \rightarrow \infty} \left(x + \frac{1}{x+1} - ax - b \right) = 4$$

$$\Rightarrow \lim_{x \rightarrow \infty} ((1-a)x - b) = 4$$

$$\Rightarrow 1-a = 0 \text{ and } -b = 4$$

$$\Rightarrow a = 1 \text{ and } b = -4$$

$$17. \text{ b. } \text{Let } 1+a=y$$

$$\Rightarrow (y^{1/3} - 1)x^2 + (y^{1/2} - 1)x + y^{1/6} - 1 = 0$$

$$\Rightarrow \left(\frac{y^{1/3} - 1}{y - 1} \right) x^2 + \left(\frac{y^{1/2} - 1}{y - 1} \right) x + \frac{y^{1/6} - 1}{y - 1} = 0$$

Now, taking $\lim_{y \rightarrow 1}$ on both sides, we get

$$\frac{1}{3}x^2 + \frac{1}{2}x + \frac{1}{6} = 0$$

$$\Rightarrow 2x^2 + 3x + 1 = 0$$

$$x = -1, -\frac{1}{2}$$

Multiple Correct Answers Type

1. a., c.
$$L = \lim_{x \rightarrow 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}$$

$$= \lim_{x \rightarrow 0} \frac{1}{x^2 \left(a + \sqrt{a^2 - x^2} \right)} - \frac{1}{4x^2}$$

$$= \lim_{x \rightarrow 0} \frac{(4-a) - \sqrt{a^2 - x^2}}{4x^2 \left(a + \sqrt{a^2 - x^2} \right)}$$

Numerator $\rightarrow 0$ if $a = 2$ and then $L = \frac{1}{64}$.

Alternative Method:

$$L = \lim_{x \rightarrow 1} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}$$

$$= \lim_{x \rightarrow 1} \frac{a - (a^2 - x^2)^{\frac{1}{2}} - \frac{x^2}{4}}{x^4}$$

$$= \lim_{x \rightarrow 1} \frac{a - a \left(1 - \frac{x^2}{a^2} \right)^{\frac{1}{2}} - \frac{x^2}{4}}{x^4}$$

$$= \lim_{x \rightarrow 1} \frac{a - a \left(1 + \frac{1}{2} \left(-\frac{x^2}{a^2} \right) + \frac{\frac{1}{2} \left(\frac{1}{2} - 1 \right)}{2!} \left(-\frac{x^2}{a^2} \right)^2 \right) - \frac{x^2}{4}}{x^4}$$

$$= \lim_{x \rightarrow 1} \frac{\left(\frac{1}{2a} - \frac{1}{4} \right) x^2 + \frac{x^4}{8a^3}}{x^4}$$

This limit exists if $\frac{1}{2a} - \frac{1}{4} = 0$ or $a = 2$.

Also, when $a = 2$ then

$$L = \lim_{x \rightarrow 1} \frac{\frac{x^4}{64}}{x^4} = \frac{1}{64}$$

Integer Answer Type

1. (2)
$$\lim_{x \rightarrow 1} \left\{ \frac{-ax + \sin(x-1) + a}{x + \sin(x-1) - 1} \right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$$

$$\Rightarrow \lim_{x \rightarrow 1} \left(\frac{\frac{\sin(x-1)}{(x-1)} - a}{\frac{\sin(x-1)}{(x-1)} + 1} \right)^{\frac{1-\sqrt{x}}{1-\sqrt{x}}} = \frac{1}{4}$$

$$\Rightarrow \left(\frac{1-a}{2} \right)^2 = \frac{1}{4}$$

$$\Rightarrow a = 0, a = 2$$

2. (2) $m \geq 2$ and $n \geq 2$

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \frac{e^{\cos(\alpha^n)} - e}{\alpha^n} &= \lim_{\alpha \rightarrow 0} \frac{e(e^{\cos(\alpha^n)-1} - 1)}{\cos(\alpha^n) - 1} \times \left(\frac{\cos(\alpha^n) - 1}{(\alpha^n)^2} \right) \frac{\alpha^{2n}}{\alpha^n} \\ &= e \times \lim_{\alpha \rightarrow 0} \left(\frac{e^{\cos(\alpha^n)-1} - 1}{\cos(\alpha^n) - 1} \right) \times \lim_{\alpha \rightarrow 0} \left(\frac{\cos(\alpha^n) - 1}{\alpha^{2n}} \right) \times \lim_{\alpha \rightarrow 0} \alpha^{2n-m} \\ &= e \times 1 \times \lim_{\alpha \rightarrow 0} \frac{-2 \sin^2 \frac{\alpha^n}{2}}{\alpha^{2n}} \times \lim_{\alpha \rightarrow 0} \alpha^{2n-m} \\ &= e \times 1 \times \left(-\frac{1}{2} \right) \times \lim_{\alpha \rightarrow 0} \alpha^{2n-m} \end{aligned}$$

Now, $\lim_{\alpha \rightarrow 0} \alpha^{2n-m}$ must be equal to 1.

$$\text{i.e., } 2n - m = 0$$

$$\text{or } \frac{m}{n} = 2$$

Fill in the Blanks Type

1.
$$\lim_{x \rightarrow 1} (1-x) \tan \frac{\pi x}{2} = \lim_{x \rightarrow 1} \frac{(1-x)}{\tan \left(\frac{\pi}{2} - \frac{\pi x}{2} \right)}$$

$$= \frac{2}{\pi} \lim_{x \rightarrow 1} \frac{\frac{\pi}{2}(1-x)}{\tan \left(\frac{\pi}{2}(1-x) \right)}$$

$$= \frac{2}{\pi}$$

Alternative Method:

$$\begin{aligned} \lim_{x \rightarrow 1} (1-x) \tan \frac{\pi x}{2} &= \lim_{x \rightarrow 1} \frac{(1-x)}{\cot \frac{\pi x}{2}} \quad \left(\frac{0}{0} \text{ form} \right) \\ &= \lim_{x \rightarrow 1} \frac{-1}{-\frac{\pi}{2} \operatorname{cosec}^2 \frac{\pi x}{2}} \quad (\text{Applying L'Hospital's Rule}) \\ &= \frac{2}{\pi} \end{aligned}$$

2.
$$f(x) = \begin{cases} \sin x, & x \neq n\pi, n \in I \\ 2, & \text{otherwise} \end{cases}$$

$$\text{and } g(x) = \begin{cases} x^2 + 1, & x \neq 0 \\ 4, & x = 0 \\ 5, & x = 2 \end{cases}$$

$$\lim_{x \rightarrow 0^+} g\{f(x)\} = g(f(0^+)) = g((\sin 0^+)) = g(0^+) = (0)^2 + 1 = 1$$

$$\lim_{x \rightarrow 0^-} g\{f(x)\} = g(f(0^-)) = g((\sin 0^-)) = g(0^-) = (0)^2 + 1 = 1$$

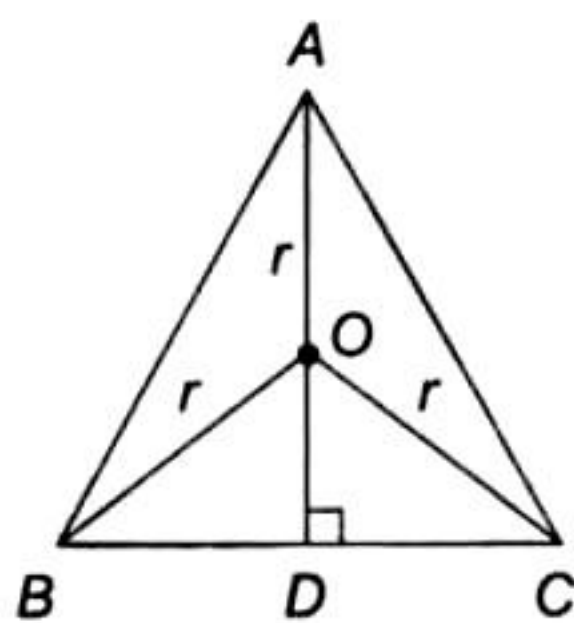
Hence, $\lim_{x \rightarrow 0} g\{f(x)\} = 1$.

$$\begin{aligned}
 3. \quad \lim_{x \rightarrow -\infty} \left[\frac{x^4 \sin\left(\frac{1}{x}\right) + x^2}{(1 + |x|^3)} \right] &= \lim_{x \rightarrow -\infty} \left[\frac{x \sin\left(\frac{1}{x}\right) + \frac{1}{x}}{\frac{1}{x^3} - 1} \right] \\
 &= \lim_{x \rightarrow -\infty} \left[\frac{\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} + \frac{1}{x}}{\frac{1}{x^3} - 1} \right] = \frac{1+0}{0-1} = -1
 \end{aligned}$$

4. In $\triangle ABC$, $AB = AC$, $AD \perp BC$ (D is the midpoint of BC).
Let r = radius of circumcircle
 $\therefore OA = OB = OC = r$

$$\text{Now, } BD = \sqrt{BO^2 - OD^2} = \sqrt{r^2 - (h-r)^2}$$

$$= \sqrt{2rh - h^2}$$



$$\therefore BC = 2\sqrt{2rh - h^2}$$

$$\therefore \text{Area of } \triangle ABC = \frac{1}{2} \times BC \times AD = h\sqrt{2rh - h^2}$$

$$\begin{aligned}
 \text{Also, } \lim_{h \rightarrow 0} \frac{A}{P^3} &= \lim_{h \rightarrow 0} \frac{h\sqrt{2rh - h^2}}{8(\sqrt{2rh - h^2} + \sqrt{2hr})^3} \\
 &= \lim_{h \rightarrow 0} \frac{h^{3/2} \sqrt{2r - h}}{8h^{3/2} (\sqrt{2r - h} + \sqrt{2r})^3} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{2r - h}}{8[\sqrt{2r - h} + \sqrt{2r}]^3} \\
 &= \frac{\sqrt{2r}}{8(\sqrt{2r} + \sqrt{2r})^3} = \frac{\sqrt{2r}}{8 \times 8 \times 2r \times \sqrt{2r}} = \frac{1}{128r}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad \lim_{x \rightarrow \infty} \left(\frac{x+6}{x+1} \right)^{x+4} &= \lim_{x \rightarrow \infty} \left(\frac{1 + \frac{6}{x}}{1 + \frac{1}{x}} \right)^{x+4} \\
 &= \lim_{x \rightarrow \infty} \left(\frac{1 + \frac{6}{x}}{1 + \frac{1}{x}} \right)^x \left(\frac{1 + \frac{6}{x}}{1 + \frac{1}{x}} \right)^4 \left[\text{Using } \lim_{x \rightarrow \infty} \left(1 + \frac{\lambda}{x} \right)^x = e^\lambda \right]
 \end{aligned}$$

$$= \frac{e^6}{e} \left(\frac{1}{1} \right)^4 = e^5$$

Alternative Method:

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \left(\frac{x+6}{x+1} \right)^{x+4} &\left(1^\infty \text{ form as } \lim_{x \rightarrow \infty} \frac{x+6}{x+1} = \lim_{x \rightarrow \infty} \frac{1 + \frac{6}{x}}{1 + \frac{1}{x}} = 1 \right) \\
 &= e^{\lim_{x \rightarrow \infty} \left(\frac{x+6}{x+1} - 1 \right)(x+4)} \\
 &= e^{\lim_{x \rightarrow \infty} \left(\frac{5(x+4)}{x+1} \right)} \\
 &= e^{\lim_{x \rightarrow \infty} 5 \left(\frac{1 + \frac{4}{x}}{1 + \frac{1}{x}} \right)} \\
 &= e^5
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \lim_{x \rightarrow 0} \left(\frac{1+5x^2}{1+3x^2} \right)^{1/x^2} &= \frac{\lim_{x \rightarrow 0} (1+5x^2)^{1/x^2}}{\lim_{x \rightarrow 0} (1+3x^2)^{1/x^2}} \\
 &= \frac{\lim_{x \rightarrow 0} \left\{ (1+5x^2)^{\frac{1}{5x^2}} \right\}^5}{\lim_{x \rightarrow 0} \left\{ (1+3x^2)^{\frac{1}{3x^2}} \right\}^3} \\
 &= e^{5-3} = e^2
 \end{aligned}$$

$$\begin{aligned}
 7. \quad \lim_{h \rightarrow 0} \frac{\ln(1+2h) - 2\ln(1+h)}{h^2} &= \lim_{h \rightarrow 0} \frac{\ln \left[\frac{(1+2h)}{(1+h)^2} \right]}{h^2} \\
 &= \lim_{h \rightarrow 0} \ln \left[\frac{1 + \frac{-h^2}{1+2h+h^2}}{1} \right] \times \frac{-1}{1+2h+h^2} \\
 &= 1 \times \lim_{h \rightarrow 0} \frac{-1}{1+2h+h^2} \left[\text{Using } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1 \right] \\
 &= -1
 \end{aligned}$$

Alternative Method:

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{\ln(1+2h) - 2\ln(1+h)}{h^2} &= \lim_{h \rightarrow 0} \frac{\frac{2}{1+2h} - \frac{2}{1+h}}{2h} \quad (\text{Applying L'Hospital's Rule}) \\
 &= \lim_{h \rightarrow 0} \frac{1+h-1-2h}{h(1+h)(1+2h)} \\
 &= \lim_{h \rightarrow 0} \frac{-1}{(1+h)(1+2h)} \\
 &= -1
 \end{aligned}$$

True/False Type

1. **False.** Consider $f(x) = \frac{|x-a|}{x-a}$, $g(x) = \frac{x-a}{|x-a|}$. Then, $\lim_{x \rightarrow a} (f(x) \times g(x))$ exists, but $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ do not exist. Therefore, statement is false.

Subjective Type

$$\begin{aligned}
 1. \quad & \lim_{x \rightarrow a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \left(\text{form } \frac{0}{0} \right) \\
 &= \lim_{x \rightarrow a} \frac{(\sqrt{a+2x} - \sqrt{3x})(\sqrt{a+2x} + \sqrt{3x})}{(\sqrt{3a+x} - 2\sqrt{x})(\sqrt{3a+x} + 2\sqrt{x})} \frac{(\sqrt{3a+x} + 2\sqrt{x})}{(\sqrt{a+2x} + \sqrt{3x})} \\
 & \quad \left(\text{form } \frac{0}{0} \right) \\
 &= \lim_{x \rightarrow a} \frac{(a+2x-3x)(\sqrt{3a+x} + 2\sqrt{x})}{(3a+x-4x)(\sqrt{a+2x} + \sqrt{3x})} \\
 &= \lim_{x \rightarrow a} \frac{\sqrt{3a+x} + 2\sqrt{x}}{3(\sqrt{a+2x} + \sqrt{3x})} \\
 &= \frac{\sqrt{3a+a} + 2\sqrt{a}}{3(\sqrt{a+2a} + \sqrt{3a})} = \frac{1}{3} \cdot \frac{4\sqrt{a}}{2\sqrt{3a}} = \frac{2}{3\sqrt{3}}
 \end{aligned}$$

Alternative Method:

$$\begin{aligned}
 & \lim_{x \rightarrow a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \left(\frac{0}{0} \text{ form} \right) \\
 &= \lim_{x \rightarrow a} \frac{\frac{1}{\sqrt{a+2x}} - \frac{\sqrt{3}}{2\sqrt{x}}}{\frac{1}{2\sqrt{3a+x}} - \frac{1}{\sqrt{x}}} \quad (\text{Applying L'Hospital's Rule}) \\
 &= \frac{\frac{1}{\sqrt{3a}} - \frac{\sqrt{3}}{2\sqrt{a}}}{\frac{1}{2\sqrt{4a}} - \frac{1}{\sqrt{a}}} \\
 &= \frac{\frac{1}{\sqrt{3}} - \frac{\sqrt{3}}{2}}{\frac{1}{4} - 1} = \frac{-\frac{1}{2\sqrt{3}}}{-\frac{3}{4}} = \frac{2}{3\sqrt{3}}
 \end{aligned}$$

$$2. \quad f(x) = \int \frac{2 \sin x - \sin 2x}{x^3} dx, x \neq 0$$

$$\therefore f'(x) = \frac{2 \sin x - \sin 2x}{x^3}, x \neq 0$$

$$\begin{aligned}
 \therefore \lim_{x \rightarrow 0} f'(x) &= \lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x^3} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin x (1 - \cos x)(1 + \cos x)}{x^3 (1 + \cos x)}
 \end{aligned}$$

$$= \lim_{x \rightarrow 0} 2 \times \frac{\sin^3 x}{x^3} \times \frac{1}{1 + \cos x} = 2 \times (1)^3 \times \frac{1}{2} = 1$$

Alternative Method:

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x^3} \left(\frac{0}{0} \text{ form} \right) \\
 &= \lim_{x \rightarrow 0} \frac{2 \cos x - 2 \cos 2x}{3x^2} \quad (\text{Applying L'Hospital's Rule}) \\
 &= \lim_{x \rightarrow 0} \frac{-2 \sin x + 4 \sin 2x}{6x} \quad (\text{Applying L'Hospital's Rule}) \\
 &= \lim_{x \rightarrow 0} \frac{-2 \cos x + 8 \cos 2x}{6} \quad (\text{Applying L'Hospital's Rule}) \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \lim_{h \rightarrow 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h} \\
 &= \lim_{h \rightarrow 0} \frac{a^2 [\sin(a+h) - \sin a] + 2ah \sin(a+h) + h^2 \sin(a+h)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{a^2 \left[2 \cos \left(a + \frac{h}{2} \right) \sin \frac{h}{2} \right]}{2 \times \frac{h}{2}} + \lim_{h \rightarrow 0} 2a \sin(a+h) \\
 & \quad + \lim_{h \rightarrow 0} h \sin(a+h) \\
 &= a^2 \cos a + 2a \sin a
 \end{aligned}$$

Alternative Method:

$$\begin{aligned}
 & \lim_{h \rightarrow 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h} \left(\frac{0}{0} \text{ form} \right) \\
 &= \lim_{h \rightarrow 0} \frac{2(a+h) \sin(a+h) + (a+h)^2 \cos(a+h)}{1} \\
 & \quad (\text{Applying L'Hospital's Rule}) \\
 &= 2a \sin a + a^2 \cos a
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & \lim_{x \rightarrow 0} \frac{2^x - 1}{\sqrt{1+x} - 1} = \lim_{x \rightarrow 0} \frac{2^x - 1}{\sqrt{1+x} - 1} \times \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1} \\
 &= \lim_{x \rightarrow 0} \frac{(2^x - 1)(\sqrt{1+x} + 1)}{1+x-1} \\
 &= \lim_{x \rightarrow 0} \frac{2^x - 1}{x} \lim_{x \rightarrow 0} (\sqrt{1+x} + 1) \\
 &= \ln 2 (1+1) = 2 \ln 2
 \end{aligned}$$

$$\begin{aligned}
 5. \quad & \lim_{x \rightarrow 0} \left\{ \tan \left(\frac{\pi}{4} + x \right) \right\}^{1/x} = \lim_{x \rightarrow 0} \left\{ \frac{1 + \tan x}{1 - \tan x} \right\}^{1/x} \\
 &= \frac{\lim_{x \rightarrow 0} \left[(1 + \tan x)^{1/\tan x} \right]^{\frac{\tan x}{x}}}{\lim_{x \rightarrow 0} \left[(1 - \tan x)^{-1/\tan x} \right]^{\frac{-\tan x}{x}}} \\
 &= \frac{e}{e^{-1}} = e^2
 \end{aligned}$$

Alternative Method:

$$\begin{aligned} \lim_{x \rightarrow 0} \left\{ \tan \left(\frac{\pi}{4} + x \right) \right\}^{1/x} \\ = e^{\lim_{x \rightarrow 0} \left\{ \tan \left(\frac{\pi}{4} + x \right) - 1 \right\} \frac{1}{x}} \end{aligned}$$

(1^∞ form)

$$\begin{aligned} &= e^{\lim_{x \rightarrow 0} \left(\frac{2 \tan x}{1 + \tan x} \right) \frac{1}{x}} \\ &= e^{\lim_{x \rightarrow 0} \frac{\tan x}{x} \cdot \frac{2}{1 + \tan x}} \\ &= e^2 \end{aligned}$$