MATHEMATICS

TRIGONOMETRIC EQUATIONS

SOLUTIONS OF TRIGONOMETRIC EQUATIONS

NOTES

What you already know

- Trigonometric functions
- Trigonometric ratios of multiple and submultiple angles
- Transformations formulas, Periodicity of trigonometric functions
- Graphs of trigonometric functions, Transformation of graphs

What you will learn

- Trigonometric equations
- Solving trigonometric equations
- Principal solutions
- General solutions

Trigonometric equations

Any equation involving trigonometric functions of unknown angles is known as a trigonometric equation. Solving these trigonometric equations to find the unknown angles give us the solutions of respective trigonometric equations.

What is/are the solution(s) of $\sin x = \frac{1}{2}$?

a. $2n\pi + \frac{\pi}{6}\ \forall\ n \in Z$ b. $n\pi + \frac{\pi}{6}\ \forall\ n \in Z$ c. $2n\pi + \frac{5\pi}{6}\ \forall\ n \in Z$ d. $n\pi + \frac{5\pi}{6}\ \forall\ n \in Z$

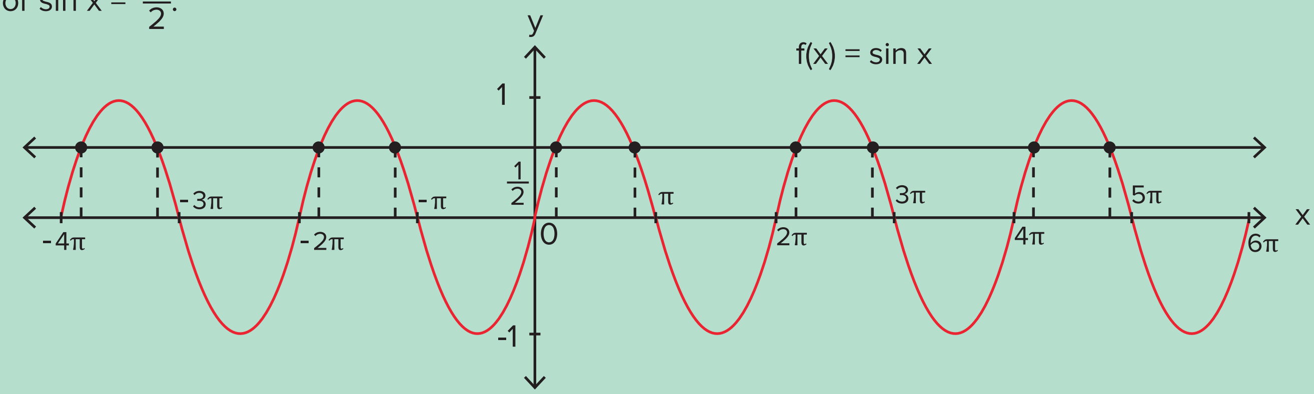
Solution

Step 1:

Given, $\sin x = \frac{1}{2}$

Let's consider $f(x) = \sin x$ and $g(x) = \frac{1}{2}$.

Now, the x coordinate of intersection points of the these two functions will give the solution for $\sin x = \frac{1}{2}$.



Step 2:

From the graph we can observe that the number of solutions of for sin x = $\frac{1}{2}$ are infinite as sin x is a periodic function and with domain R and g(x) = $\frac{1}{2}$ is a straight line parallel to x-axis with domain R.

Let us consider the points of intersection in the domain [0, 2π]. The first intersection point is at $x = \frac{\pi}{6}$. The next point of intersection will be at $x = \frac{5\pi}{6}$.

Step 3:

As we know that sin x is a periodic function with period 2π, the points of intersection repeat after every 2π. Therefore, in the domain [2π, 4π] the two points of intersection will be at $x = 2\pi + \frac{\pi}{6}$ and $x = 2\pi + \frac{5\pi}{6}$. With similar explanation in the domain [-2π, 0] the two points of intersection will be at $x = -2\pi + \frac{\pi}{6}$ and $x = -2\pi + \frac{5\pi}{6}$.

$$\left\{\frac{\pi}{6}, 2\pi + \frac{\pi}{6}, -2\pi + \frac{\pi}{6}, 4\pi + \frac{\pi}{6}, -4\pi + \frac{\pi}{6}, ...\right\} = \left\{2n\pi + \frac{\pi}{6}\ \forall\ n \in Z\right\}$$

$$\left\{\frac{5\pi}{6}, 2\pi + \frac{5\pi}{6}, -2\pi + \frac{5\pi}{6}, 4\pi + \frac{5\pi}{6}, -4\pi + \frac{5\pi}{6}, ...\right\} = \left\{2n\pi + \frac{5\pi}{6}\ \forall\ n \in Z\right\}$$

Step 4:

To generalise we can say that the points of intersection are

$x = 2n\pi + \frac{\pi}{6}$ and $x = 2n\pi + \frac{5\pi}{6}\ \forall\ n \in Z$

Therefore, options a and c are correct.

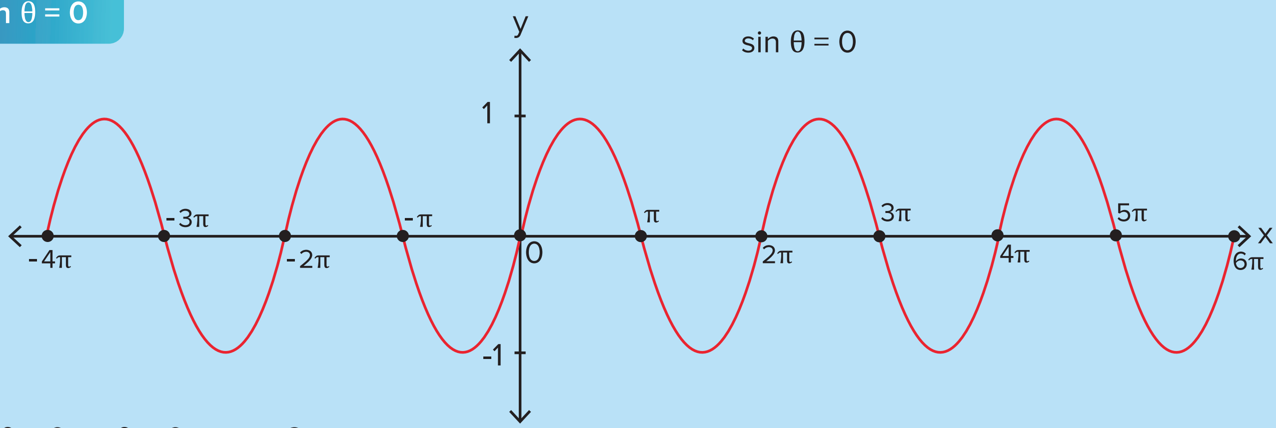
Types of Solutions

1. **Principal Solution**
   when $0 \le \theta < 2\pi$ i.e. domain of θ = [0, 2π)

2. **General Solution**
   for $\theta \in R$ i.e. domain of θ is R

General Solutions of Standard Trigonometric Equations

sin θ = 0

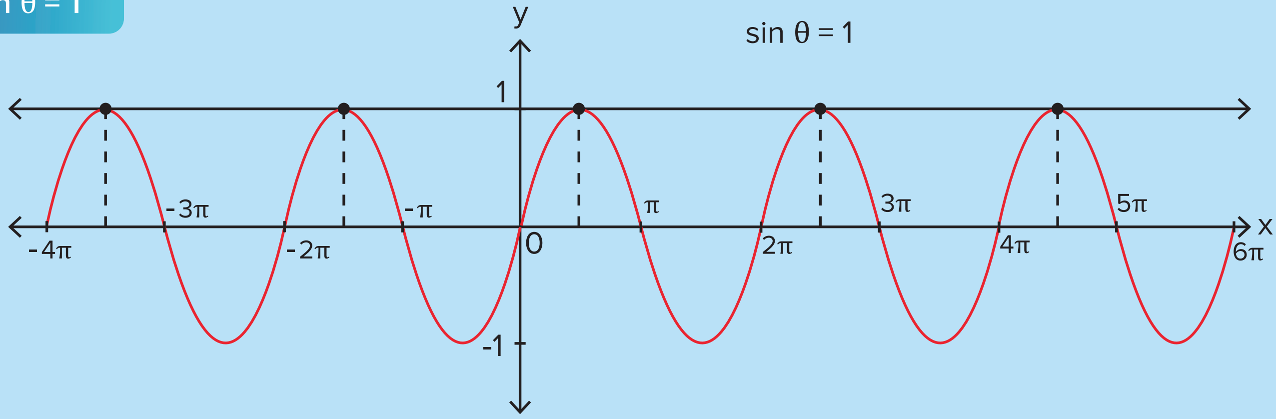


$\sin\theta = 0 \Leftrightarrow \theta = 0, \pm\pi, \pm 2\pi.......$

$\Rightarrow \sin\theta = 0 \Leftrightarrow \theta = n\pi\ \forall\ n \in Z$

Therefore, general solution of sin θ = 0 is $\theta = n\pi\ \forall\ n \in Z$.

sin θ = 1



$\sin\theta = 1 \Leftrightarrow \theta = ..., -\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}.......$

$\Rightarrow \sin\theta = 1 \Leftrightarrow \theta = (4n + 1)\frac{\pi}{2}\ \forall\ n \in Z$

Therefore, general solution of $\sin\theta = 1$ is $\theta = (4n + 1)\frac{\pi}{2}\ \forall\ n \in Z$.

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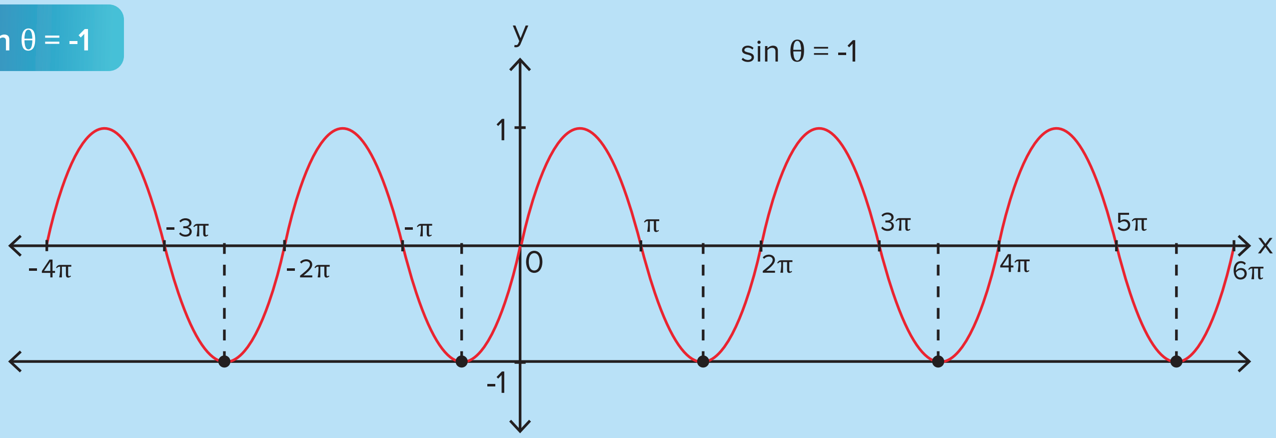
sin θ = -1



$\sin\theta = -1 \Leftrightarrow \theta = ..., -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}.......$

$\Rightarrow \sin\theta = -1 \Leftrightarrow \theta = (4n - 1)\frac{\pi}{2}\ \forall\ n \in Z$

Therefore, general solution of $\sin\theta = -1$ is $\theta = (4n - 1)\frac{\pi}{2}\ \forall\ n \in Z$.

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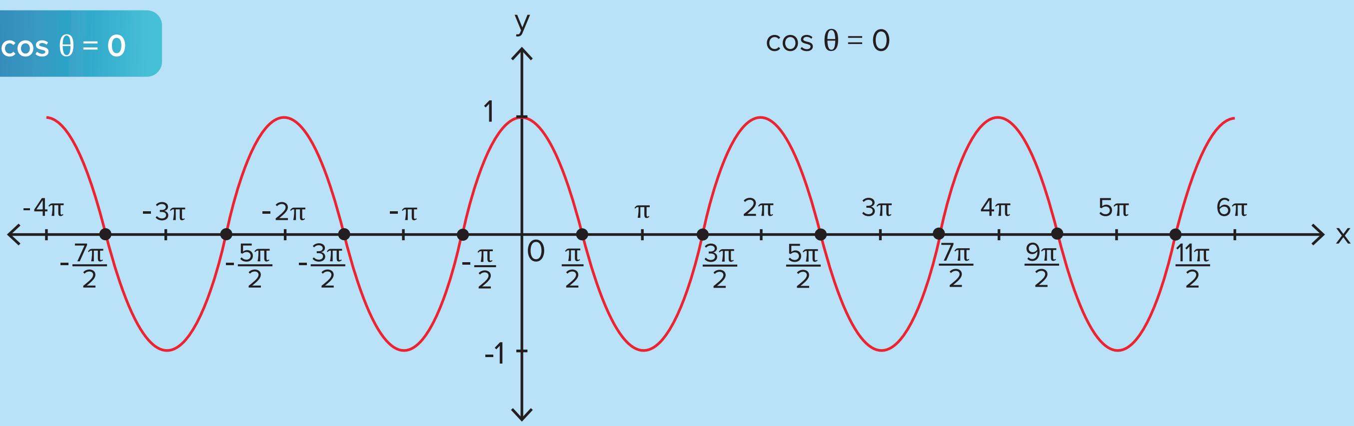
cos θ = 0

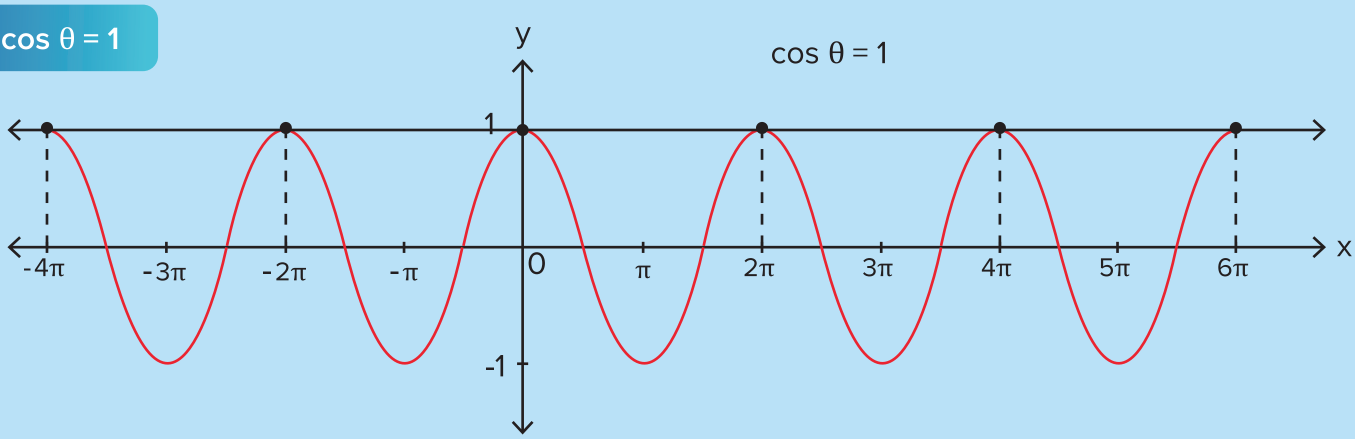


$\cos\theta = 0 \Leftrightarrow \theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2},.......$

$\Rightarrow \cos\theta = 0 \Leftrightarrow \theta = (2n + 1)\frac{\pi}{2}\ \forall\ n \in Z$

Therefore, general solution of $\cos\theta = 0$ is $\theta = (2n + 1)\frac{\pi}{2}\ \forall\ n \in Z$.

cos θ = 1



$\cos\theta = 1 \Leftrightarrow \theta = 0, \pm 2\pi, \pm 4\pi......$

$\Rightarrow \cos\theta = 1 \Leftrightarrow \theta = 2n\pi\ \forall\ n \in Z$

Therefore, general solution of $\cos\theta = 1$ is $\theta = 2n\pi\ \forall\ n \in Z$.

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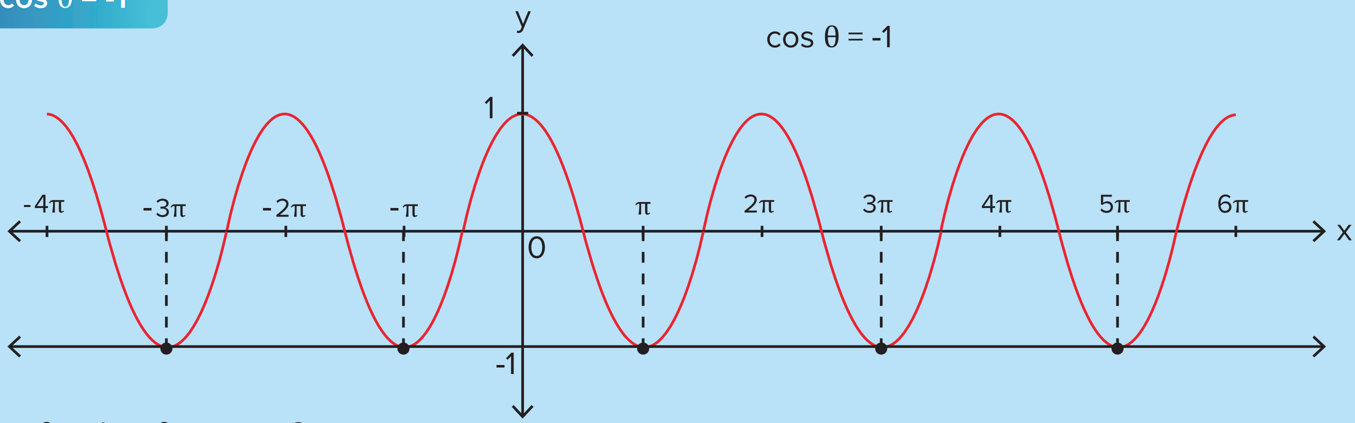
cos θ = -1



$\cos\theta = -1 \Leftrightarrow \theta = \pm\pi, \pm 3\pi......$

$\Rightarrow \cos\theta = -1 \Leftrightarrow \theta = (2n + 1)\pi\ \forall\ n \in Z$

Therefore, general solution of $\cos\theta = -1$ is $\theta = (2n + 1)\pi\ \forall\ n \in Z$.

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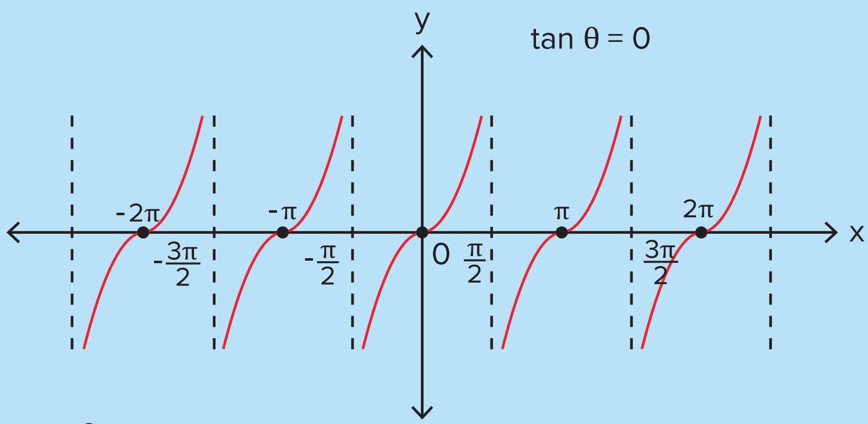
tan θ = 0



$\tan\theta = 0 \Leftrightarrow \theta = 0, \pm\pi, \pm 2\pi......$

$\Rightarrow \tan\theta = 0 \Leftrightarrow \theta = n\pi\ \forall\ n \in Z$

Therefore, general solution of $\tan\theta = 0$ is $\theta = n\pi\ \forall\ n \in Z$.

Find principal solutions of the equation $\sqrt{3}\tan x - 1 = 0$.

Solution

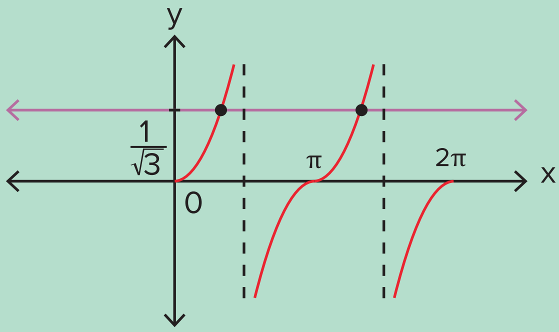
Step 1:

Given, $\sqrt{3}\tan x - 1 = 0$

As we are finding principal solutions, $x \in [0, 2\pi)$

$\Rightarrow \tan x = \frac{1}{\sqrt{3}}$

$\Rightarrow \tan x = \tan\frac{\pi}{6}$



Step 2:

And $\tan x = \tan\left(\pi + \frac{\pi}{6}\right)$

$\Rightarrow \tan x = \tan\frac{7\pi}{6}$

$\Rightarrow x = \frac{\pi}{6}, \frac{7\pi}{6}$

Therefore, principal solutions are as follows:

$x = \frac{\pi}{6}, \frac{7\pi}{6}$

Quick Query 1

The principal solution(s) of $2\sin x - \sqrt{3} = 0$ is/are

a. $x \in \left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$ b. $x \in \left\{\frac{\pi}{6}, \frac{5\pi}{3}\right\}$ c. $x \in \left\{\frac{\pi}{6}, \frac{7\pi}{6}\right\}$ d. $x \in \left\{\frac{\pi}{3}, \frac{4\pi}{3}\right\}$

Concept Check 1

The principal solution(s) of $2\cos x - 1 = 0$ is/are as follows:

a. $\frac{\pi}{3}, \frac{2\pi}{3}$ b. $\frac{\pi}{6}, \frac{5\pi}{3}$ c. $\frac{\pi}{6}, \frac{7\pi}{6}$ d. $\frac{\pi}{3}, \frac{5\pi}{3}$

$\sin\theta = \sin\alpha$

$\Rightarrow \theta = n\pi + (-1)^n\alpha;\ \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\ \forall\ n \in Z$

Proof

Step 1: Let $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

We have, $\sin\theta = \sin\alpha$

$\Rightarrow \sin\theta - \sin\alpha = 0$

$\Rightarrow 2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$ $\left[\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$

$\Rightarrow \cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Rightarrow \cos\left(\frac{\theta+\alpha}{2}\right) = 0$ or $\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

Step 2: Now this gives us two cases.

We know,

$\cos \Phi = 0$

$\Rightarrow \Phi = (2n + 1)\frac{\pi}{2}\ \forall\ n \in Z$

Similarly,

$\cos\left(\frac{\theta + \alpha}{2}\right) = 0$

$\Rightarrow \frac{\theta + \alpha}{2} = (2n + 1)\frac{\pi}{2}$

$\Rightarrow \theta = (2n + 1)\,\pi - \alpha\ \forall\ n \in Z$

n	0	1	2
θ	$\pi - \alpha$	$3\pi - \alpha$	$5\pi - \alpha$

Odd -

We know,

$\sin \Phi = 0$

$\Rightarrow \Phi = n\pi\ \forall\ n \in Z$

Similarly,

$\sin\left(\frac{\theta - \alpha}{2}\right) = 0$

$\Rightarrow \frac{\theta - \alpha}{2} = n\pi$

$\Rightarrow \theta = 2n\pi + \alpha\ \forall\ n \in Z$

n	0	1	2
θ	$0 + \alpha$	$2\pi + \alpha$	$4\pi + \alpha$

Even +

Step 3:

Observe from both the tables that when coefficient of π is odd, coefficient of α is negative which implies coefficient of α is $(-1)^{odd}$ and when coefficient of π is even, coefficient of α is positive which implies coefficient of α is $(-1)^{even}$. Let's say the coefficient of π is n and coefficient of α is $(-1)^n$, where n ∈ Z. Now, this satisfies both the cases. Hence the solution is

$\theta = n\pi + (-1)^n\alpha;\ \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\ \forall\ n \in Z.$

Solve for $\theta : \sin\left(\theta + \frac{\pi}{6}\right) = -\frac{1}{\sqrt{2}}$.

Solution

Step 1:

Given,

$\sin\left(\theta + \frac{\pi}{6}\right) = -\frac{1}{\sqrt{2}}$

$\Rightarrow \sin\left(\theta + \frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{4}\right)$

$\left\{\because -\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}$

Step 2:

We know,

$\sin\theta = \sin\alpha \Leftrightarrow \theta = n\pi + (-1)^n\alpha;\ \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\ \forall\ n \in Z$

$\Rightarrow \theta + \frac{\pi}{6} = n\pi + (-1)^n\left(-\frac{\pi}{4}\right)\ \forall\ n \in Z$

$\Rightarrow \theta = n\pi + (-1)^n\left(-\frac{\pi}{4}\right) - \frac{\pi}{6}\ \forall\ n \in Z$

Step 3: Now, we'll be dividing this in two cases i.e. when n is odd and when n is even.

Case 1: For n = 2k(even), where k ∈ Z

$\theta = 2k\pi + (-1)^{2k}\left(-\frac{\pi}{4}\right) - \frac{\pi}{6};\ k \in Z$

$\Rightarrow \theta = 2k\pi - \frac{\pi}{4} - \frac{\pi}{6};\ k \in Z$

$\Rightarrow \theta = 2k\pi - \frac{5\pi}{12};\ k \in Z$

Case 2: For n = 2k + 1(odd), where k ∈ Z

$\theta = (2k + 1)\pi + (-1)^{2k+1}\left(-\frac{\pi}{4}\right) - \frac{\pi}{6};\ k \in Z$

$\Rightarrow \theta = (2k + 1)\pi - \left(-\frac{\pi}{4}\right) - \frac{\pi}{6};\ k \in Z$

$\Rightarrow \theta = 2k\pi + \pi + \frac{\pi}{4} - \frac{\pi}{6};\ k \in Z$

$\Rightarrow \theta = 2k\pi + \frac{13\pi}{12};\ k \in Z$

Final solution set will be a union of solutions from case 1 & 2.

Therefore, $\theta \in \left\{\left(2k\pi - \frac{5\pi}{12}\right) \cup \left(2k\pi + \frac{13\pi}{12}\right)\right\}\ \forall\ k \in Z$

Concept Check 2

Solve for θ: $\sin\theta = \frac{1}{2}$

$\cos\theta = \cos\alpha$

$\Rightarrow \theta = 2n\pi \pm \alpha;\ \alpha \in [0, \pi];\ n \in Z$

Proof

Step 1:

Let $\alpha \in [0, \pi]$

$\cos\theta = \cos\alpha$

$\Rightarrow \cos\theta - \cos\alpha = 0$

$\Rightarrow -2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$ $\quad \left[\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$

$\Rightarrow \sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Rightarrow \sin\left(\frac{\theta+\alpha}{2}\right) = 0$ or $\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

Step 2:

$\sin\left(\frac{\theta+\alpha}{2}\right) = 0$

We know,

$\sin\Phi = 0 \Leftrightarrow \Phi = n\pi;\ n \in Z$

Similarly,

$\sin\left(\frac{\theta+\alpha}{2}\right) = 0$

$\Rightarrow \frac{\theta+\alpha}{2} = n\pi$

$\Rightarrow \theta = 2n\pi - \alpha\ \forall\ n \in Z$

$\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

We know,

$\sin\Phi = 0 \Leftrightarrow \Phi = n\pi;\ n \in Z$

Similarly,

$\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Rightarrow \frac{\theta-\alpha}{2} = n\pi$

$\Rightarrow \theta = 2n\pi + \alpha\ \forall\ n \in Z$

Combining both, we get

$\theta = 2n\pi \pm \alpha;\ \alpha \in [0, \pi];\ n \in Z$

What is the number of solutions of $\cos 2\theta = \frac{\sqrt{3}}{2}$ in the interval $[0, 2\pi]$?

a. 1 b. 2 c. 4 d. 6

Solution

Step 1:

Given, $\cos 2\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \cos 2\theta = \cos\left(\frac{\pi}{6}\right)$ $\left\{\because \frac{\pi}{6} \in [0, \pi]\right\}$

Step 2:

We know,

$\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi \pm \alpha;\ \alpha \in [0, \pi];\ n \in Z$

$\therefore \cos 2\theta = \cos\left(\frac{\pi}{6}\right) \Rightarrow 2\theta = 2n\pi \pm \frac{\pi}{6};\ n \in Z$

$\Rightarrow \theta = n\pi \pm \frac{\pi}{12};\ n \in Z$

Step 3: Now, for

n	0	1	2
θ	$\pm\frac{\pi}{12}$	$\frac{11\pi}{12}, \frac{13\pi}{12}$	$\frac{23\pi}{12}, \frac{25\pi}{12}$

Values of θ satisfying the equation in $[0, 2\pi]$ are $\frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

Therefore, there are 4 solutions in $[0, 2\pi]$.

Concept Check 3

If cos 3x = -1, where $0° \le x \le 360°$ then what is x?

a. $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$ b. π c. $\frac{\pi}{3}, \pi$ d. $\pi, \frac{5\pi}{3}$

$\tan\theta = \tan\alpha$

$\theta = n\pi + \alpha;\ \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right);\ n \in Z$

Proof

Step 1:

Let $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\tan\theta = \tan\alpha$

$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$

$\Rightarrow \sin\theta\cos\alpha = \sin\alpha\cos\theta$

$\Rightarrow \sin\theta\cos\alpha - \sin\alpha\cos\theta = 0$ [∵ sin A cos B - cos A sin B = sin (A - B)]

Step 2:

$\Rightarrow \sin(\theta - \alpha) = 0$

$\Rightarrow (\theta - \alpha) = n\pi \ \forall n \in Z$

$\Rightarrow \theta = n\pi + \alpha \ \forall n \in Z$

[∵ $\sin\Phi = 0 \Leftrightarrow \Phi = n\pi \ \forall n \in Z$]

If $\tan\theta\tan(120° - \theta)\tan(120° + \theta) = \frac{1}{\sqrt{3}}$

a. $\frac{n\pi}{3} - \frac{\pi}{12}; n \in \mathbb{Z}$ b. $\frac{n\pi}{3} - \frac{\pi}{18}; n \in \mathbb{Z}$ c. $\frac{n\pi}{3} + \frac{\pi}{18}; n \in \mathbb{Z}$ d. $\frac{n\pi}{3} + \frac{\pi}{12}; n \in \mathbb{Z}$

Solution

Step 1:

$\tan\theta\tan(120° - \theta)\tan(120° + \theta) = \frac{1}{\sqrt{3}}$

$\Rightarrow \tan\theta\tan[(180° - 60°) - \theta]\tan[(180° - 60°) + \theta] = \frac{1}{\sqrt{3}}$

$\Rightarrow \tan\theta\tan[180° - (60° + \theta)]\tan[180° - (60° - \theta)] = \frac{1}{\sqrt{3}}$

$\Rightarrow \tan\theta(-\tan(60° + \theta))(-\tan(60° - \theta)) = \frac{1}{\sqrt{3}}$ [∵ $\tan(\pi - x) = -\tan x$]

$\Rightarrow \tan 3\theta = \frac{1}{\sqrt{3}}$ {$\tan\Phi\tan(60° - \Phi)\tan(60° + \Phi) = \tan 3\Phi$}

Step 2:

$\Rightarrow \tan 3\theta = \tan \frac{\pi}{6}$ $\left\{\because \frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}$

$\Rightarrow 3\theta = n\pi + \frac{\pi}{6}; n \in Z$

$\Rightarrow \theta = \frac{n\pi}{3} + \frac{\pi}{18}; n \in Z$

Therefore, option c is correct.

Concept Check 4

Solve for θ: $\tan \theta = \frac{1}{\sqrt{3}}$

Concept Check 5

Solve: $\tan 5\theta = \cot 2\theta$

Summary sheet

Key Takeaways

- Principal solution lies in $[0, 2\pi)$
- General solution is for the whole domain of the function which is usually R.

Key Formulae

- General solution of $\sin \theta = \sin \alpha$ is
  $\theta = n\pi + (-1)^n \alpha; \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \forall n \in Z$
- If $\cos \theta = \cos \alpha$, then general solution is
  $\theta = 2n\pi \pm \alpha; \alpha \in [0, \pi] \forall n \in Z$
- General solution of $\tan \theta = \tan \alpha$ is
  $\theta = n\pi + \alpha; \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right); n \in Z$

Mind Map

Trigonometric Equations

- **General Solutions**
  - $\sin \theta = \sin \alpha$
    $\Rightarrow \theta = n\pi + (-1)^n \alpha; \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \forall n \in Z$
  - $\cos \theta = \cos \alpha$
    $\Rightarrow \theta = 2n\pi \pm \alpha; \alpha \in [0, \pi] \forall n \in Z$
  - $\tan \theta = \tan \alpha$
    $\Rightarrow \theta = n\pi + \alpha; \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right); n \in Z$
- **Principal Solutions**

Self-Assessment

1. Solve $2\cos^2\theta + 3\sin\theta = 0$.

2. Solve $\cos\theta + \cos 3\theta - 2\cos 2\theta = 0$.

Answers

Quick Query

1. Step 1:

Given, $2\sin x - \sqrt{3} = 0$

We know that domain for principle solution is [0, 2π). Now,

$$\Rightarrow \sin x = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \sin x = \sin\frac{\pi}{3}$$

And $\sin x = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right)$ $\quad$ ($\because \sin(\pi - \theta) = \sin\theta$)

$$\Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}$$

Therefore, principal solutions are $x = \frac{\pi}{3}, \frac{2\pi}{3}$

Concept Check

1. Step 1:

Given $2\cos x - 1 = 0$

The domain of principle solution is [0, 2π). Now,

$$\Rightarrow \cos x = \frac{1}{2}$$

$$\Rightarrow \cos x = \cos\frac{\pi}{3}$$

And $\cos x = \cos\left(2\pi - \frac{\pi}{3}\right) = \cos\left(\frac{5\pi}{3}\right)$ $\quad$ ($\because \cos(2\pi - \theta) = \cos\theta$)

$$\Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$$

Therefore, principal solutions are

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

2. Step 1:

Given

$$\sin\theta = \frac{1}{2}$$

$$\Rightarrow \sin\theta = \sin\left(\frac{\pi}{6}\right) \qquad \left\{\because \frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}$$

We know, $\sin\theta = \sin\alpha \Leftrightarrow \theta = n\pi + (-1)^n \alpha; \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \forall n \in \mathbb{Z}$

$$\Rightarrow \theta = n\pi + (-1)^n \frac{\pi}{6} \forall n \in \mathbb{Z}$$

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3. Step 1:

Given, cos 3x = -1

We know, $\cos\theta = -1 \Leftrightarrow \theta = (2n + 1)\pi;\ n \in Z$

cos 3x = -1

$\Rightarrow 3x = (2n + 1)\pi;\ n \in Z$

$\Rightarrow x = (2n + 1)\frac{\pi}{3}$

Step 2:

Now, for

n	0	1	2
x	$\frac{\pi}{3}$	π	$\frac{5\pi}{3}$

Clearly, values of x satisfying the equation in $[0, 2\pi]$ are $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$

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4. Step 1:

Given, $\tan\theta = \frac{1}{\sqrt{3}}$

$$\Rightarrow \tan\theta = \tan\frac{\pi}{6} \qquad \left\{\because \frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}$$

We know, $\tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha; \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right); n \in \mathbb{Z}$

$$\therefore \tan\theta = \tan\frac{\pi}{6} \Leftrightarrow \theta = n\pi + \frac{\pi}{6} \forall n \in \mathbb{Z}$$

$$\Rightarrow \theta = n\pi + \frac{\pi}{6} \forall n \in \mathbb{Z}$$

5. Step 1:

We have,

$$\Rightarrow \tan 5\theta = \cot 2\theta = \tan\left(\frac{\pi}{2} - 2\theta\right) \qquad \left\{\because \cot\phi = \tan\left(\frac{\pi}{2} - \phi\right)\right\}$$

$$\Rightarrow 5\theta = n\pi + \frac{\pi}{2} - 2\theta \qquad \{\because \tan\phi = \tan\alpha \Leftrightarrow \phi = n\pi + \alpha; n \in \mathbb{Z}\}$$

$$\Rightarrow 7\theta = n\pi + \frac{\pi}{2}$$

$$\Rightarrow \theta = \frac{n\pi}{7} + \frac{\pi}{14} \text{ where } n \in \mathbb{Z}$$

$$\Rightarrow \theta = (2n+1)\frac{\pi}{14}, \text{where } n \in \mathbb{Z}$$

Step 2:

But, for n = ..., 3, 10, 17,.... tan 5θ is not defined. Hence, the above solution holds for all n except these values.

$$\Rightarrow \theta = (2n+1)\frac{\pi}{14}, \text{ where } n \in \mathbb{Z}, \text{ but } n \neq \ldots, 3, 10, 17, \ldots$$

Self-Assessment

1. Step 1:

We have,

$$\Rightarrow 2(1-\sin^2\theta) + 3\sin\theta = 0 \qquad [\because \cos^2\phi = 1 - \sin^2\phi]$$

$$\Rightarrow -2\sin^2\theta + 3\sin\theta + 2 = 0$$

$$\Rightarrow 2\sin^2\theta - 3\sin\theta - 2 = 0$$

$$\Rightarrow (\sin\theta - 2)(2\sin\theta + 1) = 0$$

$$\Rightarrow \sin\theta = 2 \text{ or } \sin\theta = -\frac{1}{2}$$

Step 2:

Since, range of sin θ is [-1, 1]. Hence, sin θ = 2 is not possible.

$$\Rightarrow \sin\theta = -\frac{1}{2} = \sin\left(-\frac{\pi}{6}\right) \qquad \left\{\because -\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}$$

$$\Rightarrow \theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right); n \in \mathbb{Z} \qquad \left\{\because \sin\phi = \sin\alpha \Leftrightarrow \phi = n\pi + (-1)^n\alpha; \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]; n \in \mathbb{Z}\right\}$$

$$\theta = n\pi + (-1)^{n+1}\left(\frac{\pi}{6}\right); n \in \mathbb{Z}$$

---

2. Step 1:

We have, cos θ + cos 3θ - 2cos 2θ = 0

$$\Rightarrow 2\cos 2\theta\cos\theta - 2\cos 2\theta = 0 \qquad \left\{\because \cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right\}$$

$$\Rightarrow 2\cos 2\theta(\cos\theta - 1) = 0$$

$$\Rightarrow \cos 2\theta = 0 \text{ or } \cos\theta - 1 = 0$$

$$\Rightarrow \cos 2\theta = 0 \text{ or } \cos\theta = 1$$

Step 2:

$$\text{Cos}\,\phi = 0 \Leftrightarrow \phi = (2n+1)\frac{\pi}{2} \text{ where } n \in \mathbb{Z}$$

$$\text{Cos}\,\phi = 1 \Leftrightarrow \phi = 2n\pi \text{ where } n \in \mathbb{Z}$$

$$\Rightarrow 2\theta = (2\text{n}+1)\frac{\pi}{2}\ \forall\, n \in \mathbb{Z} \text{ or } \theta = 2\text{n}\pi\ \forall\ n \in \mathbb{Z}$$

$$\Rightarrow \theta = (2\text{n}+1)\frac{\pi}{4}\ \forall\ n \in \mathbb{Z} \text{ or } \theta = 2\text{n}\pi\ \forall\ n \in \mathbb{Z}$$

NOTES

M A T H E M A T I C S

TRIGONOMETRIC EQUATIONS

METHOD TO SOLVE TRIGONOMETRIC EQUATIONS

What you already know

- Trigonometric ratios
- Trigonometric ratios of multiple angles
- Extreme values of trigonometric functions
- Periodicity of trigonometric function
- Linear trigonometric equations

What you will learn

- Non-linear trigonometric equations
- Various types of trigonometric equations and their methods to solve (Equations reducible to quadratic form, Introducing auxiliary angle, solve by transforming sum into product and product into sum)

Common transformation formulae:

$\sin 2\theta = 2\sin\theta\cos\theta$	$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
$2\sin A\cos B = \sin(A+B) + \sin(A-B)$	$2\cos A\sin B = \sin(A+B) - \sin(A-B)$
$2\cos A\cos B = \cos(A+B) + \cos(A-B)$	$2\sin A\sin B = \cos(A-B) - \cos(A+B)$
$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$	$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$
$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$	$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

Some common trigonometric equations and their general solutions:

Trigonometric equations	General solution
$\sin x = 0$	$x = n\pi, n \in \mathbb{Z}$
$\sin x = 1$	$x = (4n+1)\frac{\pi}{2}, n \in \mathbb{Z}$
$\sin x = -1$	$x = (4n-1)\frac{\pi}{2}, n \in \mathbb{Z}$

$\cos x = 0$	$x = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$
$\cos x = 1$	$x = 2n\pi, n \in \mathbb{Z}$
$\cos x = -1$	$x = (2n+1)\pi, n \in \mathbb{Z}$
$\tan x = 0$	$x = n\pi, n \in \mathbb{Z}$

Solutions of Linear Trigonometric Equations

Form	**θ =**	**α ∈**
$\sin\theta = \sin\alpha$	$n\pi + (-1)^n \alpha, \forall n \in \mathbb{Z}$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos\theta = \cos\alpha$	$2n\pi \pm \alpha, \forall n \in \mathbb{Z}$	$[0, \pi]$
$\tan\theta = \tan\alpha$	$n\pi + \alpha, \forall n \in \mathbb{Z}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Non-Linear Trigonometric Equations

We have solved the linear trigonometric equations, now we will solve the non-linear trigonometric equations like, $\cos^2\theta = \cos^2\alpha$ etc.

$$\sin^2\theta = \sin^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha \,;\, \alpha \in \left[0, \frac{\pi}{2}\right]; \forall\, n \in \mathbb{Z}$$

Proof

Let, $\alpha \in \left[0, \frac{1}{2}\right]$

We have,

$\sin^2\theta = \sin^2\alpha$ $\qquad \left\{\sin^2\theta = \frac{1-\cos 2\theta}{2}\right\}$

$\Rightarrow \frac{1-\cos 2\theta}{2} = \frac{1-\cos 2\alpha}{2}$

$\Rightarrow \cos 2\theta = \cos 2\alpha$

Here, 2θ is unknown and 2α is known. We know,

$\left\{\begin{array}{l} \because \cos x = \cos y \\ \Leftrightarrow x = 2n\pi \pm y; y \in [0, \pi] \end{array}\right\}$

As the given angle

$\alpha \in \left[0, \frac{\pi}{2}\right]$

$\Rightarrow 0 \le 2\alpha \le \pi \qquad \Rightarrow 2\alpha \in [0, \pi]$

$\Rightarrow 2\theta = 2n\pi \pm 2\alpha$

$\Rightarrow \theta = n\pi \pm \alpha \,;\, \alpha \in \left[0, \frac{\pi}{2}\right]; n \in \mathbb{Z}$

(Hence proved)

Quick Query 1

Prove $\cos^2\theta = \cos^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha\ ; \alpha \in \left[0, \frac{\pi}{2}\right]$; $\forall\ n \in \mathbb{Z}$

Quick Query 2

Prove $\tan^2\theta = \tan^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha\ ; \alpha \in \left[0, \frac{\pi}{2}\right)$; $\forall\ n \in \mathbb{Z}$

Solve for x, $\cos^2 x = \frac{1}{2}$.

We have, $\cos^2 x = \left(\frac{1}{\sqrt{2}}\right)^2$

So we can apply the formula: $\cos^2\theta = \cos^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha;\ \alpha \in \left[0, \frac{\pi}{2}\right]; n \in \mathbb{Z}$

$\Rightarrow \cos^2 x = \left(\frac{1}{\sqrt{2}}\right)^2 = \cos^2\frac{\pi}{4}$ $\quad \left\{\because \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ and } \frac{\pi}{4} \in \left[0, \frac{\pi}{2}\right]\right\}$

$\Rightarrow x = n\pi \pm \frac{\pi}{4}; n \in \mathbb{Z}$, is the required solution.

Concept Check

Find the interval of x satisfying $4\sec^2 x = 5 + \tan^2 x$

Points to keep in mind before solving trigonometric equations

1. Random squaring should be avoided, as it leads to extraneous solution.

Example

Consider, $\sin x = 1$

CORRECT APPROACH : We know, $\sin\theta = 1 \Leftrightarrow \theta = (4n+1)\frac{\pi}{2}; n \in \mathbb{Z}$

$\therefore$ Solution set $= \left\{x : x = (4n+1)\frac{\pi}{2}\ ;\ n \in \mathbb{Z}\right\}$, this is the correct approach.

WRONG APPROACH : Suppose we square both sides,

$\Rightarrow \sin^2 x = 1 \Rightarrow \sin x = \pm 1$

$\Rightarrow \sin x = +1$ *or* $\sin x = -1$ $\quad \{'or' \text{ means } \cup\}$

$x = (4n+1)\frac{\pi}{2};\ n \in \mathbb{Z}$ *or* $x = (4n-1)\frac{\pi}{2};\ n \in \mathbb{Z}$

Here, we can observe $\left\{x = (4n-1)\frac{\pi}{2}\right\}$ is the extra solution set.

2. Never cancel the terms containing unknowns, as it leads to loss of genuine solutions.

Example

Consider, $\sin x \cos x = \cos x$

WRONG APPROACH : Dividing by cos x from both sides

$\Rightarrow \sin x \cos x = \cos x \quad \Rightarrow \sin x = 1$

$\Rightarrow x = (4n+1)\frac{\pi}{2};\ n \in \mathbb{Z}$

As we are dividing both sides with cos x,

we should not include the intervals where cos x = 0 (as denominator cannot be 0.)

CORRECT APPROACH: $\sin x \cos x = \cos x$

$\Rightarrow \sin x \cos x - \cos x = 0$

$\Rightarrow \cos x (\sin x - 1) = 0$

$\Rightarrow \cos x = 0$ or $\sin x = 1$

$x = (2n+1)\frac{\pi}{2};\ n \in \mathbb{Z}$ or $x = (4n+1)\frac{\pi}{2};\ n \in \mathbb{Z}$

$\Rightarrow x \in \left\{x : x = (2n+1)\frac{\pi}{2};\ n \in \mathbb{Z}\right\} \cup \left\{x : x = (4n+1)\frac{\pi}{2};\ n \in \mathbb{Z}\right\}$

This is the correct answer.

3. The solution should not contain values for which the equation becomes undefined.

Example

Consider, $\tan x \cot x = 1$

WRONG APPROACH : $\tan x \times cot\ x = 1$

$\Rightarrow \tan x \times \frac{1}{\tan x} = 1 \Rightarrow 1 = 1$

$\Rightarrow x \in \mathbb{R}$

Here, we did not consider the case, where $\tan x = 0$. So this is an incorrect approach.

CORRECT APPROACH : $\tan x \times \frac{1}{\tan x} = 1$

$\Rightarrow x \in \mathbb{R} - \{\text{cases where } \tan x = 0 \text{ or } \tan x = \text{undefined}\}$

$\because \tan x \neq 0 \quad \Rightarrow x \neq n\pi\ ; n \in \mathbb{Z}$

and $\tan\left((2n+1)\frac{\pi}{2}\right)$ is, undefined ; $n \in \mathbb{Z}$

Thus, $\tan x \cot x = 1$

$\Rightarrow x \in \mathbb{R} - \left\{(2n+1)\frac{\pi}{2}\ ; n \in \mathbb{Z}\right\} - \{n\pi\ ;\ n \in \mathbb{Z}\}$,

is the required solution.

4. The solution should not contain values for which the denominator is zero at any stage of solving.

Example

Consider, $\frac{\sin 2x}{\sin x} = 0$

WRONG APPROACH :

Suppose we multiply sin x on both sides,

$\Rightarrow \frac{\sin 2x}{\sin x} \times \sin x = 0 \times \sin x$, looks like a valid step

$\Rightarrow \sin 2x = 0 \qquad \{\sin\theta = 0 \Leftrightarrow \theta = n\pi\ ; n \in \mathbb{Z}\}$

$\Rightarrow 2x = n\pi\ ; n \in \mathbb{Z} \qquad \Rightarrow x = \frac{n\pi}{2}\ ; n \in \mathbb{Z}$

$\Rightarrow x = 0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \pm 2\pi.....$

This is an incorrect solution set.

CORRECT APPROACH :

$\frac{\sin 2x}{\sin x} = 0$

$\Rightarrow \sin 2x = 0$ and $\sin x \neq 0$

$\Rightarrow 2x = n\pi\ ; n \in \mathbb{Z}$ *and* $x \neq n\pi\ ;\ n \in \mathbb{Z}$

$\Rightarrow x = \frac{n\pi}{2}\ ; n \in \mathbb{Z}$ *and* $x \neq n\pi\ ;\ n \in \mathbb{Z}$

$\Rightarrow x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}.....,$

is the required solution set.

Methods to solve trigonometric equations

Type 1: Factorisation Method

The number of solutions of $2\cos x\ \cos 2x = \cos x$ in $x \in [-\pi, \pi]$ is

Solution

Given,

$2\cos x\ \cos 2x - \cos x = 0$

$\Rightarrow \cos x(2\cos 2x - 1) = 0$

We know, product of two linear factos is zero when either of them is zero.

$\Rightarrow \cos x = 0$ or $2\cos 2x - 1 = 0$

Taking each term as different cases.

Case I : $\cos x = 0$

$\left\{\because \cos\theta = 0 \Leftrightarrow \theta = (2n+1)\frac{\pi}{2}\ ; n \in \mathbb{Z}\right\}$

$\Rightarrow x = (2n+1)\frac{\pi}{2}\ ; n \in \mathbb{Z} \leftarrow$ Let this be Set A

Case II :

$2\cos 2x - 1 = 0$

$\Rightarrow \cos 2x = \frac{1}{2}$ $\left\{\because \cos\frac{\pi}{3} = \frac{1}{2} \text{ and } \frac{\pi}{3} \in [0, \pi]\right\}$

Also, $\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi \pm \alpha\ ; \alpha \in [0, \pi]\ ; n \in \mathbb{Z}$

$\Rightarrow \cos 2x = \cos\frac{\pi}{3}$

$\Rightarrow 2x = 2n\pi \pm \frac{\pi}{3}; n \in \mathbb{Z}$

$\Rightarrow x = n\pi \pm \frac{\pi}{6}; n \in \mathbb{Z} \leftarrow$ Let this be Set B

Final general solution set is given as,

$A \cup B = \left\{x : x = (2n+1)\frac{\pi}{2}\ ; \forall n \in \mathbb{Z}\right\} \cup \left\{x : x = n\pi \pm \frac{\pi}{6}\ ; \forall n \in \mathbb{Z}\right\}$

However, we consider solution only in $[-\pi, \pi]$ (Given)

Case 1: $x = (2n+1)\frac{\pi}{2}\ ; \forall n \in \mathbb{Z}$			Case 2: $x = n\pi \pm \frac{\pi}{6}\ ; \forall n \in \mathbb{Z}$		
n	0	±1	n	0	±1
x	$\frac{\pi}{2}$	$\frac{3\pi}{2}, \frac{-\pi}{2}$	x	$\pm\frac{\pi}{6}$	$\pm\frac{5\pi}{6}, \pm\frac{7\pi}{6}$

Here, $\left\{\frac{3\pi}{2}, \pm\frac{7\pi}{6} \text{ and angles at other values of } n\right\} \notin [-\pi, \pi]$

Hence, the distinct solutions satisfying the equation are,

$x = \left\{\frac{-5\pi}{6}, \frac{-\pi}{6}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}\right\}$

Thus, the trigonometric equation has 6 solutions.

Type 2: Equation reducible to quadratic form

The genral solution of $2\cos^2 x - 3\cos x - 2 = 0$ is__________.

Solution

We have, $2\cos^2 x - 3\cos x - 2 = 0$

Let, $\cos x = t$

Then the given equation can be written as,

$2t^2 - 3t - 2 = 0$ (A quadratic equation in terms of t)

We factorise using splitting the middle term,

$\Rightarrow 2t^2 - 4t + t - 2 = 0 \quad \Rightarrow 2t(t - 2) + 1(t - 2) = 0$

$\Rightarrow (2t + 1)(t - 2) = 0$

We know, product of two terms is zero only when either of them is zero.

$\Rightarrow (2t + 1) = 0$ or $(t - 2) = 0$ (Replacing t by cos x)

$\Rightarrow (2\cos x + 1) = 0$ or $(\cos x - 2) = 0$

Case 1

$(2\cos x + 1) = 0$

$\Rightarrow \cos x = \frac{-1}{2}$

$\{\because \cos\left(\pi - \left(\frac{\pi}{3}\right)\right) = -\cos\frac{\pi}{3} = \frac{-1}{2}\}$

$\Rightarrow \cos x = \cos\left(\frac{2\pi}{3}\right)$

$\{\because \cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi \pm \alpha;\ \alpha \in [0,\pi],$
$n \in Z$ and $\left(\frac{2\pi}{3}\right) \in [0,\pi]\}$

$\Rightarrow x = 2n\pi \pm \left(\frac{2\pi}{3}\right);\ n \in Z \leftarrow$ Let this be Set A

Case 2

$\cos x - 2 = 0$

$\Rightarrow \cos x = 2$ { But, $-1 \le \cos\theta \le 1;\ \forall\ \theta \in R$} So, this case is not possible.

$\Rightarrow x \in \Phi$ (Null Set) $\leftarrow$ Let this be Set B

Final solution set = $A \cup B = A \cup \Phi = \{x : x = 2n\pi \pm \left(\frac{2\pi}{3}\right);\ n \in Z\}$

Type 3: Solving by introducing auxiliary angle

Trigonometric equation of the form, $a \sin x + b \cos x = c;\ \forall\ a,b,c \in R$

Algorithm

Step 1:

Divide both sides of the equation with $\sqrt{(a^2 + b^2)}$, such that $\sqrt{(a^2 + b^2)}$ is defined and known.

Assumption: a and b are not simultaneously zero.

$\Rightarrow$if $a = 0$ then $b \neq 0$ and if $a \neq 0$ then $b = 0 \Rightarrow \sqrt{(a^2 + b^2)} > 0;\ \forall\ a,b \in R$

Step 2:

The given equation can be rewritten as, a sin x + b cos x = c

$$\Rightarrow \left(\frac{a}{\sqrt{(a^2+b^2)}}\right)\sin x + \left(\frac{b}{\sqrt{(a^2+b^2)}}\right)\cos x = \left(\frac{c}{\sqrt{(a^2+b^2)}}\right)$$

Step 3:

We construct a right angled triangle ABC with:

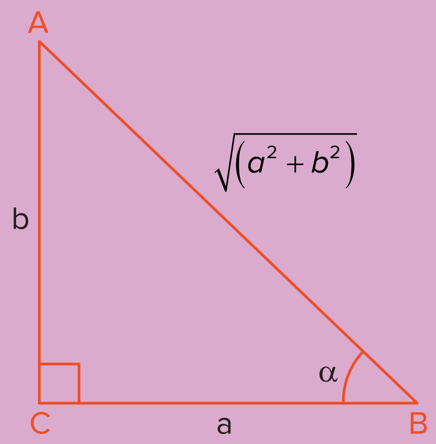
Base = $BC = a$, perpendicular = $AC = b$, $\angle B = \alpha$

Upon observing we find that:

$$\text{Hypotenuse} = \sqrt{(a^2+b^2)}$$

$$\sin\alpha = \frac{b}{\sqrt{(a^2+b^2)}}$$

$$\cos\alpha = \frac{a}{\sqrt{(a^2+b^2)}}$$



Step 4:

Putting sin α and cos α in the equation, we get:

$$\left(\frac{a}{\sqrt{(a^2+b^2)}}\right)\sin x + \left(\frac{b}{\sqrt{(a^2+b^2)}}\right)\cos x = \frac{c}{\sqrt{(a^2+b^2)}}$$

$$\Rightarrow \cos\alpha \ \sin x + \sin\alpha \ \cos x = \frac{c}{\sqrt{a^2+b^2}} \qquad \{\because \sin(A+B) = \sin A\cos B + \cos A\sin B\}$$

$$\Rightarrow \sin(x+\alpha) = \frac{c}{\sqrt{a^2+b^2}}$$

Now, RHS is written in terms of some known sine angle β and our equation becomes,

$$\Rightarrow a \ \sin x + b \ \cos x = c$$

$$\Rightarrow \sin(x+\alpha) = \sin\beta$$

Now we can use the relation for trigonometric equation in sine :

$$\sin\theta = \sin\alpha \Leftrightarrow \theta = n\pi + (-1)^n \alpha \ ; \ \alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]; n \in \mathbb{Z}$$

Note

We can also convert the equation a sin x + b cos x = c into cos (x - α) = cos β. In that case we have to use the result of the trigonometric equation in cosine.

Find the general solution of $\sqrt{3}\cos x + \sin x = -2$.

Solution

Upon comparing with the standard equation a sin x + b cos x = c, we get: $\sqrt{3}\cos x + \sin x = -2$.

$\Rightarrow \sqrt{a^2+b^2} = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$

We have, $\sqrt{3}\cos x + \sin x = -2$

Dividing both sides by $\sqrt{a^2+b^2} = 2$ we get,

$\Rightarrow \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \frac{-2}{2}$ $\quad \left\{\because \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},\ \cos\frac{\pi}{3} = \frac{1}{2}\right\}$

$\Rightarrow \sin\frac{\pi}{3}\cos x + \cos\frac{\pi}{3}\sin x = -1$ $\quad \left\{\because \sin A\cos B + \cos A\sin B = \sin(A+B)\right\}$

$\Rightarrow \sin\left(x + \frac{\pi}{3}\right) = -1$

$\left\{\because \sin\theta = -1 \Leftrightarrow \theta = (4n-1)\frac{\pi}{2};\ n \in \mathbb{Z}\right\}$

$\Rightarrow x + \frac{\pi}{3} = (4n-1)\frac{\pi}{2};\ n \in \mathbb{Z}$

$\Rightarrow x = (4n-1)\frac{\pi}{2} - \frac{\pi}{3} = 2n\pi - \frac{\pi}{2} - \frac{\pi}{3}$

$\Rightarrow x = 2n\pi - \frac{5\pi}{6};\ n \in \mathbb{Z}$,is the required general solution.

Type 4: Solving by transforming sum into product

The number of solution(s) of cos 3x + sin 2x - sin 4x = 0 in the interval $[0, 2\pi]$ is?

Solution

We have, $\cos 3x + \sin 2x - \sin 4x = 0$ $\quad \left\{\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right\}$

$\Rightarrow \cos 3x + 2\cos\left(\frac{2x+4x}{2}\right)\sin\left(\frac{2x-4x}{2}\right) = 0$

$\Rightarrow \cos 3x - 2\cos 3x\,\sin x = 0$ $\quad \left\{\because \sin(-\theta) = -\sin\theta\right\}$

$\Rightarrow \cos 3x\,(1 - 2\sin x) = 0$

$\Rightarrow \cos 3x = 0 \quad or \quad (1 - 2\sin x) = 0$

$\Rightarrow \cos 3x = 0 \quad or \quad \sin x = \frac{1}{2}$

Case 1

cos 3x = 0

$\left\{\because \cos\theta = 0 \Leftrightarrow \theta = (2n+1)\frac{\pi}{2}\,; n \in \mathbb{Z}\right\}$

$\Rightarrow 3x = (2n+1)\frac{\pi}{2}\,; n \in \mathbb{Z}$

$\Rightarrow x = (2n+1)\frac{\pi}{6}\,; n \in \mathbb{Z}$

Now 3 cases arise : $n < 0,\ n = 0,\ n > 0$

Putting $n = -1$ we get,

$x = \frac{-\pi}{6}$, but $\frac{-\pi}{6} \notin [0, 2\pi]$ (Given)

$\therefore n < 0$ is ignored.

Case 2

$\sin x = \frac{1}{2}$

$\left\{\because \sin\theta = \sin\alpha \Leftrightarrow \theta = n\pi + (-1)^n \alpha\,; \alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]; n \in \mathbb{Z}\right\}$

$\Rightarrow \sin x = \sin\frac{\pi}{6}$ $\quad \left\{\because \frac{\pi}{6} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right\}$

$\Rightarrow x = n\pi + (-1)^n \frac{\pi}{6}\,; n \in \mathbb{Z}$

Now 3 cases arise: $n < 0, n = 0, n > 0$

Putting $n = -1$ we get,

$x = -\pi + (-1)^{-1}\frac{\pi}{6} = -\pi - \frac{\pi}{6} = \frac{-7\pi}{6}$

but $\frac{-7\pi}{6} \notin [0, 2\pi]$ (Given)

$\therefore n < 0$ is ignored.

Case 1 cos3x = 0							
n	0	1	2	3	4	5	6
x	$\frac{\pi}{6}$	$\frac{3\pi}{6}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{9\pi}{6}$	$\frac{11\pi}{6}$	$\frac{13\pi}{6}$

Here, $\frac{13\pi}{6} \notin [0, 2\pi]$ (Given)

$\Rightarrow x = \left\{\frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{9\pi}{6}, \frac{11\pi}{6}\right\}$ ← Let this be Set A.

Case 2 $\sin x = \frac{1}{2}$	
n	x
0	$\frac{\pi}{6}$
1	$\frac{5\pi}{6}$
2	$\frac{13\pi}{6}$

Here, $\frac{13\pi}{6} \notin [0, 2\pi]$ (Given)

$\Rightarrow x = \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$ ← Let this be Set B.

$\therefore$ Final solution set $= A \cup B$

$= \left\{\frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{9\pi}{6}, \frac{11\pi}{6}\right\}$

Thus, there are 6 distinct solutions.

Summary sheet

Key Takeaways

- **Points to keep in mind before solving trigonometric equations:**
  (1) Random squaring should be avoided, as it leads to extraneous solutions.
  (2) Never cancel terms containing unknowns, as it leads to loss of genuine solutions..
  (3) The solution should not contain values for which the equation becomes undefined.
  (4) The solution should not contain values for which the denominator is zero at any stage.
- **Methods of solving trigonometric equations:**
  (1) Solve by factorisation
  (2) Equations reducible to quadratic form
  (3) Solve by introducing auxiliary angle
  (4) Solve by transforming sum into product

Key Results

Typical trigonometric equations and their solutions:

Form	$\sin\theta = \sin\alpha$	$\cos\theta = \cos\alpha$	$\tan\theta = \tan\alpha$	$\sin^2\theta = \sin^2\alpha$	$\cos^2\theta = \cos^2\alpha$	$\tan^2\theta = \tan^2\alpha$
$\theta =$	$n\pi + (-1)^n a \,\forall\, n \in \mathbb{Z}$	$2n\pi \pm a \,\forall\, n \in \mathbb{Z}$	$n\pi + a \,\forall\, n \in \mathbb{Z}$	$n\pi \pm a \,\forall\, n \in \mathbb{Z}$	$n\pi \pm a \,\forall\, n \in \mathbb{Z}$	$n\pi \pm a \,\forall\, n \in \mathbb{Z}$
$\alpha \in$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\left[0, \frac{\pi}{2}\right]$	$\left[0, \frac{\pi}{2}\right]$	$\left[0, \frac{\pi}{2}\right)$

Summary sheet

Self-Assessment

1. Find the general solution of $\cot x - \cos x = 1 - \cot x \cos x$.
2. Find the general solution of $\tan^2 x - \left(\sqrt{3}-1\right)\tan x - \sqrt{3} = 0$.
3. Find the general solution of $\sin x + \cos x = \sqrt{2}$.
4. The most suitable general solution of $\sin x + \sin 5x = \sin 2x + \sin 4x$ is____________.

Answers

Quick Query 1

Let $\alpha \in \left[0, \frac{\pi}{2}\right]$

$\cos^2\theta = \cos^2\alpha$

$\Rightarrow \frac{1+\cos 2\theta}{2} = \frac{1+\cos 2\alpha}{2}$ $\left\{\because \cos^2\theta = \frac{1+\cos 2\theta}{2}\right\}$

As, $\alpha \in \left[0, \frac{\pi}{2}\right] \Rightarrow 0 \le \alpha \le \frac{\pi}{2}$

$\Rightarrow 0 \le 2\alpha \le \pi \quad \Rightarrow 2\alpha \in [0, \pi]$

$\Rightarrow 2\theta = 2n\pi \pm 2\alpha$ $\left\{\begin{array}{l}\because \cos x = \cos y \\ \Leftrightarrow x = 2n\pi \pm y;\ y \in [0,\pi]\end{array}\right\}$

$\Rightarrow \theta = n\pi \pm \alpha; \alpha \in \left[0, \frac{\pi}{2}\right]; n \in \mathbb{Z}$

Quick Query 2

$\tan^2\theta = \tan^2\alpha$ $\{\tan^2 x = sec^2 x - 1\}$

$\Rightarrow sec^2\theta - 1 = sec^2\alpha - 1 \quad \Rightarrow \frac{1}{\cos^2\theta} = \frac{1}{\cos^2\alpha}$, such that $\cos^2\alpha \ne 0$

$\Rightarrow \frac{1+\cos 2\theta}{2} = \frac{1+\cos 2\alpha}{2} \quad \Rightarrow \cos 2\theta = \cos 2\alpha$ $\left\{\begin{array}{l}\because \cos x = \cos y \\ \Leftrightarrow x = 2n\pi \pm y;\ y \in [0,\pi]\end{array}\right\}$

$\Rightarrow 2\theta = 2n\pi \pm 2\alpha$

We have, $\cos^2\alpha \ne 0 \quad \Rightarrow \cos\alpha \ne 0$

$\alpha \ne (2n+1)\frac{\pi}{2}$

n	0	1
α	$\frac{\pi}{2}$	$\frac{3\pi}{2}$

$\Rightarrow \alpha \neq \frac{\pi}{2}$ as $\alpha \in \left[0, \frac{\pi}{2}\right)$. The other angle $\frac{3\pi}{2}$ is ignored as $\frac{3\pi}{2} \notin \left[0, \frac{\pi}{2}\right)$

From $\cos 2\theta = \cos 2\alpha$, we get,

$\Rightarrow 2\theta = 2n\pi \pm 2\alpha \quad \Rightarrow \theta = n\pi \pm \alpha \; ; \alpha \in \left[0, \frac{\pi}{2}\right) ; n \in \mathbb{Z}$

(Hence Proved)

Concept check

$4sec^2 x = 5 + \tan^2 x$

We know, $\tan^2 x + 1 = sec^2 x$

$\Rightarrow 4sec^2 x = 5 + \tan^2 x \qquad \Rightarrow 4\left(\tan^2 x + 1\right) = 5 + \tan^2 x$

$\Rightarrow 4 + 4\tan^2 x = 5 + \tan^2 x$

$\Rightarrow 3\tan^2 x = 1$

$\Rightarrow \tan^2 x = \frac{1}{3} = \left(\frac{1}{\sqrt{3}}\right)^2 = \tan^2 \frac{\pi}{6} \qquad \left\{\because \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \text{ and } \frac{\pi}{6} \in \left[0, \frac{\pi}{2}\right)\right\}$

So we can apply the formula :

$\tan^2 \theta = \tan^2 \alpha \quad \Leftrightarrow \theta = n\pi \pm \alpha \; ; \alpha \in \left[0, \frac{\pi}{2}\right) ; n \in \mathbb{Z}$

$\Rightarrow \tan^2 x = \tan^2 \frac{\pi}{6}$

$\Rightarrow x = n\pi \pm \frac{\pi}{6} \; ; \alpha \in \left[0, \frac{\pi}{2}\right) ; n \in \mathbb{Z}$, is the required solution.

Self-Assessment 1

We have, $\cot x - \cos x = 1 - \cot x \cos x$

Bringing all the terms to LHS $\Rightarrow \cot x - \cos x - 1 + \cot x \cos x = 0$

Rearranging the terms to get linear factors $\Rightarrow \cot x\left(1 + \cos x\right) - 1\left(\cos x + 1\right) = 0$

$\Rightarrow \left(\cot x - 1\right)\left(1 + \cos x\right) = 0$

Case 1

$\cot x = 1$ $\left\{\because \cot x = \dfrac{1}{\tan x},\ \tan x \neq 0\right\}$

$\Rightarrow \tan x = 1$ $\Rightarrow \tan x = 1 = \tan\dfrac{\pi}{4}$

$\left\{\because \tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha\ ;\ \alpha \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right); n \in \mathbb{Z}\right\}$

As $\dfrac{\pi}{4} \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$, so the formula can be applied

$\Rightarrow x = n\pi + \dfrac{\pi}{4}\ ; n \in \mathbb{Z}$ $\leftarrow$ Let this be Set A.

Case 2

$1 + \cos x = 0$

$\Rightarrow \cos x = -1$ $\{\because \cos\pi = -1\}$

$\Rightarrow \cos x = \cos\pi$ $\left\{\begin{array}{c}\because \cos\theta = \cos\alpha \\ \Leftrightarrow \theta = 2n\pi \pm \alpha\ ;\ \alpha \in [0, \pi]\ ; n \in \mathbb{Z}\end{array}\right\}$

As $\pi \in [0, \pi]$, so this formula can be applied

$\Rightarrow x = 2n\pi \pm \pi\ ; n \in \mathbb{Z}$

$\Rightarrow x = (2n \pm 1)\pi\ ; n \in \mathbb{Z}$ $\leftarrow$ Let this be called Set B

Thus, solution set = $A \cup B$

$$= \left\{x : x = n\pi + \frac{\pi}{4}\ ; n \in \mathbb{Z}\right\} \cup \{x : x = (2n \pm 1)\pi\ ; n \in \mathbb{Z}\}$$

CAUTION : We know, $\cot n\pi$ is NOT DEFINED for $n \in \mathbb{Z}$.

So, solution set B is rejected.

Hence, final solution set = $A = \left\{x : x = n\pi + \dfrac{\pi}{4}\ ; n \in \mathbb{Z}\right\}$

Self-Assessment 2

We have, $\tan^2 x - \sqrt{3}\tan x + \tan x - \sqrt{3} = 0$

Let, $\tan x = y$

Given equation can be rewritten as,

$\Rightarrow y^2 - \sqrt{3}y + y - \sqrt{3} = 0$ (A quadratic equation in terms of y)

We factorise into linear factors,

$\Rightarrow y^2 - \sqrt{3}y + y - \sqrt{3} = 0$ $\Rightarrow y\left(y - \sqrt{3}\right) + 1\left(y - \sqrt{3}\right) = 0$

$\Rightarrow \left(y+1\right)\left(y - \sqrt{3}\right) = 0$

$\Rightarrow y + 1 = 0$ *or* $y - \sqrt{3} = 0$ (Replacing y with tan x)

$\Rightarrow \tan x + 1 = 0$ *or* $\tan x - \sqrt{3} = 0$

Case 1	Case 2
$\tan x = -1$ $\left\{\because \tan\left(\frac{-\pi}{4}\right) = -\tan\frac{\pi}{4} = -1\right\}$ $\Rightarrow \tan x = \tan\left(\frac{-\pi}{4}\right)$ $\left\{\because \tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha\ ;\alpha \in \left(\frac{-\pi}{2},\frac{\pi}{2}\right);n \in \mathbb{Z}\right\}$ As, $\frac{-\pi}{4} \in \left(\frac{-\pi}{2},\frac{\pi}{2}\right)$, so formula can be applied $\Rightarrow x = n\pi - \frac{\pi}{4}\ ;n \in \mathbb{Z}$ ←Let this be Set A.	$\tan x = \sqrt{3}$ $\left\{\because \tan\left(\frac{\pi}{3}\right) = \sqrt{3}\right\}$ $\Rightarrow \tan x = \tan\left(\frac{\pi}{3}\right)$ $\left\{\because \tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha\ ;\alpha \in \left(\frac{-\pi}{2},\frac{\pi}{2}\right);n \in \mathbb{Z}\right\}$ As, $\frac{\pi}{3} \in \left(\frac{-\pi}{2},\frac{\pi}{2}\right)$, so formula can be applied $\Rightarrow x = n\pi + \frac{\pi}{3}\ ;n \in \mathbb{Z}$ ←Let this be Set B.

Final solution set $= A \cup B$

$$= \left\{x : x = n\pi - \frac{\pi}{4}\ ; n \in \mathbb{Z}\right\} \cup \left\{x : x = n\pi + \frac{\pi}{3}\ ; n \in \mathbb{Z}\right\}$$

Self-Assessment 3

We have, $\sin x + \cos x = \sqrt{2}$

Here, $a = 1, b = 1, c = \sqrt{2}$

Dividing both sides by $\sqrt{a^2 + b^2} = \sqrt{2}$ we get,

$\Rightarrow \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = 1 \qquad \left\{\because \sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right\}$

$\Rightarrow \cos\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x = 1$

$\Rightarrow \sin\left(x + \frac{\pi}{4}\right) = 1 \qquad$ [sin A cos B + cos A sin B = sin (A+B)]

$\Rightarrow x + \frac{\pi}{4} = (4n+1)\frac{\pi}{2}, n \in \mathbb{Z}$

$\Rightarrow x = (4n+1)\frac{\pi}{2} - \frac{\pi}{4} = 2n\pi + \frac{\pi}{2} - \frac{\pi}{4}$

$\Rightarrow x = 2n\pi + \frac{\pi}{4}$, $n \in \mathbb{Z}$, is the required General Solution.

Self-Assessment 4

We have, $\sin x + \sin 5x = \sin 2x + \sin 4x \qquad \left\{\because \sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right\}$

$\Rightarrow 2\sin\left(\frac{x+5x}{2}\right)\cos\left(\frac{x-5x}{2}\right) = 2\sin\left(\frac{2x+4x}{2}\right)\cos\left(\frac{2x-4x}{2}\right)$

$\Rightarrow \sin 3x \cos 2x = \sin 3x \cos x$

$\Rightarrow \sin 3x \cos 2x - \sin 3x \cos x = 0$

$\Rightarrow \sin 3x(\cos 2x - \cos x) = 0 \qquad \left\{\because \cos C - \cos D = (-)2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right\}$

$\Rightarrow \sin 3x\left(-2\sin\frac{3x}{2}\sin\frac{x}{2}\right) = 0$

$\Rightarrow \sin 3x = 0 \quad$ or $\quad \sin\frac{3x}{2} = 0 \quad$ or $\quad \sin\frac{x}{2} = 0$

$\{\because \sin\theta = 0 \Leftrightarrow \theta = n\pi\ ;\ n \in \mathbb{Z}\}$

Case 1	Case 2	Case 3
$\sin 3x = 0$ $\Rightarrow 3x = n\pi\ ; n \in \mathbb{Z}$ $\Rightarrow x = \frac{n\pi}{3}\ ; n \in \mathbb{Z}$	$\sin\frac{3x}{2} = 0$ $\Rightarrow \frac{3x}{2} = n\pi\ ; n \in \mathbb{Z}$ $\Rightarrow x = \frac{2n\pi}{3}\ ; n \in \mathbb{Z}$	$\sin\frac{x}{2} = 0$ $\Rightarrow \frac{x}{2} = n\pi\ ; n \in \mathbb{Z}$ $\Rightarrow x = 2n\pi; n \in \mathbb{Z}$

Case 1: $\sin 3x = 0$					
n	0	±1	±2	±3	....
x	0	$\pm\frac{\pi}{3}$	$\pm\frac{2\pi}{3}$	$\pm\pi$	....

Case 1: $\sin\frac{3x}{2} = 0$					
n	0	±1	±2	±3	....
x	0	$\pm\frac{2\pi}{3}$	$\pm\frac{4\pi}{3}$	$\pm 2\pi$	....

Case 3: $\sin\frac{x}{2} = 0$					
n	0	±1	±2	±3	....
x	0	$\pm 2\pi$	$\pm 4\pi$	$\pm 6\pi$	....

$\therefore$ Upon combining all these solutions, we get,

$$\Rightarrow x \in \left\{\frac{n\pi}{3}\,; n \in \mathbb{Z}\right\} \cup \left\{\frac{2n\pi}{3}\,; n \in \mathbb{Z}\right\} \cup \left\{2n\pi\,; n \in \mathbb{Z}\right\}$$

$\Rightarrow x = \frac{n\pi}{3}\,; n \in \mathbb{Z}$,is the required solution set.

MATHEMATICS

TRIGONOMETRIC EQUATIONS

SOME MORE METHODS TO SOLVE TRIGONOMETRIC EQUATIONS

NOTES

What you already know

- Trigonometric ratios
- Trigonometric ratios of multiple angles
- Extreme values of trigonometric functions
- Periodicity of trigonometric functions
- Linear trigonometric equations

What you will learn

- Non-linear trigonometric equations
- Various types of trigonometric equations and methods to solve them (Equations reducible to quadratic form
- Introducing auxiliary angle
- Solving by transforming sum into product and product into sum

Common transformation formulae

$\sin 2\theta = 2\sin\theta\cos\theta$	$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
$2\sin A\cos B = \sin(A+B) + \sin(A-B)$	$2\cos A\sin B = \sin(A+B) - \sin(A-B)$
$2\cos A\cos B = \cos(A+B) + \cos(A-B)$	$2\sin A\sin B = \cos(A-B) - \cos(A+B)$
$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$	$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$
$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$	$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

Some common trigonometric equations and their general solutions

Trigonometric equation	General solution
$\sin x = 0$	$x = n\pi,\ n \in \mathbb{Z}$
$\sin x = 1$	$x = (4n+1)\frac{\pi}{2},\ n \in \mathbb{Z}$
$\sin x = -1$	$x = (4n-1)\frac{\pi}{2},\ n \in \mathbb{Z}$
$\cos x = 0$	$x = (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}$
$\cos x = 1$	$x = 2n\pi,\ n \in \mathbb{Z}$
$\cos x = -1$	$x = (2n+1)\pi,\ n \in \mathbb{Z}$
$\tan x = 0$	$x = n\pi,\ n \in \mathbb{Z}$

Typical trigonometric equations and their solutions

Form	$\theta =$	$\alpha \in$
$\sin\theta = \sin\alpha$	$n\pi + (-1)^n \alpha\ \forall n \in \mathbb{Z}$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos\theta = \cos\alpha$	$2n\pi \pm \alpha\ \forall n \in \mathbb{Z}$	$[0, \pi]$
$\tan\theta = \tan\alpha$	$n\pi + \alpha\ \forall n \in \mathbb{Z}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\sin^2\theta = \sin^2\alpha$	$n\pi \pm \alpha\ \forall n \in \mathbb{Z}$	$\left[0, \frac{\pi}{2}\right]$
$\cos^2\theta = \cos^2\alpha$	$n\pi \pm \alpha\ \forall n \in \mathbb{Z}$	$\left[0, \frac{\pi}{2}\right]$
$\tan^2\theta = \tan^2\alpha$	$n\pi \pm \alpha\ \forall n \in \mathbb{Z}$	$\left[0, \frac{\pi}{2}\right)$

Type V: Solve by transforming product into sum

The number of solutions of sin5x cos3x = sin6x cos2x in the interval $[0, \pi]$ is

(a) 3 (b) 4 (c) 5 (d) 6

Solution

Step 1

Convert the product of trigonometric functions into sum.

We have, sin5x cos3x = sin6x cos2x $\{\because 2\sin A \cos B = \sin(A+B) + \sin(A-B)\}$

$\Rightarrow 2\sin 5x \cos 3x = 2\sin 6x \cos 2x$

$\Rightarrow \sin(5x+3x) + \sin(5x-3x) = \sin(6x+2x) + \sin(6x-2x)$

$\Rightarrow \sin 8x + \sin 2x = \sin 8x + \sin 4x$

$\Rightarrow \sin 4x - \sin 2x = 0$ $\left\{\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right\}$

Step 2

Break into linear factors and equate each factor to zero.

$\Rightarrow 2\cos\left(\frac{4x+2x}{2}\right)\sin\left(\frac{4x-2x}{2}\right) = 0$

$\Rightarrow 2\cos 3x \sin x = 0$ $\Rightarrow \cos 3x \sin x = 0$

$\Rightarrow \cos 3x = 0$ or $\sin x = 0$

CASE I : $\cos 3x = 0$

$\left\{\because \cos\theta = 0 \Leftrightarrow \theta = (2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\right\}$

$\Rightarrow 3x = (2n+1)\frac{\pi}{2}; n \in \mathbb{Z}$

$\Rightarrow x = (2n+1)\frac{\pi}{6}; n \in \mathbb{Z}$

Now 3 cases arise : $n < 0,\ n = 0, n > 0$

Putting n = -1, we get,

$x = \frac{-\pi}{6}$, but $\frac{-\pi}{6} \notin [0, \pi]$ (Given)

$\therefore n < 0$ is ignored

CASE II : $\sin x = 0$

$\{\because \sin\theta = 0 \Leftrightarrow \theta = n\pi; n \in \mathbb{Z}\}$

$\Rightarrow x = n\pi; n \in \mathbb{Z}$

Now 3 cases arise : $n < 0, n = 0, n > 0$

Putting n = -1, we get,

$x = -\pi$, but $-\pi \notin [0, \pi]$ (Given)

$\therefore n < 0$ is ignored

Step 3

Put values of n to get the values of x. Check whether the value comes in the interval or not.

CASE I : $\cos 3x = 0$

n	0	1	2	3
x	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$

Here, $\frac{7\pi}{6} \notin [0, \pi]$ (Given)

$\Rightarrow x = \left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}\right\}$ ← Let this be Set A.

CASE II : sin x = 0			
n	0	1	2
x	0	π	2π

Here, $2\pi \notin [0, \pi]$ (Given)

$\Rightarrow x = \{0, \pi\}$ ← Let this be Set B.

Step 4

Combine the solution sets using union operation to get the final solution set.

$\therefore$ Final solution set $= A \cup B$

$$= \left\{0, \frac{\pi}{6}, \frac{\pi}{2}, \pi, \frac{5\pi}{6}\right\}$$

Thus, there are 5 distinct solutions.

Option (c)

Concept Check 1

If the general solution of $8\cos x \cos 2x \cos 4x = \frac{\sin 6x}{\sin x}$ is $x = (2n+1)\frac{\pi}{a}$, the value of a is :

(a) 4 (b) 12 (c) 14 (d) 18

Type VI: Problems based on method of completing the square

The general solution of $\sin^2 x + 2\tan^2 x - 2\sin x + 4\tan x + 3 = 0$ is :

(a) $x \in \left\{(4n-1)\frac{\pi}{2} ; n \in \mathbb{Z}\right\}$ (b) $x \in \phi$ (c) $x \in \left\{n\pi + \frac{\pi}{4} ; n \in \mathbb{Z}\right\}$ (d) $x \in \left\{\frac{n\pi}{2} ; n \in \mathbb{Z}\right\}$

Solution

We have, $\sin^2 x + 2\tan^2 x - 2\sin x + 4\tan x + 3 = 0$

Step 1

Rearrange the terms in order to make perfect square terms involving trigonometric functions.

Rearranging the terms, we get,

$\Rightarrow (\sin^2 x - 2\sin x + 1) + 2(\tan^2 x + 2\tan x + 1) = 0$

$\Rightarrow (\sin x - 1)^2 + 2(\tan x + 1)^2 = 0$

Step 2

Replace the square of trigonometric functions with algebraic variables a and b.

Let, $\sin x - 1 = a$ and $\tan x + 1 = b$

$\because$ Square of any real number is non-negative

$\Rightarrow a^2 \geq 0$ and $b^2 \geq 0$

We know that, sum of two non-negative quantities is zero only when both the quantities are zero individually.

Step 3

Find the value of a and b for which the equation gets satisfied.

$\Rightarrow a^2 + 2b^2 = 0$ iff $a^2 = 0$ and $2b^2 = 0$

$\Rightarrow a = 0$ and $b = 0$

$\Rightarrow (\sin x - 1) = 0$ and $(\tan x + 1) = 0$ $\{\text{'and' means } \cap\}$

$\Rightarrow \sin x = 1$ and $\tan x = -1$

Step 4

Divide the factors into cases and solve them separately.

CASE I: $\sin x = 1$

$\left\{\because \sin\theta = 1 \Leftrightarrow \theta = (4n+1)\frac{\pi}{2} \; ; \; n \in \mathbb{Z}\right\}$

$\Rightarrow x = (4n+1)\frac{\pi}{2} \; ; \; n \in \mathbb{Z}$

However, the given trigonometric equation:

$\sin^2 x + 2\tan^2 x - 2\sin x + 4\tan x + 3 = 0$,

becomes undefined.

As tan x is NOT DEFINED for

$x = (4n+1)\frac{\pi}{2} \; ; n \in \mathbb{Z}$

Hence, $x \in \phi$ $\leftarrow$ Let this be Set A.

CASE II: $\tan x = -1$ $\left\{\because \tan\left(\frac{-\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right) = -1\right\}$

$\Rightarrow \tan x = \tan\left(\frac{-\pi}{4}\right)$

$\left\{\because \tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha \; ; \; \alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \; ; n \in \mathbb{Z}\right\}$

As, $\frac{-\pi}{4} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, we can use this relation.

$\Rightarrow x = n\pi - \frac{\pi}{4} \; ; n \in \mathbb{Z}$ $\leftarrow$ Let this be Set B.

Step 5

Combine the solution sets of both the cases using intersection operation.

$\therefore$ Final solution set $=$ Case I $\cap$ Case II

$= A \cap B = \phi \cap B$

$= \phi$

Hence, there is no solution.

Option (b)

Concept Check 2

The general solution of $2\cot^2 x + 2\sqrt{3}\cot x + 4\operatorname{cosec} x + 8 = 0$ is :

(a) $x = n\pi + \frac{\pi}{4}; n \in \mathbb{Z}$ (b) $x = 2n\pi - \frac{\pi}{4}; n \in \mathbb{Z}$ (c) $x = n\pi + \frac{\pi}{6}; n \in \mathbb{Z}$ (d) $x = 2n\pi - \frac{\pi}{6}; n \in \mathbb{Z}$

Type VII: Problems based on boundary condition

Range of trigonometric functions

Trigonometric function	Range
$\sin\theta$	$[-1,1]$
$\cos\theta$	$[-1,1]$
$\tan\theta$	$(-\infty,\infty)$
$\operatorname{cosec}\theta$	$(-\infty,-1]\cup[1,\infty)$
$\sec\theta$	$(-\infty,-1]\cup[1,\infty)$
$\cot\theta$	$(-\infty,\infty)$

Solve

The number of solutions of the equation $4\sin 2x + \cos x = 5$ is :

(a) 0 (b) 2 (c) 4 (d) ∞

Solution

We have, $4\sin 2x + \cos x = 5$

Step 1

Replace the trigonometric function with algebraic variables. Recall range of trigonometric functions involved.

Let $\sin 2x = a$ and $\cos x = b$

Equation becomes, $4a + b = 5$ $\{\because \sin\theta \in [-1,1] \quad \cos\theta \in [-1,1]\}$

$\Rightarrow -1 \le a \le 1$ and $-1 \le b \le 1$

We need to know for what values of a and b the equation gets satisfied.

By trial and error method, we get values of a and b.

Putting $a = 1, b = 1$

$LHS = 4a + b = 4(1) + (1) = 5 = RHS$

Step 2

Replace the algebraic variables with trigonometric functions and divide them into cases.

$\Rightarrow 4\sin 2x + \cos x = 5$ is possible iff $\sin 2x = 1$ and $\cos x = 1$

$\Rightarrow \sin 2x = 1$ and $\cos x = 1$ $\{\text{and means} \cap\}$

Step 3

Solve the cases separately.

CASE I : $\sin 2x = 1$

$\left\{\because \sin\theta = 1 \Leftrightarrow \theta = (4n+1)\frac{\pi}{2}\ ;\ n \in \mathbb{Z}\right\}$

$\Rightarrow 2x = (4n+1)\frac{\pi}{2}\ ;\ n \in \mathbb{Z}$

$\Rightarrow x = (4n+1)\frac{\pi}{4}\ ;\ n \in \mathbb{Z}$ $\leftarrow$ Let this be Set A.

CASE II : $\cos x = 1$

$\{\because \cos\theta = 1 \Leftrightarrow \theta = 2n\pi\ ;\ n \in \mathbb{Z}\}$

$\Rightarrow x = 2n\pi\ ;\ n \in \mathbb{Z}$ $\leftarrow$ Let this be Set B.

Step 4

Put values of n to get the values of x. Also check whether the value comes in the interval or not.

CASE I : $\sin 2x = 1$						
n	0	1	2	3	4	...
x	$\frac{\pi}{4}$	$\frac{5\pi}{4}$	$\frac{9\pi}{4}$	$\frac{13\pi}{4}$	$\frac{17\pi}{4}$	...

CASE II : $\cos x = 1$						
n	0	1	2	3	4	...
x	0	2π	4π	6π	8π	...

Step 5

Combine the set of solutions into final solution set.

Here, A = Set of odd integral multiple of $\frac{\pi}{4}$

B = Set of even integral multiple of π

$\therefore$ Final solution set $= A \cap B$

$= \phi$

Hence, the number of solutions is zero.

Option (a)

Concept Check 3

The number of solutions of the equation $\sin x \cos x \cos 2x + \frac{1}{2} = 0$ is :

(a) 0 (b) 2 (c) 4 (d) ∞

Solve

Find the general solution of the equation $\sin x\left(\cos\left(\frac{x}{4}\right) - 2\sin x\right) + \left(1 + \sin\left(\frac{x}{4}\right) - 2\cos x\right)\cos x = 0$

Solution

We have, $\sin x \cos\frac{x}{4} - 2\sin^2 x + \cos x + \cos x \sin\frac{x}{4} - 2\cos^2 x = 0$

Step 1

Rearrange the terms and use transformation formulae involving trigonometric functions.

$\Rightarrow \left(\sin x \cos\frac{x}{4} + \cos x \sin\frac{x}{4}\right) + \cos x - 2\left(\sin^2 x + \cos^2 x\right) = 0$ $\{\because \sin^2\theta + \cos^2\theta = 1\}$

$\Rightarrow \sin\left(x + \frac{x}{4}\right) + \cos x - 2 = 0$ $\{\because \sin(A + B) = \sin A\cos B + \cos A\sin B\}$

$\Rightarrow \sin\left(\frac{5x}{4}\right) + \cos x = 2$

Step 2

Replace the trigonometric functions with algebraic variables a and b. Recall range of trigonometric functions involved.

Let $\sin\left(\frac{5x}{4}\right) = a$ and $\cos x = b$

Equation becomes, $a + b = 2$

$\left\{\begin{array}{l}\because \sin\theta \in [-1,1] \\ \cos\theta \in [-1,1]\end{array}\right\}$

Step 3

Use trial and error method to get the values of a and b satisfying the equation.

$\Rightarrow -1 \le a \le 1$ and $-1 \le b \le 1$

We need to know for what values of a and b the equation gets satisfied.

By trial and error method, we get values of a and b.

Putting $a = 1,\ b = 1$

$LHS = a + b = 1 + 1 = 2 = RHS$

Step 4

Replace the algebraic variables with trigonometric functions and divide them into cases.

$\Rightarrow \sin\frac{5x}{4} + \cos x = 2$ is possible iff $\sin\frac{5x}{4} = 1$ and $\cos x = 1$

$\Rightarrow \sin\frac{5x}{4} = 1$ and $\cos x = 1$

CASE I: $\sin\frac{5x}{4} = 1$

$\left\{\because \sin\theta = 1 \Leftrightarrow \theta = (4n+1)\frac{\pi}{2}\ ;\ n \in \mathbb{Z}\right\}$

$\Rightarrow \frac{5x}{4} = (4n+1)\frac{\pi}{2}\ ;\ n \in \mathbb{Z}$

$\Rightarrow x = (4n+1)\frac{2\pi}{5}\ ;\ n \in \mathbb{Z}$ ← Let this be Set A.

CASE II: $\cos x = 1$

$\{\because \cos\theta = 1 \Leftrightarrow \theta = 2n\pi\ ;\ n \in \mathbb{Z}\}$

$\Rightarrow x = 2n\pi\ ;\ n \in \mathbb{Z}$ ← Let this be Set B.

Step 5

Put values of n to get the values of x. Also check whether the value comes in the interval or not.

CASE I: $\sin\frac{5x}{4} = 1$

n	0	1	2	3	4	5	6	...	11	...
x	$\frac{2\pi}{5}$	2π	$\frac{18\pi}{5}$	$\frac{26\pi}{5}$	$\frac{34\pi}{5}$	$\frac{42\pi}{5}$	10π	...	18π	...

CASE II: $\cos x = 1$

n	0	1	2	3	4	5	...	9	...
x	0	2π	4π	6π	8π	10π	...	18π	...

Step 6

Combine the set of solutions into final solution set.

$\therefore$ General solution = Case I $\cap$ Case II

$= \{\cdots, 2\pi, 10\pi, 18\pi, \cdots\}$

On observing the angles, we can find a pattern in terms of n as,

Let $x = (4n+1)2\pi\ ;\ n \in \mathbb{Z}$

Then $n = 0 : x = 2\pi$

$n = 1 : x = 10\pi$

$n = 2 : x = 18\pi$

$\vdots \quad \vdots$

Hence, $x = (8n+2)\pi;\ n \in \mathbb{Z}$, is the required solution.

Solve

If x, y are the solutions of the equation $3\sin x + 4\cos x = 5y^2 - 30y + 50$,

then the value of $4\tan\left(\frac{xy}{3}\right)$ is :

(a) 1 (b) 2 (c) 3 (d) 4

Solution

Step 1

Simplify the LHS using the standard form a sin x + b cos x = c.

We need to first find the value of unknowns x and y.

Given, $3\sin x + 4\cos x = 5y^2 - 30y + 50$

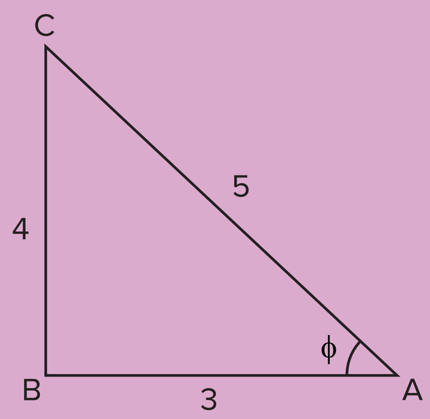
Here, L.H.S. resembles $a\sin x + b\cos x$, where $a = 3$, $b = 4$, $c = 5y^2 - 30y + 50$

Dividing both sides of the equation with $\sqrt{a^2 + b^2} = 5$,

$$\Rightarrow \frac{3}{5}\sin x + \frac{4}{5}\cos x = \frac{5y^2 - 30y + 50}{5}$$

Consider a right-angled ΔABC with $AB = 3$ units, $BC = 4$ units, $\angle CAB = \phi$.

Here, $\sin\phi = \frac{4}{5}$, $\cos\phi = \frac{3}{5}$



Step 2

Transform the equation using the trigonometric equations from the triangle.

Replacing the fractions with $\sin\phi$ and $\cos\phi$, we get,

$\Rightarrow \cos\phi \sin x + \sin\phi \cos x = y^2 - 6y + 10$

$\Rightarrow \sin(x + \phi) = y^2 - 6y + 3^2 - 3^2 + 10$ $\{\because \sin(A + B) = \sin A\cos B + \cos A\sin B\}$

$\Rightarrow \sin(x + \phi) = (y - 3)^2 + 1$ ...(1)

Step 3

Recall the range of trigonometric functions and deduce the values for terms in RHS.

Find the value of y.

Now, $\because \sin\theta \in [-1, 1] \Rightarrow \sin\theta \le 1$

$\Rightarrow \sin(x+\phi) \le 1$

$\Rightarrow (y-3)^2 + 1 \le 1$ $\quad \{\text{from the equation}\}$

$\Rightarrow (y-3)^2 \le 0$

However, square of any real quantity is always ≥ 0.

$\Rightarrow 0 \le (y-3)^2 \le 0$

$\Rightarrow (y-3)^2 = 0 \quad \Rightarrow (y-3) = 0$

$\Rightarrow y = 3 \quad \cdots(2)$

Step 4

Put the value of y and solve the trigonometric equation.

Substituting y in equation(1), we get,

$\Rightarrow \sin(x+\phi) = (y-3)^2 + 1$

$\Rightarrow \sin(x+\phi) = (3-3)^2 + 1 = 1 \quad \Rightarrow \sin(x+\phi) = 1$

$\left\{\because \sin\theta = 1 \Leftrightarrow \theta = (4n+1)\frac{\pi}{2}; n \in \mathbb{Z}\right\}$

$\Rightarrow x + \phi = (4n+1)\frac{\pi}{2}; n \in \mathbb{Z}$

$\Rightarrow x = (4n+1)\frac{\pi}{2} - \phi;\ n \in \mathbb{Z} \quad \cdots(3)$

Step 5

Substitute the values of x and y in $\tan\left(\frac{xy}{3}\right)$. Use the value from the triangle for angle $\boldsymbol{\phi}$.

Substituting the value of x and y to get the value of

$$4\tan\left(\frac{xy}{3}\right) = 4\tan\left\{\left[\left((4n+1)\frac{\pi}{2}\right) - \phi\right] \times \frac{3}{3}\right\} \quad \{\text{From equation}(2) \text{ and} (3)\}$$

$$= 4\tan\left\{(4n+1)\frac{\pi}{2} - \phi\right\}$$

$$= 4\tan\left(2n\pi + \frac{\pi}{2} - \phi\right) = 4\tan\left\{2n\pi + \left(\frac{\pi}{2} - \phi\right)\right\} \quad \{\because \tan(2n\pi + \theta) = \tan\theta\}$$

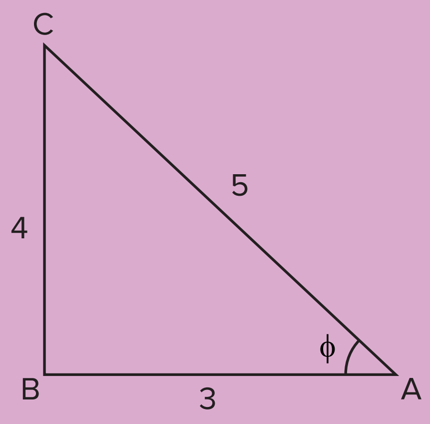
$$= 4\tan\left(\frac{\pi}{2} - \phi\right)$$

$$= 4\cot\phi \quad \left\{\because \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta\right\}$$

$= 4\frac{\cos\phi}{\sin\phi}$

$= 4 \times \frac{\frac{3}{5}}{\frac{4}{5}} = 3$, is the required solution.

Option (c)



Concept Check 4

Find the number of integral values of k for which $7\cos x + 5\sin x = 2k + 1$ has a solution.

(a) 4 (b) 8 (c) 10 (d) 12

Solve

If the number of solutions of the equation $(\sin\theta - 1)(2\sin\theta - 1)(3\sin\theta - 1)\ldots(n\sin\theta - 1) = 0$, for $n \in \mathbb{N}, \theta \in [0, \pi]$ is 9. Find the number of ordered pairs (x, y) satisfying the equation $3 + \operatorname{cosec}^2 x + 2^{\sin^2 y} = n$, if $0 \le x \le 4\pi, 0 \le y \le 4\pi$.

Solution

After finding the value of n, we will proceed with the solution of the equation involving x and y.

Step 1

Break the given equation into linear factors.

We have, $(\sin\theta - 1)(2\sin\theta - 1)(3\sin\theta - 1)\ldots(n\sin\theta - 1) = 0 \quad \cdots(1)$

$$3 + \operatorname{cosec}^2 x + 2^{\sin^2 y} = n \quad \cdots(2)$$

Rewriting the equation (1), we get,

$\Rightarrow \sin\theta - 1 = 0$ or $2\sin\theta - 1 = 0$ or $3\sin\theta - 1 = 0$ or $\ldots n\sin\theta - 1 = 0$

$\Rightarrow \sin\theta = 1$ or $\sin\theta = \frac{1}{2}$ or $\sin\theta = \frac{1}{3}$ or $\ldots \sin\theta = \frac{1}{n}$

Step 2

Assume the number of solutions for each equation.

Let number of solutions of 1st equation = m_1

number of solutions of 2nd equation = m_2

number of solutions of 3rd equation = m_3

⋮ ⋮ ⋮ ⋮

number of solutions of nth equation = m_n

Step 3

Plot the graph of the sin x and y = a type lines and check the number of intersection points.

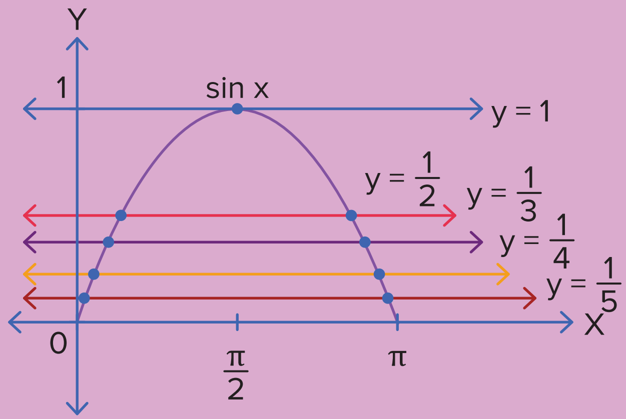
We plot the graph of sin θ in the interval [0, π].

Then we trace the graphs of R.H.S. of all the equations from

$$y = 1, \quad y = \frac{1}{2}, \quad y = \frac{1}{3}, \quad \dots, \quad y = \frac{1}{n}$$

and find the number of intersection points with the graph of $\sin\theta$.

We stop when the number of intersection points (i.e., number of solutions) reach 9.



Step 4

Stop plotting y = a type lines when the number of intersection points reaches 9.

From the graph, we get,

$y = 1$ intersects $\sin\theta$ at $\theta = \frac{\pi}{2} \Rightarrow m_1 = 1$

$y = \frac{1}{2}$ intersects $\sin\theta$ at $\theta = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow m_2 = 2$

Similarly, $y = \frac{1}{3}$ intersects $\sin\theta$ twice $\Rightarrow m_3 = 2$

$y = \frac{1}{4}$ intersects $\sin\theta$ twice $\Rightarrow m_4 = 2$

$y = \frac{1}{5}$ intersects $\sin\theta$ twice $\Rightarrow m_5 = 2$

Thus, $m_1 + m_2 + m_3 + m_4 + m_5 = 1 + 2 + 2 + 2 + 2$

$\Rightarrow m_1 + m_2 + m_3 + m_4 + m_5 = 9$

$\therefore n = 5$

Step 5

Reduce the first equation according to the value of n obtained.

So, we get 9 solutions from 5 linear factors, equation (1) becomes,

$$(\sin\theta - 1)(2\sin\theta - 1)(3\sin\theta - 1)(4\sin\theta - 1)(5\sin\theta - 1) = 0$$

$\therefore$ 9 solutions iff $n = 5$

Substituting the value of n in equation (2), we get,

$$\Rightarrow 3 + \operatorname{cosec}^2 x + 2^{\sin^2 y} = n = 5$$

$$\Rightarrow \operatorname{cosec}^2 x + 2^{\sin^2 y} = 2$$

Step 6

Replace the trigonometric functions with algebraic variables. Recall the range of trigonometric functions.

Let $\operatorname{cosec}^2 x = a$ and $2^{\frac{\sin^2 y}{2}} = b$

As $\operatorname{cosec} x \in (-\infty, -1] \cup [1, \infty)$

$\Rightarrow 1 \le \operatorname{cosec}^2 x < \infty$

$\Rightarrow 1 \le a < \infty$

As $\sin x \in [-1, 1]$

$\Rightarrow 0 \le \sin^2 x \le 1$

$\Rightarrow 2^0 \le 2^{\sin^2 y} \le 2^1$

$\Rightarrow 1 \le 2^{\sin^2 y} \le 2$

$\Rightarrow 1 \le b \le 2$

Step 7

Use trial and error method to get the values of a and b satisfying the equation.

By trial and error method, we get values of a and b.

Equation becomes, $a + b = 2$

where $a \in [1, \infty)$

$b \in [1, 2]$

We need to know for what values of a and b the equation gets satisified.

Putting $a = 1, \quad b = 1$

$\text{LHS} = a + b = 1 + 1 = 2 = \text{RHS}$

$\Rightarrow \operatorname{cosec}^2 x + 2^{\sin^2 y} = 2$, is possible iff $\operatorname{cosec}^2 x = 1$ and $2^{\sin^2 y} = 1$

$\Rightarrow \operatorname{cosec}^2 x = 1$ and $2^{\sin^2 y} = 1$ $\left\{\begin{array}{l}\text{Given, } x \in [0, 4\pi] \\ \quad y \in [0, 4\pi]\end{array}\right\}$

Step 8

Solve the cases separately and combine the solutions.

CASE I: $\operatorname{cosec}^2 x = 1$

$\Rightarrow \operatorname{cosec} x = \pm 1$

$\Rightarrow \sin x = \pm 1$ $\left\{\because \operatorname{cosec}\theta = \frac{1}{\sin\theta}\right\}$

As, $x \in [0, 4\pi]$

$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ ← Let this be Set A.

CASE II: $2^{\sin^2 y = 1}$

$\Rightarrow 2^{\sin^2 y} = 2^0$

We know that, when base is equal, exponents can be compared.

$\Rightarrow \sin^2 y = 0$

As, $y \in [0, 4\pi]$

$\Rightarrow y = 0, \pi, 2\pi, 3\pi, 4\pi$ ← Let this be Set B.

There are 4 values of x and 5 values of y.

$\therefore$ Total number of ordered pairs $(x, y) = 4 \times 5 = 20$

Solve

Find the number of solutions of $2(\cos x+\cos 2x)+\sin 2x(1+2\cos x)=2\sin x$ in the interval $[0,\pi]$.

Solution

Step 1

Simplify the equation using transformation formulae.

We have, $2(\cos x+\cos 2x)+\sin 2x(1+2\cos x)=2\sin x$

$$\Rightarrow 2(\cos x+2\cos^2 x-1)+2\sin x\cos x(1+2\cos x)$$

$$=2\sin x\ \{\because \sin 2\theta=2\sin\theta\cos\theta\ \&\ \cos 2\theta=2\cos^2\theta-1\}$$

$$\Rightarrow (\cos x+2\cos^2 x-1)+\sin x\cos x(1+2\cos x)=\sin x$$

$$\Rightarrow \cos x+2\cos^2 x-1+\sin x\cos x+2\sin x\cos^2 x=\sin x$$

Step 2

Replace the trigonometric functions with algebraic variables.

Let $\cos x=a$, $\sin x=b$

Equation becomes, $a+2a^2-1+ab+2a^2b=b$

$$\Rightarrow a+2a^2-1+ab+2a^2b-b=0$$

$$\Rightarrow (a+2a^2-1)+b(a+2a^2-1)=0$$

$$\Rightarrow (a+2a^2-1)(b+1)=0$$

$$\Rightarrow a+2a^2-1=0 \quad \text{or} \quad b+1=0 \qquad \{\text{'or' means } \cup\}$$

Step 3

Solve the algebraic equations and find the linear factors. Replace the algebraic variables with respective trigonometric functions and solve them separately.

CASE I: $a+2a^2-1=0$

$$\Rightarrow 2a^2+a-1=0$$

$$\Rightarrow 2a^2-a+2a-1=0$$

$$\Rightarrow a(2a-1)+1(2a-1)=0$$

$$\Rightarrow (a+1)(2a-1)=0$$

$$\Rightarrow a+1=0 \quad \text{or} \quad 2a-1=0$$

CASE II: $b+1=0$

$$\Rightarrow \sin x=-1$$

As, $x\in[0,\pi]$

$$\Rightarrow 0\le x\le\pi$$

$$\Rightarrow 0\le \sin x\le 1$$

$\therefore \sin x\ne -1$; $x\in[0,\pi]$;

Let set B be the set of solutions, then,

$B=\phi$

$$\Rightarrow a=-1 \quad \text{or} \quad a=\frac{1}{2}$$

$$\Rightarrow \cos x=-1 \quad \text{or} \quad \cos x=\frac{1}{2}$$

$$\Rightarrow x=\pi \quad \text{or} \quad x=\frac{\pi}{3} \qquad \{\text{Given, } x\in[0,\pi]\}$$

$\Rightarrow x=\frac{\pi}{3},\pi$ ← Let this be Set A.

Step 4

Combine the solutions of both the cases.

$\therefore$ Final solution set = Case I $\cup$ Case II

$$= A \cup B = A \cup \phi$$

$$= A = \left\{\frac{\pi}{3}, \pi\right\}$$

Hence, total number of solutions in $[0, \pi] = 2$.

Summary

Key Takeaways

- Range of trigonometric functions

Trigonometric function	Range
$\sin\theta$	$[-1,1]$
$\cos\theta$	$[-1,1]$
$\tan\theta$	$(-\infty,\infty)$
$\operatorname{cosec}\theta$	$(-\infty,-1]\cup[1,\infty)$
$\sec\theta$	$(-\infty,-1]\cup[1,\infty)$
$\cot\theta$	$(-\infty,\infty)$

- Replacing the trigonometric functions with common algebraic variables (like a, b) help in simplifying the equation.
- In problems based on completing the squares and boundary condition, the trigonometric equation is broken into two or more cases with generally **AND** in between. So, after finding the solutions for all the cases, we perform **INTERSECTION** operation on the solution sets to get the final solution set.
- In all other types of problems that are solved by factorisation method, using transformation of trigonometric functions formulae, the trigonometric equation is broken into two or more cases with generally **OR** in between. So, after finding the solutions for all the cases, we perform **UNION** operation on the solution sets to get the final solution set.

Mind map

Trigonometric Equation

- Solve by transforming product into sum
  - Convert the product of trigonometric function into sum
  - Simplify the equation
  - Solve using factorisation
- Solve by completing the square
  - Simplify the trigonometric eqation using properties of trigonometric function
  - Rearrange the terms to make square terms
  - Replace trigonometric function by algebraic variables
  - Break the equation into cases and solve them separately
- Solve using boundary condition
  - Simplify the trigonometric equation and replace the trigonometric functions with algebraic variables
  - Invoke boundary conditions of trigonometric equations involved
  - Break the equation into parts and look for values in boundary condition that satisfies the equation, equate to those values and solve them separately

Self-Assessment

1. Find the principal solution of the equation

$$\sqrt{\cot 3x + \sin^2 x - \frac{1}{4}} + \sqrt{\sin x + \sqrt{3}\cos x - 2} = \sin\frac{3x}{2} - \frac{\sqrt{2}}{2}.$$

2. Find the total number of solutions of $\sqrt{1-\cos x} = \sin x \; \forall \; x \in [0, 2\pi]$.

Answers

Concept Check 1

Step 1

Use the transformation formulae to simplify the equation.

We have, $8\cos x\cos 2x\cos 4x=\dfrac{\sin 6x}{\sin x}$

$\therefore \text{RHS}=\dfrac{\sin 6x}{\sin x}\ \Rightarrow \sin x\neq 0$, for the equation to exist,

$\Rightarrow x\neq n\pi$

Now, $8\cos x\cos 2x\cos 4x=\dfrac{\sin 6x}{\sin x}$

$\Rightarrow 8\sin x\cos x\cos 2x\cos 4x=\sin 6x$ $\{\sin 2\theta=2\sin\theta\cos\theta\}$

$\Rightarrow 4\sin 2x\cos 2x\cos 4x=\sin 6x$

$\Rightarrow 2\sin 4x\cos 4x=\sin 6x$

$\Rightarrow \sin 8x=\sin 6x$

$\Rightarrow \sin 8x-\sin 6x=0$ $\left\{\because \sin C-\sin D=2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)\right\}$

$\Rightarrow 2\cos\left(\dfrac{8x+6x}{2}\right)\sin\left(\dfrac{8x-6x}{2}\right)=0$

$\Rightarrow 2\cos 7x\sin x=0$

Step 2

Divide the equation into linear factors and solve them separately.

$\Rightarrow \cos 7x\sin x=0$

$\Rightarrow \cos 7x=0$ or $\sin x=0$

CASE I : $\cos 7x=0$

$\left\{\because \cos\theta=0\Leftrightarrow\theta=(2n+1)\dfrac{\pi}{2}\ ;\ n\in\mathbb{Z}\right\}$

$\Rightarrow 7x=(2n+1)\dfrac{\pi}{2}\ ;n\in\mathbb{Z}$

$\Rightarrow x=(2n+1)\dfrac{\pi}{14}\ ;\ n\in\mathbb{Z}$ ← Let this be Set A.

CASE II : $\sin x=0$

$\{\because \sin\theta=\theta\Leftrightarrow\theta=n\pi;\ n\in\mathbb{Z}\}$

$\Rightarrow x=n\pi$

But, $\sin x\neq 0$, for $\text{RHS}=\dfrac{\sin 6x}{\sin x}$ to exist

$\Rightarrow x\neq n\pi; n\in\mathbb{Z}$

$\therefore x\in\phi$ ← Let this be Set B.

Step 3

Combine the solutions to get the final solution set.

$\therefore$ Final Solution $=A\cup B=A\cup\phi=A$

$$=\left\{x: x=(2n+1)\frac{\pi}{14}\ ;\ n\in\mathbb{Z}\right\}$$

However, general solution is,

$x=(2n+1)\dfrac{\pi}{a}$ (Given)

$\Rightarrow x=(2n+1)\dfrac{\pi}{a}=(2n+1)\dfrac{\pi}{14}$

$\Rightarrow a=14$, is the required value.

Option (c)

Concept Check 2

Solution

Step 1

Use the transformation formulae to simplify the equation.

Here, we have terms containing cot x and cosec x, so we try to form squares of cot x and cosec x.

$\Rightarrow 2\cot^2 x + 2\sqrt{3}\cot x + 4\operatorname{cosec} x + 8 = 0$

$\Rightarrow \cot^2 x + \cot^2 x + 2\sqrt{3}\cot x + 4\operatorname{cosec} x + 8 = 0$ $\quad \left\{\because \cot^2 x + 1 = \operatorname{cosec}^2 x\right\}$

$\Rightarrow \cot^2 x + \operatorname{cosec}^2 x - 1 + 2\sqrt{3}\cot x + 4\operatorname{cosec} x + 8 = 0$

$\Rightarrow \cot^2 x + 2\sqrt{3}\cot x + 3 + \operatorname{cosec}^2 x + 4\operatorname{cosec} x + 4 = 0$

$\Rightarrow \left(\cot^2 x + 2\sqrt{3}\cot x + \left(\sqrt{3}\right)^2\right) + \left(\operatorname{cosec}^2 x + 4\operatorname{cosec} x + 2^2\right) = 0$

$\Rightarrow \left(\cot x + \sqrt{3}\right)^2 + \left(\operatorname{cosec} x + 2\right)^2 = 0$

Step 2

Divide the equation into linear factors and solve them separately.

$\Rightarrow \left(\cot x + \sqrt{3}\right)^2 = 0$ and $\left(\operatorname{cosec} x + 2\right)^2 = 0$ $\quad$ {'and' means $\cap$}

$\Rightarrow \left(\cot x + \sqrt{3}\right) = 0$ and $\left(\operatorname{cosec} x + 2\right) = 0$

CASE I: $\left(\cot x + \sqrt{3}\right) = 0$

$\Rightarrow \cot x = -\sqrt{3}$ $\quad \left\{\because \cot\theta = \dfrac{1}{\tan\theta}\right\}$

$\Rightarrow \tan x = \dfrac{-1}{\sqrt{3}}$

$\left\{\because \tan\left(\dfrac{-\pi}{6}\right) = -\tan\left(\dfrac{\pi}{6}\right) = \dfrac{-1}{\sqrt{3}}\right\}$

$\Rightarrow \tan x = \tan\left(\dfrac{-\pi}{6}\right)$

$\left\{\because \tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha\ ; \alpha \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right);\ n \in \mathbb{Z}\right\}$

As, $\dfrac{-\pi}{6} \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$, we can use this relation.

$\Rightarrow x = n\pi - \dfrac{\pi}{6}\ ;\ n \in \mathbb{Z}$ $\quad \leftarrow$ Let this be Set A.

CASE - II: $\left(\operatorname{cosec} x + 2\right) = 0$

$\Rightarrow \operatorname{cosec} x = -2$ $\quad \left\{\because \operatorname{cosec}\theta = \dfrac{1}{\sin\theta}\right\}$

$\Rightarrow \sin x = \dfrac{-1}{2}$

$\left\{\because \sin\left(\dfrac{-\pi}{6}\right) = -\sin\left(\dfrac{\pi}{6}\right) = \dfrac{-1}{2}\right\}$

$\Rightarrow \sin x = \sin\left(\dfrac{-\pi}{6}\right)$

$\left\{\because \sin\theta = \sin\alpha \Leftrightarrow \theta = n\pi + (-1)^n\alpha\ ; \alpha \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right];\ n \in \mathbb{Z}\right\}$

As, $\dfrac{-\pi}{6} \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$, we can use this relation.

$\Rightarrow x = n\pi + (-1)^n\left(-\dfrac{\pi}{6}\right);\ n \in \mathbb{Z}$

$\Rightarrow x = n\pi + (-1)^{n+1}\left(\dfrac{\pi}{6}\right);\ n \in \mathbb{Z}$ $\quad \leftarrow$ Let this be Set B.

Step 3

Put values of n to get the values of x. Also check whether the value comes in the interval or not.

CASE I: $\left(\cot x + \sqrt{3}\right) = 0$

n	0	1	2	3	4	...
x	$\frac{-\pi}{6}$	$\frac{5\pi}{6}$	$\frac{11\pi}{6}$	$\frac{17\pi}{6}$	$\frac{23\pi}{6}$	...

CASE II: (cosec x + 2) = 0

n	0	1	2	3	4	...
x	$\frac{-\pi}{6}$	$\frac{7\pi}{6}$	$\frac{11\pi}{6}$	$\frac{19\pi}{6}$	$\frac{23\pi}{6}$	...

Step 4

Find a pattern in the solution. Rewrite in terms of n. Combine the solutions to get the final solution set.

$\therefore$ Final general solution = Case I $\cap$ Case II

$$= A \cap B$$

$$= \left\{\frac{-\pi}{6}, \frac{11\pi}{6}, \frac{23\pi}{6}, \ldots\right\}$$

We need to find a pattern in order to write a general solution in terms of n. Combining both solution sets,

Let $x = 2n\pi - \frac{\pi}{6}; n \in \mathbb{Z}$

Then, $n = 0 : x = -\frac{\pi}{6}$

$n = 1 : x = \frac{11\pi}{6}$

$n = 2 : x = \frac{23\pi}{6}$

$\vdots \qquad \vdots$

Hence, $x = 2n\pi - \frac{\pi}{6}$; $n \in \mathbb{Z}$, is the required solution.

Option (d)

Concept Check 3

Solution

Step 1

Use the transformation formulae to simplify the equation.

We have, $\sin x \cos x \cos 2x = \frac{-1}{2}$

$\Rightarrow 2\sin x \cos x \cos 2x = -1 \qquad \{\because \sin 2\theta = 2\sin\theta \cos\theta\}$

$= \sin 2x \cos 2x \quad -1$

Multiplying and dividing by 2 in LHS,

$\Rightarrow \frac{1}{2} \times 2\sin 2x \cos 2x = -1$

Step 2
Recall the range of the trigonometric functions.
sin 4x = -2, not possible as sin ∈ [-1, 1]
Hence, no solution exists.
Option (a)

Concept Check 4

Solution

Step 1
Observe the LHS of the equation and compare with the standard form.
We have, 7 cos x + 5 sin x = 2k + 1
On observing the equation, we find that it is of the form a sin x + b cos x = c,
Where a = 5, b = 7, c = 2k + 1

Step 2
Recall the range of the standard form asin x + bcos x = c.
We know if $f(x) = a\sin x + b\cos x$
then $f \in \left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$

$\Rightarrow -\sqrt{a^2+b^2} \le 7\cos x + 5\sin x \le \sqrt{a^2+b^2}$ $\left\{\because \sqrt{a^2+b^2} = \sqrt{5^2+7^2} = \sqrt{74}\right\}$

$\Rightarrow -\sqrt{74} \le 7\cos x + 5\sin x \le \sqrt{74}$

$\Rightarrow -\sqrt{74} \le 2k+1 \le \sqrt{74}$ $\left\{\because 7\cos x + 5\sin x = 2k+1\right\}$

Step 3
Find the approximate value of the square root and solve the inequality from both sides to reach k.
⇒ -8.6 ≤ 2k + 1 ≤ 8.6
⇒ -8.6 -1 ≤ 2k ≤ 8.6 -1
⇒ -9.6 ≤ 2k ≤ 7.6
$\Rightarrow \frac{-9.6}{2} \le k \le \frac{7.6}{2}$
⇒ -4.8 ≤ k ≤ 3.8
As k ∈ Z ⇒ k = -4, -3, -2, -1, 0, 1, 2, 3
k can take 8 integral values.
Option (b)

Self-assessment 1

Solution

Step 1
Divide the whole equation into parts and simplify the terms involved using transformation formulae and the standard form asin x + bcos x = c.
For principal solution, [0, 2π)

We have, $\sqrt{\cot 3x+\sin^2 x-\frac{1}{4}}+\sqrt{\sin x+\sqrt{3}\cos x-2}=\sin\frac{3x}{2}-\frac{\sqrt{2}}{2}$

Let us consider the 2nd term of L.H.S. i.e., $\sqrt{\sin x+\sqrt{3}\cos x-2}$

Here, $\left(\sin x+\sqrt{3}\cos x\right)$ is similar to $\left(a\sin x+b\cos x\right)$ with $a=1$, $b=\sqrt{3}$

Dividing and multiplying by $\sqrt{a^2+b^2}=\sqrt{1^2+\left(\sqrt{3}\right)^2}=2$, we get,

$$\Rightarrow \sin x+\sqrt{3}\cos x=2\left(\frac{\sin x+\sqrt{3}\cos x}{2}\right)$$

$$=2\left(\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x\right)$$

$$\Rightarrow \sin x+\sqrt{3}\cos x=2\left(\sin\frac{\pi}{6}\sin x+\cos\frac{\pi}{6}\cos x\right) \qquad \left\{\because \sin\frac{\pi}{6}=\frac{1}{2},\ \cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}\right\}$$

$$\Rightarrow \sin x+\sqrt{3}\cos x=2\cos\left(x-\frac{\pi}{6}\right) \qquad \left\{\because \cos(A-B)=\cos A\cos B+\sin A\sin B\right\}$$

Step 2

Replace the term under square root with simplified expression. Enforce the conditions of square root to find the range of the expression under square root.

The second term of L.H.S becomes,

$$\Rightarrow \sqrt{\sin x+\sqrt{3}\cos x-2}=\sqrt{2\cos\left(x-\frac{\pi}{6}\right)-2}$$

$$=\sqrt{2\left(\cos\left(x-\frac{\pi}{6}\right)-1\right)}$$

As $\sqrt{2\left(\cos\left(x-\frac{\pi}{6}\right)-1\right)}$ exists iff $2\left(\cos\left(x-\frac{\pi}{6}\right)-1\right)\geq 0$

$$\Rightarrow 2\left(\cos\left(x-\frac{\pi}{6}\right)-1\right)\geq 0$$

$$\Rightarrow \cos\left(x-\frac{\pi}{6}\right)-1\geq 0$$

$\Rightarrow$ vvvvxvvvvv$\left(\frac{\pi}{6}\right)$≥vvvvvv $\qquad \theta\in[-\quad]$

$$\Rightarrow \cos\left(x-\frac{\pi}{6}\right)=1$$

$\{\because \cos\theta=1\Leftrightarrow\theta=2n\pi\ ;\ n\in\mathbb{Z}\}$

$$\Rightarrow x-\frac{\pi}{6}=2n\pi\ ;\ n\in\mathbb{Z}$$

$$\Rightarrow x=2n\pi+\frac{\pi}{6}\ ;\ n\in\mathbb{Z}$$

Step 3
Put values of n to get the values of x. Also check whether the value comes in the interval or not.

n	0	1
x	$\frac{\pi}{6}$	$\frac{13\pi}{6}$

Here, $x = \frac{13\pi}{6} \notin [0, 2\pi)$

$\therefore$ For 2nd term of L.H.S. to exist : $x = \frac{\pi}{6}$

Step 4
Check whether the value of x obtained satisfies the equation completely or not.

Let us check whether $x = \frac{\pi}{6}$ satisfies the given equation or not.

$$\therefore \text{L.H.S.} = \sqrt{\cot 3x + \sin^2 x - \frac{1}{4}} + \sqrt{2\cos\left(x - \frac{\pi}{6}\right) - 2}$$

$$= \sqrt{\cot 3 \times \frac{\pi}{6} + \sin^2 \frac{\pi}{6} - \frac{1}{4}} + \sqrt{2\cos\left(\frac{\pi}{6} - \frac{\pi}{6}\right) - 2}$$

$$= \sqrt{\cot \frac{\pi}{2} + \sin^2 \frac{\pi}{6} - \frac{1}{4}} + 0$$

$$= \sqrt{0 + \left(\frac{1}{2}\right)^2 - \frac{1}{4}} + 0 \qquad \left\{\because \cot\frac{\pi}{2} = 0,\ \sin\frac{\pi}{6} = \frac{1}{2}\right\}$$

$\Rightarrow$ L.H.S. $= 0 + 0 = 0$

$$\therefore \text{R.H.S.} = \sin\left(\frac{3x}{2}\right) - \frac{\sqrt{2}}{2}$$

$$= \sin\left(\frac{3}{2} \times \frac{\pi}{6}\right) - \frac{\sqrt{2}}{2}$$

$$= \sin\frac{\pi}{4} - \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \qquad \left\{\because \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right\}$$

$\Rightarrow$ R.H.S. $= 0$

$\therefore$ L.H.S. = R.H.S.

Hence, $x = \frac{\pi}{6}$ is the only principal solution.

Self-assessment 2

Solution

Step 1
Rewrite the equation using transformation formulae such that square root gets resolved.

We have, $\sqrt{1-\cos x} = \sin x$

$$\Rightarrow \sqrt{2\sin^2\frac{x}{2}} = 2\sin\frac{x}{2}\cos\frac{x}{2} \qquad \left\{\begin{matrix} \because \cos 2\theta = 1-2\sin^2\theta \Rightarrow 1-\cos 2\theta = 2\sin^2\theta \\ \sin 2\theta = 2\sin\theta\cos\theta \end{matrix}\right\}$$

$$\Rightarrow \sqrt{2}\left|\sin\frac{x}{2}\right| = 2\sin\frac{x}{2}\cos\frac{x}{2} \qquad \left\{\because \sqrt{y^2} = |y|\right\} \qquad \cdots(1)$$

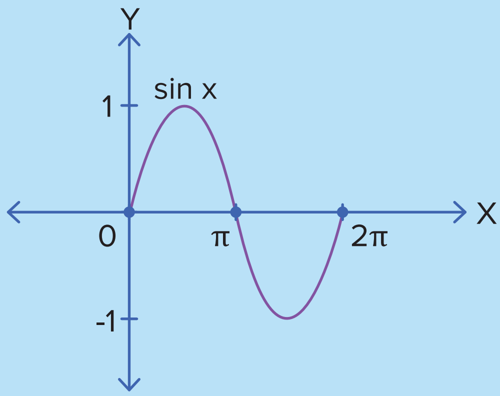
Step 2

Observe the graph of the term in modulus and decide the value in the interval concerned.

$\because 0 \le x \le 2\pi \quad$ (Given)

$$\Rightarrow 0 \le \frac{x}{2} \le \pi$$

$$\Rightarrow 0 \le \sin\frac{x}{2} \le 1 \qquad \text{(From graph)}$$



$$\Rightarrow \sin\frac{x}{2} \ge 0$$

$$\Rightarrow \left|\sin\frac{x}{2}\right| = \sin\frac{x}{2} \qquad \left\{\because |y| = \left\{\begin{matrix} y, y \ge 0 \\ -y, y < 0 \end{matrix}\right\}\right\}$$

Step 3

Replace the value in the modulus function. Simplify the equation. Divide into linear factors and solve them separately.

Putting the value of $\left|\sin\frac{x}{2}\right|$ in the equation (1) we get,

$$\Rightarrow \sqrt{2}\left|\sin\frac{x}{2}\right| = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$\Rightarrow \sqrt{2}\sin\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$\Rightarrow \sqrt{2}\sin\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2} = 0$$

$$\Rightarrow \sqrt{2}\sin\frac{x}{2}\left(1-\sqrt{2}\cos\frac{x}{2}\right) = 0$$

$$\Rightarrow \sqrt{2}\sin\frac{x}{2} = 0 \qquad \text{or} \qquad 1-\sqrt{2}\cos\frac{x}{2} = 0 \qquad \{\text{'or' means} \cup\}$$

CASE I: $\sqrt{2}\sin\frac{x}{2} = 0$

$\Rightarrow \sin\frac{x}{2} = 0$

$\Rightarrow \frac{x}{2} = n\pi$ $\{\because \sin\theta = 0 \Leftrightarrow \theta = n\pi \ ; n \in \mathbb{Z}\}$

$\Rightarrow x = 2n\pi \ ; n \in \mathbb{Z}$

n	0	1	2
x	0	2π	4π

As $x \in [0, 2\pi]$ and $4\pi \notin [0, 2\pi]$
$\Rightarrow x = \{0, 2\pi\}$ Let this be Set A

CASE II: $1 - \sqrt{2}\cos\frac{x}{2} = 0$

$\Rightarrow \cos\frac{x}{2} = \frac{1}{\sqrt{2}}$

$\Rightarrow \cos\frac{x}{2} = \cos\frac{\pi}{4}$ $\left\{\because \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right\}$

$\{\because \cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi \pm \alpha \ ; \alpha \in [0, \pi] ; n \in \mathbb{Z}\}$

$\Rightarrow \frac{x}{2} = 2n\pi \pm \frac{\pi}{4} \ ; n \in \mathbb{Z}$

$\Rightarrow x = 4n\pi \pm \frac{\pi}{2} \ ; n \in \mathbb{Z}$

n	0	1
x	$\pm\frac{\pi}{2}$	$4\pi \pm \frac{\pi}{2}$

As $x \in [0, 2\pi]$ and $\frac{-\pi}{2}, 4\pi \pm \frac{\pi}{2} \notin [0, 2\pi]$

$\Rightarrow x = \left\{\frac{\pi}{2}\right\}$ $\leftarrow$ Let this be Set B.

Step 4

Combine the solutions to get the final solution set.

$\therefore$ Final solution set = Case I $\cup$ Case II

$= A \cup B$

$= \{0, 2\pi\} \cup \left\{\frac{\pi}{2}\right\} = \left\{0, \frac{\pi}{2}, 2\pi\right\}$

Hence, the total number of solutions in $[0, 2\pi] = 3$.

MATHEMATICS

TRIGONOMETRIC EQUATIONS

TRIGONOMETRIC INEQUATIONS

NOTES

What you already know

- Trigonometric ratios
- Trigonometric ratios of multiple angles
- Extreme values of trigonometric functions
- Periodicity of trigonometric functions
- Trigonometric equations and their solutions

What you will learn

- Practice trigonometric equations' related problems
- Trigonometric inequations

$\sin(A+B) = \sin A\cos B + \cos A\sin B$	$\sin(A-B) = \sin A\cos B - \cos A\sin B$
$\cos(A+B) = \cos A\cos B - \sin A\sin B$	$\cos(A-B) = \cos A\cos B + \sin A\sin B$

Typical Trigonometric Equations and Their Solutions

Form	θ =	α ∈
$\sin\theta = \sin\alpha$	$n\pi + (-1)^n \alpha \; \forall n \in \mathbb{Z}$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos\theta = \cos\alpha$	$2n\pi \pm \alpha \; \forall n \in \mathbb{Z}$	$[0, \pi]$
$\tan\theta = \tan\alpha$	$n\pi + \alpha \; \forall n \in \mathbb{Z}$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\sin^2\theta = \sin^2\alpha$	$n\pi \pm \alpha \; \forall n \in \mathbb{Z}$	$\left[0, \frac{\pi}{2}\right]$
$\cos^2\theta = \cos^2\alpha$	$n\pi \pm \alpha \; \forall n \in \mathbb{Z}$	$\left[0, \frac{\pi}{2}\right]$
$\tan^2\theta = \tan^2\alpha$	$n\pi \pm \alpha \; \forall n \in \mathbb{Z}$	$\left[0, \frac{\pi}{2}\right)$

Find the total number of solutions of the equation $\sin\left(\frac{11}{10}\cos t\right)=\cos\left(\frac{11}{10}\sin t\right)$

Given, $\sin\left(\frac{11}{10}\cos t\right)=\cos\left(\frac{11}{10}\sin t\right)$

$\Rightarrow \sin\left(\frac{11}{10}\cos t\right)=\sin\left(\frac{\pi}{2}-\frac{11}{10}\sin t\right)$ Since, $\cos\theta=\sin\left(\frac{\pi}{2}-\theta\right)$

$\Rightarrow \frac{11}{10}\cos t=\frac{\pi}{2}-\frac{11}{10}\sin t$ $\Rightarrow \frac{11}{10}(\sin t+\cos t)=\frac{\pi}{2}$

$\Rightarrow \sin t+\cos t=\frac{\pi}{2}\times\frac{10}{11}$

$\Rightarrow \sin t\cdot\frac{1}{\sqrt{2}}+\cos t\cdot\frac{1}{\sqrt{2}}=\frac{5\pi}{11\sqrt{2}}$

$\left\{\begin{array}{l}\text{As for equations of the the form}\\ a\sin x+b\cos x=c,\\ \text{We divide both sides by }\sqrt{a^2+b^2}.\end{array}\right\}$

$\Rightarrow \sin t\cdot\cos\frac{\pi}{4}+\cos t\cdot\sin\frac{\pi}{4}=\frac{5\pi}{11\sqrt{2}}$ $\Rightarrow \sin\left(t+\frac{\pi}{4}\right)=\frac{5\pi}{11\sqrt{2}}\approx 1.01$

But the maximum value of $\sin\left(t+\frac{\pi}{4}\right)=1$

∴ The above equation does not hold true. Hence, the total number of solutions of the equation is zero.

Concept Check

Find the number of distinct solutions of the equation $\frac{5}{4}\cos^2 2x+\cos^4 x+\sin^4 x+\cos^6 x+\sin^6 x=2$ in the interval $[0, 2\pi]$.

Let S be the sum of all the distinct solutions of the equation $\sqrt{3}\sec x+\operatorname{cosec} x+2(\tan x-\cot x)=0$, where $x\in(-\pi,\pi)-\left\{-\frac{\pi}{2},0,\frac{\pi}{2}\right\}$. Find the value of S.

$\Rightarrow \sqrt{3}\sec x+\operatorname{cosec} x=2(\cot x-\tan x)$ $\Rightarrow \frac{\sqrt{3}}{\cos x}+\frac{1}{\sin x}=2\left(\frac{\cos x}{\sin x}-\frac{\sin x}{\cos x}\right)$

$\Rightarrow \frac{\sqrt{3}\sin x+\cos x}{\cos x\sin x}=2\left(\frac{\cos^2x-\sin^2x}{\sin x\cos x}\right)$

$\left\{\begin{array}{l}\because \cos 2x=\cos^2x-\sin^2x\\ x\in(-\pi,\pi)-\left\{\frac{-\pi}{2},0,\frac{\pi}{2}\right\}\\ \Rightarrow \sin x\neq 0,\ \cos x\neq 0\end{array}\right\}$

$\Rightarrow \sqrt{3}\sin x + \cos x = 2\cos 2x \quad \Rightarrow \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = \cos 2x$

$\Rightarrow \sin\frac{\pi}{3}\sin x + \cos\frac{\pi}{3}\cos x = \cos 2x$

$\Rightarrow \cos 2x = \cos\left(x - \frac{\pi}{3}\right) \Rightarrow 2x = 2n\pi \pm \left(x - \frac{\pi}{3}\right) \left(\because \cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi \pm \alpha; \alpha \in [0, \pi], n \in \mathbb{Z}\right)$

Case 1

$2x = 2n\pi + \left(x - \frac{\pi}{3}\right)$

$\Rightarrow x = 2n\pi - \frac{\pi}{3}$

n	0
x	$-\frac{\pi}{3}$

Case 2

$2x = 2n\pi - \left(x - \frac{\pi}{3}\right)$

$\Rightarrow 3x = 2n\pi + \frac{\pi}{3}$

$\Rightarrow x = (6n+1)\frac{\pi}{9}$

n	0	1	-1
x	$\frac{\pi}{9}$	$\frac{7\pi}{9}$	$\frac{-5\pi}{9}$

$\therefore$ Sum of all distinct solutions, $S = -\frac{\pi}{3} - \frac{5\pi}{9} + \frac{\pi}{9} + \frac{7\pi}{9} = 0$

Simultaneous trigonometric Equations

Find the value of $x \in \mathbb{R}$ satisfying the equation $\sin 2x = -\frac{\sqrt{3}}{2}$ and $\tan x = -\frac{1}{\sqrt{3}}$.

(1) $\sin 2x = -\frac{\sqrt{3}}{2} \quad \Rightarrow \sin 2x = \sin\left(-\frac{\pi}{3}\right)$

$\Rightarrow 2x = n\pi + (-1)^n\left(-\frac{\pi}{3}\right)$ $\quad \left\{\begin{matrix}\because \sin\theta = \sin\alpha \\ \theta = n\pi + (-1)^n \alpha \ ; \alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] ; n \in \mathbb{Z}\end{matrix}\right\}$

$\Rightarrow x = \frac{n\pi}{2} + (-1)^n\left(-\frac{\pi}{6}\right) ; n \in \mathbb{Z}$

$\therefore A = \left\{x : x = \frac{n\pi}{2} - (-1)^n \frac{\pi}{6} ; n \in \mathbb{Z}\right\}$

(2) $\tan x = -\frac{1}{\sqrt{3}}$

$$\Rightarrow \tan x = \tan\left(-\frac{\pi}{6}\right) \qquad \left\{\begin{array}{c} \because \tan\theta = \tan\alpha \\ \theta = n\pi + \alpha; \alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right); n \in \mathbb{Z} \end{array}\right\}$$

$$\Rightarrow x = n\pi - \frac{\pi}{6}; n \in \mathbb{Z}$$

$$\therefore B = \left\{x : x = n\pi - \frac{\pi}{6} \text{ where } n \in \mathbb{Z}\right\}$$

Now, we will find a pattern between the elements of set A and B.

$$x = \frac{n\pi}{2} - (-1)^n \frac{\pi}{6}; n \in \mathbb{Z}$$

n	0	1	-1	2	-2	3	-3	4	....
x	$\frac{-\pi}{6}$	$\frac{2\pi}{3}$	$\frac{-\pi}{3}$	$\frac{5\pi}{6}$	$\frac{-7\pi}{6}$	$\frac{5\pi}{3}$	$\frac{-4\pi}{3}$	$\frac{11\pi}{6}$	....

$$x = n\pi - \frac{\pi}{6}; n \in \mathbb{Z}$$

n	0	1	-1	2	-2	....
x	$\frac{-\pi}{6}$	$\frac{5\pi}{6}$	$\frac{-7\pi}{6}$	$\frac{11\pi}{6}$	$\frac{-13\pi}{6}$	....

$$\therefore A \cap B = \left\{\frac{-\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}, ...\right\} \text{ i.e., general solution: } x = n\pi + \frac{5\pi}{6}; n \in \mathbb{Z}$$

Find the least positive values of x and y satisfying the following: $x - y = \frac{\pi}{4}$ and $\cot x + \cot y = 2$

Considering cot x + cot y = 2,

$$\Rightarrow \frac{\cos x}{\sin x} + \frac{\cos y}{\sin y} = 2 \quad \Rightarrow \frac{\cos x \sin y + \cos y \sin x}{\sin x \sin y} = 2$$

$$\Rightarrow \sin(x+y) = 2\sin x \sin y \qquad \left\{\begin{array}{l} \because 2\sin A \sin B = \cos(A-B) - \cos(A+B) \\ (x-y) = \frac{\pi}{4} \text{(Given)} \end{array}\right\}$$

$$\Rightarrow \sin(x+y) = \cos(x-y) - \cos(x+y)$$

$$\Rightarrow \sin(x+y) + \cos(x+y) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{1}{\sqrt{2}}\sin(x+y) + \frac{1}{\sqrt{2}}\cos(x+y) = \frac{1}{2} \quad \Rightarrow \sin\frac{\pi}{4}\sin(x+y) + \cos\frac{\pi}{4}\cos(x+y) = \frac{1}{2}$$

$$\Rightarrow \cos\left(x + y - \frac{\pi}{4}\right) = \cos\frac{\pi}{3} \qquad \{\because \cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi \pm \alpha; \alpha \in [0, \pi], n \in \mathbb{Z}\}$$

$\Rightarrow x + y - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}\ ;\ n \in \mathbb{Z}$ $\Rightarrow x + y = 2n\pi \pm \frac{\pi}{3} + \frac{\pi}{4}\ ;\ n \in \mathbb{Z}$

We need to find the least positive value, so we put n = 0 to get,

$\Rightarrow x + y = \frac{7\pi}{12}, \frac{-\pi}{12}$ $\because x, y > 0 \Rightarrow x + y > 0$, so $\frac{-\pi}{12}$ is rejected.

$x - y = \frac{\pi}{4}$ (Given)

Solving both the equations simultaneously, we get,

$\Rightarrow x = \frac{5\pi}{12}\ ,\ y = \frac{\pi}{6}$

Hence, the least positive values of x and y are $\frac{5\pi}{12}$ and $\frac{\pi}{6}$, respectively.

If $y \in (-\pi, \pi)$, then find the total number of ordered pairs (x, y) satisfying the equation $\sec^2(x+2)y + x^2 - 1 = 0$

We have, $\sec^2(x+2)y + x^2 - 1 = 0$ $\{\because \tan^2 x + 1 = \sec^2 x\}$

$\Rightarrow \tan^2(x+2)y + x^2 = 0$, Square of any quantity is non-negative and the sum of two non-negatives is zero only when each of the quantities involved is zero.

$\Rightarrow x^2 = 0 \Rightarrow x = 0$

$\Rightarrow \tan^2(x+2)y = 0$ $\Rightarrow \tan^2 2y = 0$

$\Rightarrow \tan^2 2y = 0$ $\left\{\begin{array}{l} \because \tan^2\theta = \tan^2\alpha \\ \Leftrightarrow \theta = n\pi \pm \alpha\ \forall\ n \in \mathbb{Z} \text{ for } \alpha \in \left[0, \frac{\pi}{2}\right) \end{array}\right\}$

$\Rightarrow 2y = n\pi\ ;\ n \in \mathbb{Z}$ $\Rightarrow y = \frac{n\pi}{2}\ ;\ n \in \mathbb{Z}$

$\Rightarrow y = 0, \frac{\pi}{2}, \frac{-\pi}{2}$ $x = 0$

$\therefore$The ordered pairs are formed as $(x, y) = \left\{(0,0), \left(0, \frac{\pi}{2}\right), \left(0, \frac{-\pi}{2}\right)\right\}$. Hence, three ordered pairs are formed.

Trigonometric Inequations

To solve trigonometric equations of the type f(x) ≥ a or f(x) ≤ a where f(x) is some trigonometric function and ‘a’ is a fixed real constant, we take the following steps:

Step 1: With trigonometric function being periodic, we draw the graph of f(x) in an interval of length equal to the fundamental period of f(x). Let the fundamental period be p.

Step 2: Draw the graph of the constant function a, i.e., y = a.

Step 3: Study the interaction of both the graphs and extract the interval corresponding to which the inequation is getting satisfied. This interval is known as the partial solution set.

Step 4: Add pn(n ∈ Z) to the lower limit as well as the upper limit of the interval obtained in Step 3 and take union over the set of integers, where p the fundamental period of f(x).

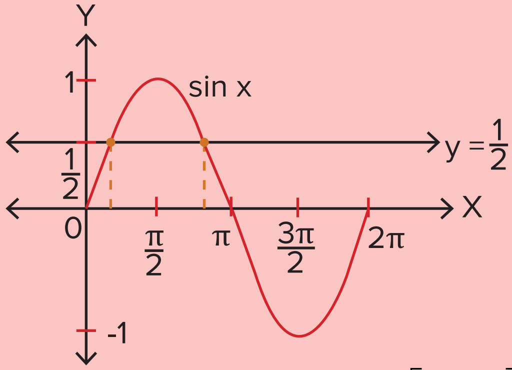
Find the solution set of the following inequalities:

(a) $\sin x \geq \frac{1}{2}$ (b) $\cos x \geq \frac{1}{2}$ (c) $\tan x > \sqrt{3}$

(a) $\sin x \geq \frac{1}{2}$

$f(x) = \sin x$

Period of $f(x) = p = 2\pi$



On observing the graph, we can find that the region satisfying the inequation has $x \in \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$. This is the partial solution set.

Thus $\frac{\pi}{6}$ is the lower limit and $\frac{5\pi}{6}$ is the upper limit, we add pn = 2πn to both the limits and get the required solution set.

$\Rightarrow \frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$ $\quad\quad \Rightarrow 2n\pi + \frac{\pi}{6} \leq x \leq 2n\pi + \frac{5\pi}{6}$; $n \in \mathbb{Z}$

$\Rightarrow x \in \left[2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right]$

$n = 0$	$x \in \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$
$n = 1$	$x \in \left[2\pi + \frac{\pi}{6}, 2\pi + \frac{5\pi}{6}\right]$
$n = -1$	$x \in \left[-2\pi + \frac{\pi}{6}, -2\pi + \frac{5\pi}{6}\right]$
$n = n$	......

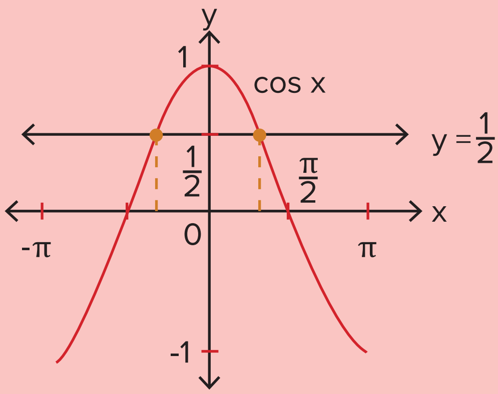
The final solution set will be the union of all these sets, written as,

$$\bigcup_{n \in \mathbb{Z}} \left[2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right]$$

(b) $\cos x \geq \frac{1}{2}$

f(x) = cos x

Period of $f(x) = p = 2\pi$



$\Rightarrow x \in \left[\frac{-\pi}{3}, \frac{\pi}{3}\right]$ is the partial solution set.

We add the n-multiple of fundamental period pn = 2πn to the partial solution set.

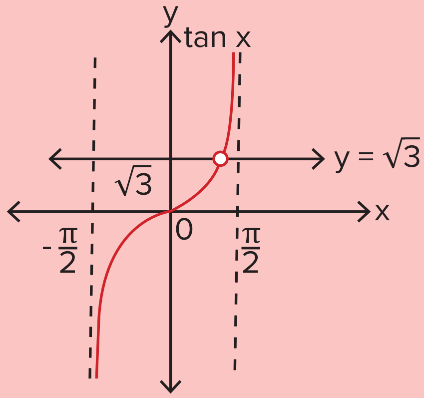
$$\Rightarrow x \in \left[2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right]$$

The final solution set will be the union of all these sets, written as, $\bigcup_{n\in\mathbb{Z}}\left[2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right]$

(c) $\tan x > \sqrt{3}$

f(x) = tan x

Period of f(x) = p = π



$$\Rightarrow x \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$$ is the partial solution set.

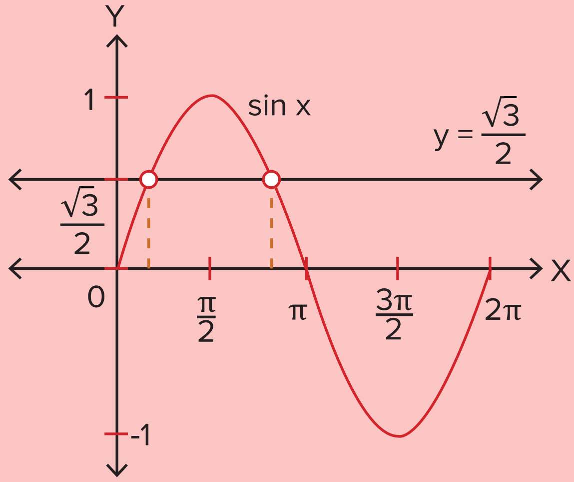
We add the n-multiple of fundamental period pn = nπ to the partial solution set.

$$\Rightarrow x \in \left(n\pi + \frac{\pi}{3}, n\pi + \frac{\pi}{2}\right)$$

The final solution set will be the union of all these sets, written as, $\bigcup_{n\in\mathbb{Z}}\left(n\pi + \frac{\pi}{3}, n\pi + \frac{\pi}{2}\right)$

Solve the following inequality: $2\sin^2 x + \sqrt{3}\sin x - 3 > 0$

We have, $2\sin^2 x + \sqrt{3}\sin x - 3 > 0$



$$\Rightarrow 2\sin^2 x + 2\sqrt{3}\sin x - \sqrt{3}\sin x - 3 > 0$$

$$\Rightarrow (2\sin x - \sqrt{3})(\sin x + \sqrt{3}) > 0$$

$$\{\because \sin x + \sqrt{3} > 0 \; \forall \; x \in \mathbb{R}\}$$

$$\Rightarrow \sin x > \frac{\sqrt{3}}{2}$$

$$\Rightarrow \frac{\sqrt{3}}{2} < \sin x \le 1 \qquad \{\because -1 \le \sin x \le 1\}$$

$$\Rightarrow \frac{\pi}{3} < x < \left(\pi - \frac{\pi}{3}\right) \qquad \Rightarrow \frac{\pi}{3} < x < \frac{2\pi}{3}$$, this is the partial solution set.

$$\Rightarrow 2n\pi + \frac{\pi}{3} < x < 2n\pi + \frac{2\pi}{3} \; ; n \in \mathbb{Z}$$

Thus, the required solution will be,

$$\bigcup_{n\in\mathbb{Z}}\left(2n\pi + \frac{\pi}{3}, 2n\pi + \frac{2\pi}{3}\right)$$

Find the solution set of the following inequation: cos x - sin x ≥ 1 in the interval [0, 2π].

We have, $\cos x - \sin x \geq 1$

$$\Rightarrow \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x \geq \frac{1}{\sqrt{2}}$$

{ As for equations of the the form asin x + bcos x = c we divide both sides by $\sqrt{a^2 + b^2}$ }

$$\Rightarrow \cos\frac{\pi}{4}\cos x - \sin\frac{\pi}{4}\sin x \geq \frac{1}{\sqrt{2}}$$

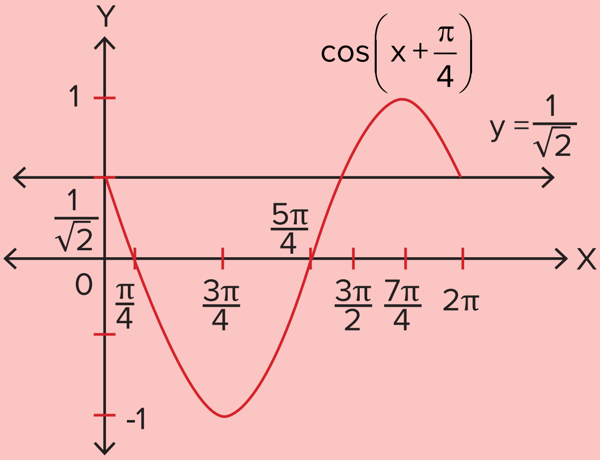
$$\Rightarrow \cos\left(x + \frac{\pi}{4}\right) \geq \frac{1}{\sqrt{2}}$$

On observing the graph, we get the interval satisfying the inequation as,

$$\Rightarrow x \in \{0\} \cup \left[\frac{3\pi}{2}, 2\pi\right]$$

As the inequation is equal to $\frac{1}{\sqrt{2}}$,

x = 0 has to be included in the solution set.



Find the sum of all the values of x ∈ [0, 314] that satisfies cos 4x + 6 = 7 cos 2x

We have, $\cos 4x + 6 = 7\cos 2x$

$\Rightarrow 2\cos^2 2x - 1 - 7\cos 2x + 6 = 0$ $\quad (\because \cos 2\theta = 2\cos^2\theta - 1)$

$\Rightarrow 2\cos^2 2x - 7\cos 2x + 5 = 0$

$\Rightarrow 2\cos 2x(\cos 2x - 1) - 5(\cos 2x - 1) = 0$

$\Rightarrow (\cos 2x - 1)(2\cos 2x - 5) = 0$

$\Rightarrow \cos 2x = \frac{5}{2}$ or $\cos 2x = 1$

$\because -1 \leq \cos\theta \leq 1 \Rightarrow \cos 2x = \frac{5}{2}$ is not possible.

$\Rightarrow \cos 2x = 1$ $\quad \{\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi \pm \alpha \,;\, \alpha \in [0, \pi],\, n \in \mathbb{Z}\}$

x = 0, π, 2π, 3π ..... 100π $\quad$ x ∈ [0,314] (Given)

{ $\Rightarrow x = 0, \pi, 2\pi, 3\pi......100\pi$
$100 \times 3.14 \neq 100\pi$
$\because 3.14 < \pi \Rightarrow 314 < 100\pi$ }

Thus $\Rightarrow x = 0, \pi, 2\pi, 3\pi......, 99\pi$

$\text{Sum} = S = 0 + \pi + 2\pi + 3\pi...... + 99\pi$

$\therefore a = 0, d = \pi, n = 100$

$$S = \frac{n}{2}(2a + (n-1)d) = \frac{100}{2}(2 \times 0 + (100-1)\pi)$$

$\Rightarrow S = 4950\pi$ is the required sum of the solutions.

Summary sheet

Key Takeaways

- While solving trigonometric equations and inequations, always find a solution in the interval asked.
- Always simplify the trigonometric expression as much as possible.
- Transformation of graphs and extreme values of trigonometric functions have to be kept in mind while solving problems.

Mind map

Simultaneous Trigonometric Equations
($f(x) = 0$ & $g(x) = 0$)

↓ **Solve $f(x) = 0$** → **Solution = Set A**

↓ **Solve $g(x) = 0$** → **Solution = Set B**

↓ **Final Solution Set = $A \cap B$**

→ **Trigonometric Equations**

↓

TRIGONOMETRIC INEQUATIONS

↓

Solving Trigonometric Inequations

↓

Draw the graph of f(x) with fundamental period p

↓

Draw the graph of y = a

↓

Extract out the interval which is satisfying the inequation

↓

Add ‘pn : $n \in Z$’ to both the limits and take union over integers

Self-Assessment

(1) Find the set of values of θ satisfying the inequation, $2\sin^2 \theta - 5\sin \theta + 2 > 0$, where $0 < \theta < 2\pi$.

(2) Find the total number of values of x in the interval $[0, 2\pi]$ satisfying $|\cos x - \sin x| \geq \sqrt{2}$.

Answers

Concept Check

We have, $\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$

$\cos^4 x + \sin^4 x = (\cos^2 x + \sin^2 x)^2 - 2\sin^2 x \cos^2 x \rightarrow 1 - 2\sin^2 x \cos^2 x$

$\cos^6 x + \sin^6 x = (\cos^2 x)^3 + (\sin^2 x)^3 \rightarrow (\cos^2 x + \sin^2 x)(\cos^4 x + \sin^4 x - \sin^2 x \cos^2 x) \rightarrow 1 - 3\sin^2 x \cos^2 x$

$$\Rightarrow \frac{5}{4}\cos^2 2x + \{1 - 2\sin^2 x\cos^2 x\} + \{1 - 3\sin^2 x\cos^2 x\} = 2$$

$$\Rightarrow \frac{5}{4}\cos^2 2x - 5\sin^2 x \cos^2 x = 0 \quad \Rightarrow \frac{5}{4}\cos^2 2x - \frac{5}{4}\sin^2 2x = 0$$

$$\Rightarrow \frac{5}{4}\left(\cos^2 2x - \sin^2 2x\right) = 0$$

$$\Rightarrow \frac{5}{4}\cos 4x = 0$$

$$\Rightarrow \cos 4x = \cos\frac{\pi}{2} \quad \Rightarrow 4x = 2n\pi \pm \frac{\pi}{2};\, n \in \mathbb{Z}$$

$$\Rightarrow x = \frac{n\pi}{2} \pm \frac{\pi}{8};\, n \in \mathbb{Z}$$

$$x = \frac{n\pi}{2} \pm \frac{\pi}{8};\, n \in \mathbb{Z}$$

$$\left\{\begin{array}{l} \sin 2\theta = 2\sin\theta\,\cos\theta \\ \cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi \pm \alpha; \alpha \in [0, \pi],\, n \in \mathbb{Z} \\ \cos 2\theta = \cos^2\theta - \sin^2\theta \end{array}\right\}$$

n = 0	n = 1	n = 2	n = 3	n = 4
$\frac{\pi}{8}$	$\frac{3\pi}{8}, \frac{5\pi}{8}$	$\frac{7\pi}{8}, \frac{9\pi}{8}$	$\frac{11\pi}{8}, \frac{13\pi}{8}$	$\frac{15\pi}{8}, \frac{17\pi}{8}$

We reject $\frac{17\pi}{8}$ and the values after that as they are out of the interval $[0, 2\pi]$.

$\therefore$ Total number of distinct solutions in $[0, 2\pi] = 8$.

Self-Assessment

(1) We have, $2\sin^2\theta - 5\sin\theta + 2 > 0$

$\Rightarrow 2\sin^2\theta - 4\sin\theta - \sin\theta + 2 > 0$

$\Rightarrow (2\sin\theta - 1)(\sin\theta - 2) > 0$ $\left\{\begin{array}{l} \because -1 \le \sin\theta \le 1 \\ \Rightarrow -3 \le \sin\theta - 2 \le -1 \\ \Rightarrow \sin\theta - 2 < 0 \end{array}\right\}$

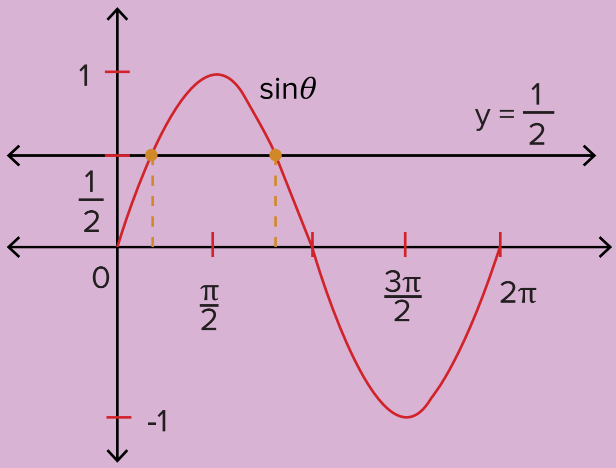
We know that, when $ab > 0$, $b < 0$

$\Rightarrow a < 0$

$\Rightarrow 2\sin\theta - 1 < 0 \Rightarrow \sin\theta < \frac{1}{2}$

From the graph, we can observe that the interval satisfying the inequation is,

$\theta \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$



(2) We have, $|\cos x - \sin x| \ge \sqrt{2}$

$|\cos x - \sin x| = \sqrt{2}\left|\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right| = \sqrt{2}\left|\cos\left(x + \frac{\pi}{4}\right)\right|$

$\because -1 \le \cos\left(x + \frac{\pi}{4}\right) \le 1, \forall\, x \in \mathbb{R}$

$\Rightarrow 0 \le \left|\cos\left(x + \frac{\pi}{4}\right)\right| \le 1$ $\Rightarrow 0 \le \sqrt{2}\left|\cos\left(x + \frac{\pi}{4}\right)\right| \le \sqrt{2}$

$\Rightarrow |\cos x - \sin x| \le \sqrt{2}$ and we have, $|\cos x - \sin x| \ge \sqrt{2}$

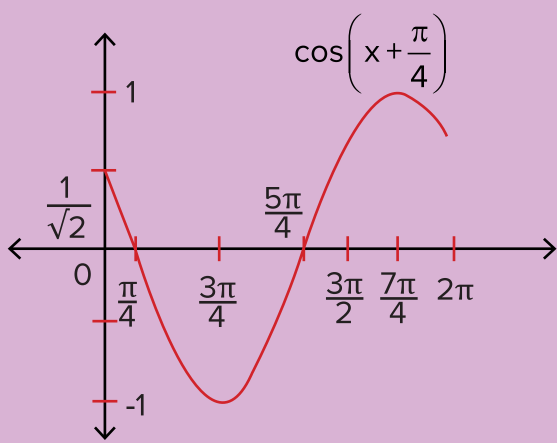
$\therefore |\cos x - \sin x| = \sqrt{2}$

$\Rightarrow \cos x - \sin x = \pm\sqrt{2} \Rightarrow \sqrt{2}\cos\left(x + \frac{\pi}{4}\right) = \pm\sqrt{2}$

$\Rightarrow \cos\left(x + \frac{\pi}{4}\right) = \pm 1$

$\therefore \cos\left(x + \frac{\pi}{4}\right) = +1, -1$

$\Rightarrow x \in \left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$



Hence, we have 2 solutions in the interval [0, 2π].