Session 3

Linear Combination of Vectors, Theorem on Coplanar & Non-coplanar Vectors, Linear Independence and Dependence of Vectors

Linear Combination of Vectors

A vector $\mathbf{r}$ is said to be a linear combination of vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$... etc., if there exist scalars x, y and z etc., such that $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + \ldots$

For examples Vectors $\mathbf{r}_1 = 2\mathbf{a} + \mathbf{b} + 3\mathbf{c}$ and $\mathbf{r}_2 = \mathbf{a} + 3\mathbf{b} + \sqrt{2}\mathbf{c}$ are linear combinations of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Collinearity and Coplanarity of Vectors

Relation between Two Collinear Vectors (or Parallel Vectors)

Let $\mathbf{a}$ and $\mathbf{b}$ be two collinear vectors and let $\hat{\mathbf{x}}$ be the unit vector in the direction of $\mathbf{a}$. Then, the unit vector in the direction of $\mathbf{b}$ is $\hat{\mathbf{x}}$ or $-\hat{\mathbf{x}}$ according as $\mathbf{a}$ and $\mathbf{b}$ are like or unlike parallel vectors. Now, $\mathbf{a} = |\mathbf{a}|\,\hat{\mathbf{x}}$ and $\mathbf{b} = \pm|\mathbf{b}|\,\hat{\mathbf{x}}$.

$$\therefore \qquad \mathbf{a} = \left(\frac{|\mathbf{a}|}{|\mathbf{b}|}\right)|\mathbf{b}|\,\hat{\mathbf{x}} \Rightarrow \mathbf{a} = \pm\left(\frac{|\mathbf{a}|}{|\mathbf{b}|}\right)\mathbf{b}$$

$$\Rightarrow \qquad \mathbf{a} = \lambda\mathbf{b}, \text{ where } \lambda = \pm\frac{|\mathbf{a}|}{|\mathbf{b}|}$$

Thus, if $\mathbf{a}$ and $\mathbf{b}$ are collinear vectors, then $\mathbf{a} = \lambda\mathbf{b}$ or $\mathbf{b} = \lambda\mathbf{a}$ for some scalar λ i.e, there exist two non-zero scalar quantities x and y so that $x\mathbf{a} + y\mathbf{b} = \mathbf{O}$

An Important Theorem

Theorem : Vectors $\mathbf{a}$ and $\mathbf{b}$ are two non-zero, non-collinear vectors and x, y are two scalars such that

$$x\mathbf{a} + y\mathbf{b} = 0$$

Then, $\quad x = 0, y = 0$

Proof It is given that $\quad x\mathbf{a} + y\mathbf{b} = 0 \quad$ ...(i)

Suppose that $x \neq 0$, then dividing both sides of (i) by the scalar x, we get

$$\mathbf{a} = -\frac{y}{x}\mathbf{b} \qquad \text{...(ii)}$$

Now, $\frac{y}{x}$ is a scalar, because x and y are scalars.

Hence, Eq. (ii) expresses $\mathbf{a}$ as product of $\mathbf{b}$ by a scalar, so that $\mathbf{a}$ and $\mathbf{b}$ are collinear. Thus, we arrive at a contradiction because $\mathbf{a}$ and $\mathbf{b}$ are given to be non-collinear.

Thus our supposition that $x \neq 0$, is wrong.

Hence, $x = 0$. Similarly, $y = 0$

Remarks

1. $x\mathbf{a} + y\mathbf{b} = 0 \Rightarrow \begin{cases} \mathbf{a} = 0, \mathbf{b} = 0 \\ \text{or} \\ x = 0, y = 0 \\ \text{or} \\ \mathbf{a} \,||\, \mathbf{b} \end{cases}$

2. If $\mathbf{a}$ and $\mathbf{b}$ are two non-collinear (or non-parallel) vectors, then $x_1\mathbf{a} + y_1\mathbf{b} = x_2\mathbf{a} + y_2\mathbf{b}$
$\Rightarrow \quad x_1 = x_2$ and $y_1 = y_2$
Proof $x_1\mathbf{a} + y_1\mathbf{b} = x_2\mathbf{b} + y_2\mathbf{b}$
$\Rightarrow \quad (x_1 - x_2)\mathbf{a} + (y_1 - y_2)\mathbf{b} = 0$
$\Rightarrow \quad x_1 - x_2 = 0$ and $y_1 - y_2 = 0$
[$\because \mathbf{a}$ and $\mathbf{b}$ are non-collinear]
$\Rightarrow \quad x_1 = x_2$ and $y_1 = y_2$
If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $b = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$, then $\mathbf{a} \,||\, \mathbf{b}$
$\Rightarrow \quad \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

Test of Collinearity of Three Points

(i) Three points A, B and C are collinear, if $\mathbf{AB} = \lambda\mathbf{BC}$

(ii) Three points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are collinear iff there exist scalars x, y and z not all zero such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = 0$, where $x + y + z = 0$

Proof Let us suppose that points A, B and C are collinear and their position vectors are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Let C divide the join of $\mathbf{a}$ and $\mathbf{b}$ in the ratio $y : x$. Then,

$$\mathbf{c} = \frac{x\mathbf{a} + y\mathbf{b}}{x + y}$$

or $\quad x\mathbf{a} + y\mathbf{b} - (x + y)\mathbf{c} = 0$

or $\quad x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = 0$, where $z = -(x + y)$

Also, $x+y+z = x+y-(x+y) = 0$

Conversely, let $x\mathbf{a}+y\mathbf{b}+z\mathbf{c}=0$, where $x+y+z=0$. Therefore,

$$x\mathbf{a}+y\mathbf{b} = -z\mathbf{c} = (x+y)\mathbf{c} \qquad (\because x+y=-z)$$

or
$$\mathbf{c} = \frac{x\mathbf{a}+y\mathbf{b}}{x+y}$$

This relation shows that **c** divides the join of **a** and **b** in the ratio $y:x$. Hence, the three points A, B and C are collinear.

(iii) If $\mathbf{a} = a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}$, $\mathbf{b} = b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}$ and $\mathbf{c}=c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}}$, then the points with position vector **a, b** and **c** will be collinear iff $\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} = 0.$

Proof The points with position vector **a**, **b** and **c** will be collinear iff there exist scalars x, y and z not all zero such that,

$x(a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}})+y(b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}})+z(c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}})=0$ and $x+y+z=0$

$$\Rightarrow \quad xa_1+yb_1+zc_1=0$$
$$xa_2+yb_2+zc_2=0$$
$$x+y+z=0$$

Thus, the points will be collinear iff the above system of equation's have non-trivial solution

Hence, the points will be collinear

$$\text{iff } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} = 0.$$

Example 34. ***Show that the vectors*** $2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ ***and*** $-4\hat{\mathbf{i}}+6\hat{\mathbf{j}}-8\hat{\mathbf{k}}$ ***are collinear.***

Sol. Let $\mathbf{a} = 2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ and $\mathbf{b} = -4\hat{\mathbf{i}}+6\hat{\mathbf{j}}-8\hat{\mathbf{k}}$

Consider, $\mathbf{b} = -4\hat{\mathbf{i}}+6\hat{\mathbf{j}}-8\hat{\mathbf{k}} = -2(2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+4\hat{\mathbf{k}}) = -2\mathbf{a}$

$\therefore$ The vectors **a** and **b** are collinear.

Example 35. ***Show that the points*** $A(1,2,3), B(3,4,7)$ ***and*** $C(-3,-2,-5)$ ***are collinear. Find the ratio in which point*** C ***divides*** AB.

Sol. Clearly, $\mathbf{AB} = (3-1)\hat{\mathbf{i}}+(4-2)\hat{\mathbf{j}}+(7-3)\hat{\mathbf{k}}$
$$= 2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$$

and $\mathbf{BC} = (-3-3)\hat{\mathbf{i}}+(-2-4)\hat{\mathbf{j}}+(-5-7)\hat{\mathbf{k}}$
$$= 6\hat{\mathbf{i}}-6\hat{\mathbf{j}}-12\hat{\mathbf{k}}$$
$$= -3(2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+4\hat{\mathbf{k}}) = -3\mathbf{AB}$$

$\because \quad \mathbf{BC} = -3\mathbf{AB}$

$\therefore$ A, B and C are collinear.

Now, let C divide AB in the ratio $k:1$, then

$$\mathbf{OC} = \frac{k\mathbf{OB}+1\cdot\mathbf{OA}}{k+1}$$

$$\Rightarrow \quad -3\hat{\mathbf{i}}-2\hat{\mathbf{j}}-5\hat{\mathbf{k}} = \frac{k(3\hat{\mathbf{i}}+4\hat{\mathbf{j}}+7\hat{\mathbf{k}})+(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}})}{k+1}$$

$$\Rightarrow \quad -3\hat{\mathbf{i}}-2\hat{\mathbf{j}}-5\hat{\mathbf{k}} = \left(\frac{3k+1}{k+1}\right)\hat{\mathbf{i}}+\left(\frac{4k+2}{k+1}\right)\hat{\mathbf{j}}+\left(\frac{7k+3}{k+1}\right)\hat{\mathbf{k}}$$

$$\Rightarrow \quad \frac{3k+1}{k+1} = -3; \ \frac{4k+2}{k+1} = -2 \text{ and } \frac{7k+3}{k+1} = -5$$

From, all relations, we get $k = \frac{-2}{3}$

Hence, C divides AB externally in the ratio $2:3$.

Example 36. ***If the position vectors of*** A, B, C ***and*** D ***are*** $2\hat{\mathbf{i}}+\hat{\mathbf{j}}$, $\hat{\mathbf{i}}-3\hat{\mathbf{j}}$, $3\hat{\mathbf{i}}+2\hat{\mathbf{j}}$ ***and*** $\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}$, ***respectively and*** $\mathbf{AB} \,||\, \mathbf{CD}$, ***then*** λ ***will be***

(a) -8 (b) -6

(c) 8 (d) 6

Sol. (b) $\mathbf{AB} = (\hat{\mathbf{i}}-3\hat{\mathbf{j}})-(2\hat{\mathbf{i}}+\hat{\mathbf{j}}) = -\hat{\mathbf{i}}-4\hat{\mathbf{j}}$;

$$\mathbf{CD} = (\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}})-(3\hat{\mathbf{i}}+2\hat{\mathbf{j}}) = -2\hat{\mathbf{i}}+(\lambda-2)\hat{\mathbf{j}};$$

$\mathbf{AB} \,||\, \mathbf{CD} \Rightarrow \mathbf{AB} = x\,\mathbf{CD}$

$$-\hat{\mathbf{i}}-4\hat{\mathbf{j}} = x\{-2\hat{\mathbf{i}}+(\lambda-2)\hat{\mathbf{j}}\}$$

$$\Rightarrow \quad -1 = -2x, -4 = (\lambda-2)x$$

$$\Rightarrow \quad x = \frac{1}{2} \text{ and } \lambda = -6$$

Example 37. ***The points with position vectors*** $60\hat{\mathbf{i}}+3\hat{\mathbf{j}}$, $40\hat{\mathbf{i}}-8\hat{\mathbf{j}}$ ***and*** $a\hat{\mathbf{i}}-52\hat{\mathbf{j}}$ ***are collinear, if*** a ***is equal to***

(a) -40 (b) 40

(c) 20 (d) None of these

Sol (a) The three points are collinear if

$$\begin{vmatrix} 60 & 3 & 1 \\ 40 & -8 & 1 \\ a & -52 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 60(-8+52)-3(40-a)+(-2080+8a) = 0$$
$$\Rightarrow \quad 2640-120+3a-2080+8a = 0$$
$$11a = -440$$
$$\Rightarrow \quad a = -40$$

Example 38. ***Let*** **a**, **b** ***and*** **c** ***be three non-zero vectors such that no two of these are collinear. If the vector*** $\mathbf{a}+2\mathbf{b}$ ***is collinear with*** **c** ***and*** $\mathbf{b}+3\mathbf{c}$ ***is collinear with a (***λ ***being some non-zero scalar), then*** $\mathbf{a}+2\mathbf{b}+6\mathbf{c}$ ***is equal to***

(a) **0** (b) $\lambda\mathbf{b}$

(c) $\lambda\mathbf{c}$ (d) $\lambda\mathbf{a}$

Sol. (a) As $\mathbf{a}+2\mathbf{b}$ and $\mathbf{c}$ are collinear $\mathbf{a}+2\mathbf{b}=\lambda\mathbf{c}$ ...(i)

Again, $\mathbf{b}+3\mathbf{c}$ is collinear with $\mathbf{a}$.

$\therefore \quad \mathbf{b}+3\mathbf{c}=\mu\mathbf{a}$ ...(ii)

Now, $\mathbf{a}+2\mathbf{b}+6\mathbf{c}=(\mathbf{a}+2\mathbf{b})+6\mathbf{c}=\lambda\mathbf{c}+6\mathbf{c}$

$=(\lambda+6)\mathbf{c}$ ...(iii)

Also, $\mathbf{a}+2\mathbf{b}+6\mathbf{c}=\mathbf{a}+2(\mathbf{b}+3\mathbf{c})=\mathbf{a}+2\mu\mathbf{a}$

$=(2\mu+1)\mathbf{a}$ ...(iv)

From Eqs. (iii) and (iv), we get

$(\lambda+6)\mathbf{c}=(2\mu+1)\mathbf{a}$

But $\mathbf{a}$ and $\mathbf{c}$ are non-zero, non-collinear vectors,

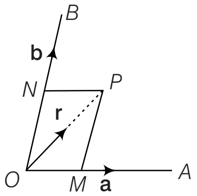
$\therefore \quad \lambda+6=0=2\mu+1$

Hence, $\quad \mathbf{a}+2\mathbf{b}+6\mathbf{c}=0$

Theorem of Coplanar Vectors

Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero, non-collinear vectors. Then any vector $\mathbf{r}$ coplanar with $\mathbf{a}$ and $\mathbf{b}$ can be uniquely expressed as a linear combination $x\mathbf{a}+y\mathbf{b}$; x and y being scalars.

Proof Let $\mathbf{a}$ and $\mathbf{b}$ be any two non-zero, non-collinear vectors and $\mathbf{r}$ be any vector coplanar with $\mathbf{a}$ and $\mathbf{b}$.
We take any point O in the plane of $\mathbf{a}$ and $\mathbf{b}$



Let $\quad \mathbf{OA}=\mathbf{a}, \mathbf{OB}=\mathbf{b}$ and $\mathbf{OP}=\mathbf{r}$

Clearly, OA, OB and OP are coplanar.

Through P, we draw lines PM and PN, parallel to OB and OA respeetively meeting OA and OB at M and N respectively.

We have, $\mathbf{OP}=\mathbf{OM}+\mathbf{MP}$

$=\mathbf{OM}+\mathbf{ON}$ [$\because MP=ON$ and $MP \,||\, ON$] ...(i)

Now, $\mathbf{OM}$ and $\mathbf{OA}$ are collinear vectors

$\mathbf{OM}=x\,\mathbf{OA}=x\mathbf{a}$, where x is scalar.

Similarly, $\mathbf{ON}=y\mathbf{OB}=y\mathbf{b}$, where y is a scalar.

Hence, from Eq. (i), $\mathbf{OP}=x\mathbf{a}+y\mathbf{b}$ or $\mathbf{r}=x'\mathbf{a}+y'\mathbf{b}$

Uniqueness: If possible, let $\mathbf{r}=x\mathbf{a}+y\mathbf{b}$ and $\mathbf{r}=x'\mathbf{a}+y'\mathbf{b}$ be two different ways of representing r.

Then, we have $\quad x\mathbf{a}+y\mathbf{b}=x'\mathbf{a}+y'\mathbf{b}$

$\Rightarrow \quad (x-x')\mathbf{a}+(y-y')\mathbf{b}=0$

But $\mathbf{a}$ and $\mathbf{b}$ are non-collinear vectors

$\therefore \quad x-x'=0$ and $y-y'=0$

$\Rightarrow \quad x'=x$ and $y'=y$

Thus, the uniqueness in established.

Test of Coplanarity of Three Vectors

(i) Three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar iff any one of them is a linear combination of the remaining two, i.e. iff $\mathbf{a}=x\mathbf{b}+y\mathbf{c}$ where x and y are scalars.

(ii) If three points with position vectors $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$, $\mathbf{b}=b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+b_3\hat{\mathbf{k}}$ and $\mathbf{c}=c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}}+c_3\hat{\mathbf{k}}$ are coplanar,

then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0.$

If vectors $\mathbf{a}$,$\mathbf{b}$ and $\mathbf{c}$ are coplanar, then there exist scalars x and y such that $\mathbf{c}=x\mathbf{a}+y\mathbf{b}$.

Hence, $c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}}+c_3\hat{\mathbf{k}}=x(a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}})$
$+y(b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+b_3\hat{\mathbf{k}})$

Now, $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are non-coplanar and hence independent.

Then, $\quad c_1=xa_1+yb_1, c_2=xa_2+yb_2$

and $\quad c_3=xa_3+yb_3$

The above system of equations in terms of x and y is consistent. Thus,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0 \text{ or } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0.$$

Remark

If vectors $x_1\mathbf{a}+y_1\mathbf{b}+z_1\mathbf{c}$, $x_2\mathbf{a}+y_2\mathbf{b}+z_2\mathbf{c}$ and $x_3\mathbf{a}+y_3\mathbf{b}+z_3\mathbf{c}$ are coplanar(where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar).

Then, $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=0$

Test of Coplanarity of Four Points

(i) To prove that four points $A(\mathbf{a})$, $B(\mathbf{b})$, $C(\mathbf{c})$ and $D(\mathbf{d})$ are coplanar, it is just sufficient to prove that vectors $\mathbf{AB}$, $\mathbf{AC}$ and $\mathbf{AD}$ and coplanar.

(ii) Four points with position vectors $\mathbf{a}, \mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are coplanar iff there exist scalars x, y, z and u not all zero such that $x\,\mathbf{a}+y\,\mathbf{b}+z\,\mathbf{c}+u\,\mathbf{d}=0$, where $x+y+z+u=0$.

(iii) Four points with position vectors

$\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$,

$\mathbf{b}=b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+b_3\hat{\mathbf{k}}$

$\mathbf{c}=c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}}+c_3\hat{\mathbf{k}}$

and $\quad \mathbf{d}=d_1\hat{\mathbf{i}}+d_2\hat{\mathbf{j}}+d_3\hat{\mathbf{k}}$

will be coplanar, iff $\begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix} = 0$

or $\begin{vmatrix} d_1 - a_1 & d_2 - a_2 & d_3 - a_3 \\ b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \end{vmatrix} = 0$

Theorem on Non-coplanar Vectors

Theorem 1

If **a, b, c**, are three non-zero, non-coplanar vectors and x, y, z are three scalars such that

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = 0.$$

Then $x = y = z = 0.$

Proof It is given that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = 0$ ...(i)

Suppose that $x \neq 0$

Then Eq. (i) can be written as

$$x\mathbf{a} = -y\mathbf{b} - z\mathbf{c}$$

$$\Rightarrow \quad \mathbf{a} = -\frac{y}{x}\mathbf{b} - \frac{z}{x}\mathbf{c} \quad \text{...(ii)}$$

Now, $\frac{y}{x}$ and $\frac{z}{x}$ are scalars because x, y and z are scalars.

Thus, Eq. (ii) expresses **a** as a linear combination of **b** and **c**.

Hence, **a** is coplanar with **b** and **c** which is contrary to our hypothesis because **a**,**b** and **c** are given to be non-coplanar.

Thus, our supposition that $x \neq 0$ is wrong.

Hence, $x = 0$

Similarly, we can prove that $y = 0$ and $z = 0$

Theorem 2

If **a**,**b** and **c** are non-coplanar vectors, then any vector **r** can be uniquely expressed as a linear combination $x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$; x, y and z being scalars.

or

Any vector in space can be expressed as a linear combination of three non-coplanar vectors.

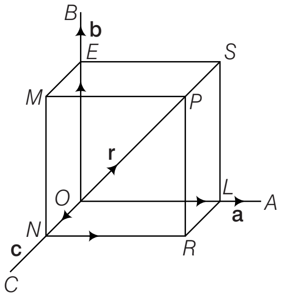
Proof Take any point O.

Let **a, b, c** be any three non-coplanar vectors and **r** be any vector in space.

Let $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$, $\mathbf{OC} = \mathbf{c}$, $\mathbf{OP} = \mathbf{r}$

Here, the three lines OA, OB, OC are not coplanar. Hence, they determine three different planes BOC, COA and AOB when taken in pairs.

Through P, draw planes parallel to these planes BOC, COA and AOB meeting OA, OB and OC in L, E and N respectively. Thus we obtain a parallelopiped with OP as diagonal and three coterminous edges OL, OE and ON along OA, OB and OC, respectively.



$\therefore$ **OL** is collinear with **OA**.

$\therefore$ $\mathbf{OL} = x\mathbf{OA} = x\mathbf{a}$, where x is a scalar.

Similarly, $\mathbf{OE} = y\,\mathbf{b}$ and $\mathbf{ON} = z\mathbf{c}$,

where y and z are scalars.

Now, $\mathbf{OP} = \mathbf{OR} + \mathbf{RP} = (\mathbf{ON} + \mathbf{NR}) + \mathbf{RP}$

$= \mathbf{ON} + \mathbf{OL} + \mathbf{OE}$ $\quad [\because \mathbf{NR} = \mathbf{OL}$ and $\mathbf{RP} = \mathbf{OE}]$

$= \mathbf{OL} + \mathbf{OE} + \mathbf{ON} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$

Thus, $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$

Hence, **r** can be expressed as a linear combination of **a**, **b** and **c**.

Uniqueness If possible let

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$$

and $\quad \mathbf{r} = x'\mathbf{a} + y'\mathbf{b} + z'\mathbf{c}$

be two different ways of representing **r**, then we have

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = x'\mathbf{a} + y'\mathbf{b} + z'\mathbf{c}$$

$$\Rightarrow \quad (x - x')\mathbf{a} + (y - y')\mathbf{b} + (z - z')\mathbf{c} = 0$$

Now **a**, **b** and **c** are non-coplanar vectors

$$\therefore \quad x - x' = 0, y - y' = 0 \text{ and } z - z' = 0$$

$$\Rightarrow \quad x = x', y = y' \text{ and } z = z'$$

Hence, the uniqueness is established.

Remark

If **a**, **b**, **c** are any three non-coplanar vectors in space, then

$$x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c} = x_2\mathbf{a} + y_2\mathbf{b} + z_2\mathbf{c}$$

$$\Rightarrow \quad x_1 = x_2, y_1 = y_2, z_1 = z_2$$

Proof $x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c} = x_2\mathbf{a} + y_2\mathbf{b} + z_2\mathbf{c}$

$$\Rightarrow \quad (x_1 - x_2)\mathbf{a} + (y_1 - y_2)\mathbf{b} + (z_1 - z_2)\mathbf{c} = 0$$

$$\Rightarrow \quad x_1 - x_2 = 0, y_1 - y_2 = 0 \text{ and } z_1 - z_2 = 0$$

$$\Rightarrow \quad x_1 = x_2, y_1 = y_2 \text{ and } z_1 = z_2$$

Example 39. *Check whether the given three vectors are coplanar or non-coplanar.*

$-2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}}, -2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-2\hat{\mathbf{k}}, 4\hat{\mathbf{i}}-2\hat{\mathbf{j}}-2\hat{\mathbf{k}}$

Sol. Let $\mathbf{a} = -2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

$\mathbf{b} = -2\hat{\mathbf{i}}+4\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ and $\mathbf{c} = 4\hat{\mathbf{i}}-2\hat{\mathbf{j}}-2\hat{\mathbf{k}}$

Now, consider

$$\begin{vmatrix} -2 & -2 & 4 \\ -2 & 4 & -2 \\ 4 & -2 & -2 \end{vmatrix} = -2(-8-4)+2(4+8)+4(4-16)$$

$$= 24 + 24 - 48 = 0$$

$\therefore$ The vectors are coplanar.

Example 40. *If the vectors* $4\hat{\mathbf{i}}+11\hat{\mathbf{j}}+m\hat{\mathbf{k}}$, $7\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}}$ *and* $\hat{\mathbf{i}}+5\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ *are coplanar, then* m *is equal to*

(a) 38 (b) 0
(c) 10 (d) −10

Sol. (c) Since the three vectors are coplanar, one will be a linear combination of the other two.

$\therefore \quad 4\hat{\mathbf{i}}+11\hat{\mathbf{j}}+m\hat{\mathbf{k}} = x(7\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}})+y(\hat{\mathbf{i}}+5\hat{\mathbf{j}}+4\hat{\mathbf{k}})$

$$\Rightarrow \quad 4 = 7x + y \quad \text{...(i)}$$
$$11 = 2x + 5y \quad \text{...(ii)}$$
$$m = 6x + 4y \quad \text{...(iii)}$$

From Eqs. (i) and (ii), we get

$$x = \frac{3}{11} \text{ and } y = \frac{23}{11}$$

From Eq. (iii), we get

$$m = 6 \times \frac{3}{11} + 4 \times \frac{23}{11} = 10$$

Trick Since, vectors $4\hat{\mathbf{i}}+11\hat{\mathbf{j}}+m\hat{\mathbf{k}}, 7\hat{\mathbf{i}}+2\hat{\mathbf{j}}+6\hat{\mathbf{k}}$ and $\hat{\mathbf{i}}+5\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ are coplanar.

$$\therefore \quad \begin{vmatrix} 4 & 11 & m \\ 7 & 2 & 6 \\ 1 & 5 & 4 \end{vmatrix} = 0$$

$$\Rightarrow \quad 4(8-30) - 11(28-6) + m(35-2) = 0$$
$$\Rightarrow \quad -88 - 11 \times 22 + 33m = 0$$
$$\Rightarrow \quad -8 - 22 + 3m = 0$$
$$\Rightarrow \quad 3m = 30 \Rightarrow m = 10$$

Example 41. *If* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *are non-coplanar vectors, prove that* $3\mathbf{a}-7\mathbf{b}-4\mathbf{c}$, $3\mathbf{a}-2\mathbf{b}+\mathbf{c}$ *and* $\mathbf{a}+\mathbf{b}+2\mathbf{c}$ *are coplanar.*

Sol. Let $\alpha = 3\mathbf{a}-7\mathbf{b}-4\mathbf{c}, \beta = 3\mathbf{a}-2\mathbf{b}+\mathbf{c}$

and $\quad \gamma = \mathbf{a}+\mathbf{b}+2\mathbf{c}$

Also, let $\quad \alpha = x\beta + y - \gamma$

$$\Rightarrow \quad 3\mathbf{a}-7\mathbf{b}-4\mathbf{c} = x(3\mathbf{a}-2\mathbf{b}+\mathbf{c})+y(\mathbf{a}+\mathbf{b}+2\mathbf{c})$$
$$= (3x+y)\mathbf{a}+(-2x+y)\mathbf{b}+(x+2y)\mathbf{c}$$

Since, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors.

Therefore,

$$3x+y=3, -2x+y=-7$$

and $\quad x+2y=-4$

Solving first two, we find that $x=2$ and $y=-3$. These values of x and y satisfy the third equation as well.

So, $x+2$ and $y=-3$ is the unique solution for the above system of equation.

$$\Rightarrow \quad \alpha = 2\beta - 3\gamma$$

Hence, the vectors α, β and γ are coplanar, because α is uniquely written as linear combination of other two.

Trick For the vectors α, β, γ to be coplanar, we must have

$$\begin{vmatrix} 3 & -7 & -4 \\ 3 & -2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 0, \text{ which is true}$$

Hence, α, β, γ are coplanar.

Example 42. *The value of* λ *for which the four points* $2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-\hat{\mathbf{k}}$, $\hat{\mathbf{i}}+2\hat{\mathbf{j}}+3\hat{\mathbf{k}}$, $3\hat{\mathbf{i}}+4\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ *and* $\hat{\mathbf{i}}-\lambda\hat{\mathbf{j}}+6\hat{\mathbf{k}}$ *are coplanar*

(a) 8 (b) 0
(c) −2 (d) 6

Sol. (c) The given four points are coplanar.

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 3 & 2 & 4 & \lambda \\ -1 & 3 & -2 & 6 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & 0 & 0 & -(\lambda+2) \\ -1 & 3 & -2 & 6 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

Operating $(R_2 \to R_2 - R_1 - R_4)$

$$\Rightarrow \quad -(\lambda+2)\begin{vmatrix} 2 & 1 & 3 \\ -1 & 3 & -2 \\ 1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow \lambda = -2$$

Example 43. *Show that the points* $P(\mathbf{a}+2\mathbf{b}+\mathbf{c})$, $Q(\mathbf{a}-\mathbf{b}-\mathbf{c})$, $R(3\mathbf{a}+\mathbf{b}+2\mathbf{c})$ *and* $S(5\mathbf{a}+3\mathbf{b}+5\mathbf{c})$ *are coplanar given that* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *are non-coplanar.*

Sol. To show that P, Q, R, S are coplanar, we will show that $\mathbf{PQ}$, $\mathbf{PR}$, $\mathbf{PS}$ are coplanar.

$$\mathbf{PQ} = -3\mathbf{b}-2\mathbf{c}$$
$$\mathbf{PR} = 2\mathbf{a}-\mathbf{b}+\mathbf{c}$$
$$\mathbf{PS} = 4\mathbf{a}+\mathbf{b}+4\mathbf{c}$$

Let $\quad \mathbf{PQ} = x\mathbf{PR}+y\mathbf{PS}$

$$\Rightarrow \quad -3\mathbf{b}-2\mathbf{c} = x(2\mathbf{a}-\mathbf{b}+\mathbf{c})+y(4\mathbf{a}+\mathbf{b}+4\mathbf{c})$$
$$\Rightarrow \quad -3\mathbf{b}-2\mathbf{c} = (2x+4y)\mathbf{a}+(-x+y)\mathbf{b}+(x+4y)\mathbf{c}$$

As the vectors **a, b, c** are non-coplanar, we can equate their coefficients.

$\Rightarrow \quad 0 = 2x + 4y$

$\Rightarrow \quad -3 = -x + y$

$\Rightarrow \quad -2 = x + 4y$

$x = 2, y = -1$ is the unique solution for the above system of equations.

$\Rightarrow \quad \mathbf{PQ} = 2\mathbf{PR} - \mathbf{PS}$

PQ, **PR**, **PS** are coplanar because **PQ** is a linear combination of **PR** and **PS**

$\Rightarrow$ The points **P, Q, R, S** are also coplanar.

Trick For the vectors **PQ, PR** and **PS** to be coplanar, we must have $\begin{vmatrix} 0 & -3 & -2 \\ 2 & -1 & 1 \\ 4 & 1 & 4 \end{vmatrix} = 0$ which is true

$\therefore$ The **PQ, PR, PS** are coplanar.

Hence, the points P, Q, R, S are also coplanar.

Linear Independence and Dependence of Vectors

1. Linearly Independent Vectors

A set of non-zero vectors $\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_n$ is said to be linearly independent, if

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = 0$$

$$\Rightarrow \quad x_1 = x_2 = \ldots = x_n = 0.$$

2. Linearly Dependence Vectors

A set of vector $\mathbf{a}_1, \mathbf{a}_1 \ldots, \mathbf{a}_n$ is said to be linearly dependent, if there exist scalars $x_1, x_2, \ldots, x_n$ not all zero such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = 0$

Properties of Linearly Independent and Dependent Vectors

(i) A super set of a linearly dependent set of vectors is linearly dependent.

(ii) A subset of a linearly independent set of vectors is linearly independent.

(iii) Two non-zero, non-collinear vectors are linearly independent.

(iv) Any two collinear vectors are linearly dependent.

(v) Any three non-coplanar vectors are linearly independent.

(vi) Any three coplanar vectors are linearly dependent.

(vii) Three vectors $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ and $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$ will be linearly dependent vectors iff $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$

(viii) Any four vectors in 3-dimensional space are linearly dependent.

Example 44. ***Show that the vectors*** $\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}, 2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - \hat{\mathbf{k}}$ ***and*** $3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ ***and linearly independent.***

Sol. Let $\alpha = \hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

$\beta = 2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - \hat{\mathbf{k}}$

and $\gamma = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$

Also, let $x\alpha + y\beta + z\gamma = 0$

$\therefore \quad x(\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + y(2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - \hat{\mathbf{k}}) + z(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}) = 0$

or $(x + 2y + 3z)\hat{\mathbf{i}} + (-3x - 4y + 2z)\hat{\mathbf{j}} + (2x - y - z)\hat{\mathbf{k}} = 0$

Equating the coefficient of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, we get

$$x + 2y + 3z = 0$$
$$-3x - 4y + 2z = 0$$
$$2x - y - z = 0$$

Now, $\begin{vmatrix} 1 & 2 & 3 \\ -3 & -4 & 2 \\ 2 & -1 & -1 \end{vmatrix} = 1(4+2) - 2(3-4) + 3(3+8) = 41 \neq 0$

$\therefore$ The above system of equations have only trivial solution.

Thus, $x = y = z = 0$

Hence, the vectors α, β and γ are linearly independent.

Trick Consider the determinant of coefficients of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$

i.e. $\begin{vmatrix} 1 & -3 & 2 \\ 2 & -4 & -1 \\ 3 & 2 & -1 \end{vmatrix} = 1(4 + 2) + 3(-2 + 3) + 2(4 + 12)$

$= 6 + 3 + 32 = 41 \neq 0$

$\therefore$ The given vectors are non-coplanar. Hence, the vectors are linearly independent.

Example 45. ***If*** $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}, \mathbf{b} = 4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ ***and*** $\mathbf{c} = \hat{\mathbf{i}} + \alpha\hat{\mathbf{j}} + \beta\hat{\mathbf{k}}$ ***are linearly dependent vectors and*** $|\mathbf{c}| = \sqrt{3}$, ***then***

(a) $\alpha = 1, \beta = -1$ (b) $\alpha = 1, \beta = \pm 1$

(c) $\alpha = -1, \beta = \pm 1$ (d) $\alpha = \pm 1, \beta = 1$

Sol. (d) The given vectors are linearly dependent, hence there exist scalars x, y and z not all zero, such that

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = 0$$

i.e. $x(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) + y(4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) + z(\hat{\mathbf{i}} + \alpha\hat{\mathbf{j}} + \beta\hat{\mathbf{k}}) = 0$

i.e. $(x+4y+z)\hat{\mathbf{i}}+(x+3y+\alpha z)\hat{\mathbf{j}}+(x+4y+\beta z)\hat{\mathbf{k}}=0$

$\Rightarrow \quad x+4y+z=0,\ x+3y+\alpha z=0,\ x+4y+\beta z=0$

For non-trivial solution $\begin{vmatrix} 1 & 4 & 1 \\ 1 & 3 & \alpha \\ 1 & 4 & \beta \end{vmatrix}=0 \Rightarrow \beta=1$

$|c|^2=3 \Rightarrow 1+\alpha^2+\beta^2=3$

$\Rightarrow \quad \alpha^2=2-\beta^2=2-1=1$

$\therefore \quad \alpha=\pm 1$

Trick $\quad |c|=\sqrt{1+\alpha^2+\beta^2}=\sqrt{3}$

$\Rightarrow \quad \alpha^2+\beta^2=2$

$\because$ **a**, **b** and **c** are linearly dependent, hence $\begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 4 & \alpha & \beta \end{vmatrix}=0$

$\Rightarrow \quad \beta=1$

$\therefore \quad \alpha^2=1 \quad \Rightarrow \alpha=\pm 1$

Exercise for Session 3

1. Show that the points $A(1, 3, 2), B(-2, 0, 1)$ and $C(4, 6, 3)$ are collinear.
2. If the position vectors of the points A, B and C be $\mathbf{a}$, $\mathbf{b}$ and $3\mathbf{a}-2\mathbf{b}$ respectively, then prove that the points A, B and C are collinear.
3. The position vectors of four points P, Q, R and S are $2\mathbf{a}+4\mathbf{c}$, $5\mathbf{a}+3\sqrt{3}\mathbf{b}+4\mathbf{c}$, $-2\sqrt{3}\mathbf{b}+\mathbf{c}$ and $2\mathbf{a}+\mathbf{c}$ respectively, prove that PQ is parallel to RS.
4. If three points A, B and C have position vectors $(1, x, 3)$, $(3, 4, 7)$ and $(y, -2, -5)$, respectively and if they are collinear, then find (x, y).
5. Find the condition that the three points whose position vectors, $\mathbf{a}=a\hat{\mathbf{i}}+b\hat{\mathbf{j}}+c\hat{\mathbf{k}}$, $\mathbf{b}=\hat{\mathbf{i}}+c\hat{\mathbf{j}}$ and $\mathbf{c}=-\hat{\mathbf{i}}-\hat{\mathbf{j}}$ are collinear.
6. Vectors $\mathbf{a}$ and $\mathbf{b}$ are non-collinear. Find for what values of x vectors $\mathbf{c}=(x-2)\mathbf{a}+\mathbf{b}$ and $\mathbf{d}=(2x+1)\mathbf{a}-\mathbf{b}$ are collinear?
7. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three vectors of which every pair is non-collinear. If the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{b}+\mathbf{c}$ are collinear with $\mathbf{c}$ and $\mathbf{a}$ respectively, then find $\mathbf{a}+\mathbf{b}+\mathbf{c}$.
8. Show that the vectors $\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}}$, $2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $7\hat{\mathbf{i}}+3\hat{\mathbf{j}}-4\hat{\mathbf{k}}$ are coplanar.
9. If the vectors $2\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}}+a\hat{\mathbf{j}}+5\hat{\mathbf{k}}$ are coplanar, then prove that $a=4$.
10. Show that the vectors $\mathbf{a}-2\mathbf{b}+3\mathbf{c}$, $-2\mathbf{a}+3\mathbf{b}-4\mathbf{c}$ and $-\mathbf{b}+2\mathbf{c}$ are coplanar vector, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar vectors.
11. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors, then prove that the four points $2\mathbf{a}+3\mathbf{b}-\mathbf{c}$, $\mathbf{a}-2\mathbf{b}+3\mathbf{c}$, $3\mathbf{a}+4\mathbf{b}-2\mathbf{c}$ and $\mathbf{a}-6\mathbf{b}+6\mathbf{c}$ are coplanar.

Answers

Exercise for Session 3

4. $(2, -3)$ **5.** $a-2b=1$

6. $x=\dfrac{1}{3}$ **7.** 0