2] LTI SYSTEM :-

Linear Time Invariant System.

- A system is said to be LTI system if it possess linearity property and if it is time invariant.

$$x(t) \longrightarrow \boxed{\text{LTI system}} \longrightarrow y(t) = T\{x(t)\}$$

$h(t) \rightarrow$

$$\underset{\text{F.P.}}{y(t)} = \underset{\text{S.P.}}{h(t)}$$

$$X(s) \longrightarrow \boxed{\frac{1}{s+1}} \longrightarrow Y(s)$$

$x(t)$ $H(s)$, $h(t)$ $y(t)$

$$Y(s) = \frac{1}{s+1} X(s)$$

$x(t) = \text{impulse}$

$y(t) = x(t) * h(t)$

$x(t) = \text{unit step}$

$Y[s] = X[s] \cdot H[s]$

① $x(t) = u(t) \longrightarrow X(s) = \frac{1}{s}$

$$Y(s) = \left(\frac{1}{s+1}\right) X(s)$$

$$Y(s) = \frac{1}{s} \cdot \frac{1}{s+1}$$

$$\downarrow$$

$$Y(s) = \frac{1}{s} - \frac{1}{s+1}$$

$$\downarrow L^{-1}$$

$$y(t) = (1 - e^{-t})\, u(t)$$

$$\underline{y(t)} = u(t) - u(t)\, e^{-t}$$

$\downarrow$

Unitstep response

② $x(t) = \delta(t) \Rightarrow X(s) = 1$

$$y[s] = \frac{1}{s+1} \cdot 1$$

$$\downarrow L^{-1}$$

$$y(t) = e^{-t} \cdot u(t)$$

$\downarrow$

S.R. (system response)

$$y(t) = T\{x(t)\}$$

$$y(t) = T\{\underset{\substack{\downarrow \\ x(t) = \delta(t)}}{x(t)}\} = h(t)$$

$$y(t) = h(t) = T\{\delta(t)\}$$

$$t \to \tau$$

$$h(\tau) = T\{\delta(\tau)\}$$

$$h(t-\tau) = T\{\delta(t-\tau)\}$$

Shifting Property.

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t-\tau) \cdot d\tau$$

$$y(t) = T\{x(t)\}$$

$$= T\left\{ \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t-\tau) d\tau \right\}$$

$$= \int_{-\infty}^{\infty} x(\tau) \cdot T\{\delta(t-\tau) d\tau\}$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau$$

$$\boxed{y(t) = \underbrace{x(t)}_{\text{i/p}} \underbrace{*}_{\text{convolution}} \underbrace{h(t)}_{\text{system}}}$$

Now,

$$y(t) = x(t) * h(t)$$

If $x(t) = \delta(t)$

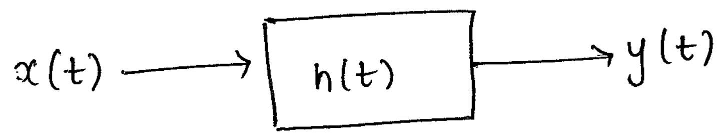
$$\boxed{y(t) = x(t) * h(t) = \delta(t) * h(t) = h(t)}$$

* Property of convolution *

$\Rightarrow \delta(t) * h(t) = h(t)$

$\Rightarrow \delta(t-t_0) * h(t) = h(t-t_0)$

$\Rightarrow \delta(t-t_0) \cdot h(t) = h(t_0) \cdot \delta(t-t_0)$



$$y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

* Properties

1] Commutative

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

$$y(t) = h(t) * x(t)$$

$$= \int_{-\infty}^{\infty} h(\tau) \cdot x(t-\tau)\, d\tau$$

2] Associative

$$y(t) = x(t) * (h_1(t) + h_2(t))$$

$$= (x(t) * h_1(t)) * h_2(t)$$

3] Distributive

$$y(t) = x(t) * (h_1(t) + h_2(t))$$
$$= [x(t) * h_1(t)] + [x(t) * h_2(t)]$$

4] $x(t) + \delta(t) = ?$

$$= \int_{\tau=-\infty}^{\tau=\infty} x(\tau) \cdot \delta(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t-\tau)\, d\tau$$

$$= x(t)$$

$$\boxed{x(t) + \delta(t) = x(t)}$$

i] $x(t-t_0) * \delta(t) = x(t-t_0)$

ii] $x(t) * \delta(t-t_0) = x(t-t_0)$

iii] $x(t-T_1) * \delta(t-T_2) = x(t-T_1-T_2)$.

5] Differentiation

$$y(t) = x(t) * h(t)$$

$$\frac{dy(t)}{dt} = \frac{dx(t)}{dt} * h(t)$$

$$\frac{dy(t)}{dt} = x(t) * \frac{dh(t)}{dt}$$

$$\boxed{\frac{d^m x(t)}{dt^m} + \frac{d^n h(t)}{dt^n} = \frac{d^{m+n} y(t)}{dt^{m+n}}}$$

6] $\boxed{x(at) + h(at) = \frac{1}{|a|} y(t)}$

There are two different method to solve continuous time convolution
① Graphical
② Analytical.

(1) Graphical Method

$$y(t) = x(t) * h(t)$$

$$y(t) = \int_{\tau=-\infty}^{\tau=\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

OR

$$y(t) = \int_{\tau=-\infty}^{\tau=\infty} h(\tau) \cdot x(t-\tau)\, d\tau$$

* Steps to find y(t) *

S_1: Find time limits of y(t)

Sum of lower limits of x(t) & h(t) $\leq t \leq$ sum of upper limits of x(t) & h(t)

S_2: Change the axis

$t \rightarrow \tau$

$x(\tau)$, $h(\tau)$

S_3: Time Reversal

$x(-\tau)$ $h(-\tau)$

S_4: Shifting

$x(t-\tau)$ $h(t-\tau)$

S_5: Multiplication

$x(\tau) \cdot h(t-\tau)$ $h(\tau) \cdot x(t-\tau)$

S_6: Integration

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

OR

$$y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot x(t-\tau)\, d\tau$$

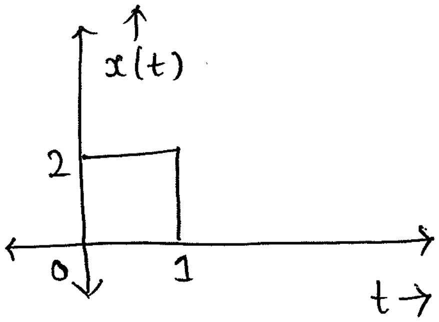
$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\cdot h(t-\tau)\,d\tau}$$

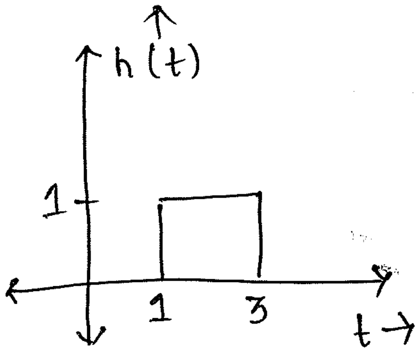
Time shifting

Time reversal

integration multiplication

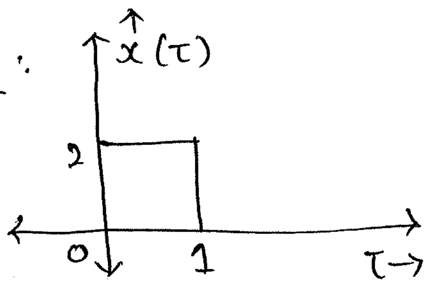
Q:- Find $x(t) * h(t)$

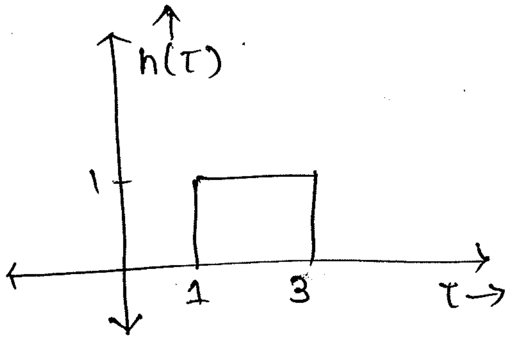




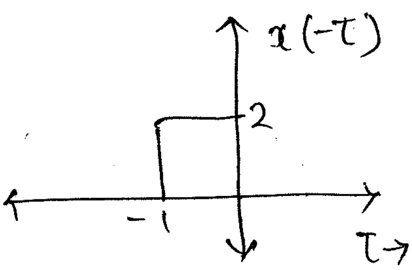
Sol:- S_1: $[1, 4]$

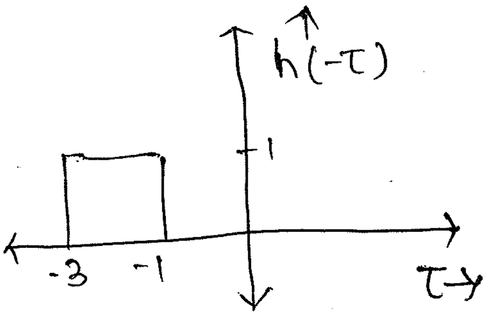
S_2:



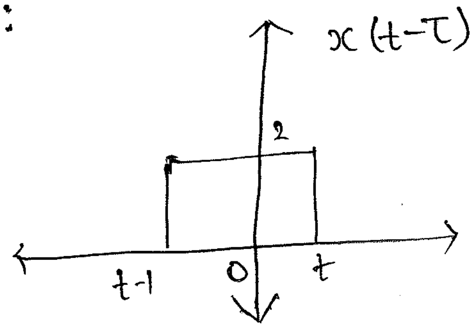


S_3:



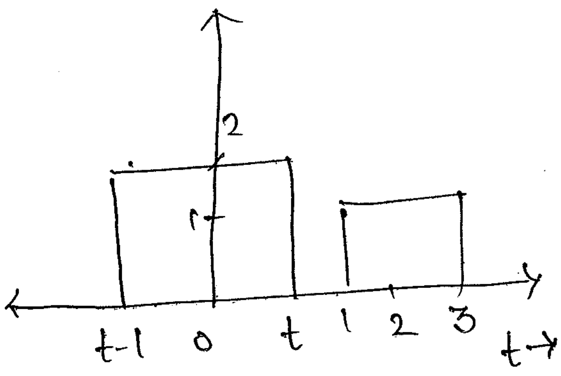


S_4:



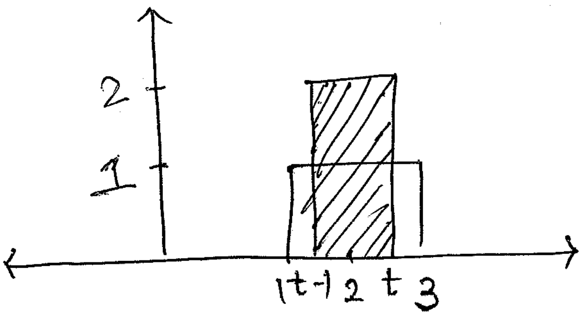
S_5:

(i) $0 < t < 1$



$$y(t) = 0 \quad , 0 < t < 1$$

(iii) $2 < t < 3$



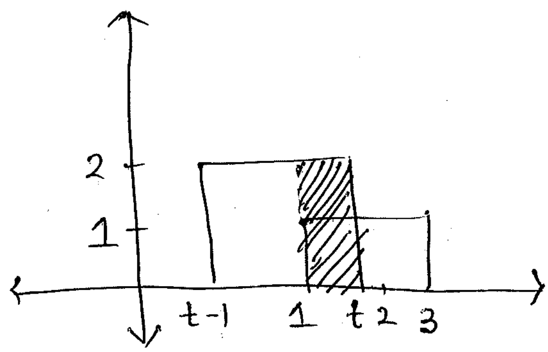
$$y(t) = \int_{t-1}^{t} 1 \cdot 2 \, d\tau$$

$$= \left[2\tau\right]_{t-1}^{t}$$

$$= 2\left[\cancel{t} - \cancel{t} + 1\right]$$

$$y(t) = 2 \quad , 2 < t < 3$$

(ii) $1 < t < 2$



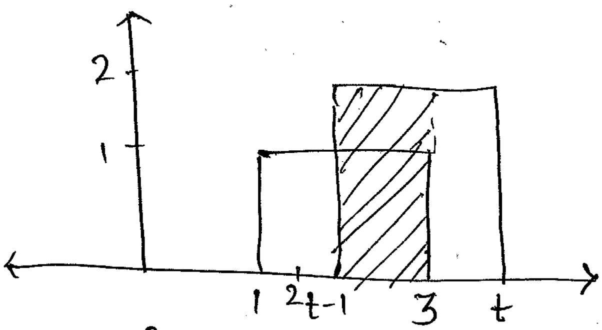
$$y(t) = \int_{1}^{\tau} h(\tau) \cdot x(t-\tau) \, d\tau$$

$$= \int_{1}^{t} 1 \cdot 2 \, d\tau$$

$$= \left[2\tau\right]_{1}^{t}$$

$$y(t) = 2t - 2 \quad , 1 < t < 2$$

(iv) $3 < t < 4$



$$y(t) = \int_{t-1}^{3} 1 \cdot 2 \, d\tau$$

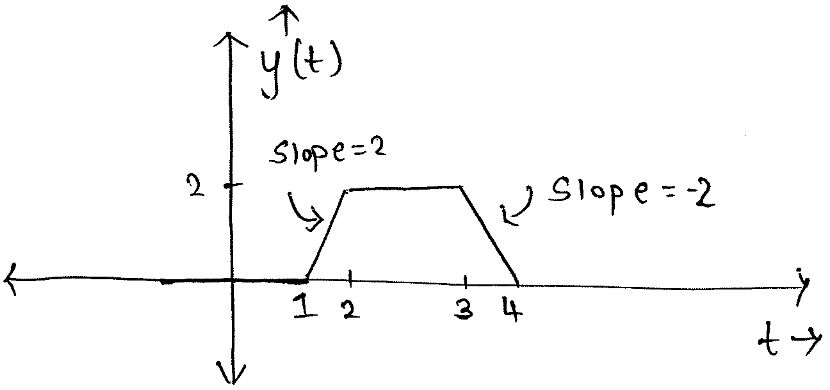
$$= \left[2\tau\right]_{t-1}^{3}$$

$$= 2\left[3 - (t-1)\right]$$

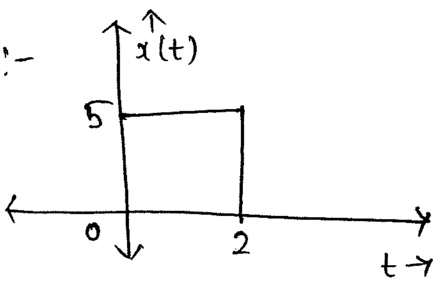
$$= 2\left[4 - t\right]$$

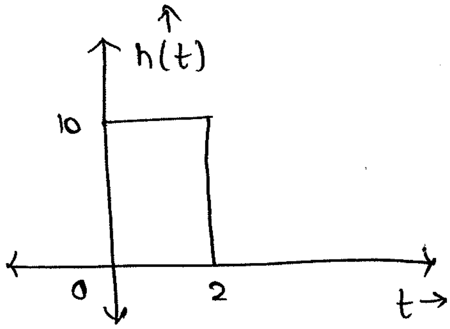
$$= 8 - 2t \quad , 3 < t < 4$$

ANS:
$$y(t) = \begin{cases} 0, & t < 1 \\ 2t - 2, & 1 \le t \le 2 \\ 2, & 2 \le t \le 3 \\ -2t + 8, & 3 \le t \le 4 \\ 0, & t > 4 \end{cases}$$



Q:-

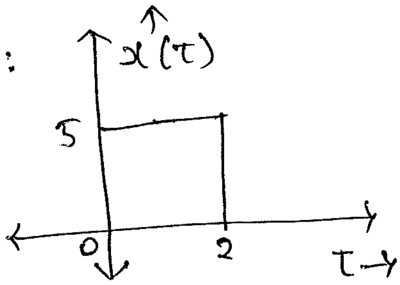


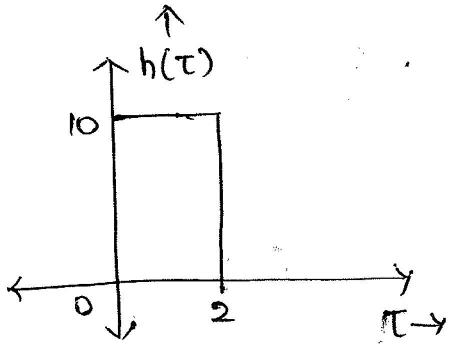


$x(t) * h(t) = ?$

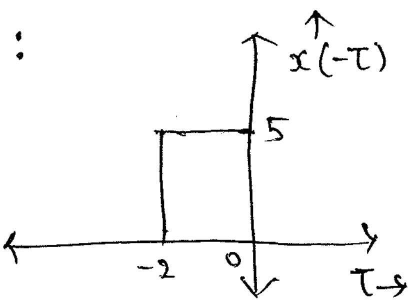
Sol:- $S_1 : [0, 4]$

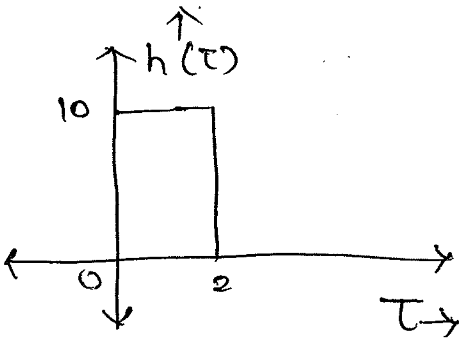
S_2 :



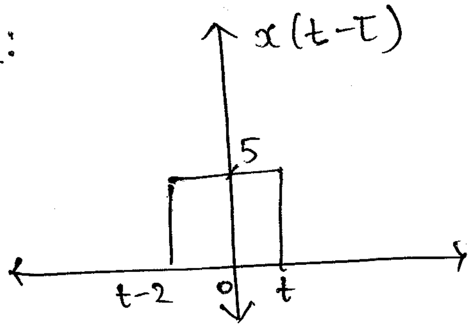


S_3 :

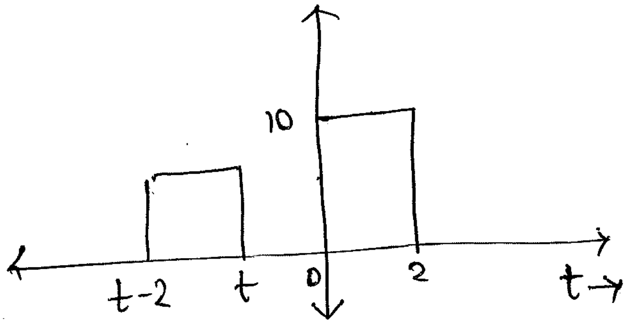




S_4:

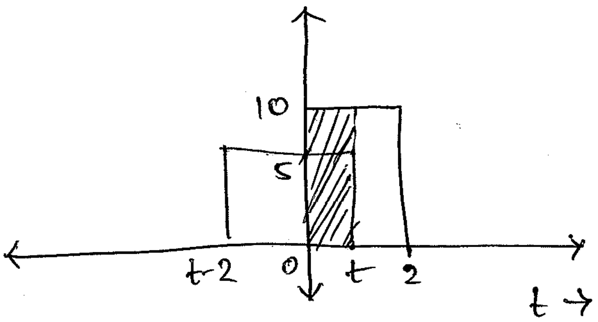


Case:- (i) $t<0$



$$y(t) = 0 \;, \; t<0$$

Case:- (ii) $0<t<2$



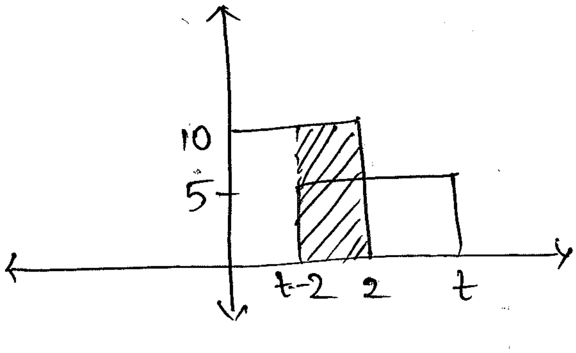
$$y(t) = \int_0^t 5 \cdot 10 \, d\tau$$

$$= [50\tau]_0^t$$

$$= 50t - 0$$

$$y(t) = 50t \;, \; 0<t<2$$

Case:- (iii) $2<t<4$



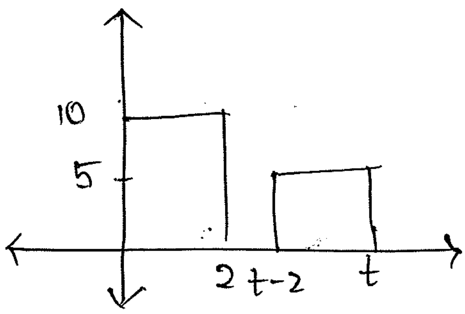
$$y(t) = \int_{t-2}^{2} 5 \cdot 10 \, d\tau$$

$$= [50\tau]_{t-2}^{2}$$

$$= 100 - [50t - 100]$$

$$y(t) = -50t + 200 \;, \; 2<t<4$$

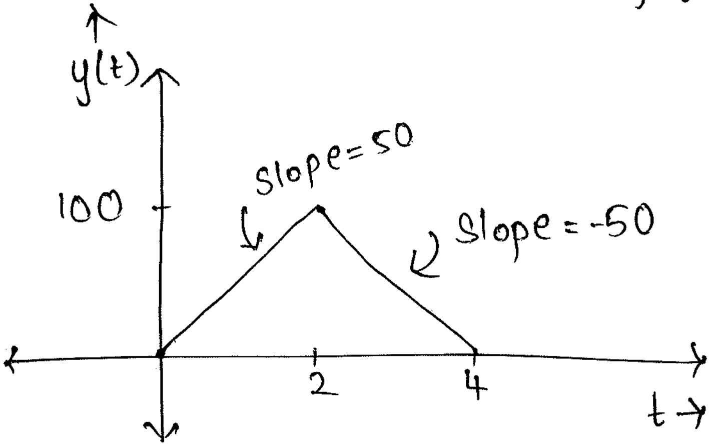
Case :- 4 $\quad t > 4$



$y(t) = 0, \; t > 4$

<u>ANS:</u>

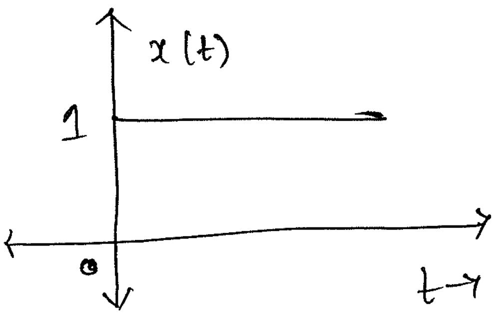
$$y(t) = \begin{cases} 0 & , t < 0 \\ 50t & , 0 \le t \le 2 \\ -50t + 200 & , 2 < t \le 4 \\ 0 & , t > 4 \end{cases}$$

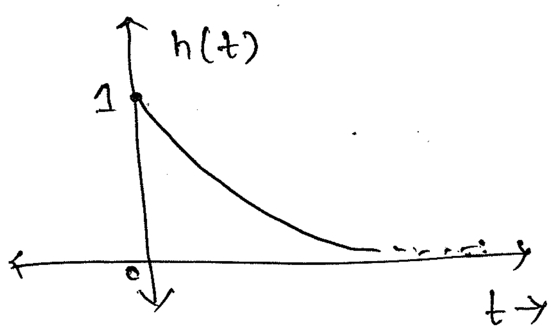


Q:- $x(t) = u(t)$

$h(t) = e^{-at} u(t)$

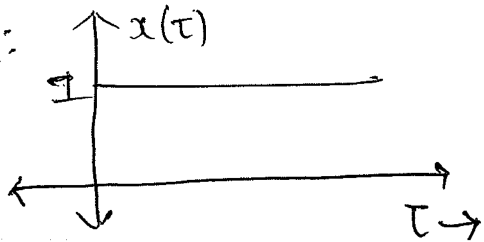
Find $x(t) * h(t)$

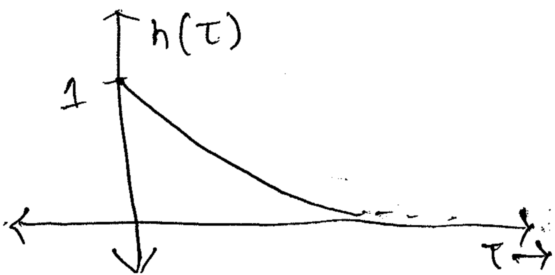




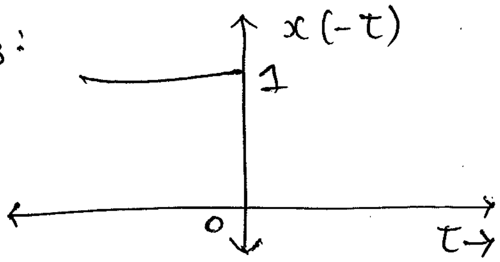
S_1 : $[0, \infty)$

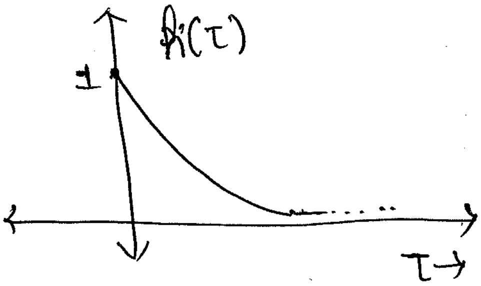
S_2:



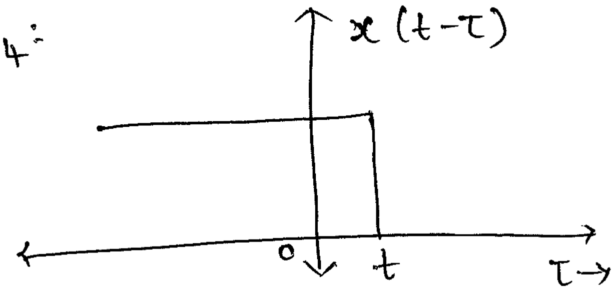


S_3:



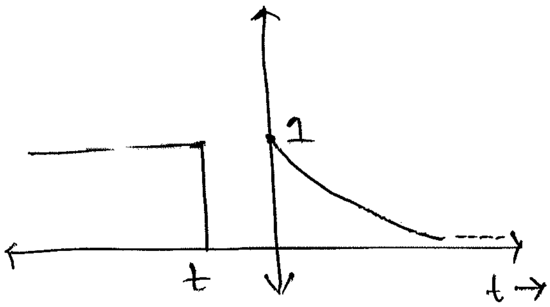


S_4:



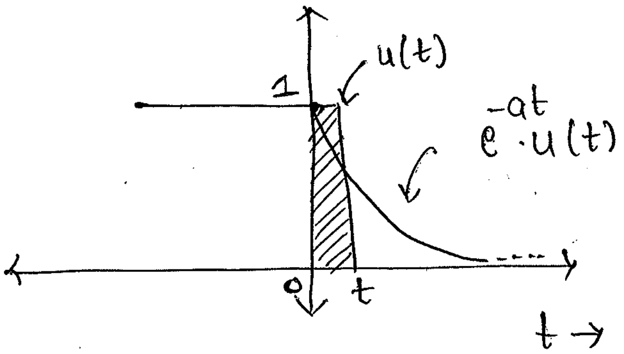
S_5:

Case:- 1 $\quad t < 0$



$$y(t) = 0, \quad t < 0$$

Case:- 2 $\quad t > 0$



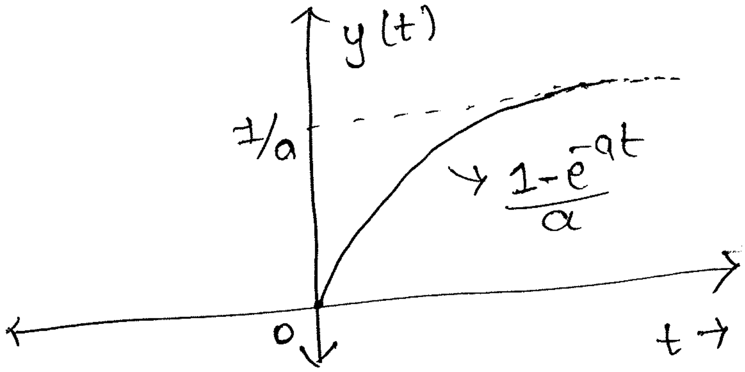
$$y(t) = \frac{1 - e^{-at}}{a}, \quad t > 0$$

$$y(t) = \int_0^t 1 \cdot e^{-a\tau} \cdot d\tau$$

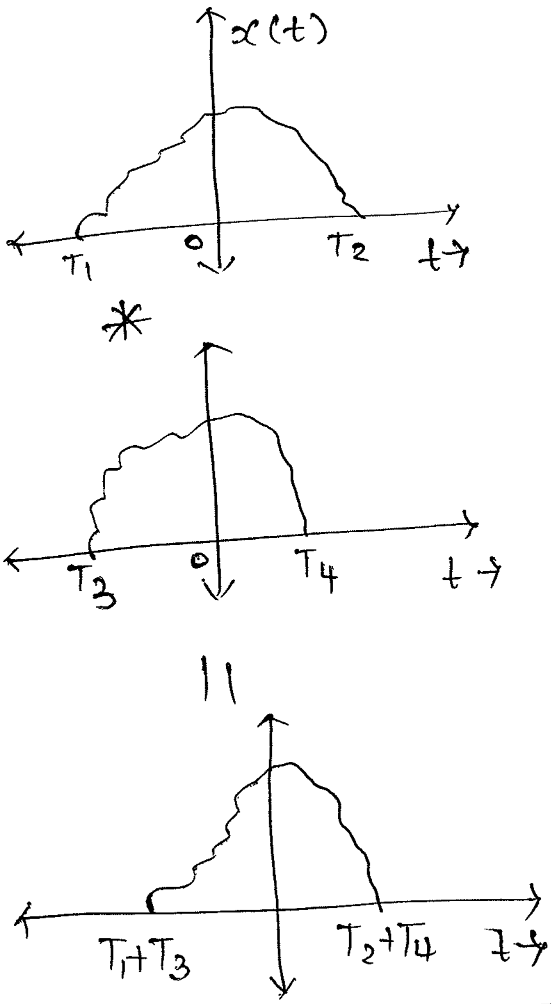
$$y(t) = \left[\frac{e^{-a\tau}}{-a}\right]_0^t = \frac{e^{-at} - 1}{-a} = \frac{1 - e^{-at}}{a}$$

$$y(t) = 0 \quad , t < 0$$

$$= \frac{1 - e^{-at}}{a} \quad , t \geq 0$$



NOTE :-



NOTE :-

$$\boxed{\begin{aligned} &(1) \sum_{k=m}^{k=n} (r)^k = \frac{(r)^{n+1} - (r)^m}{r-1} \\ &(2)\ 1+2+3+\ldots\ldots+n = \frac{n(n+1)}{2} \\ &(3)\ 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6} \end{aligned}}$$

* Change of limit according to causality of signals *

$$y(t)=\int_{\tau=-\infty}^{\tau=\infty} x(\tau)\cdot h(t-\tau)\,d\tau$$

① $x(t)$ ② $h(t)$

1]. $x(t)$ and $h(t)$ both general

$$y(t)=\int_{-\infty}^{\infty} x(\tau)\cdot h(t-\tau)\,d\tau$$

2] $x(t)$ is causal and $h(t)$ is general

$x(t)=0,\ t<0$

$x(\tau)=0,\ \tau<0$

$$y(t)=\int_{\tau=-\infty}^{\tau=\infty} x(\tau)\cdot h(t-\tau)\,d\tau$$

$$\downarrow$$

$x(\tau)=0,\ \tau<0$

$$y(t)=\int_{0}^{\infty} x(\tau)\cdot h(t-\tau)\,d\tau$$

3] $x(t)$ is general and $h(t)$ is causal

$h(t)=0,\ t<0$

$h(t-\tau)=0,\ t-\tau<0$

$h(t-\tau)=0,\ t<\tau,\ \tau>t$

$$y(t)=\int_{-\infty}^{\infty} x(\tau)\cdot h(t-\tau)\,d\tau$$

$$\downarrow$$

$h(t-\tau)=0,\ \tau>t$

$$y(t)=\int_{-\infty}^{t} x(\tau)\cdot h(t-\tau)\,d\tau$$

Upper limit changed

4] $x(t)$ and $h(t)$ both are causal

$x(t) = 0, \ t < 0$

$x(\tau) = 0, \ \tau < 0$

$h(t) = 0, \ t < 0$

$h(t-\tau) = 0, \ \tau > t$

$0 < \tau < t$

$$y(t) = \int_{\tau=-\infty}^{\tau=\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

$h(t-\tau) = 0, \ \tau > t$

$x(\tau) = 0, \ \tau < 0$

$$\boxed{y(t) = \int_{0}^{t} x(\tau) \cdot h(t-\tau)\, d\tau}$$

Both limits are changed.

* Convolution of standard signals *

① Impulse function

$y(t) = x(t) * \delta(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \cdot \delta(\tau - t)\, d\tau$$

$y(t) = x(t)$

$\left[\because \int_{-\infty}^{\infty} x(t) \cdot \delta(t-t_0)\, dt = x(t_0)\right]$

① $x(t) * \delta(t) = x(t)$

② $x(t \pm T_1) * \delta(t \pm T_2) = x(t \pm T_1 \pm T_2)$

Q:- Solve given convolution:-

① $\delta(t-1) * \delta(3-2t)$

Let $x(t) = \delta(t)$

$x(t-1) = \delta(t-1)$

$\therefore x(t-1) * \delta(3-2t)$

$x(t-1) * \frac{1}{2}\delta(t-3/2)$ $[\because \delta(-t) = \delta(t)]$

$\frac{1}{2} x(t-1-3/2) = \frac{1}{2} x(t-5/2)$

$\frac{1}{2}\delta(t-5/2)$

② $u(t+1) * \delta(3-t)$

$u(t+1) * \delta(t-3)$

$u(t+1-3)$

$u(t-2)$

③ $\delta(-t+2) * \delta(t-5)$

$\delta(t-2) * \delta(t-5)$

$\delta(t-5-2)$

$\delta(t-7)$

④ $u(-t+2) * \delta(t-5)$ $= \cancel{u(t-2) * \delta(t-5)}$

$u(t+2-5)$

$u(-t-3)$

⑤ $e^{-3t} \cdot u(t-2) * \delta(1.5-t)$

$e^{-3t} \cdot u(t-2) * \delta(t-1.5)$

$e^{-3(t-1.5)} \cdot u(t-3.5)$

② ~~Impulse~~ Unit step Function

① $u(t) * u(t)$

$$= \int_{\tau=-\infty}^{\tau=\infty} u(\tau) \cdot u(t-\tau)\, d\tau$$

$u(t-\tau) = 1 \;, \tau < t$

$u(\tau) = 1 , \tau > 0$

$$= \int_{0}^{t} 1 \cdot 1 \, d\tau$$

$$= [\tau]_0^t = t \;, \; t \geq 0$$

$$= r(t)$$

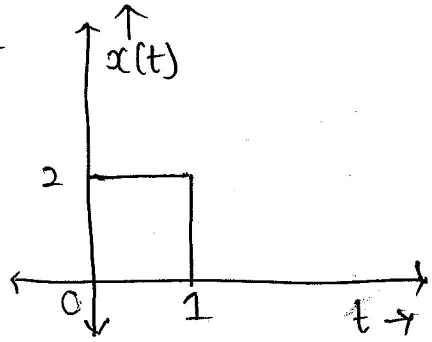
$$\boxed{\begin{aligned} u(t) * u(t) = t \cdot u(t) = r(t) & \\ &= t, t \geq 0 \\ &= 0, t < 0 \end{aligned}}$$

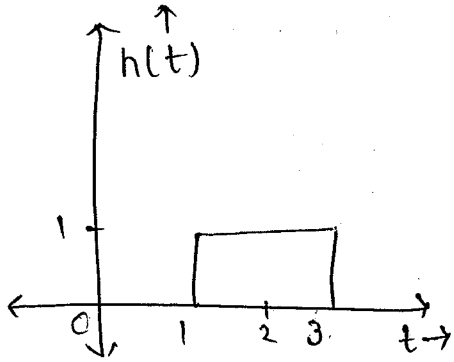
$$u(t \pm T_1) * u(t \pm T_2) = r(t \pm T_1 \pm T_2)$$

$$= t \pm T_1 \pm T_2 \quad ; \quad t \geq T_1 \mp T_2$$

$$= 0 \quad ; \quad t < T_1 \mp T_2$$

Q:-





Find $x(t) * h(t)$

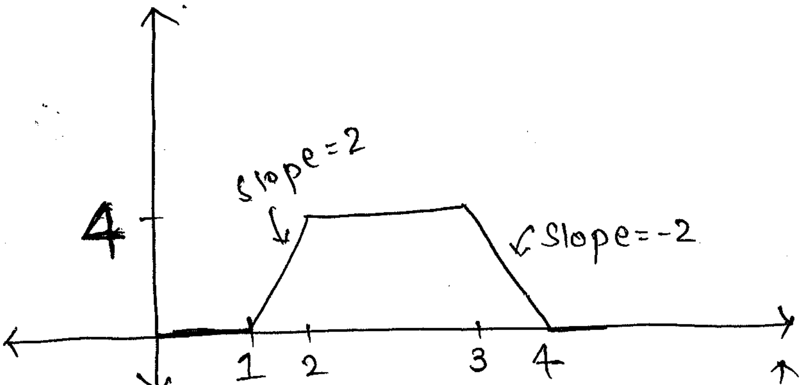
Sol:- $x(t) = 2u(t) - 2u(t-1)$ $\qquad$ $h(t) = 1\cdot u(t-1) - u(t-3)$

$$x(t) * h(t) = (2u(t) - 2u(t-1)) * [u(t-1) - u(t-3)]$$

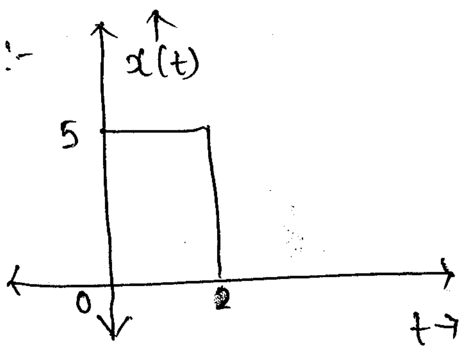
$$x(t) * h(t) = 2u(t) * u(t-1) - 2u(t-1) * u(t-1) + 2u(t)u(t-1) + 2u(t-1)u(t-3)$$

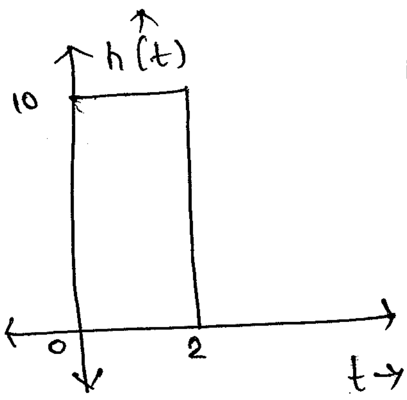
$$= 2r(t-1) + 2r(t-1) - 2r(t-1-1) + 2r(t-1-3)$$

$$x(t) * h(t) = 2r(t-1) - 2r(t-2) + 2r(t-1) + 2r(t-4)$$



Q:-





Find $x(t) * h(t)$

Sol:- $x(t) = 5u(t) - 5u(t-2)$ $\qquad$ $h(t) = 10u(t) - 10u(t-2)$

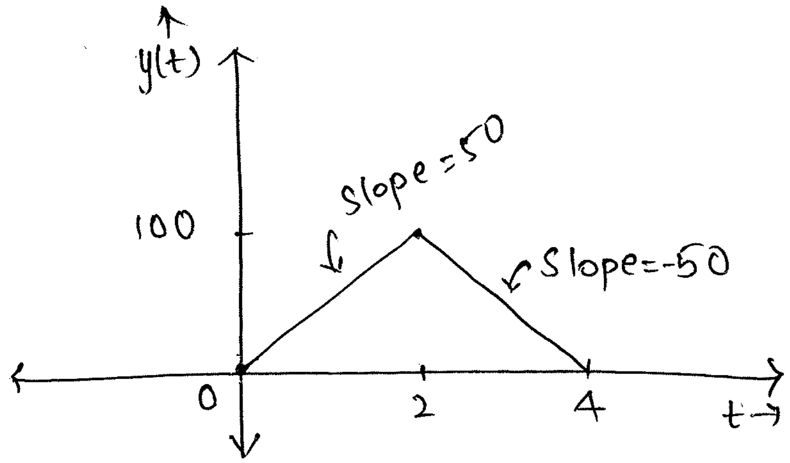
$$y(t) = x(t) * h(t)$$

$$= (5u(t) - 5u(t-2)) * (10u(t) - 10u(t-2))$$

$$= 50(u(t) * u(t)) - 5u(t-2) * 10u(t) + 50u(t-2) * u(t-2)$$
$$+ 50u(t) * u(t-2)$$

$$= 50r(t) - 50r(t-2) + 50r(t-4) + 50r(t-2)$$

$$y(t) = 50r(t) - 50r(t-2) + 50r(t-2) + 50r(t-4)$$



slope = 50

$\frac{dy}{dx} = 50$

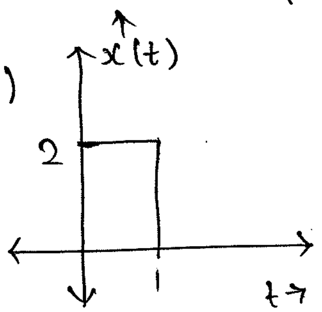
$\frac{dy}{2} = 50$

$dy = 100.$

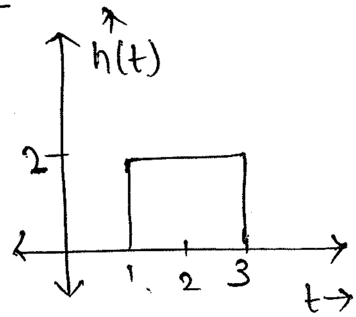
* Short-cut method *

$A_y = A_x \times A_h = 2 \times 4 = 8$

(1)



*



=

$A_x = 2$

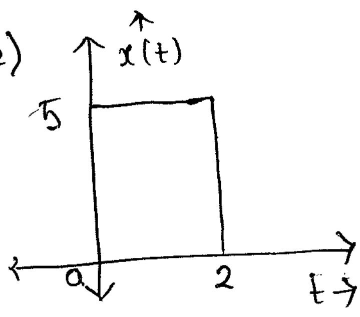
[0, 1]

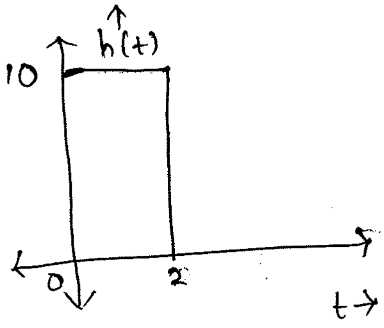
$A_y = 4$

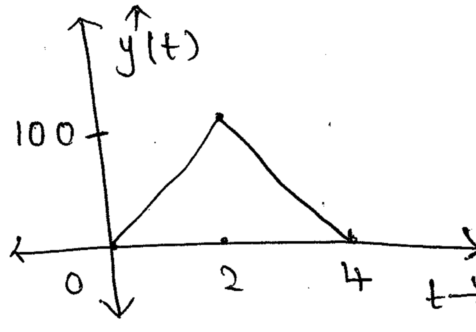
[1, 3]

[1 2 3 4]

(2)







$A_x = 10$

[0, 2]

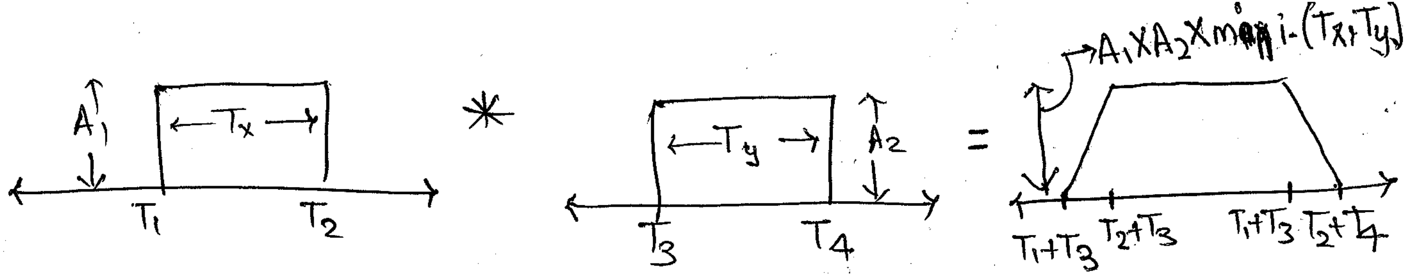
$A_h = 20$

[0, 2]

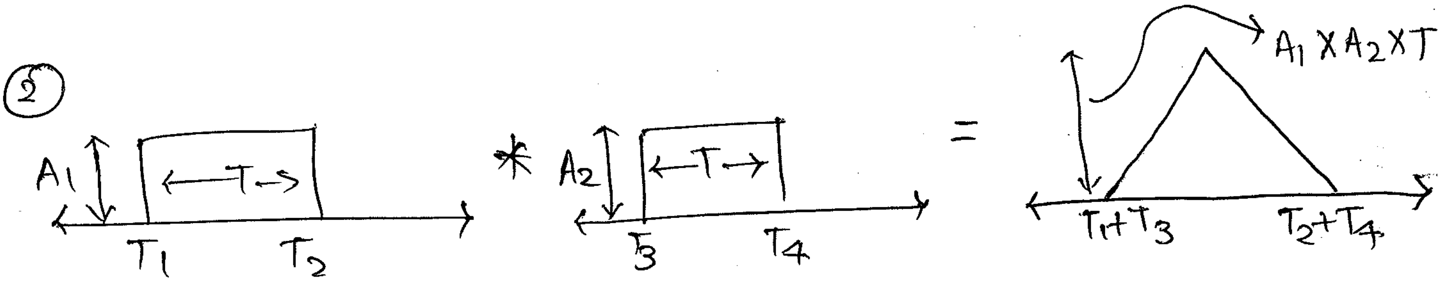
$A_y = A_x \times A_h$
$= 10 \times 20$
$= 200$

[0 2 2 4]

(1)



(2)



NOTE:-

Convolution of two impulses of unequal width produces trapezoidal signal and amplitude of output is given by $A_1 \times A_2 \times \min(T_x, T_y)$ where A_1 = amp. of 1st pulse

A_2 = amp. of 2nd pulse

T_x = width of 1st pulse

T_y = width of 2nd pulse

Convolution of two pulses of equal width produces triangular signal and amplitude of output is given by $A_1 \times A_2 \times T$ where A_1 = amp. of 1st pulse

A_2 = amp. of 2nd pulse

T = width of any signal

$y(t) = x(t) * h(t)$

$\downarrow$ A_y $\downarrow$ A_x $\downarrow$ A_h

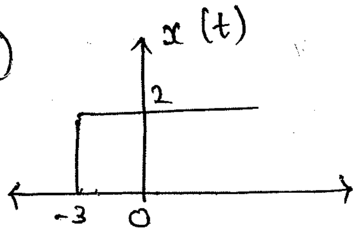
$$\boxed{A_x \times A_h = A_y}$$

A_x: area of $x(t)$

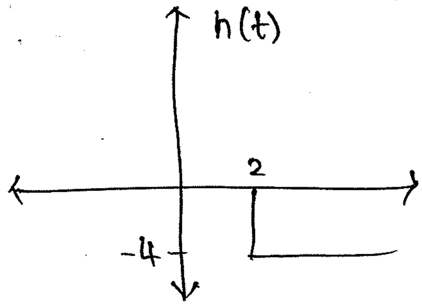
A_h: area of $h(t)$

A_y: area of $y(t)$

1



$*$

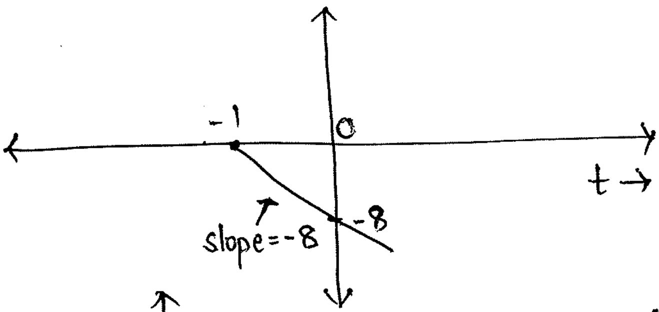


$x(t) = 2u(t+3)$

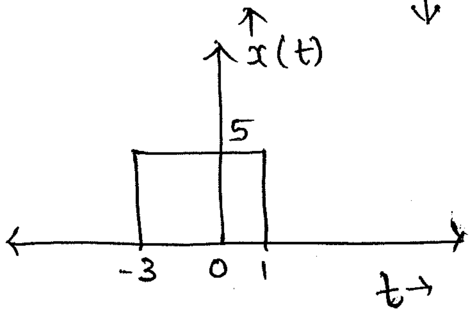
$h(t) = -4u(-t-2)$

$x(t) * h(t) = 2u(t+3) * (-4u(-t-2))$

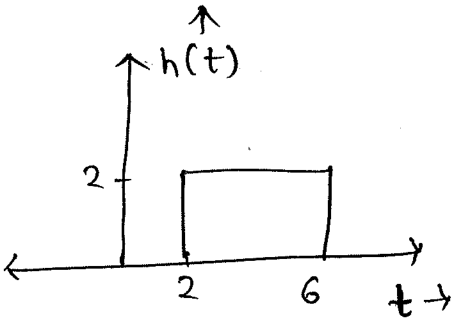
$x(t) * h(t) = -8r(t+1)$



2



$*$



$\longleftrightarrow$ 4

$[-3, 1]$

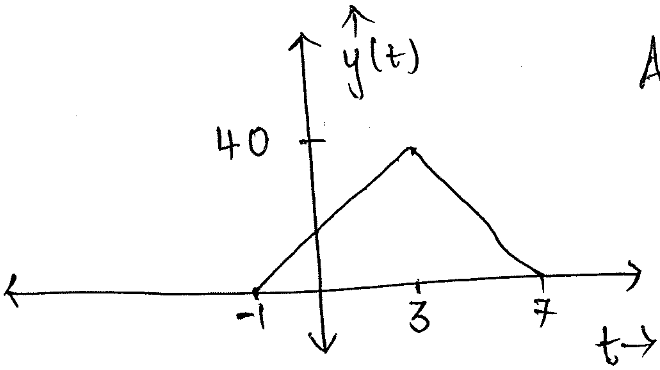
$\longleftrightarrow$ 4

$[2, 6]$

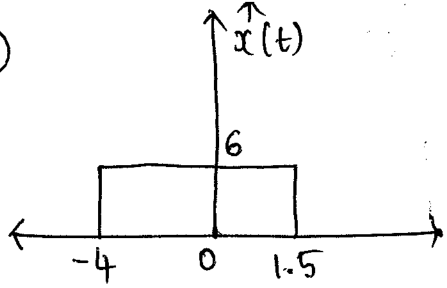
$[-1, 7]$

$-1, 3, 3, 7$

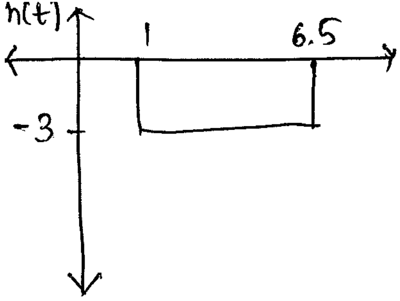
Amp:- $5 \cdot 2 \cdot 4 = 40$



(3)



*

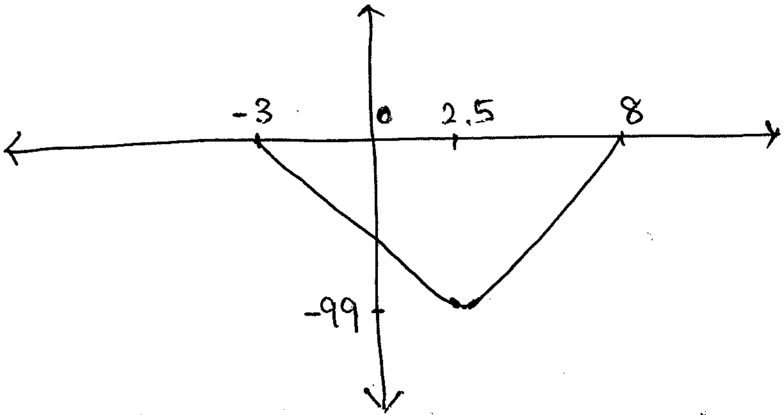


5.5   5.5

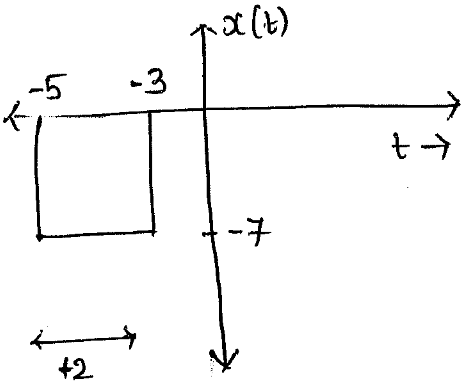
[-4, 1.5]   [1, 6.5]

[-3, 2.5, 2.5, 8]

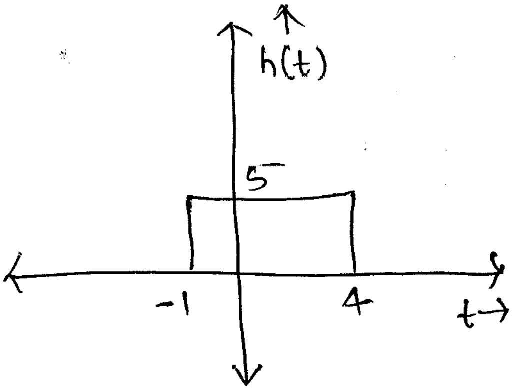
$\text{Amp.} = (\underset{3}{\cancel{6}})(-3)\left(\frac{\overset{11}{\cancel{5.5}}}{\cancel{10}\ \cancel{2}}\right) = -99$



(4)



*

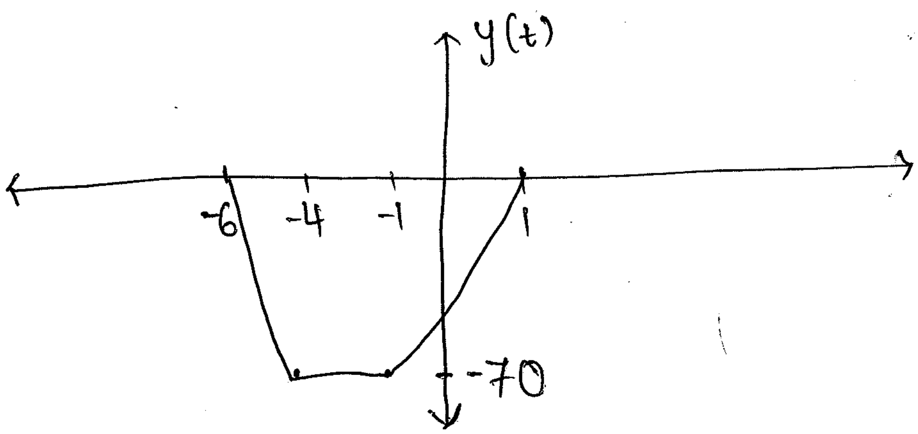


+2   5

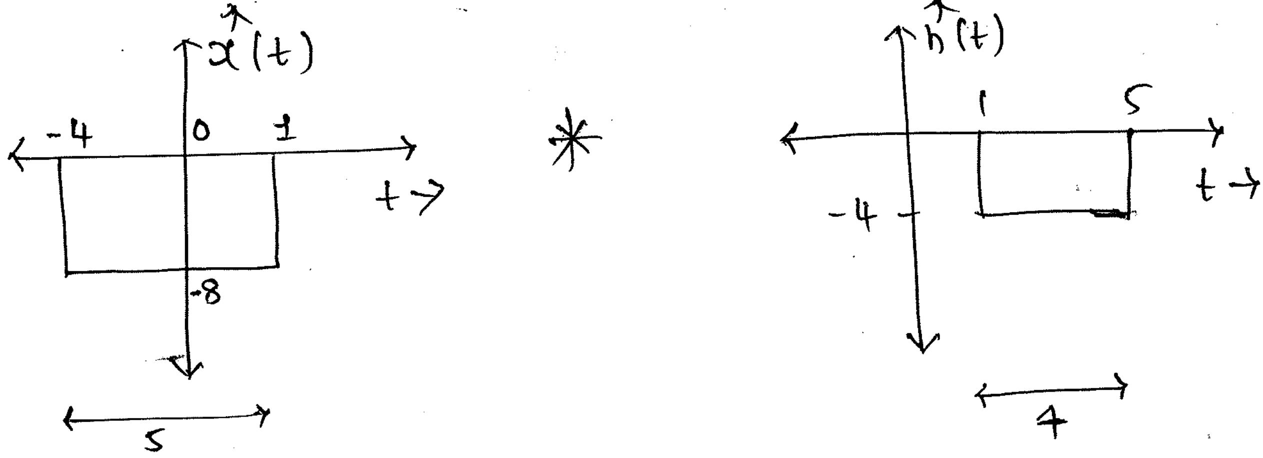
[-5, -3]   [-1, 4]

[-6, -4, -1, 1]

$\text{Amp.} = (-7)(5)\cdot 2\ (\text{min.})$
$= -70$



(5)



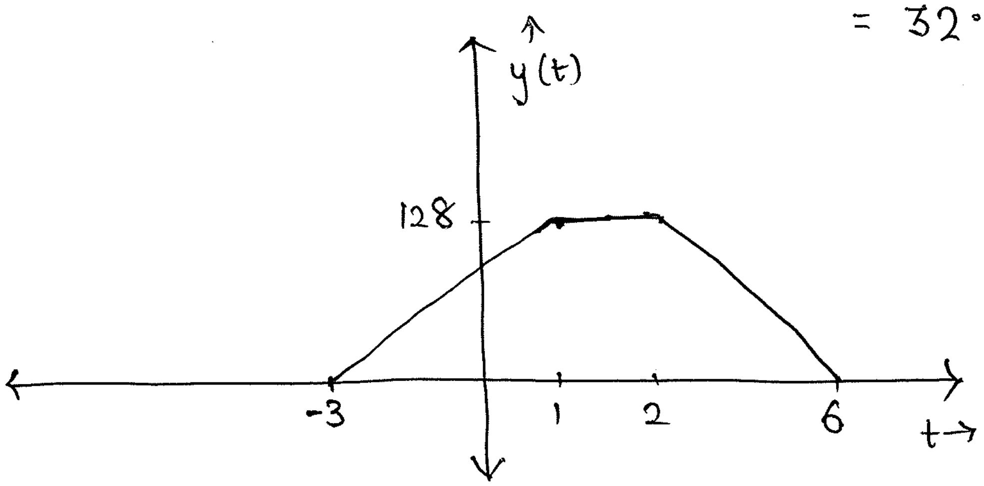
[-4,1]

[1,5]

[-3, 1, 2, 6]

Amp. = (-8) (-4) min (5, 4)

= 32 · 4 = 128



(3) Step and Ramp Function.

$u(t) * r(t) = y(t)$

$$y(t) = \int_{-\infty}^{\infty} u(\tau) \cdot r(t-\tau)\, d\tau$$

$u(\tau) = 1, \tau > 0$

$r(t-\tau) = t-\tau, t-\tau > 0$

$= t-\tau, t > \tau$

$$y(t) = \int_{0}^{t} 1 \cdot (t-\tau)\, d\tau$$

$$= t\,[\tau]_0^t - \left[\frac{\tau^2}{2}\right]_0^t \quad ; t \geq 0$$

$$= t^2 - \frac{t^2}{2}$$

$$y(t) = \frac{t^2}{2}, \quad t \geq 0$$

$$\boxed{u(t) * r(t) = y(t) = \frac{t^2}{2} u(t)}$$

$$u(t-T_1) * r(t-T_2) = \frac{(t-T_1-T_2)^2}{2} u(t-T_1-T_2)$$

④ Exponential and Step Function

$$u(t) * e^{-at} \cdot u(t)$$

$$y(t) = \int_{-\infty}^{\infty} u(\tau) \cdot e^{-a(t-\tau)} u(t-\tau)\, d\tau$$

$\to u(t-\tau) = 1 \;, t-\tau > 0$

$= 1 \;, t > \tau$

$$= \int_{0}^{t} 1 \cdot e^{-a(t-\tau)} \cdot d\tau$$

$$= \left[\frac{e^{-a(t-\tau)}}{+a} \right]_0^t$$

$$y(t) = \frac{1 - e^{-at}}{a} \quad , \quad t \geq 0$$

$$= 0 \quad , \quad t < 0$$

① $u(t) * e^{-3t} \cdot u(t)$

$$\frac{1 - e^{-3t}}{3} u(t)$$

② $u(t) * e^{3t} \cdot u(t)$

$$\frac{1 - e^{+3t}}{-3} u(t)$$

③ $u(t-2) * e^{-3t} u(t)$

$$\frac{1 - e^{-3(t-2)}}{3} u(t-2)$$

⑤ Ramp and Ramp function

$$y(t) = r(t) * r(t)$$

$$y(t) = \int_{-\infty}^{\infty} r(\tau) \cdot r(t-\tau)\, d\tau$$

$$= \int_{0}^{t} \tau \cdot (t-\tau)\, d\tau$$

$$= \int_{0}^{t} (\tau \cdot t - \tau^2)\, d\tau$$

$$= t \left[\frac{\tau^2}{2} \right]_0^t - \left[\frac{\tau^3}{3} \right]_0^t$$

$$y(t) = t^3/6 \quad , t > 0$$

$$= 0 \quad , t \leq 0$$

$$r(t) * r(t) = \frac{t^3}{6} u(t)$$

$$r(t-T_1) * r(t-T_2) = \frac{(t-T_1-T_2)^3}{6} u(t-T_1-T_2)$$

NOTE:

① $t^0 u(t) + t^0 u(t) = t^{0+0+1} \cdot u(t) = r(t)$

② $t^1 u(t) + t^1 u(t) = \frac{t^{1+1+1}}{(1+1+1)!} u(t) = \frac{t^3}{6} u(t)$

③ $t^2 u(t) + t^2 u(t) = \frac{t^{2+2+1}}{(2+2+1)!} u(t) = \frac{t^5}{5!} u(t)$

④ $t^m \cdot u(t) + t^n u(t) = \frac{t^{m+n+1}}{(m+n+1)!} u(t)$

⑥ Exponential with exponential function

$$e^{-at} \cdot u(t) * e^{-bt} u(t)$$

$$y(t) = \int_{-\infty}^{\infty} e^{-a\tau} \cdot u(\tau) \cdot e^{-b(t-\tau)} \cdot u(t-\tau)\, d\tau$$

$$= \int_0^t e^{-a\tau} \cdot e^{-b(t-\tau)}\, d\tau$$

$$= \int_0^t e^{-a\tau + (-bt) + b\tau}\, d\tau$$

$$= \int_0^t e^{(-a+b)\tau} \cdot e^{-bt} \cdot d\tau$$

$$= e^{-bt} \left[\frac{e^{(b-a)t} - 1}{b-a} \right] = \frac{e^{-bt} \cdot e^{bt} \cdot e^{-at} - e^{-bt}}{b-a} \cdot u(t)$$

$$= \frac{e^{-at} - e^{-bt}}{b-a} \cdot u(t)$$

NOTE:-

$$\boxed{e^{-at}\cdot u(t) * e^{-bt}\cdot u(t) = \frac{e^{-at}-e^{-bt}}{b-a}\cdot u(t)}, \; a \neq b$$

$$= e^{-at}\cdot t\,u(t), \; a=b$$

$$= e^{-ab}\,r(t), \; a=b$$

① $e^{-3t}\cdot u(t) * e^{-5t}u(t) = \frac{e^{-3t}-e^{-5t}}{5-3}u(t) = \frac{e^{-3t}-e^{-5t}}{2}u(t)$

② $e^{-3t}u(t) * e^{3t}u(t) = \frac{e^{-3t}-e^{3t}}{-6}u(t) = \frac{e^{3t}-e^{-3t}}{6}u(t)$

③ $e^{-3t}u(t) * e^{-3t}u(t) = e^{-3t}tu(t)$

④ $e^{5t}u(t) * e^{3t}u(t) = \frac{e^{5t}-e^{3t}}{+2}u(t)$

⑤ $e^{5t}\cdot u(t) * e^{5t}u(t) = e^{5t}tu(t)$

NOTE:-

① $x(t) * \delta(t) = x(t)$

② $x(t-T_1) * \delta(t-T_2) = x(t-T_1-T_2)$

③ $\delta(-\alpha t+\beta) * \delta(t-t_0) = \frac{1}{\alpha}\delta(t-t_0-\beta/\alpha)$

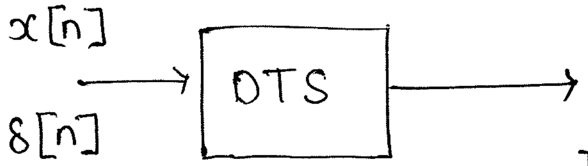
④ $u(t) * u(t) = r(t)$

⑤ $u(t-T_1) * u(t-T_2) = r(t-T_1-T_2)$

⑥ $e^{-at}u(t) * e^{-bt}u(t) = \frac{e^{-at}-e^{-bt}}{b-a}u(t)$

⑦ $t^m\cdot u(t) * t^n\cdot u(t) = \frac{t^{m+n+1}}{(m+n+1)!}u(t)$

* Discrete Time Signal



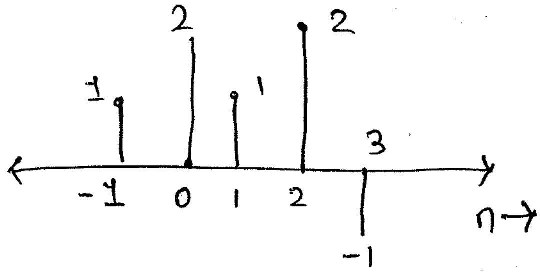
$y[n] = T\{x[n]\}$

$T\{\delta[n]\} = h[n] = y[n]$

$h[n] = T\{\delta[n]\}$

$h[n] = T\{\delta[n-k]\}$

$x[n] = x[-1]\,\delta[n+1] + x[0]\,\delta[n] + x[1]\,\delta[n-1]$
$\quad + x[2]\,\delta[n-2] + x[3]\,\delta[n-3]$



$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\cdot\delta[n-k]$$

$$y[n] = T\{x[n]\}$$

$$= T\left\{\sum_{k=-\infty}^{\infty} x[k]\cdot\delta[n-k]\right\}$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\cdot h[n-k] = x[n] * h[n]$$

If $y[n] = x[n] * h[n]$
if $x[n] = \delta[n]$

$y[n] = \delta[n] * h[n]$

then $y[n] = h[n]$

$$\boxed{\begin{aligned} &x[n] * \delta[n] = x[n] \\ &x[n-N_1] * \delta[n-N_2] = x[n-N_1-N_2] \end{aligned}}$$

$t \longrightarrow n$

$(\,) \longrightarrow [\,]$

$\int \longrightarrow \sum$

$\frac{d}{dt} \longrightarrow \nabla$

$$\boxed{\sum_{k=m}^{k=n} (r)^k = \frac{(r)^{n+1} - (r)^m}{r-1}}$$

* Convolution of standard signals

1) $x[n] * \delta[n-n_0] = x[n-n_0]$

2) $x[n] * \delta[n-n_0] = x[n-n_0]$

3) $\delta[n-1] * \delta[n+5] = \delta[n+4]$

4) $u(n-1) * \delta(n-3) = u(n-4)$

5) $(3)^n u(n-2) * \delta(n-2) = 3^{n-2} \cdot u(n-4)$

* Step with step

$$u[n] * u[n] = \sum_{k=-\infty}^{\infty} u[k] \cdot u[n-k]$$

$u[k] = 1, k \geq 0$

$u[n-k] = 1, n-k \geq 0$
$= 0, k \leq n$

$$= \sum_{k=0}^{n} 1 \cdot 1$$

$$= n+1$$

1) $u[n] * u[n] = (n+1)u[n]$

2) $u[n-N_1] * u[n-N_2] = (n-N_1-N_2+1)u[n-N_1-N_2]$

③ $u[n] * r[n] = \sum_{k=-\infty}^{\infty} u[k] * r[n-k]$

$u[k]=1 \ , k \geq 0$

$r[n-k] = n-k \ , n-k \geq 0$

$= n-k \ , k \leq n$

$$= \sum_{k=0}^{n} 1 \cdot (n-k)$$

$$= \sum_{k=0}^{k=n} \cdot n - \sum_{k=0}^{n} k$$

$$= n \sum_{k=0}^{n} 1 - \sum_{k=0}^{n} k$$

$$u[n] * r[n] = n(n+1) - \frac{n(n+1)}{2} \quad , n \geq 0$$

$$u[n] * r[n] = \frac{n(n+1)}{2} u[n]$$

$$u(t) * r(t) = \frac{t^2}{2} u(t)$$

$$u[n-N_1] * r[n-N_2] = \frac{(n-N_1-N_2)(n-N_1-N_2+1) u[n-N_1-N_2]}{2}$$

④ $u[n] * a^n u[n]$

$$= \sum_{k=-\infty}^{\infty} u[k] \cdot a^{n-k} \cdot u[n-k]$$

$u[k]=1 \ , k \geq 0$

$u[n-k]=1 \ , n-k \geq 0$

$=1 \ , n \geq k$

$$= \sum_{k=0}^{\infty} 1 \cdot a^{n-k} \cdot 1$$

$$= a^n \sum_{k=0}^{\infty} a^{-k}$$

$$= a^n \sum_{k=0}^{\infty} \left(\frac{1}{a}\right)^k$$

$m=0 \ , r = 1/a$

$$a^n \left(\frac{(1/a)^{n+1} - 1}{1/a - 1} \right) \ ; n \geq 0$$

$$\frac{a^{n+1}-1}{a-1} = \frac{1-a^{n+1}}{1-a} u[n]$$

Eg:- (1) $(2)^n \cdot u(n) * u(n)$

Here $a = 2$

$$\frac{1-(2)^{n+1}}{1-2} u(n)$$

$$(-1+2^{n+1})u(n)$$

(2) $\left(\frac{1}{3}\right)^n u(n) * u(n)$

Here $a = 1/3$

$$= \frac{1-(1/3)^{n+1}}{1-1/3} u(n) \quad = \frac{3}{2}\left(1-(1/3)^{n+1}\right)u(n)$$

* Unit ramp with unit ramp

$r[n] * r[n]$

$$= \sum_{k=-\infty}^{\infty} r[k] * r[n-k]$$

$\hookrightarrow k, k \geq 0$

$\rightarrow n-k, -k+n \geq 0$

$n-k, k \leq n$

$$= \sum_{k=0}^{n} k \cdot (n-k)$$

$$= n \sum_{k=0}^{n} k - \sum_{k=0}^{n} k^2$$

$$= n\left[\frac{n(n+1)}{2}\right] - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n^2(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{3n^2(n+1) - n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)[3n - 2n - 1]}{6}$$

$$= \frac{n(n+1)(n-1)}{6} u[n]$$

Eg:-

$$a^n \cdot u[n] * b^n \cdot u[n]$$

$$= \sum_{k=-\infty}^{\infty} a^k \cdot u[k] \cdot b^{n-k} \cdot u[n-k]$$

$$u[k] = 1 \qquad u[n-k] = 1, \; n-k \geq 0$$
$$= 1, \; n \geq k$$

$$= \sum_{k=0}^{n} a^k \cdot b^{n-k}$$

$$= b^n \sum_{k=0}^{n} \frac{a^k}{b^k}$$

$$= b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k$$

$$= b^n \left[\frac{(a/b)^{n+1} - (a/b)^0}{a/b - 1} \right]$$

$$= b^n \cdot b \left[\frac{(a/b)^{n+1} - 1}{a - b} \right]$$

$$= \frac{\cancel{b^{n+1}}}{a-b} \cdot \frac{a^{n+1} - b^{n+1}}{\cancel{b^{n+1}}}$$

$$\boxed{\begin{aligned} &= \frac{a^{n+1} - b^{n+1}}{a-b} u[n] \quad , \; a \neq b \\ &= a^n \cdot (n+1) u[n] \quad ; \; a = b \end{aligned}}$$

Eg:- (1) $2^n \cdot u[n] * 3^n u[n]$

$$= \frac{2^{n+1} - 3^{n+1}}{2-3} u[n]$$

$$= (-2^{n+1} + 3^{n+1}) u[n]$$

(2) $2^n u[n] * (-2)^n \cdot u[n]$

$$= \frac{2^{n+1} - (-2)^{n+1}}{2 - (-2)} \cdot u[n]$$

$$= \frac{2^{n+1} - (-2)^{n+1}}{4} \cdot u[n]$$

(3) $2^n u[n] * 2^n u[n]$

$$= 2^n \left[u[n] * u[n] \right]$$

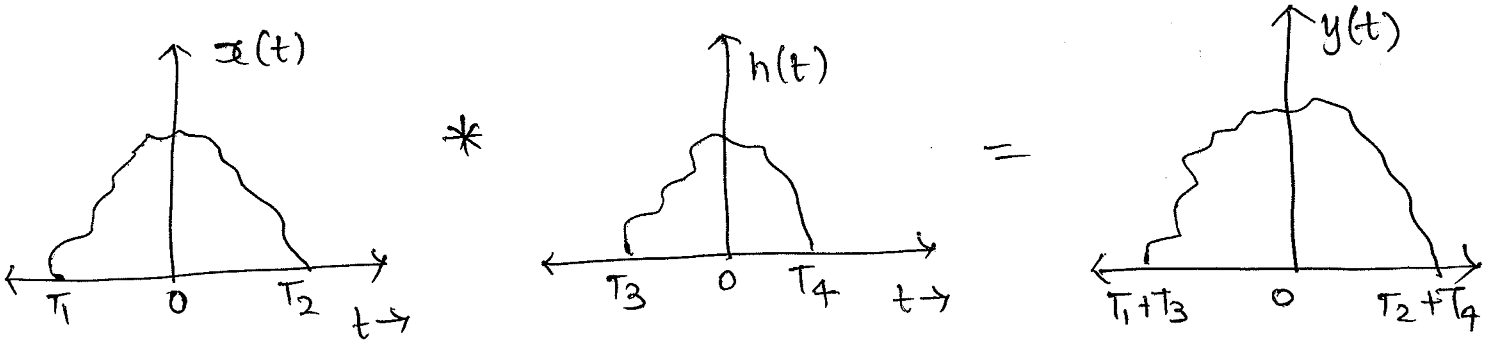
$$= 2^n (n+1) u[n]$$

(4) $\left(\frac{1}{3}\right)^n u[n] * \left(\frac{3}{5}\right)^n \cdot u[n]$

$$= \frac{\left(\frac{1}{3}\right)^{n+1} - \left(\frac{3}{5}\right)^{n+1}}{1/3 - 3/5} \cdot u[n]$$

$$= \frac{15}{4} \left[\left(\frac{3}{5}\right)^{n+1} - \left(\frac{1}{3}\right)^{n+1} \right] \cdot u[n]$$

* Convolution of finite time sequence.



$x[n] * h[n] = y[n] \longrightarrow [n_1+n_3, n_2+n_4]$

N_1 → $[n_1, n_2]$ N_2 → $[n_3, n_4]$

Linear Convolution	Circular convolution
N_1+N_2-1	max. (N_1, N_2)

Q:- Find linear convolution of given signals:-

(1) $x[n] = \{3, \underset{\uparrow}{-5}, 2, 4\}$

$h[n] = \{5, \underset{\uparrow}{2}, 4\}$

Sol^n:-

		width	limit
$x[n]$ limits	$[-1, 2]$	$N_1 = 4$	$y[n] = [-2, 3]$
$h[n]$ "	$[-1, 1]$	$N_2 = 3$.	

	3	-5	2	4
5	15	-25	10	20
2	6	-10	4	8
4	12	-20	8	16

$y[n] = [15, -19, \underset{\uparrow}{12}, 4, 16, 16]$

add diagonally

* De-convolution [System Identification]

Q :- $x[n] * h[n] = y[n]$

Sol:- $x[n] = \{3, \underset{\uparrow}{-5}, 2, 4\}$

$h[n] = ?$

$y[n] = \{15, -19, \underset{\uparrow}{12}, 4, 16, 16\}$

$[-1, 2] \rightarrow 4$
+
$[-1, 1] \rightarrow 3$
=
$[-2, 3] \rightarrow 6$

Solⁿ:-

Assume 3 cons.

	3	-5	2	4
a	3a	-5a	2a	4a
b	3b	-5b	2b	4b
c	3c	-5c	2c	4c

$3a = 15$

$\boxed{a = 5}$

$3b - 5a = -19$

$3b - 25 = -19$

$3b = 6$

$\boxed{b = 2}$

$4c = 16$

$\boxed{c = 4}$

$\therefore h[n] = [a, \underset{\uparrow}{b}, c] = [5, \underset{\uparrow}{2}, 4]$

Q:- $x(n) = \{1, \underset{\uparrow}{2}, 3, 4\}$

$h(n) = \{2, \underset{\uparrow}{3}, 5\}$. Find linear ~~circular~~ convolution

Solⁿ:- $[-1, 2]$ $[-1, 1]$

$[-2, 3] \leftarrow y[n]$ limit

→ * Circular convolution always start with 0.

	1	2	3	4
2	2	4	6	8
3	3	6	9	12
5	5	10	15	20

$y[n] = \{2, 7, \underset{\uparrow}{17}, 27, 27, 20\}$

Q:- Find from above example find 5 point <u>linear</u> convolution.

Sol:- $x(n) = \{1, \underset{\uparrow}{2}, 3, 4\}$

$h(n) = \{2, \underset{\uparrow}{3}, 5\}$

For 5 point set five points add 0 in $x(n)$ & $h(n)$

$x(n) = \{1, \underset{\uparrow}{2}, 3, 4, 0\}$ [-1, 3]

$h(n) = \{2, \underset{\uparrow}{3}, 5, 0, 0\}$ [-1, 3]

$\Rightarrow$ [-2, 6]

	1	2	3	4	0
2	2	4	6	8	0
3	3	6	9	12	0
5	5	10	15	20	0
0	0	0	0	0	0
0	0	0	0	0	0

$y[n] = \{2, 7, \underset{\uparrow}{17}, 27, 27, 20, 0, 0, 0\}$

Q:- Find circular convolution

$x(n) = \{2, -3, 4, 5\}$

$h(n) = \{5, 4, 1, 2\}$

Sol:- By default, first index is zero.

$$\begin{bmatrix} 2 & 5 & 4 & -3 \\ -3 & 2 & 5 & 4 \\ 4 & -3 & 2 & 5 \\ 5 & 4 & -3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 1 \\ 2 \end{bmatrix} = [\underset{\uparrow}{28}, 6, 20, 42]$$

$\therefore y[n] = \{28, 6, 20, 42\}$

- Same question, if we have to find <u>linear</u> <u>convolution</u>.

	2	-3	4	5
5	10	-15	20	25
4	8	-12	16	20
1	2	-3	4	5
2	4	-6	8	10

$$h[n] = \{\underset{\uparrow}{10}, -7, 10, 42, 18, 13, 10\}$$

Now,

→ From linear convolution we can obtain circular convolution by folding

$$h[n] = \{\underset{\uparrow}{10}, -7, 10, 42, 18, 13, 10\}$$

$$h[n] = \{28, 6, 20, 42\}$$

Q:- Find circular convolution

$x(n) = \{1, \underset{\uparrow}{2}, 3, 4\}$ $\qquad [-1, 2]$

$h(n) = \{2, \underset{\uparrow}{3}, 5\}$ $\qquad [-1, 1]$

<u>Sol</u>:- Circular convolution starts ~~to~~ from zero.

$h(n) = \{2, 3, 5, 0\}$

$$\begin{bmatrix} 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 0 \\ 2 \end{bmatrix} = [\underset{\uparrow}{17}, 27, 29, 27]$$

$$y[n] = \{\underset{\substack{\uparrow \\ n=0}}{17}, 27, 29, 27\}$$

Linear: $y[n] = \{2, 7, \underset{\uparrow}{17}, 27, 27, 20\}$

Q:- Find 5 point circular convolution.

$x(n) = \{1, \underset{\uparrow}{2}, 3, 4\}$

$h(n) = \{2, \underset{\uparrow}{3}, 5\}$

Sol^n:- For 5 point circular convolution add 0 (zeros) in $x(n)$ & $h(n)$ to make 5 index (i.e. 5 number).

$\therefore\ x(n) = \{1, \underset{\uparrow}{2}, 3, 4, 0\}$

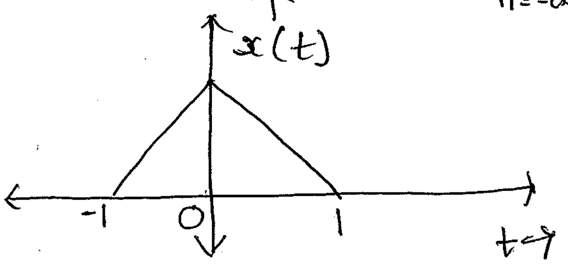
$h(n) = \{2, \underset{\uparrow}{3}, 5, 0, 0\}$

$$\begin{bmatrix} 2 & 1 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 \\ 0 & 4 & 3 & 2 & 1 \\ 1 & 0 & 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$y[n] = \{\underset{\uparrow}{17}, 27, 27, 22, 7\}$

* Miscellaneous Examples *

Q:- An input signal $x(t)$ shown in figure is applied to the system with impulse response $h(t)$. If $h(t) = \sum_{n=-\infty}^{\infty} \delta(t-3n)$ Find the output.

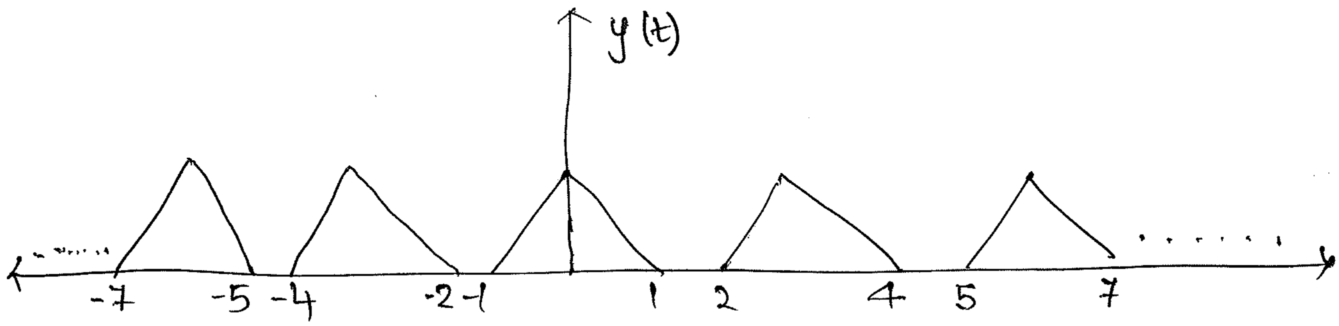


Sol^n:- $y(t) = x(t) * h(t)$

$= x(t) * \sum_{n=-\infty}^{\infty} \delta(t-3n)$

$= x(t) * [\ldots + \delta(t+6) + \delta(t+3) + \delta(t) + \delta(t-3) + \delta(t-6) + \ldots]$

$y(t) = \ldots\ldots + x(t+6) + x(t+3) + x(t) + x(t-3) + x(t-6) + \ldots\ldots$



NOTE:-

- Convolution of general signal with periodic train of impulse is periodic repeatation of general signal.

- System length (T_s) should be more than or equal to input signal length so that we can detect the information without distortion.

Q:- Consider the signal $h[n] = \left[\frac{1}{2}\right]^{n-1}(u(n-3) - u(n-10))$ such that

$$h(n-k) = \begin{cases} \left(\frac{1}{2}\right)^{n-k-1} & ; A \leq k \leq B \\ 0 & ; \text{othegwise} \end{cases}$$

Find A and B.

Sol^n: $h(n) = \left(\frac{1}{2}\right)^{n-1}(u(n-3) - u(n-10))$

$$h(n) = \left(\frac{1}{2}\right)^{n-1} \quad ; \quad 3 \leq n \leq 9$$

$$= 0 \quad ; \quad \text{otherwise}$$

$$h(n-k) = \left(\frac{1}{2}\right)^{n-k-1} ; \quad 3 \leq n-k \leq 9$$

$$3-n \leq -k \leq 9-n$$

$$9-n \geq k \geq 3-n$$

$$n-9 \leq k \leq n-3$$

$\therefore \boxed{A = n-9}$ & $\boxed{B = n-3}$

Q:- A linear system with input $x(n)$ and output $y(n)$. $y(n) = \sum_{K=-\infty}^{\infty} x(K)\cdot g(n-2K)$ where $g(n) = u(n) - u(n-4)$

Find $y(n)$. When $X(n) = \delta(n-2)$

Sol:- $X(n) = \delta(n-2) = 1, n=2$
$= 0, n \neq 2$

$x(K) = \delta(K-2) = 1, K=2$
$= 0, K \neq 2$

$y(n) = X(2)\cdot g(n-4)$

$y(n) = g(n-4) = u(n-4) - u(n-8)$

If $y(n) \rightarrow [4, 7]$

Q:- Given $x = [a, b, c, d]$ as an input to LTI system produces an output of $y = [x, x, x, \dots \text{ntimes}]$
The impulse/system response of $h[n]$ is _____.

Sol:- $h[n] = x[n] * y[n]$

Repeatation upto N times so we will take 0 to (N-1) times

$T_s \geq T$

$y = [\underset{0}{a}, \underset{1}{b}, \underset{2}{c}, \underset{3}{d}, \underset{\underline{4}}{a}, \underset{5}{b}, \underset{6}{c}, \underset{7}{d}, \underset{\underline{8}}{a}, b, c, d, \dots, a, b, c, d]$
ntimes

$= \delta(n) + \delta(n-4) + \delta(n-8) + \dots$

$y = \sum_{k=0}^{k=n-1} \delta(n-4k)$

If $\delta(n-3K)$

a b c d
a b c d
a b c d
....

ANS: $(a, b, c, a+d, b, c, a+d, \dots)$

Q:- Suppose in LTI system if input applied is $x[n] = \delta[n] - \delta[n-1]$ the o/p is observed to be $\delta[n] - \delta[n-1] + 2\delta[n-3]$. Find the output due to the input $7\delta[n] - 7\delta[n-2]$.

Sol^n:- Suppose, $x_1[n] = \delta[n] - \delta[n-1]$ —— (1)

& $x_2[n] = 7\delta[n] - 7\delta[n-2]$

$x_1[n-1] = \delta[n-1] - \delta[n-2]$ —— (2)

$x[n] + x[n-1] = \delta[n] - \delta[n-2]$

$7x[n] + 7x[n-1] = 7\delta[n] - 7\delta[n-2]$

$x[n]$	$y[n]$
$7x[n] + 7x[n-1]$	$7y[n] + 7y[n-1]$

$y[n] = 7\delta[n] - \cancel{7\delta[n-1]} + 14\delta[n-3] + \cancel{7\delta[n-1]} - 7\delta[n-2] + 14\delta[n-4]$

$y[n] = 7\delta[n] + 14\delta[n-3] - 7\delta[n-2] + 14\delta[n-4]$

* Properties of LTI system

(1) Static & Dynamic System
(memoryless) (memory)

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n]$$

$$y[n] = h[n] * x[n] \quad (\because \text{Commutative property}).$$

$y[n] = x[n]$ → present i/p & present o/p

$y[n] = x[n-1]$

$y[n] = x[n+1]$

$y[n,k] = x[n-k]$, $k > 0 \Rightarrow$ past input

$k < 0 \Rightarrow$ future input

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k]$$

$$= \underbrace{\sum_{k=-\infty}^{k=-1} h[k] \cdot x[n-k]}_{\text{future i/p}} + \underbrace{h[0]\, x[n]}_{\text{present i/p}} + \underbrace{\sum_{k=1}^{k=\infty} h[k]\, x[n-k]}_{\text{past i/p}}$$

$h[k] \neq 0$, $k = 0$

$= 0$, $k \neq 0$

$$\boxed{h[k] = 0 , k < 0}$$
↓
causal

$$\boxed{h[n] \neq 0 , n = 0}$$
↓
static

$h[k] = 0$, $k > 0$

$h[k] \neq 0$, $k = 0$

$h(t) = k\,\delta(t) = k \;, t=0$
$= 0 \;, t \neq 0$

$h[n] = k\,\delta[n] = k \;, n=0$
$= 0, n \neq 0$

$$\boxed{\begin{array}{l} h[n] = 0 \;, n<0 \\ h[t] = 0 \;, t<0 \end{array}} \Rightarrow \underline{\text{Causal.}}$$

② Causal and Non-causal

$$y[n] = \underbrace{\sum_{k=-\infty}^{-1} h[k]\cdot x[n-k]}_{\text{future i/p}} + \underbrace{h[0]\,x[n]}_{\text{present i/p}} + \underbrace{\sum_{k=1}^{\infty} h[k]\cdot x[n-k]}_{\text{past i/p}}$$

$h[k] \neq 0 \;, k=0$
$= 0 \;, k \neq 0$

$h[k] = 0 \;, k<0 \leftarrow$ Causal

$$\boxed{\left.\begin{array}{l} h[n] = 0 \;, n<0 \\ h(t) = 0 \;, t<0 \end{array}\right\} \rightarrow \text{Causal.}}$$

③ Stable and Unstable System

DTS (discrete time system):

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$y[n] = x[n] * h[n]$
$y[n] = h[n] * x[n]$

CTS (continuous time system):

$|x(t)| \leq M_x < \infty$

$|y(t)| \leq M_y < \infty$

$$|y(t)| = \int_{-\infty}^{\infty} |h(\tau) \cdot x(t-\tau)|\, d\tau$$

$\downarrow$ some finite value

$$\boxed{\begin{aligned} &\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty \rightarrow \text{stable} \\ &\int_{-\infty}^{\infty} |h(t)|\, dt < \infty \rightarrow \text{stable} \\ &\sum_{n=-\infty}^{\infty} |h(n)| < \infty \rightarrow \text{stable} \end{aligned}}$$

4 Invertible and Non-invertible system

$$x(t) \longrightarrow \boxed{h(t)} \xrightarrow{y(t)} \boxed{h_{inv.}(t)} \longrightarrow x(t)$$

$$y(t) = x(t) * h(t)$$

$$x(t) = y(t) * h_{inv.}(t)$$

$$\therefore\ x(t) = x(t) * [h(t) * h_{inv.}(t)]$$

$$\boxed{h(t) * h_{inv.}(t) = \delta(t)}$$

* STEP RESPONSE

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$\delta(t)$

$u(t)$

$y(t) = h(t) * \delta(t) = h(t) \rightarrow$ impulse response

$y(t) = h(t) * u(t) \rightarrow$ step response

$y(t) = x(t) * h(t)$

i/p o/p

$\delta(t)$ $y(t) = h(t)$ → impulse response

$u(t)$ $y(t) = u(t) * h(t)$ → step response

$s(t) = u(t) * h(t)$

↓

step response

$$\frac{ds(t)}{dt} = \frac{du(t)}{dt} * h(t)$$

$$\frac{ds(t)}{dt} = \delta(t) * h(t)$$

$$\boxed{\frac{ds(t)}{dt} = h(t)}$$ → Impulse Response

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

For DTS :-

$s[n]$ ← step signal.

$$s[n] - s[n-1] = h[n]$$

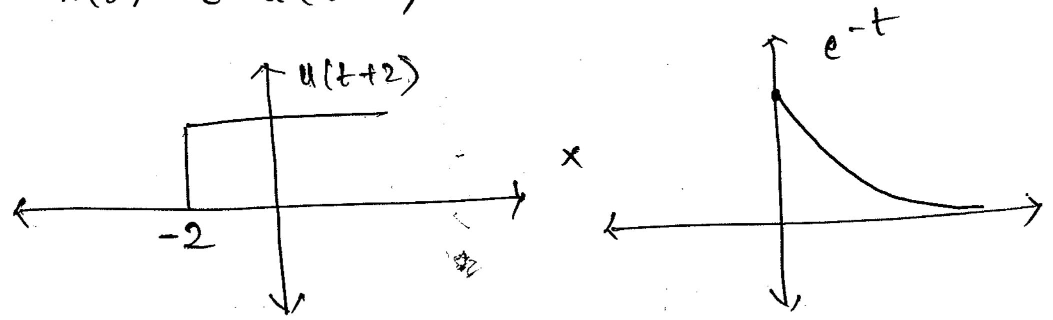
$$\boxed{s[n] = \sum_{k=-\infty}^{k=n} h[k]}$$

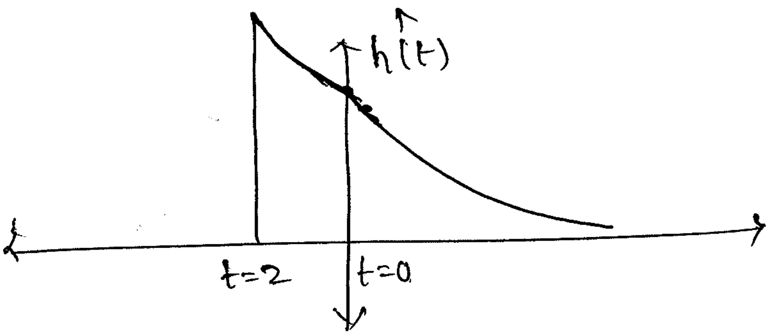
$$s[n] = \sum_{k=0}^{n} h[n-k]$$

Q:- Check causality and stability of given signal,

$h(t) = e^{-t} \cdot u(t+2)$

Sol^n:





$h(t) = 0 \,,\; t < 0$ → non-causal

bcz. condition is not satisfied

$\int_{-\infty}^{\infty} h(\tau) \cdot d\tau = \int_{-2}^{t} e^{-t} \cdot dt = [e^{-t}]_{-2}^{t} = -e^{-t} + e^{2}$

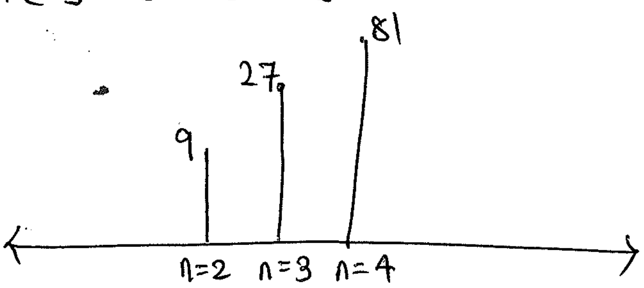
$= e^{2} - e^{-t} < \infty$

⇓

Stable

Q:- $h[n] = 3^{n} \cdot u[n-2]$

Sol^n:



$h[n] = 0 \,,\; n < 0$ → causal

unstable

dynamic.

Q:- find impulse response of system if step response is $s(t) = \cos w_0 t \cdot u(t)$

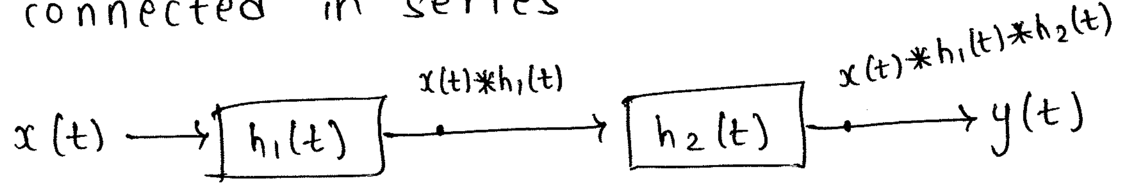
Sol^n:

$\frac{ds(t)}{dt} = h(t)$

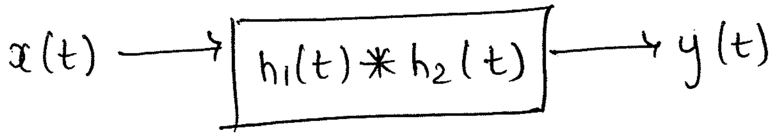
$s(t) = \int_{-\infty}^{t} h(t)\, dt$

$h(t) = w_0 \cdot u(t) \, [-\sin w_0 t] + \cos w_0 t \cdot \delta(t)$

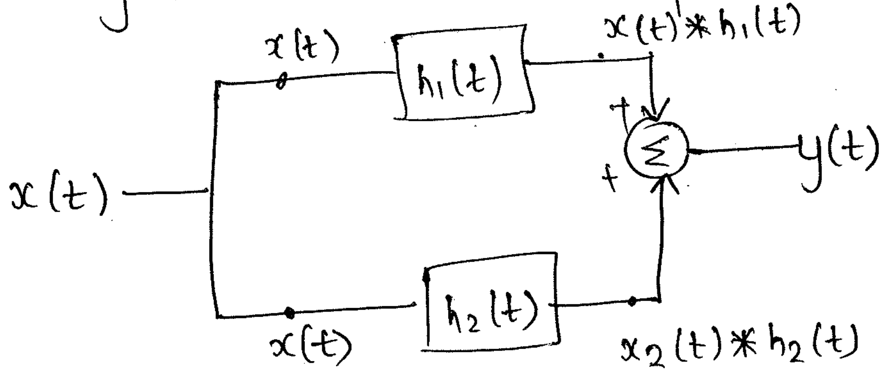
If the interconnected system is there and they are connected in series



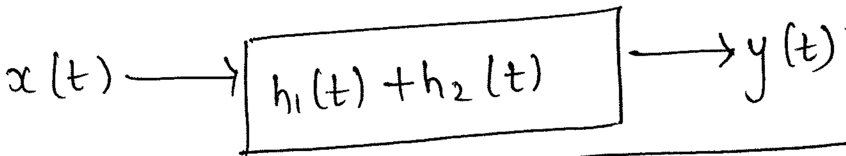
$$x(t) * (h_1(t) * h_2(t)) = y(t)$$



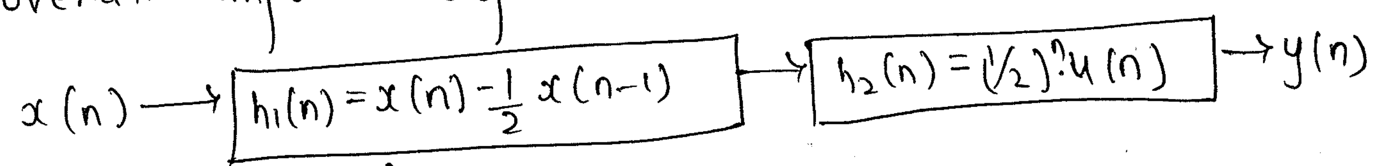
If they are connected in parallel,



$$y(t) = x(t) * (h_1(t) + h_2(t))$$



Q:- for the interconnected system shown in figure find overall impulse response.



Sol: Let $x(n) = \delta(n)$

$$h(n) = \left[x(n) - \frac{1}{2} x(n-1)\right] * \left[\left(\frac{1}{2}\right)^n u(n)\right]$$

$$= \left[\delta(n) - \frac{1}{2}\delta(n-1)\right] * \left[\left(\frac{1}{2}\right)^n u(n)\right]$$

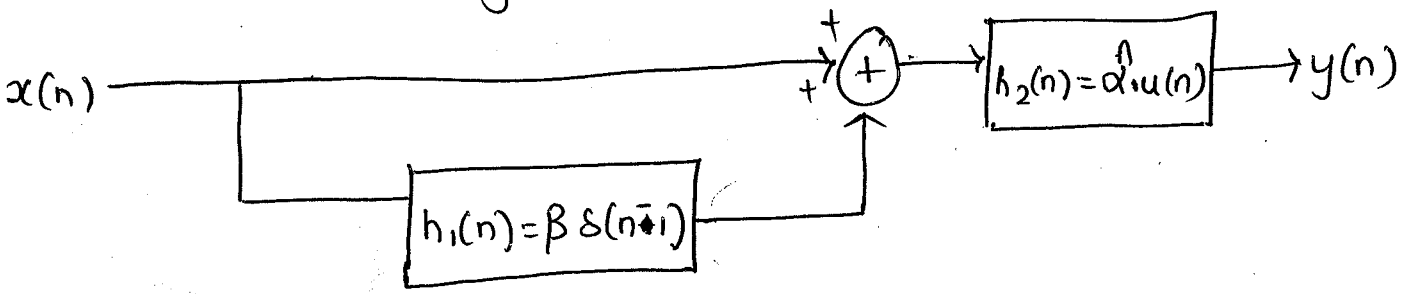
$$= \delta(n) * \left(\tfrac{1}{2}\right)^n \cdot u(n) - \tfrac{1}{2}\,\delta(n-1) * \left(\tfrac{1}{2}\right)^n \cdot u(n)$$

$$= \left(\tfrac{1}{2}\right)^n \cdot u(n) - \tfrac{1}{2}\left(\tfrac{1}{2}\right)^{n-1} \cdot u(n-1)$$

$$= \left(\tfrac{1}{2}\right)^n \left[u(n) - u(n-1)\right]$$

$$h(n) = \left(\tfrac{1}{2}\right)^n \cdot \delta(n)$$

Q:- Consider the system:-

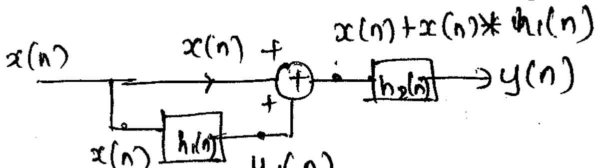


① Find impulse response of overall system.

② Is this system causal?

③ Under what condition system is stable?

Sol:-



$\therefore\ y_1(n) = x(n) * h_1(n)$

$$= \left(x(n) + x(n) * h_1(n)\right) * \alpha^n \cdot u(n)$$

for impulse response, $x[n] = \delta[n]$

$$y(n) = \left[\delta(n) + \delta[n] * \beta\delta(n-1)\right] * \alpha^n \cdot u(n)$$

$$= \alpha^n \cdot u(n) + \beta\delta(n-1) * \alpha^n \cdot u(n)$$

$$\boxed{y(n) = \alpha^n \cdot u(n) + \beta\,\alpha^{n-1} \cdot u(n-1)}$$

OR

Make equivalent system,

$$x(n) \longrightarrow \underset{h_1}{\boxed{\delta(n)+\beta\delta(n-1)}} \longrightarrow \underset{h_2}{\boxed{\alpha^n \cdot u(n)}} \longrightarrow y(n)$$

$$h(n) = h_1(n) * h_2(n)$$

$$= \left(\delta(n)+\beta\delta(n-1)\right) * \alpha^n \cdot u(n)$$

$$= \delta(n) * \alpha^n \cdot u(n) + \beta\delta(n-1) * \alpha^n \cdot u(n)$$

① $\boxed{h(n) = \underbrace{\alpha^n \cdot u(n)}_{n > 0} + \beta \cdot \alpha^{n-1} \cdot \underbrace{u(n-1)}_{n \geq 1}}$

② Causal.

③ Stable [but for stable $|\alpha^n| < 1$ bcz. it should be decreasing function if increasing then integration will become ∞]

④ Dynamic.

Q:- Consider discrete time system S_1 with impulse function response $h(n) = \left(\frac{1}{5}\right)^n u(n)$. Find 'A' such that $h(n) - Ah(n-1) = \delta(n)$

Sol^n:- $h(n) - Ah(n-1) = \delta(n)$

$$\left(\frac{1}{5}\right)^n \cdot u(n) - A\left(\frac{1}{5}\right)^{n-1} u(n-1) = \delta(n)$$

$$\left(\frac{1}{5}\right)^n = A\left(\frac{1}{5}\right)^{n-1} \qquad \left[\because u(n) - u(n-1) = \delta(n)\right]$$

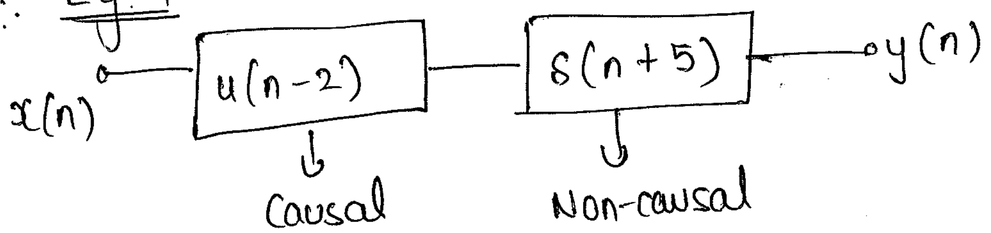
$$\cancel{\left(\frac{1}{5}\right)^n} = A\,\cancel{\left(\frac{1}{5}\right)^n} \cdot \left(\frac{1}{5}\right)^{-1}$$

$$\boxed{A = \frac{1}{5}}$$

Q:-Determine whether each of the following statements are true or false?

① The cascade (series) of non-causal LTI system with causal one is necessarily be non-causal.

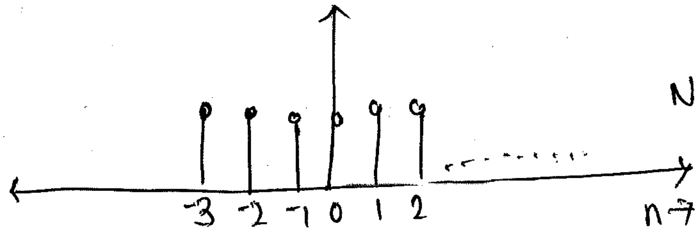
Sol^n: Eg:-1



$y(n) = u(n-2) * \delta(n+5)$

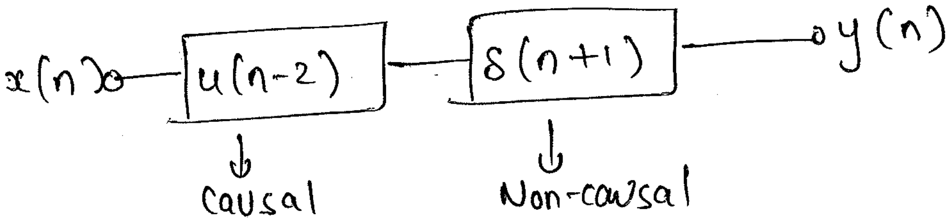
$= u(n-2+5)$

$y(n) = u(n+3)$



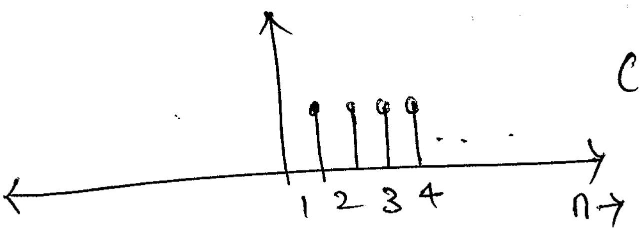
Non-causal.

Eg:-2



$y(n) = u(n-2) * \delta(n+1)$

$= u(n-1)$



Causal

∴ The above statement is false.

Q:- If an LTI system is causal than it is always stable. True or False.

Sol^n:- $h(t) = u(t) \longrightarrow$ LTI & causal

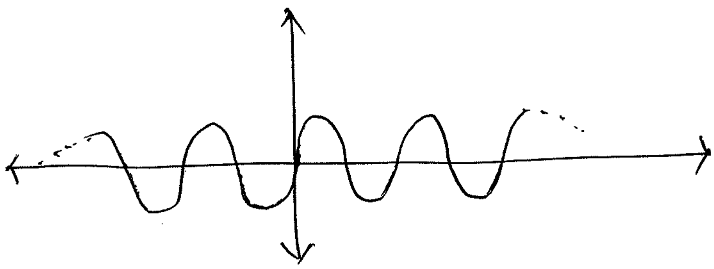
$$\int_{-\infty}^{\infty} h(t)\,d\tau < \infty \Rightarrow \text{for stable}$$

$$\int 1\,d\tau = \infty \Rightarrow \text{unstable}$$

The above statement is FALSE.

Q:- If $h(t)$ is an impulse response of an LTI system which is periodic and non-zero, system is unstable.

Sol^n:- All periodic signals are everlasting signals



$$\int_{-\infty}^{\infty} |h(t)|\,d\tau = \infty \Rightarrow \text{unstable}$$

The above statement is TRUE

*STEP RESPONSE

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

$h(t) \rightarrow$ impulse response

$y(t) \rightarrow y(t) = h(t)$

$x(t) =$ step signal

$x(t) = u(t)$

Total internal behaviour of system is chara. by I.R.
Sudden change in system behaviour is chara. by S.R.

S.R: $s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$

$\downarrow$

system response

$$\boxed{\delta(t) = \frac{d}{dt} u(t)}$$

$$\boxed{s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau}$$

$$\boxed{h(t) = \frac{d}{dt} s(t)}$$

Q:- If $x(t) = u(t-2)$ & $h(t) = e^{-3t} \cdot u(t)$. Find ~~y(n)~~ $y(t)$.

Sol^n:-

$$y(t) = \int_{-\infty}^{t} u(t)\, e^{-3t}\, dt$$

$$= \int_{0}^{t} 1 \cdot e^{-3t}\, dt$$

$$= \left[\frac{e^{-3t}}{-3}\right]_0^t \quad , t > 0$$

$$y(t) = \frac{1 - e^{-3t}}{3} \cdot u(t)$$

$x(t)$	o/p
$u(t)$	$y(t)$
$u(t-2)$	$y(t-2)$
$y(t-2)$	$\frac{1 - e^{-3(t-2)}}{3} \cdot u(t-2)$

NOTE: ★ Whenever the one signal is step signal then we can find o/p by just integrating other ★ signal.

Q:- If unit step response of system is $(1-e^{-\alpha t})\cdot u(t)$ Then its unit impulse response of system? $\parallel$ $h(t)$

<u>Sol</u>:- $s(t) = (1-e^{-\alpha t})\cdot u(t)$

$\downarrow$

unit step response

$$I.R. = \frac{d\,(S.R.)}{dt}$$

$$\therefore h(t) = \frac{du(t)}{dt}$$

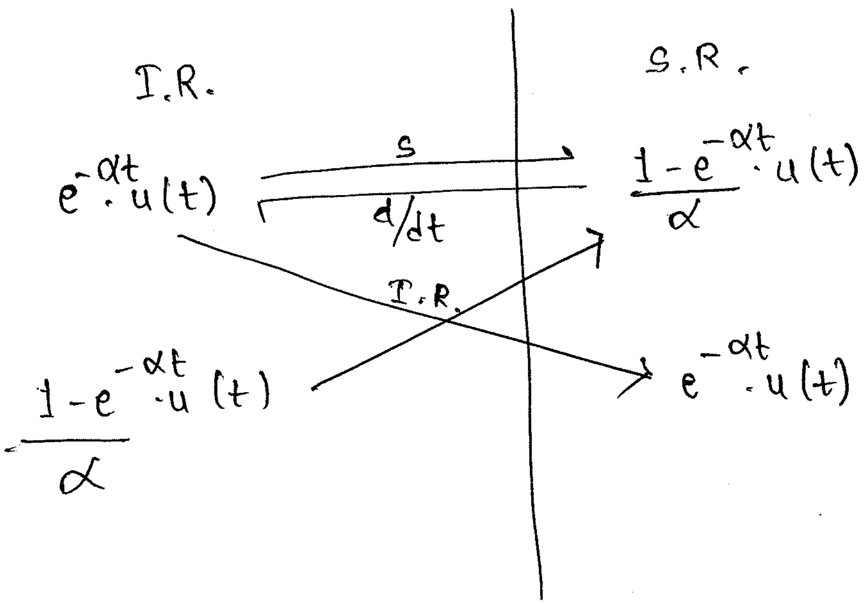
$$= \frac{d}{dt}(1-e^{-\alpha t})u(t)$$

$$= \frac{d}{dt}\left(u(t) - u(t)e^{-\alpha t}\right)$$

$$h(t) = \delta(t) - e^{-\alpha t}\cdot\delta(t) + e^{-\alpha t}\cdot u(t)\cdot\alpha$$

$$= \delta(t) - \delta(t) + \alpha e^{-\alpha t} u(t)$$

$$\boxed{h(t) = \alpha e^{-\alpha t}\cdot u(t)}$$



Q:- Given $h(t) = e^{\alpha t} u(t) + e^{\beta t} u(-t)$. Find α and β condition for system to be stable.

Sol^n:- $h(t) = e^{\alpha t} \cdot u(t) + e^{\beta t} u(-t)$

$\downarrow$ $t > 0$ $\qquad$ $\downarrow$ $t < 0$

$e^{\alpha t} \cdot u(t) \leftarrow$ decreasing

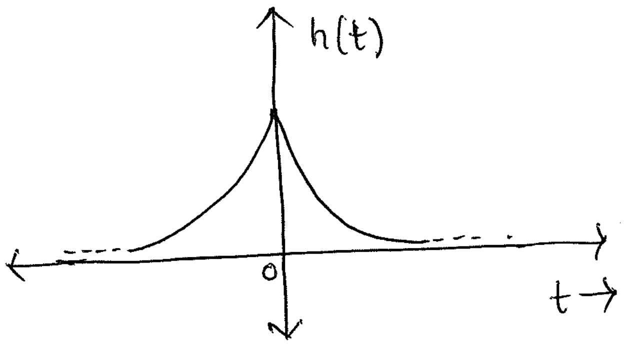
$t > 0$

$\boxed{|\alpha| < 0}$

$e^{\beta t} \cdot u(-t) \leftarrow$ increasing

$t < 0$

$\boxed{|\beta| > 0}$



FOURIER SERIES.

- Continuous time Fourier Series (CTFS)

(1) Introduction

(2) Different forms of Fourier series → Trigo., Expo., Polar.

(3) Symmetry Conditions in CTFS → Odd, Even, Half-wave

(4) Dirichlet Condition

(5) Properties of Fourier Series

(6) System Response

(7) Limitation of CTFS