13

Angles and their Measurement

13.01 Trigonometry

The word "trigonometry" is derived from a combination of three words 'tri', 'gon', and metron. 'Tri' means three, 'gon' means sides and 'metron' means ' a measure'. Thus, trigonometry deals with the measurment of sides (and angles) of a triangle. The relations between sides and angles of a triangle are used to find distance, height and areas etc. which can not be easily measured. We used the trigonometry in finding the distance of the earth from the sun and moon. Field like physics. Navy or Engineering the knowledge of trigonometry is very useful.

13.02 Positive and Negative distances

XOX' and YOY' are two mutually perpendicular lines, intersecting at O. Thus, the plane is divided into four parts. In such case

(i) The distance measured form O along OX are considered positive and the distance measured along OX' are considered negative.

(ii) The distances measured form O along OY are considered positive and those measured along OY' are considered negative. XOY, YOX', $X'OY'$ and $Y'OX$ are called the first, second, third and fourth quadrants respectively. It should be noted that this order is in anticlockwise direction.

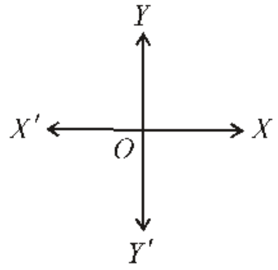


Fig. 13.01

13.03 Angle

The amount of rotation produced by the revolving the ray in moving form its initial position OX to the present position OA is called angle. Thus, in fig. 13.02, XOA is an angle. We use the symbol $\angle$ to denote an angle. Generally the capital letters A, B, C, ... of the English alphabet are used to denote the vertices of the angles and the measures of the angles are denoted by α (alpha), β(beta), γ(gamma), θ(theta), ϕ(phi), ψ(psi) etc.

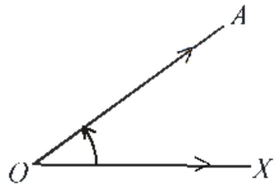


Fig. 13.02

Positive and Negative Angles :

If the ray OA, starting form its initial position OX revolves in the anticlockwise direction, the angle so formed is considered positive. In fig. 13.03, $\angle XOA$ is a positive angle.

If the ray OA, starting from its initial position OX revolves in clockwise direction, the angle so formed is considered negative. In fig. 13.03, $\angle XOA'$ is a negative angle.

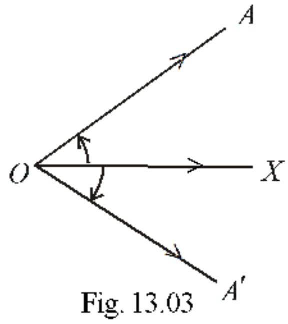


Fig. 13.03

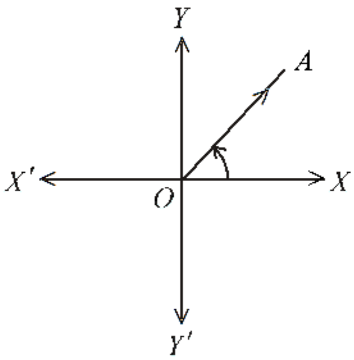
13.04 Angles of any magnitude :

(i) If the revolving ray OA, starting from its initial position OX revolving in anticlockwise direction and makes certain angle in the first quadrant, then this angle lies between $0°$ and $90°$. For example $\angle XOA$ in fig. 13.04 (i).

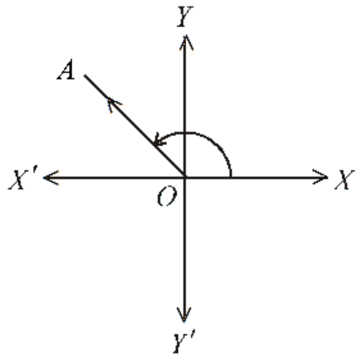
(ii) The angle in which the revolving ray OA, starting from its initial position OX and revolving in anticlockwise direction, makes in the second quadrant lies between $90°$ and $180°$. For example $\angle XOA$ in fig. 13.04 (ii).

(iii) The angle in which the revolving ray OA, starting form its initial positive OX and revolving in anticlockwise direction, makes in the third quadrant lies between $180°$ and $270°$. For example $\angle XOA$ in fig. 13.04 (iii).

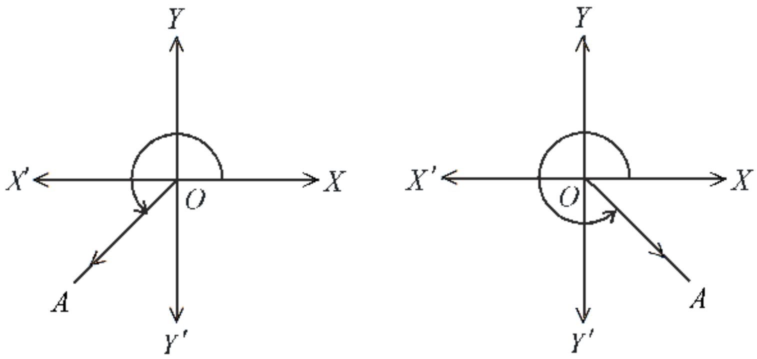
(iv) The angle in which the revolving ray OA, starting form its initial positive OX and revolving in anticlockwise direction makes, in the fourth quadrant lies between $270°$ and $360°$. For example $\angle XOA$ in fig, 13.04 (iv).



(i)



(ii)



(iii) (iv)

Fig. 13.04

If the ray OA, moving in anticlockwise direction, makes a complete revolution and comes back to its original position OX, then it describes an angle of $360°$.

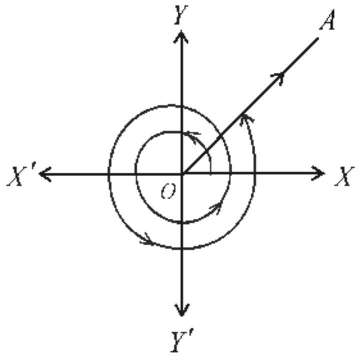


Fig. 13.05

So far we have seen that the maximum measure of an angle can be $360°$ or 4 right angles but angles greater than $360°$ are also possible. The revolving ray, revolving about its original position, described an angle of $360°$ in each complete rotation.

If the revolving ray, starting from its original position OX and revolving about the point O, make two complete rotations, then it describes an angle of $2 \times 360° = 720°$. If, after two complete rotations, come back to the position OA again then the angle so described $= 2 \times 360° + \angle XOA$ (Fig. 13.05).

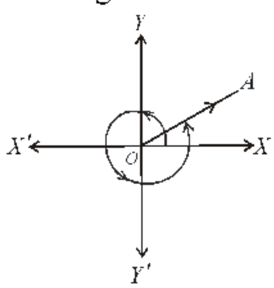


Fig. 13.06

Example 1. Display an angle of $390°$ with the help of a figure.

Solution : $390° = 1 \times 360° + 30°$

The revolving ray OA, starting form its initial position OX and revolving in anticlockwise direction point O makes one complete rotation and moves through $30°$ in the same direction and comes to the position OA in the first quadrant, as shown in figure 13.06.

Example 2. Display an angle of $-750°$ with the help of a figure.

Solution : $-750° = -(2 \times 360°) - 30°$

The revolving ray OA, starting form its initial position OX and revolving in clockwise direction about the point O, makes two complete rotation and moves in the same direction through an angle of $30°$. Thus its final position in the fourth quadrant will be as shown in the figure 13.07.

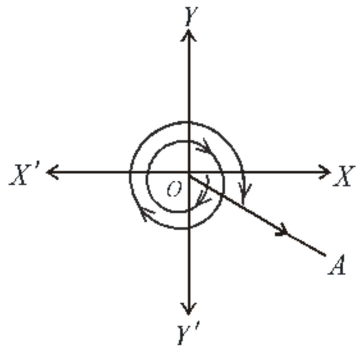


Fig. 13.07

13.05 Measurement of Angles

(i) Sexagesimal system

(ii) Centesimal system

(iii) Circular system

(i) Sexagesimal system : In this system, angles are measured in degrees, minutes and seconds. They are related as follows :

$$1 \text{ right angle} = \text{ninety degrees} = 90°$$

$$1\text{degree } (1°) = \text{ sixty minutes } = 60'$$

$$1 \text{ minute } (1') = \text{sixtey seconds} = 60''$$

(ii) Centesimal system : This system is also known as French system. In this system, the angles are measured in grades, minutes and seconds. They are related as follows :

$$1 \text{ right angle} = 100 \text{ grade} = 100^g$$

$$1 \text{ grade } (1^g) = 100 \text{ minutes} = 100'$$

$$\text{one minute } (1') = 100 \text{ seconds} = 100''$$

Note : This system is not in practice.

(iii) Circular system : In this system the unit of angle is radian. "The angle subtended by an arc of a circle, whose length is equal to radius, at the centre of the circle, is called one radian". Let O be the centre and r be the radius of the circle. Draw an arc AB whose length is r. Thus the $\angle AOB$ is called one radian. The angle of 1 radian is denoted as 1^c. Hence in Fig. 13.08, $\angle AOB = 1^c$.

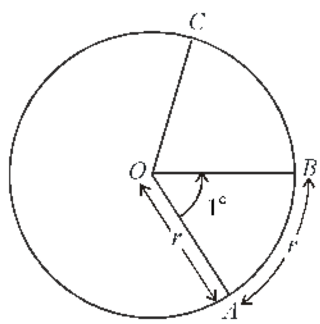


Fig. 13.08

Let C be any other point on the circumference of the circle.

Then

$$\frac{\angle AOC}{\angle AOB} = \frac{\text{arc } AC}{\text{arc } AB}$$

or $$\frac{\angle AOC}{\text{arc } AC} = \frac{\angle AOB}{\text{arc } AB}$$

$$\angle AOC = \frac{\text{arc } AC}{\text{arc } AB} \times \angle AOB$$

$$= \frac{\text{arc } AC}{\text{arc } AB} \times 1^c$$

$$= \frac{\text{arc } AC}{\text{arc } AB} \text{ radian} \quad \cdots(1)$$

If $\angle AOC = \theta^c$ (θ radian) and arc $AC = x$ then from equation (1)

$$\theta^c = \frac{x}{r} = \frac{\text{arc}}{\text{radius}} \text{ radian}$$

13.06 The Value of π

The ratio of the circumference and the diameter of a circle is always constant. This constant quantity is denoted by the Greek letter π upto 8 places of decimal is 3.14159265.

In fractional form, its value is considered as $\frac{22}{7}$.

13.07 Value of 1 radian

We know that

$$\pi = \frac{\text{circumference of the circle}}{\text{diameter of the circle}}$$

If the radius of the circle is r, its diameter is $2r$ and its circumference is $2\pi r$. We also known that there is a definite relationship between the length of an arc and the angle subtended by that arc at the centre form Fig. 13.08.

$$\frac{\angle AOB}{360°} = \frac{\text{arc } AB}{\text{circumference of the circle}}$$

or $$\frac{\angle AOB}{\text{arc } AB} = \frac{360°}{\text{circumference}}$$

or $$\frac{1^c}{r} = \frac{360°}{2\pi r}$$

or $$1^c = \frac{180°}{\pi} \quad \ldots(1)$$

or $\quad 1^c = 57°17'45''$ apporximately

(Taking $\pi = 3.1416$)

Thus we find that the value of radian does not depend on the radius of the circle. Hence the value of one radian is constant for all circles.

From equation (1)

$$\pi^c = 180°$$

In practice, we leave '*c*' and instead of writing π^c, we simply write π.

Hence $180° = \pi$.

Example 3. Convert 60° into radians.

Solution : We know that $180° = \pi$ radian

$\therefore \quad 1° = \frac{\pi}{180}$ radian

$\therefore \quad 60° = \frac{\pi}{3}$ radian.

Example 4. Convert $\frac{\pi}{4}$ radian into degrees.

Solution : We know that $1^c = \frac{180°}{\pi}$

$\therefore \quad \frac{\pi}{4} = \frac{180°}{4} = 45°$

Example 5. Find the length of the arc subtending an angle of $\frac{\pi}{3}$ radians at the centre of the circle. $\therefore$

Solution : If r be the radius of the circle and x be the length of the arc subtending an angle of $\frac{\pi}{3}$ radians at the centre, then

$$\frac{\pi}{3} = \frac{x}{r}$$

or $\quad x = \frac{\pi}{3} r.$

Example 6. Find the angle subtended by the whole of the circumference at the centre of the circle.

Solution : We known that the circumference of a circle of radius r is $2\pi r$ and the angle subtended by an arc of length r at the centre is 1 radian.

$\because$ Angle subtended at the centre by an arc of length r = 1 radian

$\therefore$ Angle subtended at the centre by an arc of length 1 $= \frac{1}{r}$ radian

$\therefore$ Angle subtended at the centre by an arc of length $2\pi r = \frac{1}{r} \times 2\pi r$ radian $= 2\pi$ radian.

Hence the whole circumference of a circle subtends an angle of 2π radians at the centre.

Example 7. How much time does the minute hand of a watch take to describe an angle of $\frac{3\pi}{2}$ radians.

Solution : We know that

4 right angles = 2π radians

The time taken by the minute hand of the watch to describe an angle of 2π radians = 1 hour

$\therefore$ The time for describing 1 radian $= \frac{1}{2\pi}$ hours

$\therefore$ Time for describing an angle of $\frac{3\pi}{2}$ radian $= \frac{1}{2\pi} \times \frac{3\pi}{2}$ hours

$= \frac{3}{4}$ hours $= \frac{3}{4} \times 60$ minutes $= 45$ minutes

Important Points

1. When the revolving ray revolves in anticlockwise direction, the angle thus formed is positive and if it revolves in clockwise direction, the angle so formed is negative.
2. The following system are used to measure angles :
   (i) Sexagesimal system
   (ii) Centesimal system
   (iii) Circular system.
3. In sexagesimal system, the unit of angle is 'degree'
   1 right angle $= 90°$, $1° = 60'$ and $1' = 60''$.
4. The angle subtended at the centre of the circle by an arc whose length is equal to its radius is called an angle of one radian.
5. The angle subtended at the centre by the circumference of the circle is 2π radians.
6. The relation between the sexagesimal and circular system is $D = \left(\frac{180°}{\pi}\right)R$, where D and R are the measures of the angle in degrees and radians respectively.

Miscellaneous Exercise 13

Objective questions [1-5]

1. The line describing an angle of $750°$, lies in :

   (A) First quadrant (B) Second quadrant

   (C) Third quadrant (D) Fourth quadrant

2. The number of radians in angle $30°$ is :

   (A) $\frac{\pi}{3}$ radian (B) $\frac{\pi}{4}$ radian

   (C) $\frac{\pi}{6}$ radian (D) $\frac{3\pi}{4}$ radian

3. The value of $\frac{3\pi}{4}$ in sexagesimal system is :

   (A) $75°$ (B) $135°$

   (C) $120°$ (D) $220°$

4. How much time the minute hand of a watch will take to describe an angle of $\frac{\pi}{6}$ radians :

   (A) 10 minutes (B) 20 minutes

   (C) 15 minutes (D) 5 minutes

5. The value of the angle, in radian, subtended at the centre of the circle of radius 100 meters by an arc of length 25π meters is :

   (A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$

   (C) $\frac{\pi}{6}$ (D) $\frac{3\pi}{4}$

6. In which quadrant does the revolving ray lie when it makes the following angles :

   (i) $240°$ (ii) $425°$ (iii) $-580°$

   (iv) $1280°$ (v) $-980°$

7. Convert the following angles in radians :

   (i) $45°$ (ii) $120°$

   (iii) $135°$ (iv) $540°$

8. Express the following angles in sexagesimal system :

(i) $\frac{\pi}{2}$ (ii) $\frac{2\pi}{5}$

(iii) $\frac{5}{6}\pi$ (iv) $\frac{\pi}{15}$

9. Find the angle in radians, subtended at the centre of a circle of radius 5 cm by an arc of the circle whose length is 12cm.

10. How much time the minute hand of a watch will take to describe an angle of $\frac{3\pi}{2}$ radians ?

11. How much time the minute hand of a watch will take to describe an angle of $120°$?

12. In a circle, the angle subtended at the centre by an arc of length 10 cm is $60°$. Find the radius of the circle.

13. If the minute hand of a watch has revolved through 30 right angles, just after the mid day, then what is the time by the watch ?

14. The angles of a triangle are in ratio 2 : 3 : 4. Find the all three angles in radians.

15. Express $\frac{3}{5}\pi$ into sexagesimal system.

Answers

Miscellaneous Exercise 13

1. (A) **2.** (C) **3.** (B)

4. (D) **5.** (A)

6. (i) Third (ii) First (iii) Second (iv) Third (v) Second

7. (i) $\frac{\pi}{4}$ (ii) $\frac{2\pi}{3}$ (iii) $\frac{3\pi}{4}$ (iv) 3π

8. (i) 90° (ii) 72° (iii) 150° (iv) 12°

9. $\frac{12}{5}$ radian

10. 45 minute

11. 20 minute

12. $\frac{30}{\pi}$ cm

13. 7.30 P.M.

14. $\frac{2\pi}{9}, \frac{\pi}{3}, \frac{4\pi}{9}$

15. 108°

❑