09

Applications of Derivatives

TOPIC 1

Rate of Change of Quantity, Approximation and Errors

01 If the surface area of a cube is increasing at a rate of $3.6 \text{ cm}^2/\text{sec}$, retaining its shape; then the rate of change of its volume (in cm^3/sec), when the length of a side of the cube is 10 cm, is **[2020, 3 Sep. Shift-II]**

(a) 18 (b) 10 (c) 9 (d) 20

Ans. *(c)*

Since, surface area of cube, $A = 6a^2 \text{ cm}^2$.

It is given, $\frac{dA}{dt} = 3.6 \text{ cm}^2/\text{sec}$

$$\Rightarrow \quad 12a\frac{da}{dt} = 3.6 \text{ cm}^2/\text{sec} \quad \text{...(i)}$$

Now, as volume of cube, $v = a^3 \text{ cm}^3$

$$\therefore \quad \frac{dv}{dt} = 3a^2\frac{da}{dt} = 3a^2\frac{3.6}{12a} \quad [\text{from Eq. (i)}]$$

So, at $a = 10$ cm, $\frac{dv}{dt} = 0.9 \times 10 = 9 \text{ cm}^3/\text{sec}$

Hence option (c) is correct.

02 The position of a moving car at time t is given by $f(t) = at^2 + bt + c$, $t > 0$, where a, b and c are real numbers greater than 1. Then, average speed of the car over the time interval $[t_1, t_2]$ is attained at the point **[2020, 6 Sep. Shift-I]**

(a) $(t_2 - t_1)/2$ (b) $a(t_2 - t_1) + b$
(c) $(t_1 + t_2)/2$ (d) $2a(t_1 + t_2) + b$

Ans. *(c)*

The average speed of the car, for time interval $[t_1, t_2]$ is

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1} = \frac{a(t_2^2 - t_1^2) + b(t_2 - t_1)}{t_2 - t_1}$$

$$= \frac{d(f(t))}{dt}$$

$$\therefore \quad 2at + b = a(t_2 + t_1) + b$$

$$\Rightarrow \quad t = \frac{t_1 + t_2}{2}$$

$\therefore$ The average speed of the car over the time interval $[t_1, t_2]$ is attained at the point $\frac{t_1 + t_2}{2}$.

03 A spherical iron ball of 10 cm radius is coated with a layer of ice of uniform thickness that melts at a rate of $50 \text{ cm}^3/\text{min}$. When the thickness of ice is 5 cm, then the rate (in cm/ min.) at which of the thickness of ice decreases, is **[2020, 9 Jan. Shift-I]**

(a) $\frac{5}{6\pi}$ (b) $\frac{1}{54\pi}$
(c) $\frac{1}{36\pi}$ (d) $\frac{1}{18\pi}$

Ans. *(d)*

It is given that, a spherical iron ball of 10 cm radius is coated with a layer of ice of uniform thickness, let the thickness is 'x' cm, then volume of the ball is

$$V = \frac{4}{3}\pi(10 + x)^3$$

On differentiating w.r.t. 't', we get

$$\frac{dV}{dt} = 4\pi(10 + x)^2\frac{dx}{dt}, \quad \text{... (i)}$$

where t is time in min.

It is given, the $\frac{dV}{dt} = -50 \text{ cm}^3/\text{min}$,

Now when x is 5 cm, then

$$-50 = 4\pi(10 + 5)^2\frac{dx}{dt} \quad [\text{from Eq. (i)}]$$

$$\Rightarrow \quad \frac{dx}{dt} = -\frac{50}{4\pi(225)} = -\frac{1}{18\pi} \text{ cm/min}$$

Negative sign indicates the thickness of ice layer decreases with time. Hence, option (d) is correct.

04 A spherical iron ball of radius 10 cm is coated with a layer of ice of uniform thickness that melts at a rate of $50 \text{ cm}^3/\text{min}$. When the thickness of the ice is 5 cm, then the rate at which the thickness (in cm/min) of the ice decreases, is **[2019, 10 April Shift-II]**

(a) $\frac{1}{9\pi}$ (b) $\frac{1}{18\pi}$
(c) $\frac{1}{36\pi}$ (d) $\frac{5}{6\pi}$

Ans. *(b)*

Let the thickness of layer of ice is x cm, the volume of spherical ball (only ice layer) is

$$V = \frac{4}{3}\pi[(10 + x)^3 - 10^3] \quad \text{...(i)}$$

On differentiating Eq. (i) w.r.t. 't', we get

$$\frac{dV}{dt} = \frac{4}{3}\pi(3(10 + x)^2)\frac{dx}{dt} = -50 \quad [\text{given}]$$

[– ve sign indicate that volume is decreasing as time passes].

$$\Rightarrow \quad 4\pi(10 + x)^2\frac{dx}{dt} = -50$$

At $x = 5$ cm

$$\frac{dx}{dt}[4\pi(10+5)^2] = -50$$

$$\Rightarrow \quad \frac{dx}{dt} = -\frac{50}{225(4\pi)}$$

$$= -\frac{1}{9(2\pi)} = -\frac{1}{18\pi} \text{ cm/min}$$

So, the thickness of the ice decreases at the rate of $\frac{1}{18\pi}$ cm/min.

04 A spherical balloon is filled with 4500π cu m of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cu m / min, then the rate (in m/min) at which the radius of the balloon decreases 49 min after the leakage began is **[AIEEE 2012]**

(a) $\frac{9}{7}$ (b) $\frac{7}{9}$

(c) $\frac{2}{9}$ (d) $\frac{9}{2}$

Ans. *(c)*

Given

(i) Volume ($V = 4500\ \pi\,\text{m}^3$/min) of the helium gas filled in a spherical balloon.

(ii) Due to a leak, the gas escapes the balloon at the rate of $72\ \pi\,\text{m}^3$/min.

$\therefore$ Rate of decrease of volume of the balloon is

$$\frac{dV}{dt} = -72\ \pi\,\text{m}^3/\text{min}$$

To find The rate of decrease of the radius of the balloon 49 min after the leakage started.

i.e., $\frac{dr}{dt}$ at $t = 49$ min

[assuming that the leakage started at time $t = 0$]

Now, the balloon is spherical in shape, hence the volume of the balloon is $V = \frac{4}{3}\pi r^3$.

On differentiating both sides w.r.t. t, we get

$$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2 \times \frac{dr}{dt}\right)$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{dV/dt}{4\pi r^2} \qquad \text{...(i)}$$

Now, to find $\frac{dr}{dt}$ at $t = 49$ min, we require $\frac{dV}{dt}$ and the radius (r) at that stage,

$$\frac{dV}{dt} = -72\ \pi\,\text{m}^3/\text{min}$$

$\therefore$ Amount of volume lost in 49 min

$$= 72\ \pi \times 49\,\text{m}^3$$

$\therefore$ Final volume at the end of 49 min

$$= (4500\ \pi - 3528\ \pi)\,\text{m}^3$$
$$= 972\ \pi\,\text{m}^3$$

If r is the radius at the end of 49 min, then

$$\frac{4}{3}\pi r^3 = 972\pi$$

$$\Rightarrow \quad r^3 = 729 \Rightarrow r = 9$$

$\therefore$ Radius of the balloon at the end of 49 min = 9 m

Hence, from Eq. (i), we get

$$\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2}$$

$$\Rightarrow \quad \left(\frac{dr}{dt}\right)_{t=49} = \frac{(dV/dt)_{t=49}}{4\pi(r^2)_{t=49}}$$

$$= \frac{72\pi}{4\pi(9^2)} = \frac{2}{9}\text{ m / min}$$

06 A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3/min. When the thickness of ice is 15 cm, then the rate at which the thickness of ice decreases, is **[AIEEE 2005]**

(a) $\frac{5}{6\pi}$ cm/min (b) $\frac{1}{54\pi}$ cm/min

(c) $\frac{1}{18\pi}$ cm/min (d) $\frac{1}{36\pi}$ cm/min

Ans. *(c)*

Given that, $\frac{dV}{dt} = 50\,\text{cm}^3$/min

$$\therefore \quad \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 50$$

$$\Rightarrow \quad 3r^2\frac{dr}{dt} = \frac{50 \times 3}{4\pi}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{50}{4\pi r^2}$$

$$\Rightarrow \quad \left(\frac{dr}{dt}\right)_{r=15} = \frac{50}{4\pi \times 225} = \frac{1}{18\pi}\text{ cm/min}$$

07 A lizard, at an initial distance of 21 cm behind an insect, moves from rest with an acceleration of 2 cm/s^2 and pursues the insect which is crawling uniformly along a straight line at a speed of 20 cm/s. Then, the lizard will catch the insect after **[AIEEE 2005]**

(a) 24 s (b) 21 s

(c) 1 s (d) 20 s

Ans. *(b)*

Let lizard catch the insect C.

And distance covered by insect $= S$

Time taken by insect, $t = \frac{S}{20}$ ...(i)

Distance covered by lizard $= 21 + S$

$$\therefore \quad 21 + S = \frac{1}{2}(2)\cdot t^2 \qquad \text{...(ii)}$$

$\left[\because S = ut + \frac{1}{2}at^2;\ \text{here } u = 0, a = 2\,\text{cm/s}^2 \text{ and } S = 20t\right]$

$$\Rightarrow \quad 21 + 20t = t^2 \qquad [\text{from Eq. (i)}]$$
$$\Rightarrow \quad t^2 - 20t - 21 = 0$$
$$\Rightarrow \quad t^2 - 21t + t - 21 = 0$$
$$\Rightarrow \quad t(t-21) + 1(t-21) = 0$$
$$\Rightarrow \quad (t+1)(t-21) = 0$$
$$\Rightarrow \quad t = -1, 21$$
$$\therefore \quad t = 21\text{ s} \quad [\because \text{neglecting } t = -1]$$

08 A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa, is **[AIEEE 2004]**

(a) (2, 4) (b) (2, – 4)

(c) $\left(-\frac{9}{8}, \frac{9}{2}\right)$ (d) $\left(\frac{9}{8}, \frac{9}{2}\right)$

Ans. *(d)*

Equation of parabola is $y^2 = 18x$.

On differentiating w.r.t. t, we get

$$2y\frac{dy}{dt} = 18\frac{dx}{dt}$$

$$\Rightarrow \quad 2\cdot 2y = 18 \quad \left[\because \frac{dy}{dt} = 2\frac{dx}{dt}, \text{given}\right]$$

$$\Rightarrow \quad y = \frac{9}{2}$$

From equation of parabola,

$$\left(\frac{9}{2}\right)^2 = 18x \Rightarrow \frac{81}{4} = 18x \Rightarrow x = \frac{9}{8}$$

Hence, required point is $\left(\frac{9}{8}, \frac{9}{2}\right)$.

TOPIC 2

Increasing and Decreasing Functions, Rolle's Theorem, Mean Value Theorem

09 The function $f(x) = x^3 - 6x^2 + ax + b$ is such that $f(2) = f(4) = 0$. Consider two statements.
(S_1) there exists $x_1, x_2 \in (2,4), x_1 < x_2$, such that $f'(x_1) = -1$ and $f'(x_2) = 0$. (S_2) there exists $x_3, x_4 \in (2,4)$, $x_3 < x_4$, such

that f is decreasing in $(2, x_4)$, increasing in $(x_4, 4)$ and $2f'(x_3)=\sqrt{3}f(x_4)$. Then,

[2021, 01 Sep. Shift-II]

(a) both (S_1) and (S_2) are true

(b) (S_1) is false and (S_2) is true

(c) both (S_1) and (S_2) are false

(d) (S_1) is true and (S_2) is false

Ans. *(a)*

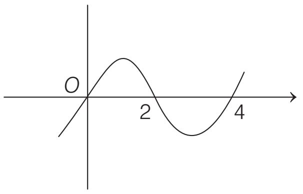
$f(x)=x^3-6x^2+ax+b$

$\because \quad f(2)=0 \Rightarrow 2a+b=16$

and $\quad f(4)=0 \Rightarrow 4a+b=32$

On solving $a=8, b=0$

$\therefore \quad f(x)=x^3-6x^2+8x=x(x-2)(x-4)$



$f'(x)=3x^2-12x+8$

$f'(x)=0 \Rightarrow x=2\pm\dfrac{2}{\sqrt{3}}$

$\because \quad f'(x_2)=0$ and $x_2\in(2,4)$

$\Rightarrow \quad x_2=2+\dfrac{2}{\sqrt{3}}$ and $f'(x_1)=-1$

$\Rightarrow \quad 3x_1^2-12x_1+8=0$

$\Rightarrow \quad x_1=1,3\ (S_1$ is true)

Now, $2(3x^2-12x+8)$

$=\sqrt{3}\left(2+\dfrac{2}{\sqrt{3}}\right)\left(\dfrac{2}{\sqrt{3}}\right)\left(\dfrac{2}{\sqrt{3}}-2\right)$

$\Rightarrow \quad x=\dfrac{8}{3},\dfrac{4}{3}\ (S_2$ is true)

10 The number of real roots of the equation $e^{4x}+2e^{3x}-e^x-6=0$ is

[2021, 31 Aug. Shift-I]

(a) 2 (b) 4 (c) 1 (d) 0

Ans. *(c)*

$f(x)=e^{4x}+2e^{3x}-e^x-6$

$f'(x)=4e^{4x}+6e^{3x}-e^x$

$=e^x(4e^{3x}+6e^{2x}-1)$

Let $g(x)=e^x$

$h(x)=4e^{3x}+6e^{2x}-1$

$g(x)>0, \forall x\in R$

$h'(x)=12e^{3x}+12e^{2x}=12e^{2x}(e^x+1)$

$h'(x)>0, \forall x\in R$

$h(x)$ is an increasing function.

Minimum value of $h(x)$ will be when $x\to-\infty$ at $[h(x)]_{\min}=-1$ and $[h(x)]_{\max}=\infty$

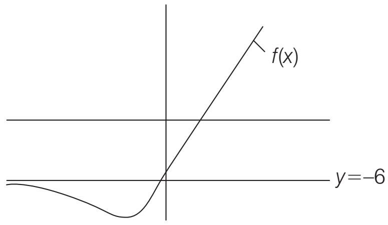
$f'(x)=g(x)\cdot h(x)$

Now, $h(x)$ is an increasing function and $h(x)$ varies from -1 to $+\infty$. So, this implies that $h(x)$ cuts the X-axis at one point and which further implies $f(x)$ changes its sign only at one point. Let's say at $x=\alpha$

$f(x)=e^{4x}+2e^{3x}-e^x-6$

When, $x\to-\infty; f(x)\to-6$

$x\to+\infty; f(x)\to+\infty$



So, $f(x)$ cuts the X-axis at a single point.

11 If R is the least value of a such that the function $f(x)=x^2+ax+1$ is increasing on $[1, 2]$ and S is the greatest value of a such that the function $f(x)=x^2+ax+1$ is decreasing on $[1, 2]$, then the value of $|R-S|$ is

[2021, 31 Aug. Shift-I]

Ans. *(2)*

$f(x)=x^2+ax+1$

$f'(x)=2x+a$

According to the question, $f'(x)\ge 0$ for $x\in[1,2]$

For the least value $2x+a\ge 0$

$\Rightarrow \quad a\ge-2x \Rightarrow a\ge-2 \Rightarrow R=-2$

For the greatest value $2x+a\le 0$

$\Rightarrow \quad a\le-2x \quad \{x\in[1,2]\}$

$\Rightarrow \quad a\le-4$

$\Rightarrow \quad S=-4$

$|R-S|=|-2+4|=2$

12 Let f be any continuous function on $[0, 2]$ and twice differentiable on $(0, 2)$. If $f(0)=0$, $f(1)=1$ and $f(2)=2$, then **[2021, 31 Aug. Shift-II]**

(a) $f''(x)=0$ for all $x\in(0, 2)$

(b) $f''(x)=0$ for some $x\in(0, 2)$

(c) $f'(x)=0$ for some $x\in[0, 2]$

(d) $f''(x)>0$ for all $x\in(0, 2)$

Ans. *(b)*

$f(0)=0, f(1)=1$ and $f(2)=2$

Let $h(x)=f(x)-x$

Clearly $h(x)$ is continuous and twice differentiable on $(0,2)$

Also, $h(0)=h(1)=h(2)=0$

$\therefore \ h(x)$ satisfies all the condition of Rolle's theorem.

$\therefore$ There exist $C_1\in(0,1)$ such that $h'(c_1)=0$

$\Rightarrow \quad f'(c_1)-1=0$

$\Rightarrow \quad f'(c_1)=1$

also there exist $c_2\in(1,2)$ such that $h'(c_2)=0$

$\Rightarrow \quad f'(c_2)=1$

Now, using Rolle's theorem on $[c_1, c_2]$ for $f'(x)$

We have $f''(c)=0, c\in(c_1, c_2)$

Hence, $f''(x)=0$ for some $x\in(0,2)$.

13 Let $f(x)=3\sin^4 x+10\sin^3 x+6\sin^2 x-3$ $x\in\left[-\dfrac{\pi}{6},\dfrac{\pi}{2}\right]$. Then, f is

[2021, 25 July Shift-I]

(a) increasing in $\left(-\dfrac{\pi}{6},\dfrac{\pi}{2}\right)$

(b) decreasing in $\left(0,\dfrac{\pi}{2}\right)$

(c) increasing in $\left(-\dfrac{\pi}{6},0\right)$

(d) decreasing in $\left(-\dfrac{\pi}{6},0\right)$

Ans. *(d)*

$f(x)=3\sin^4 x+10\sin^3 x+6\sin^2 x-3$

$\Rightarrow f'(x)$

$=\cos x(12\sin^3 x+30\sin^2 x+12\sin x)$

$\Rightarrow f'(x)=6\sin x\cos x(2\sin^2 x+5\sin x+2)$

$\Rightarrow f'(x)=3\sin 2x(2\sin x+1)(\sin x+2)$

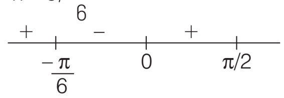
$f'(x)=0$

$\sin 2x=0$ or $2\sin x+1=0 \quad [\because \sin x\ne-2]$

$\Rightarrow \quad x=0$ or $x=n\pi+(-1)^n\left(-\dfrac{\pi}{6}\right)$

As, $x\in\left[\dfrac{-\pi}{6},\dfrac{\pi}{2}\right]$

$x=0,\dfrac{-\pi}{6}$



So, $f(x)$ is increasing in the interval

$x\in\left(0,\dfrac{\pi}{2}\right)$

And $f(x)$ is decreasing in the interval

$x\in\left(-\dfrac{\pi}{6},0\right)$

14 Let $f: R\to R$ be defined as

$$f(x)=\begin{cases}-\dfrac{4}{3}x^3+2x^2+3x, & x>0\\ 3xe^x, & x\le 0\end{cases}.$$

Then, f is increasing function in the interval **[2021, 22 July Shift-II]**

(a) $\left(-\dfrac{1}{2},2\right)$ (b) $(0, 2)$

(c) $\left(-1,\dfrac{3}{2}\right)$ (d) $(-3,-1)$

Ans. *(c)*

$$f(x)=\begin{cases}\frac{-4}{3}x^3+2x^2+3x & ,x>0\\ 3xe^x & ,x\le 0\end{cases}$$

$$f'(x)=\begin{cases}-4x^2+4x+3 & ,x>0\\ 3e^x(x+1) & ,x\le 0\end{cases}$$

$$f'(x)=\begin{cases}-(2x+1)(2x-3) & ,x>0\\ 3e^x(x+1) & ,x\le 0\end{cases}$$

$f'(x)<0$ $\quad$ $f'(x)>0$ $\quad$ $f'(x)>0$ $\quad$ $f'(x)<0$

-1 $\quad$ $1/2$ $\quad$ 0 $\quad$ $3/2$

$$f'(x)>0\ \Rightarrow\ x\in\left(-1,\frac{3}{2}\right)$$

15 Let f be a real valued function, defined on $R-\{-1,1\}$ and given by

$$f(x)=3\log_e\left|\frac{x-1}{x+1}\right|-\frac{2}{x-1}.$$

Then, in which of the following intervals, function $f(x)$ is increasing? **[2021, 16 March Shift-II]**

(a) $(-\infty,-1)\cup\left(\left[\frac{1}{2},\infty\right)-\{1\}\right)$

(b) $(-\infty,\infty)-\{-1,1\}$

(c) $\left(-1,\frac{1}{2}\right]$

(d) $\left(-\infty,\frac{1}{2}\right]-[-1]$

Ans. *(a)*

Given, $f(x)=3\log_e\left|\frac{x-1}{x+1}\right|-\frac{2}{x-1}$

$$\Rightarrow\quad f'(x)=\frac{3}{\left(\frac{x-1}{x+1}\right)}\cdot\frac{(x+1)\cdot 1-(x-1)\cdot 1}{(x+1)^2}+\frac{2}{(x-1)^2}$$

$$\Rightarrow\quad f'(x)=3\left(\frac{x+1}{x-1}\right)\left[\frac{2}{(x+1)^2}\right]+\frac{2}{(x-1)^2}$$

$$\Rightarrow\quad f'(x)=\left(\frac{2}{x-1}\right)\left(\frac{3}{x+1}+\frac{1}{x-1}\right)$$

$$\Rightarrow\quad f'(x)=\left(\frac{2}{x-1}\right)\left[\frac{3x-3+x+1}{(x-1)(x+1)}\right]$$

$$\Rightarrow\quad f'(x)=\frac{2\cdot 2\cdot(2x-1)}{(x-1)^2(x+1)}$$

So, $f'(x)=\dfrac{4(2x-1)}{(x-1)^2\,(x+1)}$

Now,

$+$ $\quad$ $-$ $\quad$ $+$ $\quad$ $+$

-1 $\quad$ $1/2$ $\quad$ 1

For $f(x)$ to be an increasing function, $f'(x)>0$.

And $f'(x)>0$ at $x\in(-\infty,-1]\cup\left[\frac{1}{2},\infty\right)$

But domain of $f(x)$ is $x\in(-\infty,-1)\cup(1,\infty)$

So, $f'(x)>0$ at $x\in(-\infty,-1)\cup\left[\frac{1}{2},1\right)\cup(1,\infty)$

or $x\in(-\infty,-1)\cup\left(\left[\frac{1}{2},\infty\right)-\{1\}\right)$

16 Let a be an integer, such that all the real roots of the polynomial $2x^5+5x^4+10x^3+10x^2+10x+10$ lie in the interval $(a,a+1)$. Then, $|a|$ is equal to ______. **[2021, 26 Feb. Shift-II]**

Ans. *(2)*

Let

$f(x)=2x^5+5x^4+10x^3+10x^2+10x+10$

Using hit and trial method,

$f(-2)=-34<0$ and $f(-1)=3>0$

Hence, $f(x)$ has a root in $(-2,-1)$.

Again,

$$f'(x)=10x^4+20x^3+30x^2+20x+10$$

$$=10x^2\left(x^2+2x+3+\frac{2}{x}+\frac{1}{x^2}\right)$$

$$=10x^2\left[\left(x^2+\frac{1}{x^2}\right)+2\left(x+\frac{1}{x}\right)+3\right]$$

$$=10x^2\left[\left(x+\frac{1}{x}\right)^2+1+2\left(x+\frac{1}{x}\right)\right]$$

$$=10x^2\left[\left(x+\frac{1}{x}+1\right)^2\right]>0,\ \forall x$$

$\Rightarrow f(x)$ is strictly increasing function, since degree of $f(x)$ is odd.

$\therefore$ It has exactly on real root.

Therefore, $f(x)$ has atleast one root in

$(-2,-1)\qquad=(a,a+1)$

$\Rightarrow\qquad |a|=|-2|=2$

17 If Rolle's theorem holds for the function $f(x)=x^3-ax^2+bx+4$, $x\in[1,2]$ with $f'\left(\frac{4}{3}\right)=0$, then ordered pair (a,b) is equal to **[2021, 25 Feb. Shift-I]**

(a) $(5,8)$ $\qquad$ (b) $(-5,8)$

(c) $(5,-8)$ $\qquad$ (d) $(-5,-8)$

Ans. *(a)*

Given, $f(x)=x^3-ax^2+bx+4$, $x\in[1,2]$

Here, $f(1)=f(2)$

$\Rightarrow\qquad 1-a+b-4=8-4a+2b-4$

$\Rightarrow\qquad 3a-b=7\qquad$ ... (i)

Also, $f'(x)=3x^2-2ax+b$

According to the question, $f'\left(\frac{4}{3}\right)=0$

$$\Rightarrow\qquad 3\times\left(\frac{4}{3}\right)^2-2a\left(\frac{4}{3}\right)+b=0$$

$\Rightarrow\qquad -8a+3b=-16\qquad$ ... (ii)

From Eqs. (i) and (ii),

$$a=5,b=8$$

$\therefore\quad (a,b)=(5,8)$

18 Let $f:R\to R$ be defined as

$$f(x)=\begin{cases}-55x, & \text{if}\quad x<-5\\ 2x^3-3x^2-120x, & \text{if}\quad -5\le x\le 4\\ 2x^3-3x^2-36x-336, & \text{if}\quad x>4\end{cases}$$

Let $A=\{x\in R: f \text{ is increasing}\}$. Then, A is equal to **[2021, 24 Feb. Shift-II]**

(a) $(-\infty,-5)\cup(4,\infty)$ $\qquad$ (b) $(-5,\infty)$

(c) $(-\infty,-5)\cup(-4,\infty)$ $\qquad$ (d) $(-5,-4)\cup(4,\infty)$

Ans. *(d)*

Given,

$$f(x)=\begin{cases}-55x & , & x<-5\\ 2x^3-3x^2-120x & , & -5\le x<4\\ 2x^3-3x^2-36x+10 & , & x\ge 4\end{cases}$$

$$\therefore\quad f'(x)=\begin{cases}-55 & , & x<-5\\ 6(x^2-x-20) & , & -5\le x<4\\ 6(x^2-x-6) & , & x\ge 4\end{cases}$$

$$f'(x)=\begin{cases}-55 & , & x<-5\\ 6(x-5)(x+4) & , & -5\le x<4\\ 6(x-3)(x+2) & , & x\ge 4\end{cases}$$

For f to be increasing, $f'(x)>0$.

Now, $f'(x)=-55$ is always less than zero.

$f'(x)=6(x-5)(x+4)<0,\ -5\le x<4$

Critical points $=5,-4$

$-\infty$ $\quad$ $+$ $\quad$ $-$ $\quad$ $+$ $\quad$ $+\infty$

-5 $\quad$ -4 $\quad$ 4 $\quad$ 5

$x\in(-5,-4)\qquad$ ...(i)

and $f'(x)=6(x-3)(x+2)<0,\ x\ge 4$

Critical point, $=3,-2$

$-\infty$ $\quad$ $+$ $\quad$ $-$ $\quad$ $+$ $\quad$ $+\infty$

-2 $\quad$ 3 $\quad$ 4

$x\in(4,\infty)\qquad$ ... (ii)

From Eqs. (i) and (ii), $f(x)$ is increasing in $x\in(-5,-4)\cup(4,\infty)$.

19 The function

$$f(x)=\frac{4x^3-3x^2}{6}-2\sin x+(2x-1)\cos x$$

[2021, 24 Feb. Shift-I]

(a) increases in $\left[\frac{1}{2},\infty\right)$

(b) decreases in $\left[\frac{1}{2},\infty\right)$

(c) increases in $\left(-\infty,\frac{1}{2}\right]$

(d) decreases in $\left(-\infty,\frac{1}{2}\right]$

Ans. *(a)*

Given,

$f(x)=\frac{4x^3-3x^2}{6}-2\sin x+(2x-1)\cos x$

$f'(x)=\frac{12x^2-6x}{6}-2\cos x+(2x-1)$

$(-\sin x)+\cos x(2)$

$=(2x^2-x)-2\cos x-2x\sin x$

$+\sin x+2\cos x$

$=2x^2-x-2x\sin x+\sin x$

$=2x(x-\sin x)-1(x-\sin x)$

$f'(x)=(2x-1)(x-\sin x)$

for $x>0$

$x-\sin x>0$

$x<0, x-\sin x<0$

for $x\in(-\infty,0]\cup\left[\frac{1}{2},\infty\right), f'\geq 0$

for $x\in\left[0,\frac{1}{2}\right], f'(x)\leq 0$

Hence, $f(x)$ increases in $\left[\frac{1}{2},\infty\right)$.

20 Let $f:(-1,\infty)\to R$ be defined by $f(0)=1$ and $f(x)=\frac{1}{x}\log_e(1+x), x\neq 0$. Then the function f

[2020, 2 Sep. Shift-II]

(a) decreases in $(-1,0)$ and increases in $(0,\infty)$
(b) increases in $(-1,\infty)$
(c) increases in $(-1,0)$ and decreases in $(0,\infty)$
(d) decreases in $(-1,\infty)$

Ans. *(d)*

Given function $f(x)=\begin{cases}\frac{1}{x}\log_e(1+x), & x\neq 0\\ 1, & x=0\end{cases}$

for $x\in(-1,\infty)$

Now, $f'(x)=\frac{1}{x(1+x)}-\frac{\log_e(1+x)}{x^2}$,

for $x\in(-1,\infty)-\{0\}=\frac{x-(1+x)\log_e(1+x)}{x^2(1+x)}$

Let another function

$g(x)=x-(1+x)\log_e(1+x)$

$\because\ g'(x)=1-1-\log_e(1+x)=-\log_e(1+x)$

Since, for $x\in(-1,0)$, $g'(x)>0$, So $g(x)$ is increasing function for $x\in(-1,0)$ but as

$g(x)<g(0), \forall x\in(-1,0)$

$\therefore\quad g(x)<0,\ \forall x\in(-1,0)$

$\therefore\quad f'(x)=\frac{g(x)}{x^2}<0$

$\Rightarrow f(x)$ is decreasing function for $x\in(-1,0)$.

Similarly, for $x\in(0,\infty)$, $g'(x)<0$, so $g(x)$ is decreasing function for $x\in(0,\infty)$.

So, $\quad g(x)<g(0)$

$\Rightarrow\quad g(x)<0, \forall x\in(0,\infty)$

$\therefore\quad f'(x)=\frac{g(x)}{x^2}<0$

$\Rightarrow f(x)$ is decreasing function for $x\in(0,\infty)$.

$\therefore$ The given function $f(x)$ is decreasing function for $(-1,\infty)$.

Hence, option (d) is correct.

21 The function, $f(x)=(3x-7)x^{2/3}$, $x\in R$, is increasing for all x lying in

[2020, 3 Sep. Shift-I]

(a) $(-\infty,0)\cup\left(\frac{14}{15},\infty\right)$
(b) $(-\infty,0)\cup\left(\frac{3}{7},\infty\right)$
(c) $\left(-\infty,\frac{14}{15}\right)$
(d) $\left(-\infty,-\frac{14}{15}\right)\cup(0,\infty)$

Ans. *(a)*

Since, the given function $f(x)=(3x-7)x^{2/3}$ is increasing for $x\in R$

$\therefore\ f'(x)\geq 0$

$\Rightarrow\ \frac{2}{3}x^{-1/3}(3x-7)+x^{2/3}(3)\geq 0, x\neq 0$

$\Rightarrow\ \frac{2(3x-7)+9x}{x^{1/3}}\geq 0, x\neq 0$

$\Rightarrow\ \frac{15x-14}{x^{1/3}}\geq 0\Rightarrow x\in(-\infty,0)\cup\left(\frac{14}{15},\infty\right)$

$\because\ (-\infty,0)\cup\left(\frac{14}{15},\infty\right)\subset(-\infty,0)\cup\left(\frac{14}{15},\infty\right)$

Hence, option (a) is correct.

22 Let the function, $f:[-7,0]\to R$ be continuous on $[-7,0]$ and differentiable on $(-7,0)$. If $f(-7)=-3$ and $f'(x)\leq 2$, for all $x\in(-7,0)$, then for all such functions f, $f(-1)+f(0)$ lies in the interval

[2020, 7 Jan. Shift-I]

(a) $(-\infty,11]$ (b) $[-3,11]$
(c) $[-6,20]$ (d) $(-\infty,20]$

Ans. *(d)*

If function $f:[-7,0]\to R$ be continuous on $[-7,0]$ and differentiable on $(-7,0)$, then according to LMVT, we have

$\frac{f(-1)-f(-7)}{(-1)-(-7)}=f'(x)\leq 2$,

$\forall x\in(-7,-1)$

$\Rightarrow\quad \frac{f(-1)-(-3)}{6}\leq 2$

$\Rightarrow\quad f(-1)\leq 9 \quad$ ...(i)

Similarly, $\frac{f(0)-f(-7)}{0-(-7)}=f'(x)\leq 2$,

$\forall x\in(-7,0)$

$\Rightarrow\quad \frac{f(0)-(-3)}{7}\leq 2$

$\Rightarrow\quad f(0)\leq 11 \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$f(-1)+f(0)\leq 20$

$\therefore f(-1)+f(0)$ lies in the interval $(-\infty,20]$.

23 The value of c in the Lagrange's mean value theorem for the function $f(x)=x^3-4x^2+8x+11$, when $x\in[0,1]$ is **[2020, 7 Jan. Shift-II]**

(a) $\frac{\sqrt{7}-2}{3}$ (b) $\frac{2}{3}$
(c) $\frac{4-\sqrt{5}}{3}$ (d) $\frac{4-\sqrt{7}}{3}$

Ans. *(d)*

Given function $f(x)=x^3-4x^2+8x+11$, when $x\in[0,1]$ is a continuous function in interval $x\in[0,1]$ and differentiable in interval $x\in(0,1)$, so according to Lagrange's mean value theorem for $x=c\in(0,1)$

$f'(c)=\frac{f(1)-f(0)}{1-0}$

$\Rightarrow(3x^2-8x+8)_{x=c}=\frac{(1-4+8+11)-11}{1-0}$

$\Rightarrow\quad 3c^2-8c+8=5$

$\Rightarrow\quad 3c^2-8c+3=0$

$\Rightarrow\quad c=\frac{8-\sqrt{64-36}}{6}=\frac{4-\sqrt{7}}{3}\quad [\because c\in(0,1)]$

24 If c is a point at which Rolle's theorem holds for the function, $f(x)=\log_e\left(\frac{x^2+\alpha}{7x}\right)$ in the interval $[3,4]$, where $\alpha\in R$, then $f''(c)$ is equal to **[2020, 8 Jan. Shift-I]**

(a) $-\frac{1}{24}$ (b) $-\frac{1}{12}$ (c) $\frac{1}{12}$ (d) $\frac{\sqrt{3}}{7}$

Ans. *(c)*

The given function $f(x)=\log_e\left(\frac{x^2+\alpha}{7x}\right)$ holds the Rolle's theorem for the interval $[3,4]$, so

$f(3)=f(4)$

$\Rightarrow\quad \log_e\left(\frac{9+\alpha}{21}\right)=\log_e\left(\frac{16+\alpha}{28}\right)$

$\Rightarrow\quad \frac{9+\alpha}{3}=\frac{16+\alpha}{4}$

$\Rightarrow\quad 36+4\alpha=48+3\alpha$

$\Rightarrow\quad \alpha=12 \quad$ ...(i)

and $f'(c)=0$, for some $c\in(3,4)$

$\Rightarrow\left[\frac{7x}{x^2+\alpha}\times\frac{7x(2x+0)-(x^2+\alpha)7}{(7x)^2}\right]_{x=c}=0$

$$\Rightarrow \quad \frac{c(2c)-(c^2+12)}{(c^2+12)c}=0$$

$$\Rightarrow \quad c^2-12=0 \qquad \because c\in(3,4)$$

$$\Rightarrow \quad c=\sqrt{12} \qquad \because c\in(3,4)$$

$\therefore \; f''(c=\sqrt{12})$

$$=\frac{c(c^2+12)(2c)-(c^2-12)(3c^2+12)}{((c^2+12)c)^2}$$

$$=\frac{(2\times12\times24)-(0\times48)}{(24)^2(12)}=\frac{1}{12}$$

Hence, option (c) is correct.

25 Let $f(x)=x\cos^{-1}(-\sin|x|)$, $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, then which of the following is true?

[2020, 8 Jan. Shift-I]

(a) f' is decreasing in $\left(-\frac{\pi}{2},0\right)$ and increasing in $\left(0,\frac{\pi}{2}\right)$

(b) f' is increasing in $\left(-\frac{\pi}{2},0\right)$ and decreasing in $\left(0,\frac{\pi}{2}\right)$

(c) f is not differentiable at $x=0$

(d) $f'(0)=-\frac{\pi}{2}$

Ans. *(a)*

Given function

$$f(x)=x\cos^{-1}(-\sin|x|),\; x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$$=x(\pi-\cos^{-1}(\sin|x|)) \qquad \{\because \cos^{-1}(-x)=\pi-\cos^{-1}x\}$$

$$=x\left[\pi-\left(\frac{\pi}{2}-\sin^{-1}\sin|x|\right)\right] \qquad \left\{\because \cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x\right\}$$

$$=x\left[\frac{\pi}{2}+|x|\right] \qquad \{\because \sin^{-1}\sin x=x\}$$

$$=\begin{cases} x\left(\frac{\pi}{2}-x\right), & x\in\left(-\frac{\pi}{2},0\right)\\ x\left(\frac{\pi}{2}+x\right), & x\in\left(0,\frac{\pi}{2}\right)\end{cases}$$

$$\text{So, } f'(x)=\begin{cases} \frac{\pi}{2}-2x, & x\in\left(-\frac{\pi}{2},0\right)\\ \frac{\pi}{2}+2x, & x\in\left(0,\frac{\pi}{2}\right)\end{cases}$$

$$\Rightarrow f''(x)=\begin{cases} -2, & x\in\left(-\frac{\pi}{2},0\right)\\ 2, & x\in\left(0,\frac{\pi}{2}\right)\end{cases}$$

$\therefore f'$ is decreasing in $\left(-\frac{\pi}{2},0\right)$ and increasing in $\left(0,\frac{\pi}{2}\right)$.

Hence, option (a) is correct.

26 Let S be the set of all functions $f:[0,1]\to R$, which are continuous on $[0,1]$ and differentiable on $(0,1)$. Then for every f in S, there exists a $c\in(0,1)$, depending on f, such that

[2020, 8 Jan. Shift-II]

(a) $\frac{f(1)-f(c)}{1-c}=f'(c)$

(b) $|f(c)-f(1)|<|f'(c)|$

(c) $|f(c)+f(1)|<(1+c)|f'(c)|$

(d) $|f(c)-f(1)|<(1-c)|(f'(c)|$

Ans. *(*)*

Since, the functions $f:[0,1]\to R$ which are continuous on $[0,1]$ and differentiable on $(0,1)$.

If f is a constant function then options (b), (c) and (d) are incorrect.

According to LMVT, for $c'\in(C,1)$

$$f'(c')=\frac{f(1)-f(c)}{1-c}$$

but $c'\neq c$, so option (a) is also incorrect.

27 Let f be any function continuous on $[a,b]$ and twice differentiable on (a,b). If for all $x\in(a,b)$, $f'(x)>0$ and $f''(x)<0$, then for any $c\in(a,b)$, $\frac{f(c)-f(a)}{f(b)-f(c)}$ is greater than

[2020, 9 Jan. Shift-I]

(a) $\frac{b-c}{c-a}$ (b) 1 (c) $\frac{c-a}{b-c}$ (d) $\frac{b+a}{b-a}$

Ans. *(c)*

It is given that a function f is continuous on $[a,b]$ and twice differentiable on (a,b), such that for all $x\in(a,b)$, $f'(x)>0$ and $f''(x)<0$. Now, by LMVT for $c\in(a,b)$, there is $\alpha\in(a,c)$, such that

$$f'(\alpha)=\frac{f(c)-f(a)}{c-a} \qquad \text{... (i)}$$

and there is $\beta\in(c,b)$, such that

$$f'(\beta)=\frac{f(b)-f(c)}{b-c} \qquad \text{... (ii)}$$

$\because f''(x)<0\;\forall x\in(a,b)$, then $f'(x)$ is a decreasing function, so

$$f'(\beta)<f'(\alpha) \qquad [\because \alpha<\beta]$$

From Eqs. (i) and (ii), on putting the values of $f'(\alpha)$ and $f'(\beta)$, we get

$$\frac{f(b)-f(c)}{b-c}<\frac{f(c)-f(a)}{c-a}$$

$$\Rightarrow \quad \frac{f(c)-f(a)}{f(b)-f(c)}>\frac{c-a}{b-c}$$

$[\because f'(x)>0$, so $f(x)$ is an increasing function $\forall x\in(a,b)$ and $a<c<b]$

Hence, option (c) is correct.

28 Let $f:[0,2]\to R$ be a twice differentiable function such that $f''(x)>0$, for all $x\in(0,2)$. If $\phi(x)=f(x)+f(2-x)$, then ϕ is

[2019, 8 April Shift-I]

(a) increasing on $(0,1)$ and decreasing on $(1,2)$

(b) decreasing on $(0,2)$

(c) decreasing on $(0,1)$ and increasing on $(1,2)$

(d) increasing on $(0,2)$

Ans. *(c)*

Given, $\phi(x)=f(x)+f(2-x),\;\forall x\in(0,2)$

$$\Rightarrow \quad \phi'(x)=f'(x)-f'(2-x) \qquad \text{...(i)}$$

Also, we have $f''(x)>0\;\forall x\in(0,2)$

$\Rightarrow$ $f'(x)$ is a strictly increasing function $\forall x\in(0,2)$.

Now, for $\phi(x)$ to be increasing,

$$\phi'(x)\geq0$$

$$\Rightarrow \quad f'(x)-f'(2-x)\geq0 \qquad [\text{using Eq. (i)}]$$

$$\Rightarrow f'(x)\geq f'(2-x)\Rightarrow x>2-x$$

$[\because f'$ is a strictly increasing function$]$

$$\Rightarrow \quad 2x>2\Rightarrow x>1$$

Thus, $\phi(x)$ is increasing on $(1,2)$.

Similarly, for $\phi(x)$ to be decreasing,

$$\phi'(x)\leq0$$

$$\Rightarrow \quad f'(x)-f'(2-x)\leq0 \qquad [\text{using Eq. (i)}]$$

$$\Rightarrow f'(x)\leq f'(2-x)$$

$$\Rightarrow \quad x<2-x$$

$[\because f'$ is a strictly increasing function$]$

$$\Rightarrow \quad 2x<2\Rightarrow x<1$$

Thus, $\phi(x)$ is decreasing on $(0,1)$.

29 Let $f(x)=e^x-x$ and $g(x)=x^2-x$, $\forall x\in R$. Then, the set of all $x\in R$, where the function $h(x)=(fog)(x)$ is increasing, is **[2019, 10 April Shift-I]**

(a) $\left[0,\frac{1}{2}\right]\cup[1,\infty)$ (b) $\left[-1,\frac{-1}{2}\right]\cup\left[\frac{1}{2},\infty\right)$

(c) $[0,\infty)$ (d) $\left[\frac{-1}{2},0\right]\cup[1,\infty)$

Ans. *(a)*

The given functions are

$$f(x)=e^x-x \text{ and } g(x)=x^2-x,\;\forall x\in R$$

Then, $h(x)=(fog)(x)=f(g(x))$

Now, $h'(x)=f'(g(x))\cdot g'(x)$

$$=(e^{g(x)}-1)\cdot(2x-1)$$

$$=(e^{(x^2-x)}-1)(2x-1)$$

$$=(e^{x(x-1)}-1)(2x-1)$$

$\because$ It is given that $h(x)$ is an increasing function, so $h'(x)\geq0$

$\Rightarrow \quad (e^{x(x-1)}-1)(2x-1)\ge 0$

Case I $(2x-1)\ge 0$ and $(e^{x(x-1)}-1)\ge 0$

$\Rightarrow \quad x\ge \frac{1}{2}$ and $x(x-1)\ge 0$

$\Rightarrow x\in[1/2,\infty)$ and $x\in(-\infty,0]\cup[1,\infty)$, so $x\in[1,\infty)$

Case II $(2x-1)\le 0$ and $[e^{x(x-1)}-1]\le 0$

$\Rightarrow \quad x\le \frac{1}{2}$ and $x(x-1)\le 0$

$\Rightarrow \quad x\in\left(-\infty,\frac{1}{2}\right]$ and $x\in[0,1]$ So, $x\in\left[0,\frac{1}{2}\right]$

From, the above cases, $x\in\left[0,\frac{1}{2}\right]\cup[1,\infty)$.

30 If m is the minimum value of k for which the function $f(x)=x\sqrt{kx-x^2}$ is increasing in the interval $[0,3]$ and M is the maximum value of f in the interval $[0,3]$ when $k=m$, then the ordered pair (m,M) is equal to **[2019, 12 April Shift-I]**

(a) $(4,3\sqrt{2})$ (b) $(4,3\sqrt{3})$

(c) $(3,3\sqrt{3})$ (d) $(5,3\sqrt{6})$

Ans. ***(b)***

Given function, $f(x)=x\sqrt{kx-x^2}$ ... (i)

the function $f(x)$ is defined if

$$kx-x^2\ge 0$$

$\Rightarrow \quad x^2-kx\le 0$

$\Rightarrow \quad x\in[0,k]$ ... (ii)

because it is given that $f(x)$ is increasing in interval $x\in[0,3]$, so k should be positive.

Now, on differentiating the function $f(x)$ w.r.t. x, we get

$$f'(x)=\sqrt{kx-x^2}+\frac{x}{2\sqrt{kx-x^2}}\times(k-2x)$$

$$=\frac{2(kx-x^2)+kx-2x^2}{2\sqrt{kx-x^2}}=\frac{3kx-4x^2}{2\sqrt{kx-x^2}}$$

as $f(x)$ is increasing in interval $x\in[0,3]$, so

$$f'(x)\ge 0\ \forall\, x\in(0,3)$$

$\Rightarrow \quad 3kx-4x^2\ge 0$

$\Rightarrow \quad 4x^2-3kx\le 0$

$\Rightarrow \quad 4x\left(x-\frac{3k}{4}\right)\le 0$

$\Rightarrow \quad x\in\left[0,\frac{3k}{4}\right]$ (as k is positive)

So, $\quad 3\le\frac{3k}{4}\Rightarrow k\ge 4$

$\Rightarrow$ Minimum value of $k=m=4$

and the maximum value of f in $[0,3]$ is $f(3)$.

$\because f$ is increasing function in interval $x\in[0,3]$

$\because M=f(3)=3\sqrt{4\times3-3^2}=3\sqrt{3}$

Therefore, ordered pair $(m,M)=(4,3\sqrt{3})$

Hence, option (b) is correct.

31 Let $f(x)=\dfrac{x}{\sqrt{a^2+x^2}}-\dfrac{d-x}{\sqrt{b^2+(d-x)^2}}$, $x\in R$, where a, b and d are non-zero real constants. Then, **[2019, 11 Jan. Shift-II]**

(a) f is an increasing function of x

(b) f' is not a continuous function of x

(c) f is a decreasing function of x

(d) f is neither increasing nor decreasing function of x

Ans. ***(a)***

We have,

$$f(x)=\frac{x}{(a^2+x^2)^{1/2}}-\frac{(d-x)}{(b^2+(d-x)^2)^{1/2}}$$

Differentiating above w.r.t. x, we get

$$f'(x)=\frac{(a^2+x^2)^{1/2}-x\frac{1}{2}\frac{2x}{(a^2+x^2)^{1/2}}}{(a^2+x^2)}$$

$$-\frac{(b^2+(d-x)^2)^{1/2}(-1)-(d-x)\frac{2(d-x)(-1)}{2(b^2+(d-x)^2)^{1/2}}}{(b^2+(d-x)^2)}$$

[by using quotient rule of derivative]

$$=\frac{a^2+x^2-x^2}{(a^2+x^2)^{3/2}}+\frac{b^2+(d-x)^2-(d-x)^2}{(b^2+(d-x)^2)^{3/2}}$$

$$=\frac{a^2}{(a^2+x^2)^{3/2}}+\frac{b^2}{(b^2+(d-x)^2)^{3/2}}>0,\ \forall\, x\in R$$

Hence, $f(x)$ is an increasing function of x.

32 f f and g are differentiable functions in $(0,1)$ satisfying $f(0)=2=g(1), g(0)=0$ and $f(1)=6$, then for some $c\in]0,1[$ **[JEE Main 2014]**

(a) $2f'(c)=g'(c)$ (b) $2f'(c)=3g'(c)$

(c) $f'(c)=g'(c)$ (d) $f'(c)=2g'(c)$

Ans. ***(d)***

Here, $f(0)=2=g(1), g(0)=0$ and $f(1)=6$

$\because f$ and g are differentiable in $(1,0)$.

Let $\quad h(x)=f(x)-2g(x)$

$h(0)=f(0)-2g(0)$

$h(0)=2-0=2$

Now, $\quad h(1)=f(1)-2g(1)=6-2(2)$

$h(1)=2, h(0)=h(1)=2$

Hence, using Rolle's theorem,

There exists $c\in]0,1[$, such that

$h'(c)=0$

$\Rightarrow \quad f'(c)-2g'(c)=0$, for some $c\in]0,1[$

$\Rightarrow \quad f'(c)=2g'(c)$

33 How many real solutions does the equation $x^7+14x^5+16x^3+30x-560=0$ have? **[AIEEE 2008]**

(a) 5 (b) 7 (c) 1 (d) 3

Ans. ***(c)***

Let $f(x)=x^7+14x^5+16x^3+30x-560$

$\therefore \quad f'(x)=7x^6+70x^4+48x^2+30>0$, $\forall\, x\in R$

So, $f(x)$ is increasing.

Hence, $f(x)=0$ has only one solution.

34 The function $f(x)=\tan^{-1}(\sin x+\cos x)$ is an increasing function in **[AIEEE 2007]**

(a) $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2},\frac{\pi}{4}\right)$

(c) $\left(0,\frac{\pi}{2}\right)$ (d) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Ans. ***(b)***

Since, $f(x)=\tan^{-1}(\sin x+\cos x)$

On differentiating w.r.t. x, we get

$$f'(x)=\frac{1}{1+(\sin x+\cos x)^2}(\cos x-\sin x)$$

$$=\frac{\sqrt{2}\left\{\cos x\cdot\cos\frac{\pi}{4}-\sin x\cdot\sin\frac{\pi}{4}\right\}}{1+(\sin x+\cos x)^2}$$

$$=\frac{\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)}{1+(\sin x+\cos x)^2}$$

For $f(x)$ to be increasing,

$$-\frac{\pi}{2}<x+\frac{\pi}{4}<\frac{\pi}{2}\Rightarrow -\frac{3\pi}{4}<x<\frac{\pi}{4}$$

Hence, option (b) is correct, which lies in the above interval.

35 A function is matched below against an interval, where it is supposed to be increasing. Which of the following pair is incorrectly matched? **[AIEEE 2005]**

	Interval	Function
(a)	$(-\infty,-4)$	x^3+6x^2+6
(b)	$\left(-\infty,\frac{1}{3}\right]$	$3x^2-2x+1$
(c)	$[2,\infty)$	$2x^3-3x^2-12x+6$
(d)	$(-\infty,\infty)$	x^3-3x^2+3x+3

Ans. ***(b)***

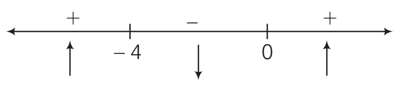
(a) Let $\quad f(x)=x^3+6x^2+6$

On differentiating w.r.t. x, we get

$$f'(x)=3x^2+12x=3x(x+4)$$

For $f(x)$ to be increasing,

$$f'(x)>0$$



$\Rightarrow \quad x\in(-\infty,-4)\cup(0,\infty)$

(b) Let $\quad f(x)=3x^2-2x+1$

$\therefore \quad f'(x) = 6x - 2$

$\Rightarrow \quad f'(x) > 0, \ \forall x \in \left(\frac{1}{3}, \infty\right)$

Since, this is wrong.

Hence, option (b) is the required answer.

35 If $2a+3b+6c=0$, then atleast one root of the equation $ax^2+bx+c=0$ lies in the interval **[AIEEE 2004]**

(a) (0, 1) (b) (1, 2)
(c) (2, 3) (d) (1, 3)

Ans. *(a)*

Let $f'(x) = ax^2 + bx + c$

On integrating both sides, we get

$$f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + d$$

$$\Rightarrow \quad f(x) = \frac{2ax^3 + 3bx^2 + 6cx + 6d}{6}$$

Since, $f(x)$ is a polynomial function and is continuous as well as differentiable in its entire real set.

$$\Rightarrow \quad f(1) = \frac{2a + 3b + 6c + 6d}{6} = \frac{6d}{6} = d$$

$[\because 2a + 3b + 6c = 0, \text{given}]$

and $f(0) = \frac{6d}{6} = d \ \therefore \ f(0) = f(1)$

Hence, according to Rolle's theorem, atleast one root of $ax^2 + bx + c = 0$ lies between 0 and 1.

37 The function $f(x) = \cot^{-1} x + x$ increases in the interval **[AIEEE 2002]**

(a) $(1, \infty)$ (b) $(-1, \infty)$
(c) $(-\infty, \infty)$ (d) $(0, \infty)$

Ans. *(c)*

Since, $\quad f(x) = \cot^{-1} x + x$

On differentiating w.r.t. x, we get

$$f'(x) = -\frac{1}{1+x^2} + 1 = \frac{x^2}{1+x^2} \geq 0$$

Hence, $f(x)$ is increasing function for all $x \in (-\infty, \infty)$.

TOPIC 3
Tangent and Normal, Maxima and Minima

38 Let A be the set of all points (α, β) such that the area of triangle formed by the points $(5, 6), (3, 2)$ and (α, β) is 12 sq units. Then, the least possible length of a line segment joining the origin to a point in A, is **[2021, 31 Aug. Shift-II]**

(a) $\frac{4}{\sqrt{5}}$ (b) $\frac{16}{\sqrt{5}}$
(c) $\frac{8}{\sqrt{5}}$ (d) $\frac{12}{\sqrt{5}}$

Ans. *(c)*

Area = 12 sq units

$$\Rightarrow \begin{vmatrix} \alpha & \beta & 1 \\ 5 & 6 & 1 \\ 3 & 2 & 1 \end{vmatrix} = \pm 24$$

$\Rightarrow \quad 4\alpha - 2\beta - 8 = \pm 24$

$\Rightarrow \quad 4\alpha - 2\beta = 32, 4\alpha - 2\beta + 16 = 0$

$\Rightarrow \quad 2\alpha - \beta - 16 = 0, 2\alpha - \beta + 8 = 0$

Distance from origin when $\beta = 2\alpha + 8$ is

$$D = \sqrt{\alpha^2 + (2\alpha + 8)^2} = \sqrt{5\alpha^2 + 32\alpha + 64}$$

$\Rightarrow \quad D^2 = 5\alpha^2 + 32\alpha + 64$

Now, $\frac{d}{d\alpha}(D^2) = 0$

$\Rightarrow \quad 10\alpha + 32 = 0 \ \Rightarrow \alpha = -\frac{16}{5}$

$\Rightarrow \quad \beta = -\frac{32}{5} + 8 = \frac{8}{5}$

$$\therefore \quad D = \sqrt{\left(\frac{-16}{5}\right)^2 + \left(\frac{8}{5}\right)^2} = \frac{8}{5}\sqrt{5} = \frac{8}{\sqrt{5}}$$

Similarly, if $\beta = 2\alpha - 16$

$$D = \frac{16}{\sqrt{5}}$$

So, least possible length of line segment $= 8/\sqrt{5}$

39 Let $f(x)$ be a cubic polynomial with $f(1) = -10, f(-1) = 6$, and has a local minima at $x = 1$, and $f'(x)$ has a local minima at $x = -1$. Then $f(3)$ is equal to **[2021, 31 Aug. Shift-II]**

Ans. *(22)*

Let $f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$f'(x)$ has local minima at $x = -1$, so

$\because \quad f''(-1) = 0$

$\Rightarrow \ -6a + 2b = 0$

$\Rightarrow \quad b = 3a \quad$ ...(i)

$f(x)$ has local minima at $x = 1$

$f'(1) = 0$

$\Rightarrow 3a + 6a + c = 0$

$\Rightarrow \quad c = -9a \quad$ ...(ii)

$f(1) = -10$

$\Rightarrow \quad -5a + d = -10 \quad$ ...(iii)

$f(-1) = 6$

$\Rightarrow \quad 11a + d = 6 \quad$ ...(iv)

Solving Eqs. (iii) and (iv)

$a = 1, d = -5$

From Eqs. (i) and (ii),

$b = 3, c = -9$

$\therefore \quad f(x) = x^3 + 3x^2 - 9x - 5$

So, $\quad f(3) = 27 + 27 - 27 - 5 = 22$

40 An angle of intersection of the curves, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $x^2 + y^2 = ab$, $a > b$, is **[2021, 31 Aug. Shift-II]**

(a) $\tan^{-1}\left(\frac{a+b}{\sqrt{ab}}\right)$ (b) $\tan^{-1}\left(\frac{a-b}{2\sqrt{ab}}\right)$
(c) $\tan^{-1}\left(\frac{a-b}{\sqrt{ab}}\right)$ (d) $\tan^{-1}(2\sqrt{ab})$

Ans. *(c)*

Given curves

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

and $x^2 + y^2 = ab, \quad$ ...(ii)

From Eqs. (ii)

$$y^2 = ab - x^2$$

From Eq. (i),

$$b^2x^2 + a^2(ab - x^2) = a^2b^2$$

$$(b^2 - a^2)x^2 = a^2b(b - a)$$

$$\Rightarrow \quad x^2 = \frac{a^2b}{a+b},$$

$$y^2 = ab - \frac{a^2b}{a+b} = \frac{ab^2}{a+b}$$

Point of intersection $\left(\sqrt{\frac{a^2b}{a+b}}, \sqrt{\frac{ab^2}{a+b}}\right)$

Now, differentiating Eq. (i) w.r.t. x, we have

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y} = m_1 \quad \text{(Let)}$$

and differentiating Eq. (ii) w.r.t. x,

$$\frac{dy}{dx} = \frac{-x}{y} = m_2 \quad \text{(Let)}$$

Let angle be θ.

Then,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{\frac{-b^2x}{a^2y} + \frac{x}{y}}{1 + \frac{b^2x^2}{a^2y^2}}\right|$$

$$= \left|\frac{xy(a^2 - b^2)}{a^2b^2}\right|$$

$$= \left|\sqrt{\frac{a^3b^3}{(a+b)^2}} \cdot \frac{(a^2 - b^2)}{a^2b^2}\right|$$

$$= \left|\frac{a-b}{\sqrt{ab}}\right|$$

$$\Rightarrow \ \theta = \tan^{-1}\left(\frac{a-b}{\sqrt{ab}}\right)$$

41 A wire of length 20 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a regular hexagon. Then, the length of the side (in m) of the the hexagon, so that the combined area of the square and the hexagon is minimum, is **[2021, 27 Aug. Shift-I]**

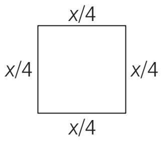
(a) $\dfrac{5}{2+\sqrt{3}}$ (b) $\dfrac{10}{2+3\sqrt{3}}$

(c) $\dfrac{5}{3+\sqrt{3}}$ (d) $\dfrac{10}{3+2\sqrt{3}}$

Ans. *(d)*

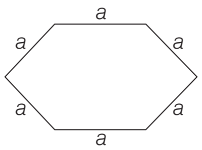
Let two pieces of wire one of length x and other of the length $20-x$.

Wire of length x is made into a square.



$\therefore$ Area of square $=\left(\dfrac{x}{4}\right)^2 = A_S$ (let)

Wire of length $(20-x)$ is made into a regular hexagon.



Area of hexagon $= 6\times\dfrac{\sqrt{3}}{4}a^2$ (Let)

$$A_H = \frac{3\sqrt{3}}{2}\left(\frac{20-x}{6}\right)^2 \quad \left[\because a = \frac{20-x}{6}\right]$$

Sum of both area

$$A = A_S + A_H = \frac{x^2}{16} + \frac{\sqrt{3}}{24}(20-x)^2$$

$$\frac{dA}{dx} = \frac{x}{8} - \frac{\sqrt{3}}{12}(20-x)$$

$$= \frac{3x - 40\sqrt{3} + 2\sqrt{3}x}{24}$$

$$\frac{dA}{dx} = 0 \Rightarrow x = \frac{40\sqrt{3}}{3+2\sqrt{3}} = \frac{40}{\sqrt{3}+2} = 40(2-\sqrt{3})$$

$$\frac{d^2A}{dx^2} = \frac{3+2\sqrt{3}}{24} > 0$$

$\Rightarrow$ Area will be minimum, when

$$x = 40(2-\sqrt{3})$$

$$\therefore \text{ Side of hexagon} = \frac{20-40(2-\sqrt{3})}{6}$$

$$= \frac{20\sqrt{3}-30}{3}$$

$$= \frac{10(2\sqrt{3}-3)}{3} = \frac{10}{2\sqrt{3}+3}$$

42 A box open from top is made from a rectangular sheet of dimension $a \times b$ by cutting squares each of side x from each of the four corners and folding up the flaps. If the volume of the box is maximum, then x is equal to **[2021, 27 Aug. Shift-II]**

(a) $\dfrac{a+b-\sqrt{a^2+b^2-ab}}{12}$

(b) $\dfrac{a+b-\sqrt{a^2+b^2+ab}}{6}$

(c) $\dfrac{a+b-\sqrt{a^2+b^2-ab}}{6}$

(d) $\dfrac{a+b+\sqrt{a^2+b^2-ab}}{6}$

Ans. *(c)*

Length of box $= a-2x$

Breadth of box $= b-2x$

Height of box $= x$

Volume of box, $V = (a-2x)(b-2x)x$

$\Rightarrow \quad V = 4x^3 - 20x^2 - 2bx^2 + abx$

Differentiating V w.r.t. x,

$V'_{(x)} = 12x^2 - 4(a+b)x + ab$

Critical Point, $V'_{(x)} = 0$

$\Rightarrow \quad 12x^2 - 4(a+b)x + ab = 0$

$$\Rightarrow \quad x = \frac{4(a+b) \pm \sqrt{16(a+b)^2 - 4\cdot 12\cdot ab}}{2(12)}$$

$$x = \frac{(a+b) \pm \sqrt{a^2+b^2-ab}}{6}$$

$V''(x) = 24x - 4(a+b)$

For $x = \dfrac{(a+b)-\sqrt{a^2+b^2-ab}}{6}$, $V''(x) < 0$

Hence, for maximum volume

$$x = \frac{(a+b)-\sqrt{a^2+b^2-ab}}{6}$$

43 A wire of length 36 m is cut into two pieces, one of the pieces is bent to form a square and the other is bent to form a circle. If the sum of the areas of the two figures is minimum and the circumference of the circle is k (m), then $\left(\dfrac{4}{\pi}+1\right)k$ is equal to **[2021, 26 Aug. Shift-I]**

Ans. *(36)*

Let $x + y = 36$

where, x is perimeter of square and y is perimeter of circle.

Then, side of square $= \dfrac{x}{4}$ and radius of circle $= \dfrac{y}{2\pi}$

Now, Sum of areas of square and circle,

$$A = \frac{x^2}{16} + \frac{y^2}{4\pi}$$

$$\Rightarrow \quad A = \frac{x^2}{16} + \frac{(36-x)^2}{4\pi} \qquad [\because y = 36-x]$$

For minimum area,

$$\frac{dA}{dx} = 0$$

Now, $\quad \dfrac{dA}{dx} = \dfrac{2x}{16} + \dfrac{-2(36-x)}{4\pi} = 0$

$$\Rightarrow \quad x = \frac{144}{\pi+4}$$

Circumference of circle $= y$

$$= (36-x)$$

$$= 36 - \frac{144}{\pi+4} = \frac{36\pi}{\pi+4}$$

According to the question,

$$k = \frac{36\pi}{\pi+4}$$

$$\Rightarrow \left(\frac{4}{\pi}+1\right)k = \left(\frac{4}{\pi}+1\right)\frac{36\pi}{\pi+4} = 36$$

44 The local maximum value of the function $f(x) = \left(\dfrac{2}{x}\right)^{x^2}$, $x > 0$ **[2021, 26 Aug. Shift-II]**

(a) $(2\sqrt{e})^{\frac{1}{e}}$ (b) $\left(\dfrac{4}{\sqrt{e}}\right)^{\frac{e}{4}}$

(c) $(e)^{\frac{2}{e}}$ (d) 1

Ans. *(c)*

$f(x) = \left(\dfrac{2}{x}\right)^{x^2}$; $x > 0$

$\therefore \quad \log f(x) = x^2(\log 2 - \log x)$

$f'(x) = f(x)[-x + (\log 2 - \log x)2x]$

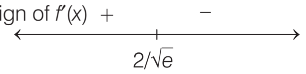
$f'(x) = f(x)\cdot x(2\log 2 - 2\log x - 1)$

For maxima or minima put $f'(x) = 0$, we get

$$2\log 2 - 2\log x - 1 = \log\left(\frac{4}{x^2}\right) - 1 = 0$$

$$\Rightarrow \quad x = \frac{2}{\sqrt{e}}$$

Sign of $f'(x)$

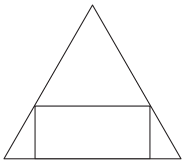


$\therefore$ At $x = \dfrac{2}{\sqrt{e}}$, $f(x)$ has maximum value.

$$\text{Maximum} = \left(\frac{2}{\frac{2}{\sqrt{e}}}\right)^{\frac{4}{e}} = e^{\frac{2}{e}}$$

45 If a rectangle is inscribed in an equilateral triangle of side length $2\sqrt{2}$ as shown in the figure, then the square of the largest area of such a rectangle is ....

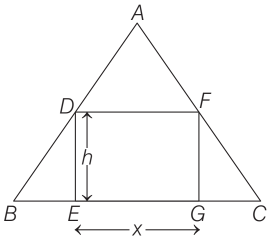
[2021, 25 July Shift-II]



Ans. *(3)*

Let $EG = x$ and $DE = h$

Area of rectangle $(A) = xh$ ...(i)



In ΔBDE,

$$h = BE \tan 60°$$

$$\Rightarrow \quad h = \left(\frac{2\sqrt{2} - x}{2}\right) \cdot \sqrt{3} \quad ...(ii)$$

From Eqs. (i) and (ii),

$$\text{Area}\,(A) = \frac{\sqrt{3}}{2}(2\sqrt{2} - x) \cdot x$$

$$\Rightarrow \quad A = \frac{\sqrt{3}}{2}(2\sqrt{2}x - x^2)$$

Differentiating w.r.t. 'x',

$$\frac{dA}{dx} = \frac{\sqrt{3}}{2}(2\sqrt{2} - 2x)$$

Substitute $\frac{dA}{dx} = 0$. To determine the critical points.

$$\Rightarrow \frac{\sqrt{3}}{2}(2\sqrt{2} - 2x) = 0 \Rightarrow x = \sqrt{2}$$

For maxima, $\frac{d^2A}{dx^2} = \frac{\sqrt{3}}{2}(-2) < 0$

From Eq. (i),

$$h = \left(\frac{\sqrt{2}}{2}\right)\sqrt{3} = \sqrt{\frac{3}{2}}$$

So, area $= \sqrt{2} \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$

Squaring both sides, $(\text{area})^2 = 3$

46 Let a be a real number such that the function $f(x) = ax^2 + 6x - 15$, $x \in R$ is increasing in $\left(-\infty, \frac{3}{4}\right)$ and decreasing in $\left(\frac{3}{4}, \infty\right)$. Then, the function $g(x) = ax^2 - 6x + 15$, $x \in R$ has a **[2021, 20 July Shift-I]**

(a) local maximum at $x = -\frac{3}{4}$

(b) local minimum at $x = -\frac{3}{4}$

(c) local maximum at $x = \frac{3}{4}$

(d) local minimum at $x = \frac{3}{4}$

Ans. *(a)*

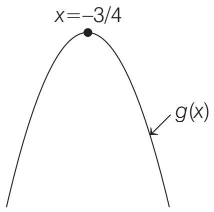
$$f(x) = ax^2 + 6x - 15$$
$$f'(x) = 2ax + 6$$
$$f'(x) = 2(ax + 3)$$

$f(x)$ is increasing for $\left(-\infty, \frac{3}{4}\right)$.

So, $\quad x = -\frac{3}{a} = \frac{3}{4}$

$$a = -4$$
$$g(x) = ax^2 - 6x + 15$$
$$g'(x) = 2(ax - 3)$$

If $\quad g'(x) = 0$



$$x = \frac{3}{a} = -\frac{3}{4}$$

$g(x)$ is maximum at $x = -\frac{3}{4}$.

47 The sum of all the local minimum values of the twice diffrentiable function $f: R \to R$ defined by $f(x) = x^3 - 3x^2 - \frac{3f''(2)}{2}x + f''(1)$ is

[2021, 20 July Shift-II]

(a) –22 (b) 5 (c) –27 (d) 0

Ans. *(c)*

Given, $f(x) = x^3 - 3x^2 - \frac{3}{2}f''(2) \cdot x + f''(1)$

Then, $f'(x) = 3x^2 - 6x - \frac{3}{2}f''(2)$

$$f''(x) = 6x - 6$$

Put $\quad x = 1$ and $x = 2$,

$$f''(2) = 12 - 6 = 6 \text{ and } f''(1) = 0$$

Therefore, $f(x) = x^3 - 3x^2 - \frac{3}{2} \times 6x + 0$

$\Rightarrow \quad f(x) = x^3 - 3x^2 - 9x$ ...(i)

and $\quad f'(x) = 3x^2 - 6x - 9$

and $\quad f''(x) = 6x - 6$

Equate $\quad f'(x) = 0$ gives,

$\Rightarrow \quad 3x^2 - 6x - 9 = 0$

$\Rightarrow \quad 3(x^2 - 2x - 3) = 0$

$3(x + 1)(x - 3) = 0 \Rightarrow x = -1, x = 3$

Now, $f''(-1) = 6(-1) - 6 = -12 < 0$

$f''(3) = 6(3) - 6 = 12 > 0$

$\therefore$ (-1) is maxima. 3 is minima.

Local minimum value $= f(3)$

$f(3) = (3)^3 - 3(3)^2 - 9(3)$ [using Eq. (i)]

$= 27 - 27 - 27$

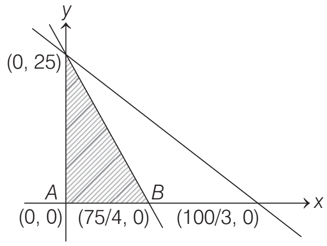
$f(3) = -27$

48 The maximum value of z in the following equation $z = 6xy + y^2$, where $3x + 4y \leq 100$ and $4x + 3y \leq 75$ for $x \geq 0$ and $y \geq 0$ is ....

[2021, 17 March shift-I]

Ans. *(904)*

$$z = 6xy + y^2$$
$$3x + 4y \leq 100$$
$$4x + 3y \leq 75$$
$$x, y \geq 0$$
$$z = y(6x + y)$$



z will be maximum at the corner points.

$$x \leq \frac{75 - 3y}{4}$$
$$z = y(6x + y)$$
$$z \leq y\left[6\left(\frac{75 - 3y}{4}\right) + y\right]$$
$$z \leq \frac{1}{2}(225 - 7y^2)$$

$(225y - 7y^2)$ is a quadratic in y whose maximum value is $\frac{-D}{4a}$.

Here, $D = \frac{225^2 - 4 \cdot 0 \cdot (-7)}{4(-7)}$

$\therefore \quad z \leq \frac{225^2}{2 \cdot 4 \cdot 7} = \frac{50625}{56} \simeq 904$

49 Let $f: [-1, 1] \to R$ be defined as $f(x) = ax^2 + bx + c$ for all $x \in [-1, 1]$, where $a, b, c \in R$, such that $f(-1) = 2, f'(-1) = 1$ and for $x \in (-1, 1)$ the maximum value of $f''(x)$ is $\frac{1}{2}$. If $f(x) \leq \alpha$, $x \in [-1, 1]$, then the least value of α is equal to ....

[2021, 17 March Shift-II]

Ans. *(5)*

Given, $f : [-1, 1] \rightarrow R$

and $f(x) = ax^2 + bx + c$

$$f(-1) = a - b + c = 2 \text{(given)} \quad \text{...(i)}$$

$$f'(-1) = -2a + b = 1 \text{(given)} \quad \text{...(ii)}$$

$$f''(x) = 2a$$

$$\therefore \quad f''_{max}(x) = 2a$$

Also, given maximum value of $f''(x) = \frac{1}{2}$

i.e. $2a = \frac{1}{2} \Rightarrow a = \frac{1}{4}$

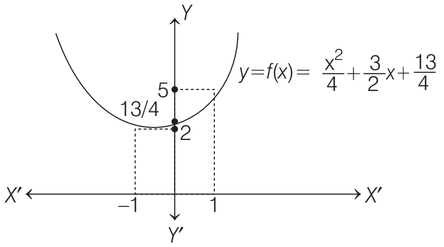
From Eq. (ii), $b = \frac{3}{2}$

From Eq. (i), $c = \frac{13}{4}$

$$\therefore \quad f(x) = \frac{x^2}{4} + \frac{3}{2}x + \frac{13}{4}$$

Here, $f(-1) = \frac{1}{4} - \frac{3}{2} + \frac{13}{4} = 2$

and $\quad f(1) = \frac{1}{4} + \frac{3}{2} + \frac{13}{4} = 5$



For $x \in [-1, 1]$

$f(x) \in [2, 5]$

$\therefore$ Least value of α is 5.

50 The range of $a \in R$ for which the function $f(x) = (4a - 3)(x + \log_e 5) + 2(a - 7) \cot\left(\frac{x}{2}\right) \sin^2\left(\frac{x}{2}\right)$, $x \neq 2n\pi$, $n \in N$, has critical points, is **[2021, 16 March Shift-I]**

(a) $(-3, 1)$ (b) $\left[-\frac{4}{3}, 2\right]$

(c) $[1, \infty)$ (d) $(-\infty, -1]$

Ans. *(b)*

Given, $f(x) = (4a - 3)(x + \log_e 5) + 2(a - 7) \cot\frac{x}{2} \cdot \sin^2\frac{x}{2}$

$$\Rightarrow \quad f(x) = (4a - 3)(x + \log_e 5) + 2(a - 7) \cos\frac{x}{2} \sin\frac{x}{2}$$

$$\Rightarrow \quad f(x) = (4a - 3)(x + \log_e 5) + (a - 7) \sin x$$

$$\Rightarrow \quad f'(x) = (4a - 3)(1 + 0) + (a - 7) \cos x$$

$$\Rightarrow \quad f'(x) = (4a - 3) + (a - 7) \cos x$$

When $f'(x) = 0$, $(4a - 3) + (a - 7) \cos x = 0$

$$\Rightarrow \quad \cos x = \frac{4a - 3}{7 - a}$$

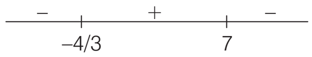
As, $-1 \leq \cos x \leq 1$

So, $-1 \leq \frac{4a - 3}{7 - a} \leq 1$

$$\Rightarrow \quad \frac{4a - 3}{7 - a} + 1 \geq 0$$

$$\Rightarrow \quad \frac{4a - 3 + 7 - a}{7 - a} \geq 0$$

$$\Rightarrow \quad \frac{3a + 4}{7 - a} \geq 0$$

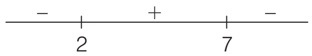


$$a \in [-4/3, 7) \quad \text{...(i)}$$

Now, $\frac{4a - 3}{7 - a} - 1 \leq 0$

$$\Rightarrow \quad \frac{4a - 3 - 7 + a}{7 - a} \leq 0$$

$$\Rightarrow \quad \frac{5a - 10}{7 - a} \leq 0$$



$$a \in (-\infty, 2] \cup [7, \infty) \quad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$a \in \left[-\frac{4}{3}, 2\right]$$

51 The triangle of maximum area that can be inscribed in a given circle of radius r is **[2021, 26 Feb. Shift-II]**

(a) an isosceles triangle with base equal to $2r$

(b) an equilateral triangle of height $\frac{2r}{3}$

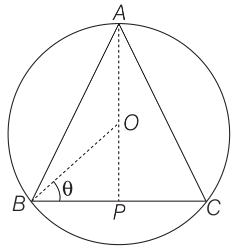
(c) an equilateral triangle having each of its side of length $\sqrt{3}r$

(d) a right angle triangle having two of its sides of length $2r$ and r

(e) Let a ΔABC inscribed in a circle with centre O and radius r.

Ans. *(c)*

Let a ΔABC inscribed in a circle with centre O and radius r.



Let $\angle OBC = \theta$

Now, area of $\Delta ABC = \frac{1}{2} \times \text{Base} \times \text{Height}$

$$A = \frac{1}{2} \times (BC) \times (AP) \quad \text{...(i)}$$

Now, $BC = 2BP$

Consider ΔOBP, where $OB = r$

Then, $BP = r \cos\theta$

Hence, $BC = 2r \cos\theta$

Again, $AP = AO + OP$

where, $AO = r$

Consider ΔOBP, where $OB = r$

Then, $OP = r \sin\theta$

$$\Rightarrow \quad AP = r + r \sin\theta$$

From Eq. (i), we get

Area $= \frac{1}{2} \times (2r \cos\theta) \times (r + r \sin\theta)$

$$A = r^2 \cos\theta(1 + \sin\theta)$$

Now,

$$\frac{dA}{d\theta} = r^2(-\sin\theta)(1 + \sin\theta) + r^2 \cos^2\theta$$

$$= r^2(\cos^2\theta - \sin\theta - \sin^2\theta)$$

$$= r^2(1 - 2\sin^2\theta - \sin\theta)$$

$$= r^2(1 + \sin\theta)(1 - 2\sin\theta)$$

Equate $\frac{dA}{d\theta} = 0$

$$\Rightarrow \quad r^2(1 + \sin\theta)(1 - 2\sin\theta) = 0$$

$$\Rightarrow \quad \sin\theta = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{6}$$

Now, $\frac{d^2A}{d\theta^2} < 0$, when $\theta = \frac{\pi}{6}$

$\Rightarrow A$ is maximum, when $\theta = \frac{\pi}{6}$

$\therefore$ Maximum area

$$= r^2 \cos\left(\frac{\pi}{6}\right)\left(1 + \sin\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{4} r^2$$

Height $= AP = \frac{3}{2} r$

Consider ΔABP,

$$(AB)^2 = (AP)^2 + (BP)^2$$

$$= \left(\frac{3}{2} r\right)^2 + \left(\frac{\sqrt{3}}{2} r\right)^2 \quad [\because BP = r \cos\theta]$$

$$= \frac{9}{4} r^2 + \frac{3}{4} r^2 = 3r^2 \Rightarrow AB = \sqrt{3} r$$

Hence, the ΔABC is an equilateral triangle with side $\sqrt{3}r$.

52 The maximum slope of the curve $y = \frac{1}{2} x^4 - 5x^3 + 18x^2 - 19x$ occurs at the point **[2021, 26 Feb. Shift-I]**

(a) $(2, 2)$ (b) $(0, 0)$

(c) $(2, 9)$ (d) $\left(3, \frac{21}{2}\right)$

Ans. *(a)*

Given, curve is

$$y=\frac{1}{2}x^4-5x^3+18x^2-19x \quad \ldots(i)$$

First, find the slope of given curve i.e. dy/dx,

Differentiate Eq. (i),

$$\frac{dy}{dx}=\frac{1}{2}(4x^3)-5(3x^2)+18(2x)-19$$
$$=2x^3-15x^2+36x-19$$

Now, let $f(x)=2x^3-15x^2+36x-19$ is slope of the curve and find its maximum value as follows,

$$f'(x)=2(3x^2)-15(2x)+36$$
$$=6x^2-30x+36$$

Equate $f'(x)=0$ and solve for 'x',

$$6x^2-30x+36=0$$
$$\Rightarrow \quad x^2-5x+6=0$$
$$\Rightarrow \quad x^2-3x-2x+6=0$$
$$\Rightarrow \quad (x-3)(x-2)=0$$
$$\Rightarrow \quad x=2 \text{ and } 3$$

Now, $f''(x)=\dfrac{d}{dx}(6x^2-30x+36)$
$$=12x-30$$

Then, $f''(2)=12(2)-30=24-30$
$$=-6<0$$

and $f''(3)=12(3)-30=6>0$

$\because f''(2)<0$, this implies '2' is point of maxim a.

$\therefore$ At $x=2$, slope will be maximum.

Since, at $x=2$, slope will be maximum, then y-coordinate will be,

$$y=\frac{1}{2}(2)^4-5(2)^3+18(2)^2-19(2)$$
$$=8-40+72-38$$
$$=72-70=2$$

$\therefore$ Maximum slope occurs at point (2, 2).

53 The shortest distance between the line $x-y=1$ and the curve $x^2=2y$ is **[2021, 25 Feb. Shift-II]**

(a) $\dfrac{1}{\sqrt{2}}$ (b) $\dfrac{1}{2\sqrt{2}}$ (c) 0 (d) $\dfrac{1}{2}$

Ans. *(b)*

Let (x, y) be any arbitrary point on curve $x^2=2y$ and find the tangent line equation at this point, such that tangent line at (x, y) is parallel to line $x-y=1$.

To find tangent equation, differentiate the following equation so that we can find slope,

$$x^2-2y=0 \quad \ldots(i)$$

$$2x-2\frac{dy}{dx}=0 \text{ gives } \frac{dy}{dx}=x$$

Slope (say m_1) $=x$

Also, slope of line $x-y=1$ or $y=x-1$ is 1 (say m_2). Since, $x-y=1$ and tangent line is parallel.

Therefore, their slope be equal.

Hence, $m_1=m_2$ gives, $x=1$

Put $x=1$ in Eq. (i), we get $y=\dfrac{1}{2}$

Thus, $(x, y)=\left(1, \dfrac{1}{2}\right)$

Perpendicular distance between line $x-y=1$ and point $\left(1, \dfrac{1}{2}\right)$ is given as,

$$P=\left|\frac{(1)(1)+\left(\frac{1}{2}\right)(-1)-1}{\sqrt{(1)^2+(-1)^2}}\right|$$
$$=\left|\frac{-1}{2\sqrt{2}}\right|$$

[$\therefore$ using perpendicular distance formula]

$$=\frac{1}{2\sqrt{2}}$$

54 If the curves $x=y^4$ and $xy=k$ cut at right angles, then $(4k)^6$ is equal to .... **[2021, 25 Feb. Shift-II]**

Ans. *(4)*

If the curves cut at right angle, then product of slopes will be -1.

First curve $x=y^4$

Differentiate it, we get

$$1=4y^3\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{4y^3}$$

Slope of first curve $(m_1)=\dfrac{1}{4y_1^3}$ [at point (x_1, y_1)]

Second curve $xy=k$

Differentiate it, $0=x\dfrac{dy}{dx}+y$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{-y}{x}$$

Slope of second curve $(m_2)=\dfrac{-y_1}{x_1}$ [at (x_1, y_1)]

$$\Rightarrow \quad m_1.m_2=-1$$
$$\Rightarrow \quad \frac{1}{4y_1^3}\left(\frac{-y_1}{x_1}\right)=-1$$
$$\Rightarrow \quad \frac{-1}{4y_1^2x_1}=-1$$
$$\Rightarrow \quad \frac{-1}{4(y_1)^6}=-1 \quad [\text{using } x_1=y_1^4]$$
$$\Rightarrow \quad y_1^6=\frac{1}{4}$$

Also, $x_1y_1=k$, using $x_1=y_1^4$, we get $k=y_1^5$ or $k^6=(y_1)^{30}$

$$\because \quad y_1^6=\frac{1}{4}, \text{ then } y_1^{30}=\left(\frac{1}{4}\right)^5$$
$$\therefore (4k)^6=4^6.k^6=4^6(y_1)^{30}$$
$$=4^6\left(\frac{1}{4}\right)^5=4$$
$$\therefore (4k)^6=4$$

55 Let $f(x)$ be a polynomial of degree 6 in x, in which the coefficient of x^6 is unity and it has extrema at $x=-1$ and $x=1$. If $\lim\limits_{x\to 0}\dfrac{f(x)}{x^3}=1$, then $5\cdot f(2)$ is equal to .... **[2021, 25 Feb. Shift-I]**

Ans. *(144)*

$$f(x)=x^6+ax^5+bx^4+cx^3+dx^2+ex+f$$

As, $\lim\limits_{x\to 0}\dfrac{f(x)}{x^3}=1$ non-zero finite

So, $d=e=f=0$

and $f(x)=x^3(x^3+ax^2+bx+c)$

Hence, $\lim\limits_{x\to 0}\dfrac{f(x)}{x^3}=c=1$

Now, as $f(x)=x^6+ax^5+bx^4+x^3$

and $f'(x)=0$ at $x=1$ and $x=-1$

i.e. $f'(x)=6x^5+5ax^4+4bx^3+3x^2$

Now, $f'(1)=0$

$$\Rightarrow \quad 6+5a+4b+3=0$$
$$\Rightarrow \quad 5a+4b=-9 \quad \ldots(i)$$
$$\text{and} \quad f'(-1)=0$$
$$\Rightarrow \quad -6+5a-4b+3=0$$
$$\Rightarrow \quad 5a-4b=3 \quad \ldots(ii)$$

From Eqs. (i) and (ii),

$$a=-3/5 \text{ and } b=-3/2$$
$$\therefore \quad f(x)=x^6-\frac{3}{5}x^5-\frac{3}{2}x^4+x^3$$
$$\therefore \quad 5f(2)=5\left[2^6-\frac{3}{5}(2)^5-\frac{3}{2}(2)^4+(2)^3\right]$$
$$=5\left[64-\frac{3\times 32}{5}-\frac{3\times 16}{2}+8\right]$$
$$=320-96-120+40=144$$

56 If the tangent to the curve $y=x^3$ at the point $P(t, t^3)$ meets the curve again at Q, then the ordinate of the point which divides PQ internally in the ratio 1 : 2 is **[2021, 24 Feb. Shift-I]**

(a) 0 (b) $2t^3$
(c) $-t^3$ (d) $-2t^3$

Ans. *(d)*

Given, curve $\Rightarrow y=x^3 \quad \ldots(i)$

$P(t, t^3)$

Equation of tangent at $P(t, t^3)$

$(y - t^3) = 3t^2(x - t)$ ... (ii)

From Eqs. (i) and (ii),

$$x^3 - t^3 = 3t^2(x - t)$$

$\Rightarrow (x - t)(x^2 + t^2 + xt) = 3t^2(x - t)$

$\Rightarrow \quad x^2 + xt - 2t^2 = 0$

$\Rightarrow \quad (x - t)(x + 2t) = 0$

$\Rightarrow \quad x = t$ or $x = -2t$

This is not possible.

Now, the coordinate of $Q = (x, y)$

$= (-2t, (-2t)^3)$ $[\because y = x^3]$

$\therefore \quad Q = (-2t, -8t^3)$

$\therefore$ Ordinate of the point dividing PQ in the ratio 1 : 2 is

$$\frac{2t^3 + (-8t^3)}{1+2} = -2t^3$$

57 If $p(x)$ be a polynomial of degree three that has a local maximum value 8 at $x = 1$ and a local minimum value 4 at $x = 2$; then $p(0)$ is equal to **[2020, 2 Sep. Shift-I]**

(a) -24 (b) 6

(c) 12 (d) -12

Ans. *(d)*

Since, $p'(x) = 0$ at $x = 1$ and $x = 2$ and $p(x)$ is cubic polynomial, so

$$p'(x) = a(x-1)(x-2) = a(x^2 - 3x + 2)$$

$$\therefore \quad p(x) = a\left(\frac{x^3}{3} - \frac{3}{2}x^2 + 2x\right) + b$$

According to the question,

$$p(1) = 8 \Rightarrow a\left(\frac{1}{3} - \frac{3}{2} + 2\right) + b = 8$$

$\Rightarrow \quad a\left(\frac{1}{3} + \frac{1}{2}\right) + b = 8 \Rightarrow 5a + 6b = 48$ ... (i)

and $p(2) = 4 \Rightarrow a\left(\frac{8}{3} - 6 + 4\right) + b = 4$

$\Rightarrow \quad 2a + 3b = 12$ ... (ii)

From Eqs. (i) and (ii), we get

$$a = 24, b = -12$$

$\because \quad p(0) = b = -12$

58 If the tangent to the curve $y = x + \sin y$ at a point (a, b) is parallel to the line joining $\left(0, \frac{3}{2}\right)$ and $\left(\frac{1}{2}, 2\right)$, then **[2020, 2 Sep. Shift-I]**

(a) $|b - a| = 1$ (b) $|a + b| = 1$

(c) $b = a$ (d) $b = \frac{\pi}{2} + a$

Ans. *(a)*

Given curve is $y = x + \sin y$

$\therefore$ On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 1 + \cos y \frac{dy}{dx}$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{1 - \cos y}$ ... (i)

$\because$ tangent at point (a, b) at given curve is parallel to line joining $\left(0, \frac{3}{2}\right)$ and $\left(\frac{1}{2}, 2\right)$.

So $\left.\frac{dy}{dx}\right|_{(a,b)} = \frac{2 - (3/2)}{(1/2) - 0} = 1$

$\Rightarrow \frac{1}{1 - \cos b} = 1 \Rightarrow 1 = 1 - \cos b \Rightarrow \cos b = 0$

$\Rightarrow \quad \sin b = \pm 1$

Now, as point (a, b) on the given curve,

So $\quad b = a + \sin b \Rightarrow b - a = \sin b$

$\Rightarrow \quad |b - a| = |\sin b| \quad \Rightarrow |b - a| = 1$

59 Let $P(h, k)$ be a point on the curve $y = x^2 + 7x + 2$, nearest to the line $y = 3x - 3$. Then the equation of the normal to the curve at P is **[2020, 2 Sep. Shift-I]**

(a) $x - 3y - 11 = 0$ (b) $x - 3y + 22 = 0$

(c) $x + 3y - 62 = 0$ (d) $x + 3y + 26 = 0$

Ans. *(d)*

As the point $P(h, k)$ is the nearest point on the curve $y = x^2 + 7x + 2$, to the line $y = 3x - 3$, so the tangent to the parabola $y = x^2 + 7x + 7$ at point $P(h, k)$ is parallel to the line $y = 3x - 3$

$\therefore \quad \left.\frac{dy}{dx}\right|_p = 2h + 7 = 3 \Rightarrow h = -2$ ... (i)

and the point $p(h, k)$ on the curve, so

$$k = h^2 + 7h + 2 = (-2)^2 + 7(-2) + 2$$

$\Rightarrow \quad k = 4 - 14 + 2 \Rightarrow k = -8$

$\therefore$ Point $P(-2, -8)$

Now, equation of normal to the parabola $y = x^2 + 7x + 7$ at point $P(-2, -8)$ is

$$y + 8 = \frac{-1}{\left.\frac{dy}{dx}\right|_p}(x + 2)$$

$\Rightarrow \quad y + 8 = -\frac{1}{3}(x + 2) \Rightarrow x + 3y + 26 = 0$

60 The equation of the normal to the curve $y = (1 + x)^{2y} + \cos^2(\sin^{-1} x)$ at $x = 0$ is **[2020, 2 Sep. Shift-II]**

(a) $y + 4x = 2$ (b) $y = 4x + 2$

(c) $x + 4y = 8$ (d) $2y + x = 4$

Ans. *(c)*

Equation of the given curve is

$$y = (1 + x)^{2y} + \cos^2(\sin^{-1} x)$$

$\Rightarrow \quad y = (1 + x)^{2y} + 1 - \sin^2(\sin^{-1} x)$

$\Rightarrow \quad y = (1 + x)^{2y} + 1 - x^2$ ...(i)

$[\text{as } \sin(\sin^{-1} x) = x]$

So, at $\quad x = 0, y = 2$.

Now, let a point $P(0, 2)$ on the curve.

On differentiating the Eq.(i) both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2y(1 + x)^{2y-1} + 2(1 + x)^{2y}$$

$$(\log_e(1 + x))\frac{dy}{dx} - 2x$$

So, at point $P(0, 2)$ $\left.\frac{dy}{dx}\right|_P = 4$

$\therefore$ Equation of normal to the curve at point 'P' is $\quad y - 2 = \frac{1}{-4}(x - 0)$

$\Rightarrow \quad x + 4y = 8$

Hence, option (c) is correct.

61 If the tangent to the curve, $y = e^x$ at a point (c, e^c) and the normal to the parabola, $y^2 = 4x$ at the point (1, 2) intersect at the same point the X-axis, then the value of c is .... **[2020, 3 Sep. Shift-II]**

Ans. *(4)*

The equation of tangent to the curve, $y = e^x$ at a point (c, e^c) is

$$y - e^c = e^c(x - c) \quad \text{...(i)}$$

and equation of normal to the curve, $y^2 = 4x$ at the point (1, 2) is

$$y - 2 = -1(x - 1) \quad \text{...(ii)}$$

$\because$ The lines (i) and (ii) intersect at same point on the X-axis, so put $y = 0$ in both the equation and equate, we get

$$x = 3 = c - 1 \Rightarrow c = 4$$

Hence, answer is 4.00.

62 Which of the following points lies on the tangent to the curve $x^4 e^y + 2\sqrt{y + 1} = 3$ at the point (1, 0)? **[2020, 5 Sep. Shift-II]**

(a) (2, 2) (b) (2, 6)

(c) $(-2, 6)$ (d) $(-2, 4)$

Ans. *(b)*

Equation of the given curve is

$$x^4 e^y + 2\sqrt{y + 1} = 3$$

On differentiating w.r.t. 'x', we get

$$e^y(4x^3) + x^4 e^y \frac{dy}{dx} + \frac{1}{\sqrt{y + 1}}\frac{dy}{dx} = 0$$

$\therefore$ At point $P(1, 0)$,

$$e^0(4 \times 1) + 1 . e^0 \frac{dy}{dx} + \frac{1}{\sqrt{0 + 1}}\frac{dy}{dx} = 0$$

$\Rightarrow \quad \left.\frac{dy}{dx}\right|_P = -2$

$\therefore$ Equation of tangent at point $P(1,0)$ is

$$y = -2(x-1) \Rightarrow 2x + y = 2 \quad \text{...(i)}$$

From the option point $(-2, 6)$ contain by the tangent (i).

63 If $x = 1$ is a critical point of the function $f(x) = (3x^2 + ax - 2 - a)e^x$, then **[2020, 5 Sep. Shift-II]**

(a) $x = 1$ and $x = -\frac{2}{3}$ are local minima of f

(b) $x = 1$ and $x = -\frac{2}{3}$ are local maxima of f

(c) $x = 1$ is a local maxima and $x = -\frac{2}{3}$ is a local minima of f

(d) $x = 1$ is a local minima and $x = -\frac{2}{3}$ is a local maxima of f

Ans. *(d)*

It is given that $x = 1$ is a critical point of the function $f(x) = (3x^2 + ax - 2 - a)e^x$

So,

$f'(1) = e^x(6x + a) + e^x(3x^2 + ax - 2 - a)|_{x=1} = 0$

$\Rightarrow \quad 6 + a + 3 + a - 2 - a = 0 \Rightarrow a = -7.$

$\therefore \quad f'(x) = e^x[3x^2 - x - 2] = 0$

$\Rightarrow \quad x = 1 \text{ or } \left(-\frac{2}{3}\right)$

and $f''(x) = e^x(6x - 1 + 3x^2 - x - 2)$

$\qquad = e^x(3x^2 + 5x - 3)$

$\because \quad f''(1) = 5e > 0$

$\Rightarrow x = 1$ is the point of local minima.

and $f''\left(-\frac{2}{3}\right) = e^{-2/3}\left(\frac{4}{3} - \frac{10}{3} - 3\right)$

$\qquad = -5e^{-2/3} < 0$

$\Rightarrow x = -\frac{2}{3}$ is the point of local maxima.

64 If the tangent to the curve, $y = f(x) = x\log_e x, (x > 0)$ at a point $(c, f(c))$ is parallel to the line segment joining the point $(1, 0)$ and (e, e) then c is equal to **[2020, 6 Sep. Shift-II]**

(a) $\frac{e-1}{e}$ (b) $e^{\left(\frac{1}{e-1}\right)}$

(c) $e^{\left(\frac{1}{1-e}\right)}$ (d) $\frac{1}{e-1}$

Ans. *(b)*

Equation of given curve,

$$y = f(x) = x\log_e x, (x > 0)$$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{x=c} = f'(c) = 1 + \log_e c$$

$\because$ The tangent to the given curve $y = f(x)$ at point $x = c$ is parallel to line segment joining points $(1, 0)$ and (e, e).

So, $1 + \log_e c = \frac{e}{e-1}$

$\Rightarrow \quad \log_e c = \frac{e}{e-1} - 1 = \frac{1}{e-1}$

$\Rightarrow \quad c = e^{\left(\frac{1}{e-1}\right)}$ is positive.

65 The set of all real values of λ for which the function $f(x) = (1 - \cos^2 x).(\lambda + \sin x)$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, has exactly one maxima and exactly one minima, is **[2020, 6 Sep. Shift-II]**

(a) $\left(-\frac{1}{2}, \frac{1}{2}\right) - \{0\}$ (b) $\left(-\frac{3}{2}, \frac{3}{2}\right)$

(c) $\left(-\frac{1}{2}, \frac{1}{2}\right)$ (d) $\left(-\frac{3}{2}, \frac{3}{2}\right) - \{0\}$

Ans. *(d)*

Given function,

$$f(x) = (1 - \cos^2 x)(\lambda + \sin x)$$
$$= \sin^2 x(\lambda + \sin x)$$

$\therefore \; f'(x) = \sin 2x(\lambda + \sin x) + \sin^2 x(\cos x)$

$$= \sin 2x\left[\lambda + \sin x + \frac{1}{2}\sin x\right]$$
$$= \sin 2x\left[\lambda + \frac{3}{2}\sin x\right]$$

For maxima and minima, as

$$f'(x) = 0 \Rightarrow \sin 2x\left(\lambda + \frac{3}{2}\sin x\right) = 0$$

So, either $\sin 2x = 0 \Rightarrow x = 0$ as $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

or $\lambda + \frac{3}{2}\sin x = 0$ as there must exactly one maxima and exactly one minima, so $\lambda \in \left(-\frac{3}{2}, \frac{3}{2}\right) - \{0\}$.

66 Let $f(x)$ be a polynomial of degree 3 such that $f(-1) = 10$, $f(1) = -6$, $f(x)$ has a critical point at $x = -1$ and $f'(x)$ has a critical point at $x = 1$. The $f(x)$ has a local minima at $x = $ .... **[2020, 8 Jan. Shift-II]**

Ans. *(3)*

Let a cubic polynomial

$$f(x) = ax^3 + bx^2 + cx + d$$

$\because \quad f(-1) = 10$

$\Rightarrow \quad -a + b - c + d = 10 \quad \text{...(i)}$

$\because \quad f(1) = -6$

$\Rightarrow \quad a + b + c + d = -6 \quad \text{...(ii)}$

$\because \quad f'(-1) = 0$

$\Rightarrow \quad 3a - 2b + c = 0 \quad \text{...(iii)}$

$\because \quad f''(1) = 0$

$\Rightarrow \quad 6a + 2b = 0$

$\Rightarrow \quad 3a + b = 0 \quad \text{...(iv)}$

From Eqs. (i) and (ii), we get

$$-2a - 2c = 16$$

$\Rightarrow \quad a + c = -8 \quad \text{...(v)}$

From Eqs. (iii), (iv) and (v), we get

$$3a - 2(-3a) + (-a - 8) = 0$$

$\Rightarrow \quad 8a - 8 = 0 \Rightarrow a = 1$

So, $\quad b = -3, \quad c = -9$ and $d = 5$

$\therefore \quad f(x) = x^3 - 3x^2 - 9x + 5$

$\because \quad f'(x) = 3x^2 - 6x - 9 = 0$

(for local maxima and minima)

$\Rightarrow \quad x^2 - 2x - 3 = 0 \Rightarrow x^2 - 3x + x - 3 = 0$

$\Rightarrow (x + 1)(x - 3) = 0 \Rightarrow f'(x) = 0$

$\Rightarrow \quad x = -1, 3$

$\because \quad f''(x) = 6x - 6$

$\because \quad f''(-1) = -12$ and $f''(3) = 12$

$\therefore x = 3$ is point of local minima.

Hence, answer is 3.

67 Let the normal at a point P on the curve $y^2 - 3x^2 + y + 10 = 0$ intersect the y-axis at $\left(0, \frac{3}{2}\right)$. If m is the slope of the tangent at P to the curve, then $|m|$ is equal to .... **[2020, 8 Jan. Shift-I]**

Ans. *(4)*

Let a point $P(x_1, y_1)$ on the curve $y^2 - 3x^2 + y + 10 = 0$, then slope of normal to the curve at point P is

$$m_N = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$$

On differentiating the curve w.r.t. x, we get

$$2y\frac{dy}{dx} - 6x + \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{6x}{2y + 1}$$

$$\therefore \quad m_N = -\frac{2y_1 + 1}{6x_1}$$

Now, the normal to the curve at point P intersect the Y-axis $\left(0, \frac{3}{2}\right)$, so slope of normal is $m_N = \frac{\frac{3}{2} - y_1}{0 - x_1}$

On equating the value of m_N, we get

$$\frac{3 - 2y_1}{x_1} = \frac{2y_1 + 1}{3x_1}$$

$\Rightarrow \quad 9 - 6y_1 = 2y_1 + 1 \quad (\because x_1 \neq 0)$

$\Rightarrow \quad 8y_1 = 8 \Rightarrow y_1 = 1$

$\because$ Point $P(x_1, y_1)$ on the curve, so

$$(1)^2 - 3x_1^2 + 1 + 10 = 0$$

$\Rightarrow \quad 3x_1^2 = 12$

$\Rightarrow \quad x_1^2 = 4$

$\Rightarrow \quad x_1 = \pm 2$

Now, slope of tangent to the curve at $P(x_1, y_1)$ is

$$m = -\frac{1}{m_N}$$

$$= \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = \frac{6x_1}{2y_1 + 1}$$

$$\Rightarrow \quad |m| = \left|\frac{6x_1}{2y_1+1}\right| = \left|\frac{6(\pm 2)}{2(1)+1}\right| = \frac{12}{3} = 4$$

Hence, answer is 4.

68 If S_1 and S_2 are respectively the sets of local minimum and local maximum points of the function, $f(x) = 9x^4 + 12x^3 - 36x^2 + 25$, $x \in R$, then **[2019, 8 April Shift-I]**

(a) $S_1 = \{-2\}$; $S_2 = \{0, 1\}$

(b) $S_1 = \{-2, 0\}$; $S_2 = \{1\}$

(c) $S_1 = \{-2, 1\}$; $S_2 = \{0\}$

(d) $S_1 = \{-1\}$; $S_2 = \{0, 2\}$

Ans. *(c)*

Given function is

$$f(x) = 9x^4 + 12x^3 - 36x^2 + 25 = y \text{ (let)}$$

For maxima or minima put $\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = 36x^3 + 36x^2 - 72x = 0$

$\Rightarrow \quad x^3 + x^2 - 2x = 0$

$\Rightarrow \quad x[x^2 + x - 2] = 0$

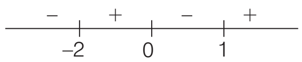
$\Rightarrow \quad x[x^2 + 2x - x - 2] = 0$

$\Rightarrow \quad x[x(x+2) - 1(x+2)] = 0$

$\Rightarrow \quad x(x-1)(x+2) = 0$

$\Rightarrow \quad x = -2, 0, 1$

By sign method, we have following



Since, $\frac{dy}{dx}$ changes it's sign from negative to positive at $x =$ '–2' and '1', so $x = -2, 1$ are points of local minima. Also, $\frac{dy}{dx}$ changes it's sign from positive to negative at $x = 0$, so $x = 0$ is point of local maxima.

$\therefore \quad S_1 = \{-2, 1\}$ and $S_2 = \{0\}$.

69 The height of a right circular cylinder of maximum volume inscribed in a sphere of radius 3 is **[2019, 8 April Shift-II]**

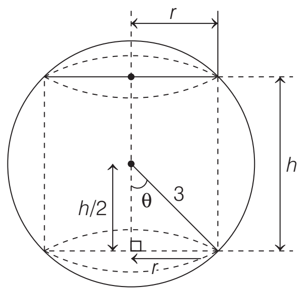
(a) $\sqrt{6}$ (b) $2\sqrt{3}$ (c) $\sqrt{3}$ (d) $\frac{2}{3}\sqrt{3}$

Ans. *(b)*

Key Idea

(i) Use formula of volume of cylinder, $V = \pi r^2 h$ where, r = radius and h = height

(ii) For maximum or minimum, put first derivative of V equal to zero



Let a sphere of radius 3, which inscribed a right circular cylinder having radius r and height is h, so

From the figure, $\frac{h}{2} = 3\cos\theta \Rightarrow h = 6\cos\theta$

and $\quad r = 3\sin\theta \quad$ ...(i)

$\because$ Volume of cylinder $V = \pi r^2 h$

$= \pi(3\sin\theta)^2(6\cos\theta)$

$= 54\pi\sin^2\theta\cos\theta$.

For maxima or minima, $\frac{dV}{d\theta} = 0$

$\Rightarrow 54\pi[2\sin\theta\cos^2\theta - \sin^3\theta] = 0$

$\Rightarrow \quad \sin\theta[2\cos^2\theta - \sin^2\theta] = 0$

$\Rightarrow \quad \tan^2\theta = 2 \quad \left[\because \theta \in \left(0, \frac{\pi}{2}\right)\right] \Rightarrow \tan\theta = \sqrt{2}$

$\Rightarrow \quad \sin\theta = \sqrt{\frac{2}{3}}$ and $\cos\theta = \frac{1}{\sqrt{3}} \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$$h = 6\frac{1}{\sqrt{3}} = 2\sqrt{3}$$

70 If $f(x)$ is a non-zero polynomial of degree four, having local extreme points at $x = -1, 0, 1$, then the set $S = \{x \in R : f(x) = f(0)\}$ contains exactly **[2019, 9 April Shift-I]**

(a) four rational numbers

(b) two irrational and two rational numbers

(c) four irrational numbers

(d) two irrational and one rational number

Ans. *(d)*

The non-zero four degree polynomial $f(x)$ has extremum points at $x = -1, 0, 1$, so we can assume

$$f'(x) = a(x+1)(x-0)(x-1) = ax(x^2 - 1)$$

where, a is non-zero constant.

$$f'(x) = ax^3 - ax$$

$$\Rightarrow \quad f(x) = \frac{a}{4}x^4 - \frac{a}{2}x^2 + C$$

[integrating both sides]

where, C is constant of integration.

Now, since $f(x) = f(0)$

$\Rightarrow \quad \frac{a}{4}x^4 - \frac{a}{2}x^2 + C = C \Rightarrow \frac{x^4}{4} = \frac{x^2}{2}$

$\Rightarrow \quad x^2(x^2 - 2) = 0 \Rightarrow x = -\sqrt{2}, 0, \sqrt{2}$

Thus, $f(x) = f(0)$ has one rational and two irrational roots.

71 If the tangent to the curve, $y = x^3 + ax - b$ at the point $(1, -5)$ is perpendicular to the line, $-x + y + 4 = 0$, then which one of the following points lies on the curve ? **[2019, 9 April Shift-I]**

(a) $(-2, 2)$ (b) $(2, -2)$

(c) $(-2, 1)$ (d) $(2, -1)$

Ans. *(b)*

Given curve is $y = x^3 + ax - b \quad$ ...(i)

passes through point $P(1, -5)$.

$\therefore \quad -5 = 1 + a - b \Rightarrow b - a = 6 \quad$ ...(ii)

and slope of tangent at point $P(1, -5)$ to the curve (i), is

$$m_1 = \left.\frac{dy}{dx}\right|_{(1,-5)} = [3x^2 + a]_{(1,-5)} = a + 3$$

$\because$ The tangent having slope $m_1 = a + 3$ at point $P(1, -5)$ is perpendicular to line $-x + y + 4 = 0$, whose slope is $m_2 = 1$.

$\therefore \quad a + 3 = -1 \Rightarrow a = -4 \quad [\because m_1 m_2 = -1]$

Now, on substituting $a = -4$ in Eq. (ii), we get $b = 2$

On putting $a = -2$ and $b = 2$ in Eq. (i), we get

$$y = x^3 - 4x - 2$$

Now, from option $(2, -2)$ is the required point which lie on it.

72 Let S be the set of all values of x for which the tangent to the curve $y = f(x) = x^3 - x^2 - 2x$ at (x, y) is parallel to the line segment joining the points $(1, f(1))$ and $(-1, f(-1))$, then S is equal to **[2019, 9 April Shift-I]**

(a) $\left\{\frac{1}{3}, -1\right\}$ (b) $\left\{\frac{1}{3}, 1\right\}$

(c) $\left\{-\frac{1}{3}, 1\right\}$ (d) $\left\{-\frac{1}{3}, -1\right\}$

Ans. *(c)*

Given curve is $y = f(x) = x^3 - x^2 - 2x \quad$ ...(i)

So, $\quad f(1) = 1 - 1 - 2 = -2$

and $\quad f(-1) = -1 - 1 + 2 = 0$

Since, slope of a line passing through (x_1, y_1) and (x_2, y_2) is given by

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

$\therefore$ Slope of line joining points $(1, f(1))$ and $(-1, f(-1))$ is

$$m = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{-2 - 0}{1 + 1} = -1$$

Now, $\quad \dfrac{dy}{dx} = 3x^2 - 2x - 2$

[differentiating Eq. (i), w.r.t. 'x']

According to the question,

$$\frac{dy}{dx} = m \Rightarrow 3x^2 - 2x - 2 = -1$$

$\Rightarrow \quad 3x^2 - 2x - 1 = 0$

$\Rightarrow \quad (x-1)(3x+1) = 0 \Rightarrow x = 1, -\dfrac{1}{3}$

Therefore, set $S = \left\{-\dfrac{1}{3}, 1\right\}$.

73 A water tank has the shape of an inverted right circular cone, whose semi-vertical angle is $\tan^{-1}\left(\dfrac{1}{2}\right)$.

Water is poured into it at a constant rate of 5 cu m/min. Then, the rate (in m/min) at which the level of water is rising at the instant when the depth of water in the tank is 10 m is **[2019, 9 April Shift-II]**

(a) $\dfrac{2}{\pi}$ (b) $\dfrac{1}{5\pi}$ (c) $\dfrac{1}{15\pi}$ (d) $\dfrac{1}{10\pi}$

Ans. *(b)*

Key Idea Use formula :

Volume of cone $= \dfrac{1}{3}\pi r^2 h$, where r = radius and h = height of the cone.

Given, semi-vertical angle of right circular cone

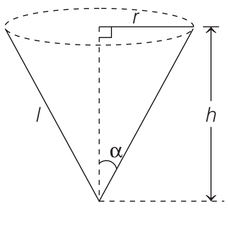
$$= \tan^{-1}\left(\frac{1}{2}\right)$$

Let $\quad \alpha = \tan^{-1}\left(\dfrac{1}{2}\right)$

$\Rightarrow \quad \tan\alpha = \dfrac{1}{2}$

$\Rightarrow \quad \dfrac{r}{h} = \dfrac{1}{2}$ [from fig. $\tan\alpha = \dfrac{r}{h}$]

$\Rightarrow \quad r = \dfrac{1}{2}h \quad$ ...(i)



$\because$ Volume of cone is $(V) = \dfrac{1}{3}\pi r^2 h$

$\therefore \quad V = \dfrac{1}{3}\pi\left(\dfrac{1}{2}h\right)^2 (h) = \dfrac{1}{12}\pi h^3$ [from Eq. (i)]

On differentiating both sides w.r.t. 't', we get

$$\frac{dV}{dt} = \frac{1}{12}\pi(3h^2)\frac{dh}{dt}$$

$\Rightarrow \quad \dfrac{dh}{dt} = \dfrac{4}{\pi h^2}\dfrac{dV}{dt}$

$\Rightarrow \quad \dfrac{dh}{dt} = \dfrac{4}{\pi h^2} \times 5$

[$\because$ given $\dfrac{dV}{dt} = 5\, m^3$/min]

Now, at $h = 10$ m, the rate at which height of water level is rising $= \left.\dfrac{dh}{dt}\right|_{h=10}$

$$= \frac{4}{\pi(10)^2} \times 5 = \frac{1}{5\pi} \text{ m/min}$$

74 If the tangent to the curve $y = \dfrac{x}{x^2 - 3}$, $x \in R$, $(x \neq \pm\sqrt{3})$, at a point $(\alpha, \beta) \neq (0, 0)$ on it is parallel to the line $2x + 6y - 11 = 0$, then **[2019, 10 April Shift-II]**

(a) $|6\alpha + 2\beta| = 19$ (b) $|6\alpha + 2\beta| = 9$
(c) $|2\alpha + 6\beta| = 19$ (d) $|2\alpha + 6\beta| = 11$

Ans. *(a)*

Equation of given curve is

$$y = \frac{x}{x^2 - 3}, x \in R, (x \neq \pm\sqrt{3}) \quad \text{...(i)}$$

On differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(x^2 - 3) - x(2x)}{(x^2 - 3)^2} = \frac{(-x^2 - 3)}{(x^2 - 3)^2}$$

It is given that tangent at a point $(\alpha, \beta) \neq (0, 0)$ on it is parallel to the line

$$2x + 6y - 11 = 0.$$

$\therefore \quad$ Slope of this line $= -\dfrac{2}{6} = \left.\dfrac{dy}{dx}\right|_{(\alpha, \beta)}$

$\Rightarrow \quad -\dfrac{\alpha^2 + 3}{(\alpha^2 - 3)^2} = -\dfrac{1}{3}$

$\Rightarrow \quad 3\alpha^2 + 9 = \alpha^4 - 6\alpha^2 + 9$

$\Rightarrow \quad \alpha^4 - 9\alpha^2 = 0 \Rightarrow \alpha = 0, -3, 3$

$\Rightarrow \quad \alpha = 3$ or -3, $\quad [\because \alpha \neq 0]$

Now, from Eq. (i),

$$\beta = \frac{\alpha}{\alpha^2 - 3}$$

$\Rightarrow \quad \beta = \dfrac{3}{9 - 3}$ or $\dfrac{-3}{9 - 3} = \dfrac{1}{2}$ or $-\dfrac{1}{2}$

According to the options, $|6\alpha + 2\beta| = 19$ at

$$(\alpha, \beta) = \left(\pm 3, \pm\frac{1}{2}\right)$$

75 The maximum volume (in cum) of the right circular cone having slant height 3m is **[2019, 9 Jan. Shift-I]**

(a) $\dfrac{4}{3}\pi$ (b) $2\sqrt{3}\pi$
(c) $3\sqrt{3}\pi$ (d) 6π

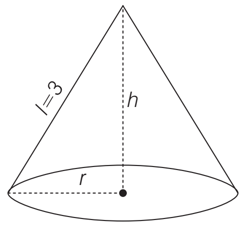
Ans. *(b)*

Let h = height of the cone,

r = radius of circular base

$= \sqrt{(3)^2 - h^2} \quad [\because l^2 = h^2 + r^2]$

$= \sqrt{9 - h^2} \quad$ ...(i)



Now, volume (V) of cone $= \dfrac{1}{3}\pi r^2 h$

$\Rightarrow \quad V(h) = \dfrac{1}{3}\pi(9 - h^2)h \quad$ [from Eq. (i)]

$= \dfrac{1}{3}\pi[9h - h^3] \quad$ ...(ii)

For maximum volume $V'(h) = 0$ and $V''(h) < 0$.

Here, $\quad V'(h) = 0 \Rightarrow (9 - 3h^2) = 0$

$\Rightarrow \quad h = \sqrt{3} \quad [\because h \nless 0]$

and $\quad V''(h) = \dfrac{1}{3}\pi(-6h) < 0$ for $h = \sqrt{3}$

Thus, volume is maximum when $h = \sqrt{3}$

Now, maximum volume

$V(\sqrt{3}) = \dfrac{1}{3}\pi(9\sqrt{3} - 3\sqrt{3})$ [from Eq. (ii)]

$= 2\sqrt{3}\pi$

76 If θ denotes the acute angle between the curves, $y = 10 - x^2$ and $y = 2 + x^2$ at a point of their intersection, then $|\tan\theta|$ is equal to **[2019, 9 Jan. Shift-I]**

(a) $\dfrac{7}{17}$ (b) $\dfrac{8}{15}$ (c) $\dfrac{4}{9}$ (d) $\dfrac{8}{17}$

Ans. *(b)*

Key Idea Angle between two curves is the angle between the tangents to the curves at the point of intersection.

Given equation of curves are

$$y = 10 - x^2 \quad \text{...(i)}$$

and $\quad y = 2 + x^2 \quad$ ...(ii)

For point of intersection, consider

$$10 - x^2 = 2 + x^2$$

$\Rightarrow \quad 2x^2 = 8$

$\Rightarrow \quad x^2 = 4$

$\Rightarrow \quad x = \pm 2$

Clearly, when $x = 2$, then $y = 6$ (using Eq. (i))

and when $x = -2$, then $y = 6$

Thus, the point of intersection are $(2, 6)$ and $(-2, 6)$.

Let m_1 be the slope of tangent to the curve (i) and m_2 be the slope of tangent to the curve (ii).

For curve (i) $\frac{dy}{dx} = -2x$ and for curve (ii) $\frac{dy}{dx} = 2x$

$\therefore$ At $(2, 6)$, slopes $m_1 = -4$ and $m_2 = 4$, and in that case

$$|\tan\theta| = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{4+4}{1-16}\right| = \frac{8}{15}$$

At $(-2, 6)$, slopes $m_1 = 4$ and $m_2 = -4$ and in that case

$$|\tan\theta| = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{-4-4}{1-16}\right| = \frac{8}{15}$$

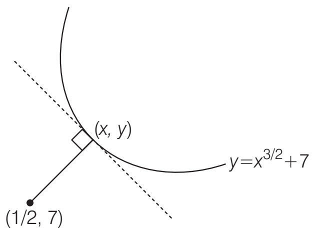
77 A helicopter is flying along the curve given by $y - x^{3/2} = 7, (x \geq 0)$. A soldier positioned at the point $\left(\frac{1}{2}, 7\right)$ wants to shoot down the helicopter when it is nearest to him. Then, this nearest distance is **[2019, 10 Jan. Shift-II]**

(a) $\frac{1}{3}\sqrt{\frac{7}{3}}$ (b) $\frac{\sqrt{5}}{6}$

(c) $\frac{1}{6}\sqrt{\frac{7}{3}}$ (d) $\frac{1}{2}$

Ans. *(c)*

The helicopter is nearest to the soldier, if the tangent to the path, $y = x^{3/2} + 7$, $(x \geq 0)$ of helicopter at point (x, y) is perpendicular to the line joining (x, y) and the position of soldier $\left(\frac{1}{2}, 7\right)$.



$\because$ Slope of tangent at point (x, y) is

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2} = m_1 \text{(let)} \quad \ldots\text{(i)}$$

and slope of line joining (x, y) and $\left(\frac{1}{2}, 7\right)$ is

$$m_2 = \frac{y-7}{x-\frac{1}{2}} \quad \ldots\text{(ii)}$$

Now, $\quad m_1 \cdot m_2 = -1$

$$\Rightarrow \frac{3}{2}x^{1/2}\left(\frac{y-7}{x-(1/2)}\right) = -1$$

[from Eqs. (i) and (ii)]

$$\Rightarrow \frac{3}{2}x^{1/2}\,\frac{x^{3/2}}{x-1/2} = -1 \quad [\because y = x^{3/2} + 7]$$

$$\Rightarrow \frac{3}{2}x^2 = -x + \frac{1}{2}$$

$$\Rightarrow 3x^2 + 2x - 1 = 0$$

$$\Rightarrow 3x^2 + 3x - x - 1 = 0$$

$$\Rightarrow 3x(x+1) - 1(x+1) = 0 \Rightarrow x = \frac{1}{3}, -1$$

$$\because \quad x \geq 0 \quad \therefore \; x = \frac{1}{3}$$

and so, $\quad y = \left(\frac{1}{3}\right)^{3/2} + 7 \quad [\because y = x^{3/2} + 7]$

Thus, the nearest point is $\left(\frac{1}{3}, \left(\frac{1}{3}\right)^{3/2} + 7\right)$

Now, the nearest distance

$$= \sqrt{\left(\frac{1}{2} - \frac{1}{3}\right)^2 + \left(7 - \left(\frac{1}{3}\right)^{3/2} - 7\right)^2}$$

$$= \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{3}\right)^3} = \sqrt{\frac{1}{36} + \frac{1}{27}}$$

$$= \sqrt{\frac{3+4}{108}} = \sqrt{\frac{7}{108}} = \frac{1}{6}\sqrt{\frac{7}{3}}$$

78 The tangent to the curve, $y = xe^{x^2}$ passing through the point $(1, e)$ also passes through the point **[2019, 10 Jan. Shift-II]**

(a) $\left(\frac{4}{3}, 2e\right)$ (b) $(3, 6e)$

(c) $(2, 3e)$ (d) $\left(\frac{5}{3}, 2e\right)$

Ans. *(a)*

Given equation of curve is

$$y = xe^{x^2} \quad \ldots\text{(i)}$$

Note that $(1, e)$ lie on the curve, so the point of contact is $(1, e)$.

Now, slope of tangent, at point $(1, e)$, to the curve (i) is

$$\left.\frac{dy}{dx}\right|_{(1,e)} = \left(x(2x)\,e^{x^2} + e^{x^2}\right)_{(1,e)}$$

$$= 2e + e = 3e$$

Now, equation of tangent is given by

$$(y - y_1) = m(x - x_1)$$

$$y - e = 3e(x - 1) \Rightarrow y = 3ex - 2e$$

On checking all the options, the option $\left(\frac{4}{3}, 2e\right)$ satisfy the equation of tangent.

79 The maximum value of the function $f(x) = 3x^3 - 18x^2 + 27x - 40$ on the set $S = \{x \in R : x^2 + 30 \leq 11x\}$ is **[2019, 11 Jan. Shift-I]**

(a) 122 (b) -122 (c) -222 (d) 222

Ans. *(a)*

We have, $f(x) = 3x^3 - 18x^2 + 27x - 40$

$\Rightarrow f'(x) = 9x^2 - 36x + 27$

$= 9(x^2 - 4x + 3) = 9(x-1)(x-3) \ldots$(i)

Also, we have

$$S = \{x \in R : x^2 + 30 \leq 11x\}$$

Clearly, $\quad x^2 + 30 \leq 11x$

$\Rightarrow \quad x^2 - 11x + 30 \leq 0$

$\Rightarrow \quad (x-5)(x-6) \leq 0 \Rightarrow x \in [5, 6]$

So, $\quad S = [5, 6]$

Note that $f(x)$ is increasing in $[5, 6]$

$[\because f'(x) > 0$ for $x \in [5, 6]]$

$\therefore f(6)$ is maximum, where

$$f(6) = 3(6)^3 - 18(6)^2 + 27(6) - 40 = 122$$

80 If the function f given by $f(x) = x^3 - 3(a-2)x^2 + 3ax + 7$, for some $a \in R$ is increasing in $(0, 1]$ and decreasing in $[1, 5)$, then a root of the equation, $\frac{f(x) - 14}{(x-1)^2} = 0 (x \neq 1)$ is **[2019, 12 Jan. Shift-I]**

(a) -7 (b) 6

(c) 7 (d) 5

Ans. *(c)*

Given that function,

$f(x) = x^3 - 3(a-2)x^2 + 3ax + 7$, for some $a \in R$ is increasing in $(0, 1]$ and decreasing in $[1, 5)$.

$f'(1) = 0$ [$\because$ tangent at $x = 1$ will be parallel to X-axis]

$\Rightarrow \quad (3x^2 - 6(a-2)x + 3a)_{x=1} = 0$

$\Rightarrow \quad 3 - 6(a-2) + 3a = 0$

$\Rightarrow \quad 3 - 6a + 12 + 3a = 0$

$\Rightarrow \quad 15 - 3a = 0 \Rightarrow a = 5$

So, $\quad f(x) = x^3 - 9x^2 + 15x + 7$

$\Rightarrow \quad f(x) - 14 = x^3 - 9x^2 + 15x - 7$

$\Rightarrow \quad f(x) - 14 = (x-1)(x^2 - 8x + 7)$

$= (x-1)(x-1)(x-7)$

$$\Rightarrow \frac{f(x) - 14}{(x-1)^2} = (x - 7) \quad \ldots\text{(i)}$$

Now, $\quad \frac{f(x) - 14}{(x-1)^2} = 0, (x \neq 1)$

$\Rightarrow \quad x - 7 = 0 \quad$ [from Eq. (i)]

$\Rightarrow \quad x = 7$

81 The tangent to the curve $y = x^2 - 5x + 5$, parallel to the line $2y = 4x + 1$, also passes through the point **[2019, 12 Jan. Shift-II]**

(a) $\left(\frac{1}{4}, \frac{7}{2}\right)$ (b) $\left(\frac{7}{2}, \frac{1}{4}\right)$

(c) $\left(-\frac{1}{8}, 7\right)$ (d) $\left(\frac{1}{8}, -7\right)$

Ans. *(d)*

The given curve is $y = x^2 - 5x + 5$ ...(i)

Now, slope of tangent at any point (x, y) on the curve is

$$\frac{dy}{dx} = 2x - 5 \quad \text{...(ii)}$$

[on differentiating Eq. (i) w.r.t. x]

$\because$ It is given that tangent is parallel to line

$$2y = 4x + 1$$

So, $\frac{dy}{dx} = 2$

[$\because$ slope of line $2y = 4x + 1$ is 2]

$\Rightarrow \quad 2x - 5 = 2 \Rightarrow 2x = 7 \Rightarrow x = \frac{7}{2}$

On putting $x = \frac{7}{2}$ in Eq. (i), we get

$$y = \frac{49}{4} - \frac{35}{2} + 5 = \frac{69}{4} - \frac{35}{2} = -\frac{1}{4}$$

Now, equation of tangent to the curve (i) at point $\left(\frac{7}{2}, -\frac{1}{4}\right)$ and having slope 2, is

$$y + \frac{1}{4} = 2\left(x - \frac{7}{2}\right) \Rightarrow y + \frac{1}{4} = 2x - 7$$

$\Rightarrow \quad y = 2x - \frac{29}{4}$ ...(iii)

On checking all the options, we get the point $\left(\frac{1}{8}, -7\right)$ satisfy the line (iii).

82 The equation of a tangent to the parabola, $x^2 = 8y$, which makes an angle θ with the positive direction of X-axis, is **[2019, 12 Jan. Shift-II]**

(a) $y = x\tan\theta - 2\cot\theta$
(b) $x = y\cot\theta + 2\tan\theta$
(c) $y = x\tan\theta + 2\cot\theta$
(d) $x = y\cot\theta - 2\tan\theta$

Ans. *(b)*

Given parabola is $x^2 = 8y$ ...(i)

Now, slope of tangent at any point (x, y) on the parabola (i) is

$$\frac{dy}{dx} = \frac{x}{4} = \tan\theta$$

[$\because$ tangent is making an angle θ with the positive direction of X-axis]

So, $x = 4\tan\theta \Rightarrow 8y = (4\tan\theta)^2$

[on putting $x = 4\tan\theta$ in Eq. (i)]

$\Rightarrow \quad y = 2\tan^2\theta$

Now, equation of required tangent is

$$y - 2\tan^2\theta = \tan\theta(x - 4\tan\theta)$$

$\Rightarrow \quad y = x\tan\theta - 2\tan^2\theta$

$\Rightarrow \quad x = y\cot\theta + 2\tan\theta$

83 If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is **[JEE Main 2018]**

(a) 6 (b) $\frac{7}{2}$ (c) 4 (d) $\frac{9}{2}$

Ans. *(d)*

We have, $y^2 = 6x \Rightarrow 2y\frac{dy}{dx} = 6 \Rightarrow \frac{dy}{dx} = \frac{3}{y}$

Slope of tangent at (x_1, y_1) is $m_1 = \frac{3}{y_1}$

Also, $9x^2 + by^2 = 16$

$\Rightarrow \quad 18x + 2by\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-9x}{by}$

Slope of tangent at (x_1, y_1) is $m_2 = \frac{-9x_1}{by_1}$

Since, these are intersection at right angle.

$\therefore \quad m_1 m_2 = -1$

$\Rightarrow \quad \frac{27x_1}{by_1^2} = 1$

$\Rightarrow \quad \frac{27x_1}{6bx_1} = 1 \quad [\because y_1^2 = 6x_1]$

$\Rightarrow \quad b = \frac{9}{2}$

84 Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$, $x \in \mathbf{R} - \{-1, 0, 1\}$. If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of $h(x)$ is **[JEE Main 2018]**

(a) 3 (b) -3 (c) $-2\sqrt{2}$ (d) $2\sqrt{2}$

Ans. *(d)*

We have,

$$f(x) = x^2 + \frac{1}{x^2} \text{ and } g(x) = x - \frac{1}{x}$$

$\Rightarrow h(x) = \frac{f(x)}{g(x)}$

$$\therefore \; h(x) = \frac{x^2 + \frac{1}{x^2}}{x - \frac{1}{x}} = \frac{\left(x - \frac{1}{x}\right)^2 + 2}{x - \frac{1}{x}}$$

$$\Rightarrow h(x) = \left(x - \frac{1}{x}\right) + \frac{2}{x - \frac{1}{x}}$$

$$x - \frac{1}{x} > 0, \left(x - \frac{1}{x}\right) + \frac{2}{x - \frac{1}{x}} \in [2\sqrt{2}, \infty)$$

$$x - \frac{1}{x} < 0, \left(x - \frac{1}{x}\right) + \frac{2}{x - \frac{1}{x}} \in (-\infty, 2\sqrt{2}]$$

$\therefore$ Local minimum value is $2\sqrt{2}$.

85 If 20 m of wire is available for fencing off a flower-bed in the form of a circular sector, then the maximum area (in sq m) of the flower-bed is **[JEE Main 2017]**

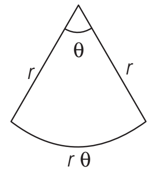
(a) 12.5 (b) 10
(c) 25 (d) 30

Ans. *(c)*

Total length $= 2r + r\theta = 20$

$\Rightarrow \quad \theta = \frac{20 - 2r}{r}$

Now, area of flower-bed,



$$A = \frac{1}{2}r^2\theta$$

$\Rightarrow \quad A = \frac{1}{2}r^2\left(\frac{20 - 2r}{r}\right)$

$\Rightarrow \quad A = 10r - r^2$

$\therefore \quad \frac{dA}{dr} = 10 - 2r$

For maxima or minima, put $\frac{dA}{dr} = 0$.

$\Rightarrow \quad 10 - 2r = 0$

$\Rightarrow \quad r = 5$

$\therefore \quad A_{\max} = \frac{1}{2}(5)^2\left[\frac{20 - 2(5)}{5}\right]$

$$= \frac{1}{2} \times 25 \times 2 = 25 \text{ sq m}$$

86 The normal to the curve $y(x-2)(x-3) = x + 6$ at the point, where the curve intersects the Y-axis passes through the point **[JEE Main 2017]**

(a) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, \frac{1}{2}\right)$

(c) $\left(\frac{1}{2}, -\frac{1}{3}\right)$ (d) $\left(\frac{1}{2}, \frac{1}{3}\right)$

Ans. *(b)*

Given curve is

$$y(x-2)(x-3) = x + 6 \quad \text{...(i)}$$

Put $x = 0$ in Eq. (i), we get

$$y(-2)(-3) = 6$$

$\Rightarrow \quad y = 1$

So, point of intersection is $(0, 1)$.

Now, $y=\dfrac{x+6}{(x-2)(x-3)}$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{1(x-2)(x-3)-(x+6)(x-3+x-2)}{(x-2)^2(x-3)^2}$

$\Rightarrow \left(\dfrac{dy}{dx}\right)_{(0,1)}=\dfrac{6+30}{4\times 9}=\dfrac{36}{36}=1$

$\therefore$ Equation of normal at (0, 1) is given by

$$y-1=\frac{-1}{1}(x-0) \Rightarrow x+y-1=0$$

which passes through the point $\left(\dfrac{1}{2},\dfrac{1}{2}\right)$.

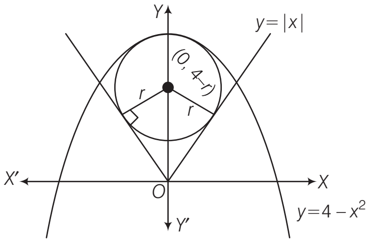
87 The radius of a circle having minimum area, which touches the curve $y=4-x^2$ and the lines $y=|x|$, is **[JEE Main 2017]**

(a) $2(\sqrt{2}+1)$ (b) $2(\sqrt{2}-1)$
(c) $4(\sqrt{2}-1)$ (d) $4(\sqrt{2}+1)$

Ans. *(c)*

Let the radius of circle with least area be r.

Then, coordinates of centre $=(0,4-r)$.



Since, circle touches the line $y=x$ in first quadrant.

$\therefore \quad \left|\dfrac{0-(4-r)}{\sqrt{2}}\right|=r$

$\Rightarrow \quad r-4=\pm r\sqrt{2}$

$\Rightarrow \quad r=\dfrac{4}{\sqrt{2}+1}$ or $\dfrac{4}{1-\sqrt{2}}$

But $\quad r\neq\dfrac{4}{1-\sqrt{2}} \quad \left[\because \dfrac{4}{1-\sqrt{2}}<0\right]$

$\therefore \quad r=\dfrac{4}{\sqrt{2}+1}=4(\sqrt{2}-1)$

88 Consider $f(x)=\tan^{-1}\left(\sqrt{\dfrac{1+\sin x}{1-\sin x}}\right)$, $x\in\left(0,\dfrac{\pi}{2}\right)$.

A normal to $y=f(x)$ at $x=\dfrac{\pi}{6}$ also passes through the point **[JEE Main 2016]**

(a) (0, 0) (b) $\left(0,\dfrac{2\pi}{3}\right)$
(c) $\left(\dfrac{\pi}{6},0\right)$ (d) $\left(\dfrac{\pi}{4},0\right)$

Ans. *(b)*

We have, $f(x)=\tan^{-1}\sqrt{\dfrac{1+\sin x}{1-\sin x}}$, $x\in\left(0,\dfrac{\pi}{2}\right)$

$$\Rightarrow \quad f(x)=\tan^{-1}\sqrt{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}$$

$$=\tan^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right)$$

$$\left(\because \cos\frac{x}{2}>\sin\frac{x}{2} \text{ for } 0<\frac{x}{2}<\frac{\pi}{4}\right)$$

$$=\tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)$$

$$=\tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]=\frac{\pi}{4}+\frac{x}{2}$$

$\Rightarrow \quad f'(x)=\dfrac{1}{2} \Rightarrow f'\left(\dfrac{\pi}{6}\right)=\dfrac{1}{2}$

Now, equation of normal at $x=\dfrac{\pi}{6}$ is given by

$$\left[y-f\left(\frac{\pi}{6}\right)\right]=-2\left(x-\frac{\pi}{6}\right)$$

$$\Rightarrow \quad \left(y-\frac{\pi}{3}\right)=-2\left(x-\frac{\pi}{6}\right)$$

$$\left[\because f\left(\frac{\pi}{6}\right)=\frac{\pi}{4}+\frac{\pi}{12}=\frac{4\pi}{12}=\frac{\pi}{3}\right]$$

which passes through $\left(0,\dfrac{2\pi}{3}\right)$.

89 A wire of length 2 units is cut into two parts which are bent respectively to form a square of side $=x$ units and a circle of radius $=r$ units. If the sum of the areas of the square and the circle so formed is minimum, then **[JEE Main 2016 (Offline)]**

(a) $2x=(\pi+4)r$ (b) $(4-\pi)x=\pi r$
(c) $x=2r$ (d) $2x=r$

Ans. *(c)*

According to given information, we have

Perimeter of square + Perimeter of circle = 2 units

$\Rightarrow \quad 4x+2\pi r=2$

$\Rightarrow \quad r=\dfrac{1-2x}{\pi} \quad$ ...(i)

Now, let A be the sum of the areas of the square and the circle. Then,

$$A=x^2+\pi r^2$$

$$=x^2+\pi\frac{(1-2x)^2}{\pi^2}$$

$\Rightarrow \quad A(x)=x^2+\dfrac{(1-2x)^2}{\pi}$

Now, for minimum value of $A(x)$, $\dfrac{dA}{dx}=0$

$\Rightarrow \quad 2x+\dfrac{2(1-2x)}{\pi}\cdot(-2)=0$

$\Rightarrow \quad x=\dfrac{2-4x}{\pi}$

$\Rightarrow \quad \pi x+4x=2$

$\Rightarrow \quad x=\dfrac{2}{\pi+4} \quad$ ...(ii)

Now, from Eq. (i), we get

$$r=\frac{1-2\cdot\frac{2}{\pi+4}}{\pi}$$

$=\dfrac{\pi+4-4}{\pi(\pi+4)}=\dfrac{1}{\pi+4} \quad$ ...(iii)

From Eqs. (ii) and (iii), we get

$$x=2r$$

90 The normal to the curve $x^2+2xy-3y^2=0$ at (1,1) **[JEE Main 2015]**

(a) does not meet the curve again
(b) meets the curve again in the second quadrant
(c) meets the curve again in the third quadrant
(d) meets the curve again in the fourth quadrant

Ans. *(d)*

Given equation of curve is

$x^2+2xy-3y^2=0 \quad$ ...(i)

On differentiating w.r.t. x, we get

$$2x+2xy'+2y-6yy'=0$$

$\Rightarrow \quad y'=\dfrac{x+y}{3y-x}$

At $x=1, y=1, y'=1$

i.e. $\quad \left(\dfrac{dy}{dx}\right)_{(1,1)}=1$

Equation of normal at (1, 1) is

$$y-1=-\frac{1}{1}(x-1)$$

$\Rightarrow \quad y-1=-(x-1)$

$\Rightarrow \quad x+y=2 \quad$ ...(ii)

On solving Eqs. (i) and (ii) simultaneously, we get

$$x^2+2x(2-x)-3(2-x)^2=0$$

$\Rightarrow \quad x^2+4x-2x^2-3(4+x^2-4x)=0$

$\Rightarrow \quad -x^2+4x-12-3x^2+12x=0$

$\Rightarrow \quad -4x^2+16x-12=0$

$\Rightarrow \quad 4x^2-16x+12=0$

$\Rightarrow \quad x^2-4x+3=0$

$\Rightarrow \quad (x-1)(x-3)=0$

$\Rightarrow \quad x=1, 3$

Now, when $x=1$, then $y=1$

and when $x=3$, then $y=-1$

$\therefore \quad P=(1,1)$ and $Q=(3,-1)$

Hence, normal meets the curve again at $(3,-1)$ in fourth quadrant.

Aliter

Given, $x^2+2xy-3y^2=0$

$\Rightarrow \quad (x-y)(x+3y)=0$

$\Rightarrow \quad x-y=0$ or $x+3y=0$

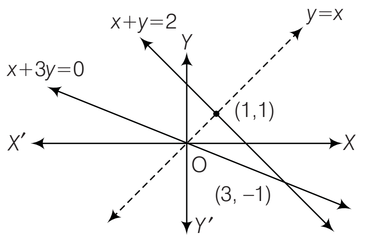
Equation of normal at $(1,1)$ is

$$y-1=-1(x-1)$$

$\Rightarrow \quad x+y-2=0$

It intersects $x+3y=0$ at $(3,-1)$

and hence normal meet the curve in fourth quadrant.



91 Let $f(x)$ be a polynomial of degree four having extreme values at $x=1$ and $x=2$. If $\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right]=3$, then $f(2)$ is equal to **[JEE Main 2015]**

(a) -8 (b) -4 (c) 0 (d) 4

Ans. *(c)*

Central Idea Any function have extreme values (maximum or minimum) at its critical points, where $f'(x)=0$.

Since, the function have extreme values at $x=1$ and $x=2$.

$\therefore \quad f'(x)=0$ at $x=1$ and $x=2$

$\Rightarrow \quad f'(1)=0$ and $f'(2)=0$

Also it is given that

$$\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right]=3$$

$\Rightarrow 1+\lim_{x\to 0}\frac{f(x)}{x^2}=3 \Rightarrow \lim_{x\to 0}\frac{f(x)}{x^2}=2$

$\Rightarrow f(x)$ will be of the form

$ax^4+bx^3+2x^2$

[$\because f(x)$ is of four degree polynomial]

Let $\quad f(x)=ax^4+bx^3+2x^2$

$\Rightarrow \quad f'(x)=4ax^3+3bx^2+4x$

$\Rightarrow \quad f'(1)=4a+3b+4=0 \quad$ ...(i)

and $\quad f'(2)=32a+12b+8=0$

$\Rightarrow \quad 8a+3b+2=0 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$a=\frac{1}{2}, b=-2$$

$\therefore \quad f(x)=\frac{x^4}{2}-2x^3+2x^2$

$\Rightarrow \quad f(2)=8-16+8=0$

92 If $x=-1$ and $x=2$ are extreme points of $f(x)=\alpha\log|x|+\beta x^2+x$, then **[JEE Main 2014]**

(a) $\alpha=-6, \beta=\frac{1}{2}$ (b) $\alpha=-6, \beta=-\frac{1}{2}$

(c) $\alpha=2, \beta=-\frac{1}{2}$ (d) $\alpha=2, \beta=\frac{1}{2}$

Ans. *(c)*

Here, $x=-1$ and $x=2$ are extreme points of $f(x)=\alpha\log|x|+\beta x^2+x$, then

$$f'(x)=\frac{\alpha}{x}+2\beta x+1$$

$\therefore \quad f'(-1)=-\alpha-2\beta+1=0 \quad$ ...(i)

[at extreme point, $f'(x)=0$]

and $f'(-2)=\frac{\alpha}{2}+4\beta+1=0 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$\alpha=2, \beta=-\frac{1}{2}$$

93 The intercepts on X-axis made by tangents to the curve, $y=\int_0^x |t|\,dt, x\in R$, which are parallel to the line $y=2x$, are equal to **[JEE Main 2013]**

(a) ± 1 (b) ± 2

(c) ± 3 (d) ± 4

Ans. *(a)*

Given, $y=\int_0^x |t|\,dt \quad \therefore \quad \frac{dy}{dx}=|x|$

Since, tangent to the curve is parallel to line $y=2x$.

$\Rightarrow \quad \frac{dy}{dx}=2$

$\therefore \quad x=\pm 2$

$\therefore$ Points, $\quad y=\int_0^{\pm 2}|t|\,dt=\pm 2$

$\therefore$ Equation of tangents are

$$y-2=2(x-2)$$
$$y+2=2(x+2)$$

For x-intercept, put $y=0$, we get

$$0-2=2(x-2)$$
$$0+2=2(x+2)$$

$\Rightarrow \quad x=\pm 1$

94 Let $a, b\in R$ be such that the function f given by $f(x)=\log|x|+bx^2+ax, x\neq 0$ has extreme values at $x=-1$ and $x=2$.

Statement I f has local maximum at $x=-1$ and at $x=2$.

Statement II $a=\frac{1}{2}$ and $b=\frac{-1}{4}$ **[AIEEE 2012]**

(a) Statement I is false, Statement II is true

(b) Statement I is true, Statement II is true; Statement II is a correct explanation of Statement I

(c) Statement I is true, Statement II is true; Statement II is not a correct explanation of Statement I

(d) Statement I is true, Statement II is false

Ans. *(c)*

Given,

(i) A function f, such that $f(x)=\log|x|+bx^2+ax, x\neq 0$

(ii) The function f has extrema at $x=-1$ and $x=2$, *i.e.*, $f'(-1)=f'(2)=0$ and $f''(-1)\neq 0\neq f''(2)$.

Now, given function f is given by

$$f(x)=\log|x|+bx^2+ax$$

$\Rightarrow \quad f'(x)=\frac{1}{x}+2bx+a$

$\Rightarrow \quad f''(x)=\frac{-1}{x^2}+2b$

Since, f has extrema at $x=-1$ and $x=2$.

Hence, $\quad f'(-1)=0=f'(2)$

$f'(-1)=0$

$\Rightarrow \quad a-2b=1 \quad$ ...(i)

and $\quad f'(2)=0$

$\Rightarrow \quad a+4b=\frac{-1}{2} \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$a=\frac{1}{2} \text{ and } b=\frac{-1}{4}$$

$\Rightarrow \quad f''(x)=\frac{-1}{x^2}+\frac{-1}{2}=-\left(\frac{x^2+2}{2x^2}\right)$

$\Rightarrow \quad f''(-1)<0$ and $f''(2)<0$

Hence, f has local maxima at both $x=-1$ and $x=2$. Hence, Statement I is correct. Also, while solving for Statement I, we found the values of a and b, which justify that Statement II is also correct.

However, Statement II does not explain Statement I in any way.

95 Let f be a function defined by

$$f(x)=\begin{cases}\dfrac{\tan x}{x}, & x\neq 0\\ 1\quad, & x=0\end{cases}$$

Statement I $x=0$ is point of minima of f.

Statement II $f'(0)=0$ **[AIEEE 2011]**

(a) Statement I is false, Statement II is true

(b) Statement I is true, Statement II is true; Statement II is correct explanation of Statement I

(c) Statement I is true, Statement II is true; Statement II is not a correct explanation of Statement I

(d) Statement I is true, Statement II is false

Ans. ***(c)***

$$f(x)=\begin{cases}\dfrac{\tan x}{x}, & x\neq 0\\ 1\ , & x=0\end{cases}$$

As, $\dfrac{\tan x}{x}>1,\ \forall x\neq 0$

$\therefore\quad f(0+h)>f(0)$ and $f(0-h)>f(0)$

At $x=0$, $f(x)$ attains minim(a)

$$\therefore\quad f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{\frac{\tan h}{h}-1}{h}$$

$$=\lim_{h\to 0}\frac{\tan h-h}{h^2}$$

$$=\lim_{h\to 0}\frac{\sec^2 h-1}{2h}\quad[\text{using L' Hospital rule}]$$

$$=\lim_{h\to 0}\frac{\tan^2 h}{2h^2}\cdot h\quad[\because \tan^2\theta=\sec^2\theta-1]$$

$$=\frac{1}{2}\cdot 0=0\quad\left[\because \lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1\right]$$

So, Statement II is true.

Hence, both statements are true but Statement II is not the correct explanation of Statement I.

96 Let $f:R\to R$ be a positive increasing function with $\lim\limits_{x\to\infty}\dfrac{f(3x)}{f(x)}=1$. Then, $\lim\limits_{x\to\infty}\dfrac{f(2x)}{f(x)}$ is equal to **[AIEEE 2010]**

(a) 1 (b) $\dfrac{2}{3}$ (c) $\dfrac{3}{2}$ (d) 3

Ans. ***(a)***

Since, $f(x)$ is a positive increasing function.

$$\Rightarrow\quad 0<f(x)<f(2x)<f(3x)$$

$$\Rightarrow\quad 0<1<\frac{f(2x)}{f(x)}<\frac{f(3x)}{f(x)}$$

$$\Rightarrow\quad \lim_{x\to\infty}1\le\lim_{x\to\infty}\frac{f(2x)}{f(x)}\le\lim_{x\to\infty}\frac{f(3x)}{f(x)}$$

By Sandwich theorem, $\lim\limits_{x\to\infty}\dfrac{f(2x)}{f(x)}=1$

97 The equation of the tangent to the curve $y=x+\dfrac{4}{x^2}$, *i.e.*, parallel to X-axis, is **[AIEEE 2010]**

(a) $y=0$ (b) $y=1$ (c) $y=2$ (d) $y=3$

Ans. ***(d)***

We have, $y=x+\dfrac{4}{x^2}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx}=1-\frac{8}{x^3}$$

Since, the tangent is parallel to X-axis, therefore

$$\frac{dy}{dx}=0\ \Rightarrow\ x^3=8$$

$$\Rightarrow\quad x=2\ \text{ and }\ y=3$$

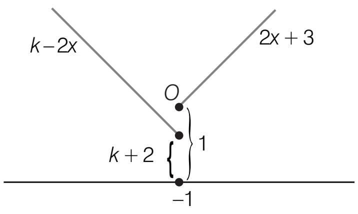
98 Let $f:R\to R$ be defined by

$$f(x)=\begin{cases}k-2x, & \text{if } x\le -1\\ 2x+3, & \text{if } x>-1\end{cases}.$$

If f has a local minimum at $x=-1$, then a possible value of k is **[AIEEE 2010]**

(a) 1 (b) 0 (c) $-\dfrac{1}{2}$ (d) -1

Ans. ***(d)***



$$\because\quad k+2\le 1$$

$$\therefore\quad k\le -1$$

99 Given, $P(x)=x^4+ax^3+bx^2+cx+d$ such that $x=0$ is the only real root of $P'(x)=0$. If $P(-1)<P(1)$, then in the interval $[-1,1]$, **[AIEEE 2009]**

(a) $P(-1)$ is the minimum and $P(1)$ is the maximum of P

(b) $P(-1)$ is not minimum but $P(1)$ is the maximum of P

(c) $P(-1)$ is the minimum and $P(1)$ is not the maximum of P

(d) Neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P

Ans. ***(b)***

Given, $P(x)=x^4+ax^3+bx^2+cx+d$

$\Rightarrow\quad P'(x)=4x^3+3ax^2+2bx+c$

Since, $x=0$ is a solution for $P'(x)=0$, then

$$c=0$$

$$\therefore\quad P(x)=x^4+ax^3+bx^2+d \quad\ldots(\text{i})$$

Also, we have $P(-1)<P(1)$

$\Rightarrow\quad 1-a+b+d<1+a+b+d$

$\Rightarrow\quad a>0$

Since, $P'(x)=0$, only when $x=0$ and $P(x)$ is differentiable in $(-1,1)$, we should have the maximum and minimum at the points $x=-1$, 0 and 1.

Also, we have $P(-1)<P(1)$

$\therefore$ Maximum of $P(x)=\text{Max}\{P(0),P(1)\}$

and minimum of $P(x)=\text{Min}\{P(-1),P(0)\}$

In the interval $[0,1]$,

$$P'(x)=4x^3+3ax^2+2bx$$

$$=x(4x^2+3ax+2b)$$

Since, $P'(x)$ has only one root $x=0$, then $4x^2+3ax+2b=0$ has no real roots.

$\therefore\quad (3a)^2-32b<0$

$\Rightarrow\quad \dfrac{3a^2}{32}<b$

$\therefore\quad b>0$

Thus, we have $a>0$ and $b>0$.

$\therefore\quad P'(x)=4x^3+4ax^2+2bx>0$, $\forall\ x\in(0,1)$

Hence, $P(x)$ is increasing in $[0,1]$.

$\therefore$ Maximum of $P(x)=P(1)$

Similarly, $P(x)$ is decreasing in $[-1,0]$.

Therefore, minimum $P(x)$ does not occur at $x=-1$.

100 If the cubic x^3-px+q has three distinct real roots, where $p>0$ and $q>0$. Then, which one of the following holds? **[AIEEE 2008]**

(a) The cubic has maxima at both $\sqrt{\dfrac{p}{3}}$ and $-\sqrt{\dfrac{p}{3}}$

(b) The cubic has minima at $\sqrt{\dfrac{p}{3}}$ and maxima at $-\sqrt{\dfrac{p}{3}}$

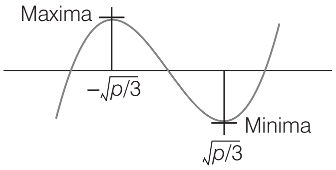
(c) The cubic has minima at $-\sqrt{\dfrac{p}{3}}$ and maxima at $\sqrt{\dfrac{p}{3}}$

(d) The cubic has minima at both $\sqrt{\dfrac{p}{3}}$ and $-\sqrt{\dfrac{p}{3}}$

Ans. *(b)*

Let $\quad f(x) = x^3 - px + q$

Then, $\quad f'(x) = 3x^2 - p$



Put $\quad f'(x) = 0 \Rightarrow x = \sqrt{\frac{p}{3}}, -\sqrt{\frac{p}{3}}$

Now, $\; f''(x) = 6x$

$\therefore$ At $x = \sqrt{\frac{p}{3}}$, $f''(x) = 6\sqrt{\frac{p}{3}} > 0 \quad$ [minima]

and at $\quad x = -\sqrt{\frac{p}{3}}$,

$f''(x) < 0 \quad$ [maxima]

101 The normal to a curve at $P(x, y)$ meets the X-axis at G. If the distance of G from the origin is twice the abscissa of P, then the curve is a **[AIEEE 2007]**

(a) ellipse (b) parabola
(c) circle (d) hyperbola

Ans. *(d)*

Let the equation of normal be

$$Y - y = -\frac{dx}{dy}(X - x)$$

It meets the X-axis at G. Therefore, coordinates of G are $\left(x + y\frac{dy}{dx}, 0\right)$.

According to given condition,

$$x + y\frac{dy}{dx} = 2x \Rightarrow y\frac{dy}{dx} = x$$

$\Rightarrow \quad y\,dy = x\,dx$

On integrating, we get

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$\Rightarrow \quad x^2 - y^2 = -2C$

which shows a hyperbola.

102 A value of C for which the conclusion of mean value theorem holds for the function $f(x) = \log_e x$ on the interval $[1, 3]$ is **[AIEEE 2007]**

(a) $2\log_3 e$
(b) $\frac{1}{2}\log_e 3$
(c) $\log_3 e$
(d) $\log_e 3$

Ans. *(a)*

Using mean value theorem,

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} \quad \left[\because f'(c) = \frac{f(b) - f(a)}{b - a}\right]$$

$\Rightarrow \quad \frac{1}{c} = \frac{\log_e 3 - \log_e 1}{2}$

$\therefore \quad c = \frac{2}{\log_e 3} = 2\log_3 e$

103 The function $f(x) = \frac{x}{2} + \frac{2}{x}$ has a local minimum at **[AIEEE 2006]**

(a) $x = -2$ (b) $x = 0$
(c) $x = 1$ (d) $x = 2$

Ans. *(d)*

$\because \quad f(x) = \frac{x}{2} + \frac{2}{x}$

$\therefore \quad f'(x) = \frac{1}{2} - \frac{2}{x^2}$

For maxima or minima, put $f'(x) = 0$

$\therefore \quad \frac{1}{2} - \frac{2}{x^2} = 0 \Rightarrow x^2 = 4$

$\Rightarrow \quad x = \pm 2$

Now, $\quad f''(x) = \frac{4}{x^3}$

$\Rightarrow \quad f''(2) = \frac{4}{8} = \frac{1}{2} > 0 \quad$ [minima]

and $\quad f''(-2) = -\frac{4}{8} = -\frac{1}{2} < 0 \quad$ [maxima]

Hence, $f(x)$ is minimum at $x = 2$.

104 Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points $(2, 0)$ and $(3, 0)$ is **[AIEEE 2006]**

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

Ans. *(a)*

$\because \quad y = x^2 - 5x + 6$

$\therefore \quad \frac{dy}{dx} = 2x - 5$

Now, $\quad m_1 = \left(\frac{dy}{dx}\right)_{(2, 0)} = 4 - 5 = -1$

and $\quad m_2 = \left(\frac{dy}{dx}\right)_{(3, 0)} = 6 - 5 = 1$

Now, $\quad m_1 m_2 = -1 \times 1 = -1$

Hence, angle between the tangents is $\frac{\pi}{2}$.

105 If x is real, then maximum value of $\frac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$ is **[AIEEE 2006]**

(a) 41 (b) 1
(c) $\frac{17}{7}$ (d) $\frac{1}{4}$

Ans. *(a)*

Let $f(x) = \frac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$

$$= 1 + \frac{10}{3\left(x^2 + 3x + \frac{7}{3}\right)}$$

[by division algorithm]

$$= 1 + \frac{10}{3\left[\left(x + \frac{3}{2}\right)^2 + \frac{1}{12}\right]}$$

Hence, $f(x)$ will be maximum at $x = -\frac{3}{2}$.

So, the maximum value of $f(x)$

$$= 1 + \frac{10}{3\left(\frac{1}{12}\right)} = 1 + 40 = 41$$

106 If $u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$, then the difference between the maximum and minimum values of u^2 is given by **[AIEEE 2004]**

(a) $2(a^2 + b^2)$ (b) $2\sqrt{a^2 + b^2}$
(c) $(a + b)^2$ (d) $(a - b)^2$

Ans. *(d)*

Given that,

$$u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$\therefore u^2 = a^2\cos^2\theta + b^2\sin^2\theta + a^2\sin^2\theta + b^2\cos^2\theta + 2\sqrt{(a^2\cos^2\theta + b^2\sin^2\theta)} \times \sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)}$$

$\Rightarrow \quad u^2 = a^2 + b^2 + 2\sqrt{x(a^2 + b^2 - x)}$

[where, $x = a^2\cos^2\theta + b^2\sin^2\theta$]

$\Rightarrow \quad u^2 = (a^2 + b^2) + 2\sqrt{(a^2 + b^2)x - x^2}$

On differentiating w.r.t. θ, we get

$$\frac{du^2}{d\theta} = \frac{2}{2\sqrt{(a^2 + b^2)x - x^2}} (a^2 + b^2 - 2x) \times \frac{dx}{d\theta}$$

and $\quad \frac{dx}{d\theta} = (b^2 - a^2)\sin 2\theta$

$[\because x = a^2\cos^2\theta + b^2\sin^2\theta]$

$$\Rightarrow \frac{du^2}{d\theta} = \frac{(a^2 + b^2 - 2x)}{\sqrt{(a^2 + b^2)x - x^2}} \times (b^2 - a^2)\sin 2\theta$$

For maxima and minima, put $\frac{du^2}{d\theta} = 0$

$\therefore \quad a^2 + b^2 = 2(a^2\cos^2\theta + b^2\sin^2\theta)$

and $\quad \sin 2\theta = 0$

$\Rightarrow \quad \cos 2\theta\,(b^2 - a^2) = 0$ and $\sin 2\theta = 0$

$\Rightarrow \quad \cos 2\theta = 0$

and $\quad \theta = 0$

$\Rightarrow \quad 2\theta = \frac{\pi}{2}$ and $\theta = 0$

$\Rightarrow \quad \theta = \frac{\pi}{4}$ and $\theta = 0$

So, u^2 will be minimum at $\theta = 0$ and will be maximum at $\theta = \frac{\pi}{4}$.

$\therefore \quad u^2_{min} = (a+b)^2$

and $\quad u^2_{max} = 2(a^2 + b^2)$

Hence, $u^2_{max} - u^2_{min} = 2(a^2 + b^2) - (a+b)^2$
$= (a-b)^2$

107 A function $y = f(x)$ has a second order derivative $f''(x) = 6(x-1)$. If its graph passes through the point (2, 1) and at that point, the tangent to the graph is $y = 3x - 5$, then the function is **[AIEEE 2004]**

(a) $(x-1)^2$ (b) $(x-1)^3$
(c) $(x+1)^3$ (d) $(x+1)^2$

Ans. ***(b)***

Since, $\quad f''(x) = 6(x-1)$

On integrating, we get

$$f'(x) = 3(x-1)^2 + C \quad \text{...(i)}$$

Also, at the point (2, 1), the tangent to graph is

$$y = 3x - 5$$

Slope of tangent = 3

$\Rightarrow \quad f'(2) = 3$

$\therefore \quad f'(2) = 3(2-1)^2 + C = 3$ [from Eq. (i)]

$\Rightarrow \quad 3 + C = 3$

$\Rightarrow \quad C = 0$

From Eq. (i),

$$f'(x) = 3(x-1)^2$$

On integrating, we get

$$f(x) = (x-1)^3 + k \quad \text{...(ii)}$$

Since, graph passes through (2, 1).

$\therefore \quad 1 = (2-1)^2 + k$

$\Rightarrow \quad k = 0$

Hence, equation of function is $f(x) = (x-1)^3$.

108 The normal to the curve $x = a(1+\cos\theta)$, $y = a\sin\theta$ at θ always passes through the fixed point **[AIEEE 2004]**

(a) $(a, 0)$ (b) $(0, a)$
(c) $(0, 0)$ (d) (a, a)

Ans. ***(a)***

Given that,

$$x = a(1+\cos\theta), \quad y = a\sin\theta$$

On differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(-\sin\theta) \text{ and } \frac{dy}{d\theta} = a\cos\theta$$

$\therefore \quad \frac{dy}{dx} = \frac{-\cos\theta}{\sin\theta}$

$\therefore$ Slope of normal $= \frac{-1}{(-\cos\theta/\sin\theta)}$

Equation of normal at the given points is

$$y - a\sin\theta = \frac{\sin\theta}{\cos\theta}[x - a(1+\cos\theta)]$$

It is clear that in the given options, normal passes through the point $(a, 0)$.

109 The real number x when added to its inverse gives the minimum value of the sum at x equal to **[AIEEE 2003]**

(a) 2 (b) 1 (c) –1 (d) –2

Ans. ***(b)***

Let $\quad f(x) = x + \frac{1}{x}$

$$f'(x) = 1 - \frac{1}{x^2}$$

For maxima and minima, put $f'(x) = 0$

$\Rightarrow \quad 1 - \frac{1}{x^2} = 0 \Rightarrow x = \pm 1$

Now, $\quad f''(x) = \frac{2}{x^3}$

At $\quad x = 1, f''(x) = +\text{ve}$ [minima]

and at $x = -1, f''(x) = -\text{ve}$ [maxima]

Thus, $f(x)$ attains minimum value at $x = 1$.

110 If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at p and q respectively, such that $p^2 = q$, then a is equal to **[AIEEE 2003]**

(a) 3 (b) 1 (c) 2 (d) 1/2

Ans. ***(c)***

$\because \quad f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$

$\therefore \quad f'(x) = 6x^2 - 18ax + 12a^2$

For maxima or minima, put $f'(x) = 0$

$\therefore \quad 6(x^2 - 3ax + 2a^2) = 0$

$\Rightarrow \quad x^2 - 3ax + 2a^2 = 0$

$\Rightarrow \quad x^2 - 2ax - ax + 2a^2 = 0$

$\Rightarrow \quad x(x-2a) - a(x-2a) = 0$

$\Rightarrow \quad (x-a)(x-2a) = 0$

$\Rightarrow \quad x = a, \ x = 2a$

Now, $\quad f''(x) = 12x - 18a$

At $\quad x = a$

$$f''(x) = 12a - 18a = -6a$$

So, $f(x)$ will be maximum at $x = a$.

i.e., $\quad p = a$

Again, at $x = 2a$

$$f''(x) = 24a - 18a = 6a$$

So, $f(x)$ will be minimum at $x = 2a$.

i.e., $\quad q = 2a$

Given, $\quad p^2 = q \Rightarrow a^2 = 2a$

$\therefore \quad a = 2$

111 The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$ **[AIEEE 2002]**

(a) cut at right angle
(b) touch each other
(c) cut at an angle $\frac{\pi}{3}$
(d) cut at an angle $\frac{\pi}{4}$

Ans. ***(a)***

The equations of two curves are

$$x^3 - 3xy^2 + 2 = 0 \quad \text{...(i)}$$

and $\quad 3x^2y - y^3 - 2 = 0 \quad$ ...(ii)

On differentiating Eqs. (i) and (ii) w.r.t. x, we get

$$\left(\frac{dy}{dx}\right)_{C_1} = \frac{x^2 - y^2}{2xy}$$

and $\quad \left(\frac{dy}{dx}\right)_{C_2} = \frac{-2xy}{x^2 - y^2}$

Now,

$$\left(\frac{dy}{dx}\right)_{C_1} \times \left(\frac{dy}{dx}\right)_{C_2} = \left(\frac{x^2 - y^2}{2xy}\right)\left(\frac{-2xy}{x^2 - y^2}\right) = -1$$

Hence, the two curves cut at right angle.

112 The greatest value of $f(x) = (x+1)^{1/3} - (x-1)^{1/3}$ on [0, 1] is **[AIEEE 2002]**

(a) 1 (b) 2
(c) 3 (d) 1/3

Ans. ***(b)***

Given that,

$$f(x) = (x+1)^{1/3} - (x-1)^{1/3}$$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{1}{3}\left[\frac{1}{(x+1)^{2/3}} - \frac{1}{(x-1)^{2/3}}\right]$$
$$= \frac{(x-1)^{2/3} - (x+1)^{2/3}}{3(x^2-1)^{2/3}}$$

Clearly, $f'(x)$ does not exist at $x = \pm 1$.

Now, put $f'(x) = 0$, then

$$(x-1)^{2/3} = (x+1)^{2/3}$$

$\Rightarrow \quad x = 0$

At $\quad x = 0$

$$f(x) = (0+1)^{1/3} - (0-1)^{1/3} = 2$$

Hence, the greatest value of $f(x)$ is 2.