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Application of Derivatives

APPLICATION OF DERIVATIVES

1. DERIVATIVE AS RATE OF CHANGE

In various fields of applied mathematics one has the quest to know the rate at which one variable is changing, with respect to other. The rate of change naturally refers to time. But we can have rate of change with respect to other variables also.

An economist may want to study how the investment changes with respect to variations in interest rates.

A physician may want to know, how small changes in dosage can affect the body's response to a drug.

A physicist may want to know the rate of change of distance with respect to time.

All questions of the above type can be interpreted and represented using derivatives.

Definition :

The **average rate of change** of a function $f(x)$ with respect to x over an interval [a, a + h] is defined as $\frac{f(a+h)-f(a)}{h}$.

Definition :

The **instantaneous rate of change** of f with respect to x is defined as

$$f'(x)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$, provided the limit exists.

Note...

To use the word 'instantaneous', x may not be representing time. We usually use the word 'rate of change' to mean 'instantaneous rate of change'.

2. EQUATIONS OF TANGENT & NORMAL

(I) The value of the derivative at P (x_1, y_1) gives the slope of the tangent to the curve at P. Symbolically

$$f'(x_1) = \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = \text{Slope of tangent at}$$

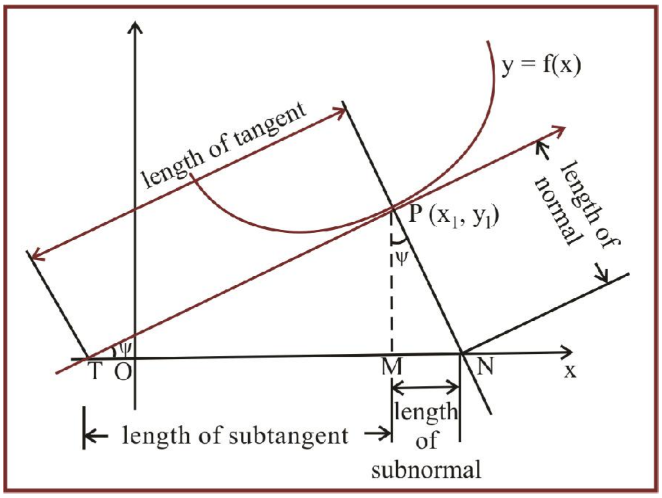
P $(x_1, y_1) = m$ (say).

(II) Equation of tangent at (x_1, y_1) is ;

$$(y-y_1)=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}\times(x-x_1)$$

(III) Equation of normal at (x_1, y_1) is ;

$$(y-y_1)=\left(\frac{-1}{\frac{dy}{dx}}\right)_{(x_1,y_1)}\times(x-x_1)$$



Note..

1. The point P (x_1, y_1) will satisfy the equation of the curve & the equation of tangent & normal line.
2. If the tangent at any point P on the curve is parallel to X-axis then dy/dx = 0 at the point P.
3. If the tangent at any point on the curve is parallel to Y-axis, then dy/dx = ∞ or dx/dy = 0.
4. If the tangent at any point on the curve is equally inclined to both the axes then dy/dx = ± 1.
5. If the tangent at any point makes equal intercept on the coordinate axes then dy/dx = ± 1.
6. Tangent to a curve at the point P (x_1, y_1) can be drawn even though dy/dx at P does not exist. e.g. x = 0 is a tangent to $y = x^{2/3}$ at (0, 0).
7. If a curve passing through the origin be given by a rational integral algebraic equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation. e.g. If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ i.e. x + y = 0 and x – y = 0.

(IV) (a) Length of the tangent (PT) = $\dfrac{y_1\sqrt{1+[f'(x_1)]^2}}{f'(x_1)}$

(b) Length of Subtangent (MT) = $\dfrac{y_1}{f'(x_1)}$

(c) Length of Normal (PN) = $y_1\sqrt{1+[f'(x_1)]^2}$

(d) Length of Subnormal (MN) = $y_1 f'(x_1)$

(V) Differential :

The differential of a function is equal to its derivative multiplied by the differential of the independent variable. Thus if, y = tan x then dy = $\sec^2 x$ dx.

In general dy = $f'(x)$ dx.

Note..

d(c) = 0 where 'c' is a constant.

$d(u + v - w) = du + dv - dw$

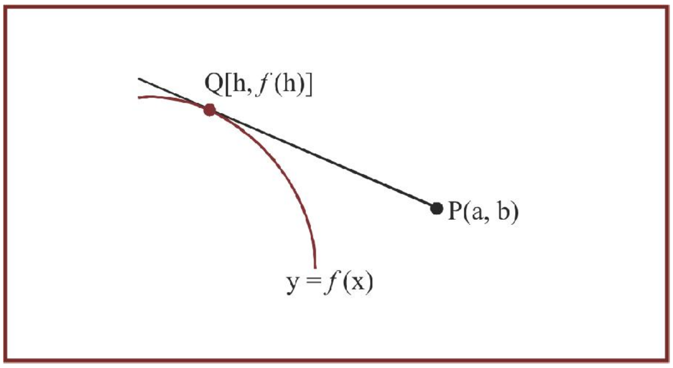
$d(uv) = udv + vdu$

* The relation $dy = f'(x)\,dx$ can be written as $\dfrac{dy}{dx} = f'(x)$; thus the quotient of the differentials of 'y' and 'x' is equal to the derivative of 'y' w.r.t. 'x'.

3. TANGENT FROM AN EXTERNAL POINT

Given a point P (a, b) which does not lie on the curve $y = f(x)$, then the equation of possible tangents to the curve $y = f(x)$, passing through (a, b) can be found by solving for the point of contact Q.

And equation of tangent is $y - b = \dfrac{f(h) - b}{h - a}(x - a)$

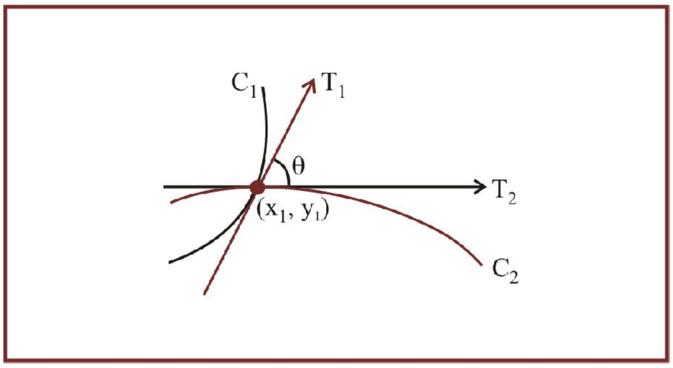


4. ANGLE BETWEEN THE CURVES

Angle between two intersecting curves is defined as the acute angle between their tangents or the normals at the point of intersection of two curves.

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

where m_1 & m_2 are the slopes of tangents at the intersection point (x_1, y_1).



Note..

(i) The angle is defined between two curves if the curves are intersecting. This can be ensured by finding their point of intersection or by graphically.

(ii) If the curves intersect at more than one point then angle between curves is found out with respect to the point of intersection.

(iii) Two curves are said to be **orthogonal** if angle between them at each point of intersection is right angle i.e. $m_1 m_2 = -1$.

5. SHORTEST DISTANCE BETWEEN TWO CURVES

Shortest distance between two non-intersecting differentiable curves is always along their common normal. (Wherever defined)

6. ERRORS AND APPROXIMATIONS

(a) Errors

Let $y = f(x)$

From definition of derivative, $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$

$\frac{\delta y}{\delta x} = \frac{dy}{dx}$ approximately

or $\delta y = \left(\frac{dy}{dx}\right) . \delta x$ approximately

Definition :

(i) δx is known as absolute error in x.

(ii) $\frac{\delta x}{x}$ is known as relative error in x.

(iii) $\frac{\delta x}{x} \times 100$ is known as percentage error in x.

Note..

δx and δy are known as differentials.

(b) Approximations

From definition of derivative,

$\therefore$ Derivative of $f(x)$ at $(x = a) = f'(a)$

or $f'(a) = \lim_{\delta x \to 0} \frac{f(a+\delta x) - f(a)}{\delta x}$

or $\frac{f(a+\delta x) - f(a)}{\delta x} \to f'(a)$ (approximately)

$f(a+\delta x) = f(a) + \delta x f'(a)$ (approximately)

7. DEFINITIONS

1. A function $f(x)$ is called an **Increasing Function** at a point x = a if in a sufficiently small neighbourhood around x = a we have

$f(a+h) > f(a)$

$f(a-h) < f(a)$

Similarly **Decreasing Function** if

$f(a+h) < f(a)$

$f(a-h) > f(a)$

Above statements hold true irrespective of whether f is non derivable or even discontinuous at x = a

2. A differentiable function is called increasing in an interval (a, b) if it is increasing at every point within the interval (but not necessarily at the end points). A function decreasing in an interval (a, b) is similarly defined.

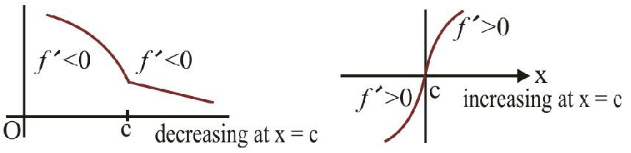
3. A function which in a given interval is increasing or decreasing is called **"Monotonic"** in that interval.

4. Tests for increasing and decreasing of a function at a point :

If the derivative $f'(x)$ is positive at a point x = a, then the function $f(x)$ at this point is increasing. If it is negative, then the function is decreasing.

Note..

Even if $f'(a)$ is not defined, f can still be increasing or decreasing. (Look at the cases below).



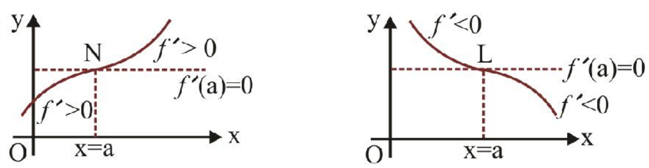
Note : $f'(c)$ is not defined in both the cases.

Note..

If $f'(a) = 0$, then for $x = a$ the function may be still increasing or it may be decreasing as shown. It has to be identified by a seperate rule.

e.g. $f(x) = x^3$ is increasing at every point.

Note that, $dy/dx = 3x^2$.



Note..

1. If a function is invertible it has to be either increasing or decreasing.
2. If a function is continuous, the intervals in which it rises and falls may be separated by points at which its derivative fails to exist.
3. If f is increasing in [a, b] and is continuous then $f(b)$ is the greatest and $f(a)$ is the least value of f in [a, b]. Similarly if f is decreasing in [a, b] then $f(a)$ is the greatest value and $f(b)$ is the least value.

5. (a) ROLLE'S Theorem :

Let $f(x)$ be a function of x subject to the following conditions :

(i) $f(x)$ is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) $f(a) = f(b)$.

Then there exists at least one point x = c such that $a < c < b$ where $f'(c) = 0$.

(b) LMVT Theorem :

Let $f(x)$ be a function of x subject to the following conditions :

(i) $f(x)$ is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

Then there exists at least one point x = c such that

$a < c < b$ where $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Geometrically, the slope of the secant line joining the curve at x = a & x = b is equal to the slope of the tangent line drawn to the curve at x = c.

Note the following : Rolles theorem is a special case of LMVT since

$$f(a) = f(b) \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a} = 0$$

Note..

Physical Enterpretation of LMVT :

Now $[f(b) - f(a)]$ is the change in the function f as x changes from a to b so that $\dfrac{f(b) - f(a)}{b - a}$ is the average rate of change of the function over the interval [a, b]. Also $f'(c)$ is the actual rate of change of the function for x = c. Thus, the theorem states that the average rate of change of a function over an interval is also the actual rate of change of the function at some point of the interval. In particular, for instance, the average velocity of a particle over an interval of time is equal to the velocity at some instant belonging to the interval.

This interpretation of the theorem justifies the name "Mean Value" for the theorem.

(c) Application of rolles theorem for isolating the real roots of an equation $f(x)=0$

Suppose a & b are two real numbers such that ;

(i) $f(x)$ & its first derivative $f'(x)$ are continuous for $a \le x \le b$.

(ii) $f(a)$ & $f(b)$ have opposite signs.

(iii) $f'(x)$ is different from zero for all values of x between a & b.

Then there is one & only one real root of the equation $f(x)=0$ between a & b.

8. HOW MAXIMA & MINIMA ARE CLASSIFIED

1. A function $f(x)$ is said to have a local maximum at $x=a$ if $f(a)$ is greater than every other value assumed by $f(x)$ in the immediate neighbourhood of $x=a$. Symbolically

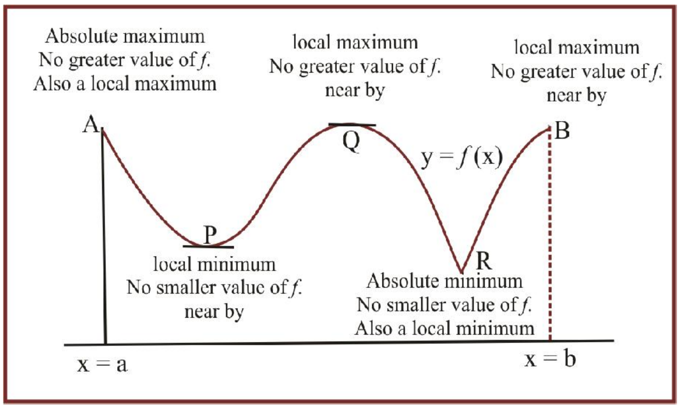
$$\left.\begin{matrix} f(a)>f(a+h) \\ f(a)>f(a-h) \end{matrix}\right] \Rightarrow x=a \text{ gives maxima}$$

for a sufficiently small positive h.

Similarly, a function $f(x)$ is said to have a local minimum value at $x=b$ if $f(b)$ is least than every other value assumed by $f(x)$ in the immediate neighbourhood at $x=b$. Symbolically if

$$\left.\begin{matrix} f(b)<f(b+h) \\ f(b)<f(b-h) \end{matrix}\right] \Rightarrow x=b \text{ gives minima for a sufficiently}$$

small positive h.



Note...

(i) The local maximum & local minimum values of a function are also known as local/relative maxima or local/relative minima as these are the greatest & least values of the function relative to some neighbourhood of the point in question.

(ii) The term 'extremum' is used both for maxima or a minima.

(iii) A local maximum (local minimum) value of a function may not be the greatest (least) value in a finite interval.

(iv) A function can have several local maximum & local minimum values & a local minimum value may even be greater than a local maximum value.

(v) Maxima & minima of a continuous function occur alternately & between two consecutive maxima there is a minima & vice versa.

2. A necessary condition for maxima & minima

If $f(x)$ is a maxima or minima at $x=c$ & if $f'(c)$ exists then $f'(c)=0$.

Note...

(i) The set of values of x for which $f'(x)=0$ are often called as stationary points. The rate of change of function is zero at a stationary point.

(ii) In case $f'(c)$ does not exist $f(c)$ may be a maxima or a minima & in this case left hand and right hand derivatives are of opposite signs.

(iii) The greatest (global maxima) and the least (global minima) values of a function f in an interval [a, b] are $f(a)$ or $f(b)$ or are given by the values of x which are critical points.

(iv) **Critical points** are those where :

(i) $\frac{dy}{dx}=0$, if it exists; (ii) or it fails to exist

3. Sufficient condition for extreme values

First Derivative Test

$$\left.\begin{array}{l} f'(c-h)>0 \\ f'(c+h)<0 \end{array}\right] \Rightarrow x=c \text{ is a point of local maxima,}$$

where h is a sufficiently small positive quantity

$$\text{Similarly } \left.\begin{array}{l} f'(c-h)<0 \\ f'(c+h)>0 \end{array}\right] \Rightarrow x=c \text{ is a point of local minima,}$$

where h is a sufficiently small positive quantity

Note :– $f'(c)$ in both the cases may or may not exist. If it exists, then $f'(c)=0$.

Note...

If $f'(x)$ does not change sign i.e. has the same sign in a certain complete neighbourhood of c, then $f(x)$ is either strictly increasing or decreasing throughout this neighbourhood implying that $f(c)$ is not an extreme value of f.

4. Use of second order derivative in ascertaining the maxima or minima

(a) $f(c)$ is a minima of the function f, if $f'(c)=0$ & $f''(c)>0$.

(b) $f(c)$ is a maxima of the function f, if $f'(c)=0$ & $f''(c)<0$.

Note...

If $f''(c) = 0$ then the test fails. Revert back to the first order derivative check for ascertaining the maxima or minima.

5. Summary-working rule

First : When possible, draw a figure to illustrate them problem & label those parts that are important in the problem. Constants & variables should be clearly distinguished.

Second : Write an equation for the quantity that is to be maximised or minimised. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. This may require some algebraic manipulations.

Third : If $y=f(x)$ is a quantity to be maximum or minimum, find those values of x for which $dy/dx = f'(x)=0$.

Fourth : Test each values of x for which $f'(x)=0$ to determine whether it provides a maxima or minima or neither. The usual tests are :

(a) If d^2y/dx^2 is positive when $dy/dx = 0$

$\Rightarrow$ y is minima.

If d^2y/dx^2 is negative when $dy/dx = 0$

$\Rightarrow$ y is maxima.

If $d^2y/dx^2 = 0$ when $dy/dx = 0$, the test fails.

(b) $$\text{If } \frac{dy}{dx} \text{ is } \left.\begin{array}{lll} \text{positive} & \text{for} & x<x_0 \\ \text{zero} & \text{for} & x=x_0 \\ \text{negative} & \text{for} & x>x_0 \end{array}\right] \Rightarrow \text{a maxima occurs at } x=x_0.$$

But if dy/dx changes sign from negative to zero to positive as x advances through x_0, there is a minima. If dy/dx does not change sign, neither a maxima nor a minima. Such points are called **INFLECTION POINTS**.

Fifth : If the function $y = f(x)$ is defined for only a limited range of values $a \le x \le b$ then examine $x = a$ & $x = b$ for possible extreme values.

Sixth : If the derivative fails to exist at some point, examine this point as possible maxima or minima.

(In general, check at all Critical Points).

Note...

- If the sum of two positive numbers x and y is constant than their product is maximum if they are equal, i.e. $x + y = c$, $x > 0$, $y > 0$, then

$$xy = \frac{1}{4}\left[(x+y)^2-(x-y)^2\right]$$

- If the product of two positive numbers is constant then their sum is least if they are equal.

i.e. $(x+y)^2=(x-y)^2+4xy$

6. Useful formulae of mensuration to remember

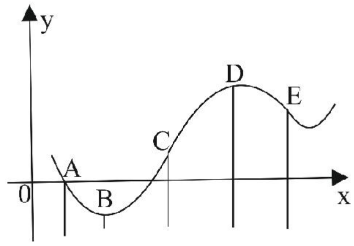
- Volume of a cuboid = $l$bh.
- Surface area of a cuboid = 2 (lb + bh + hl).
- Volume of a prism = area of the base × height.
- Lateral surface of a prism = perimeter of the base × height.
- Total surface of a prism = lateral surface + 2 area of the base

  (Note that lateral surfaces of a prism are all rectangles).
- Volume of a pyramid $= \frac{1}{3}$ area of the base × height.
- Curved surface of a pyramid = $\frac{1}{2}$ (perimeter of the base) × slant height.

  (Note that slant surfaces of a pyramid are triangles).
- Volume of a cone $= \frac{1}{3}\pi r^2 h$.
- Curved surface of a cylinder = $2\pi rh$.
- Total surface of a cylinder = $2\pi rh + 2\pi r^2$.
- Volume of a sphere $= \frac{4}{3}\pi r^3$.
- Surface area of a sphere = $4\pi r^2$.
- Area of a circular sector $= \frac{1}{2}r^2\theta$, where θ is in radians.

7. Significance of the sign of 2nd order derivative and points of inflection

The sign of the 2nd order derivative determines the concavity of the curve. Such point such as C & E on the graph where the concavity of the curve changes are called the points of inflection. From the graph we find that if :



(i) $\frac{d^2y}{dx^2} > 0 \Rightarrow$ concave upwards

(ii) $\frac{d^2y}{dx^2} < 0 \Rightarrow$ concave downwards.

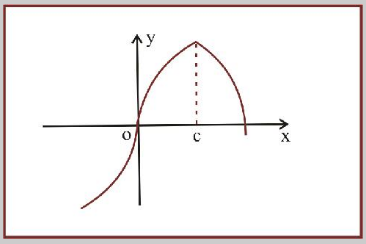
At the point of inflection we find that $\frac{d^2y}{dx^2} = 0$ and $\frac{d^2y}{dx^2}$ changes sign.

Inflection points can also occur if $\frac{d^2y}{dx^2}$ fails to exist (but changes its sign). For example, consider the graph of the function defined as,

$$f(x) = \begin{bmatrix} x^{3/5} & \text{for } x \in (-\infty, 1) \\ 2 - x^2 & \text{for } x \in (1, \infty) \end{bmatrix}$$

Note...

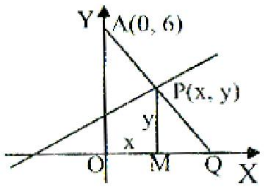
The graph below exhibits two critical points one is a point of local maximum (x = c) & the other a point of inflection (x = 0). This implies that not every Critical Point is a point of extrema.



SOLVED EXAMPLES

Example – 1

A point P (x, y) moves along the line whose equation is $x - 2y + 4 = 0$ in such a way that y increases at the rate of 3 units/sec. The point A (0, 6) is joined to P and the segment AP is prolonged to meet the x-axis in a point Q. Find how fast the distance from the origin to Q is changing when P reaches the point (4, 4).



Sol. The rate of change of y is given and it is desired to find the rate of change of OQ, which we denote by z. If MP is perpendicular to the x-axis, MP = y and OM = x.

The triangles OAQ and MPQ are similar, hence

$$\frac{z}{6} = \frac{z-x}{y} \Rightarrow yz = 6z - 6x \Rightarrow z = \frac{6x}{6-y}$$

Substituting the value of x from the equation of the given line, we have

$$z = \frac{12\,(y-2)}{6-y}$$

$$\frac{dz}{dt} = \frac{48}{(6-y)^2}\,\frac{dy}{dt}$$

Setting $y = 4$ and $\frac{dy}{dt} = 3$, we obtain $\frac{dz}{dt} = 36$ that is, z is increasing at the rate of 36 units/sec.

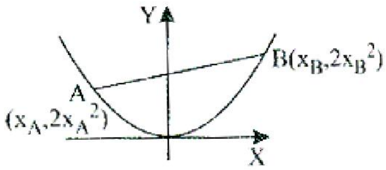
Example – 2

The ends A and B of a rod of length $\sqrt{5}$ are sliding along the curve $y = 2x^2$. Let x_A and x_B be the x-coordinate of the ends. At the moment when A is at (0, 0) and B is at (1, 2), find the value of the derivative $\frac{dx_B}{dx_A}$.

Sol. We have $y = 2x^2$

$(AB)^2 = (x_B - x_A)^2 + (2x_B^2 - 2x_A^2)^2 = 5$

or $(x_B - x_A)^2 + 4\,(x_B^2 - x_A^2)^2 = 5$



Differentiating w.r.t. x_A and denoting $\frac{dx_B}{dx_A} = D$

$2\,(x_B - x_A)\,(D - 1) + 8\,(x_B^2 - x_A^2)\,(2x_B D - 2x_A) = 0$

Put $x_A = 0$, $x_B = 1$

$2\,(1-0)\,(D-1) + 8\,(1-0)\,(2D - 0) = 0$

$2D - 2 + 16D = 0 \Rightarrow D = 1/9.$

Example – 3

Find the approximate value of $(0.007)^{1/3}$.

Sol. Let $f(x) = (x)^{1/3}$

Now, $f(x + \delta x) - f(x) = f'(x).\delta x = \frac{\delta x}{3x^{2/3}}$

we may write, $0.007 = 0.008 - 0.001$

Taking $x = 0.008$ and $\delta x = -0.001$, we have

$$f(0.007) - f(0.008) = -\frac{0.001}{3\,(0.008)^{2/3}}$$

or $$f(0.007) - (0.008)^{1/3} = -\frac{0.001}{3\,(0.2)^2}$$ or

$$f(0.007) = 0.2 - \frac{0.001}{3\,(0.04)} = 0.2 - \frac{1}{120} = \frac{23}{120}$$

Hence $(0.007)^{1/3} = \frac{23}{120}$.

Example – 4

The period T of a simple pendulum is

$$T = 2\pi\sqrt{\frac{(\ell)}{g}}$$

Find the maximum error in T due to possible errors upto 1% in *l* and 2.5% in g.

Sol. Since $T = 2\pi\sqrt{\frac{(\ell)}{g}} = 2\pi\left(\frac{\ell}{g}\right)^{1/2}$

Taking logarithm on both sides, we get

$$\ln T = \ln 2\pi + \frac{1}{2}\ln l - \frac{1}{2}\ln g$$

Differentiating both sides, we get

$$\frac{dT}{T} = 0 + \frac{1}{2}\cdot\frac{dl}{l} - \frac{1}{2}\cdot\frac{dg}{g}$$

or $$\left(\frac{dT}{T}\times 100\right) = \frac{1}{2}\left(\frac{dl}{l}\times 100\right) - \frac{1}{2}\left(\frac{dg}{g}\times 100\right)$$

$$\frac{1}{2}(1 \pm 2.5) \quad \left(\because \frac{dl}{l}\times 100 = 1 \text{ and } \frac{dg}{g}\times 100 = 2.5\right)$$

∴ Maximum error in T = 1.75%.

Example – 5

If $2a + 3b + 6c = 0$, a, b, c ∈ R then show that the equation $ax^2 + bx + c = 0$ has at least one root between 0 and 1.

Sol. Given $2a + 3b + 6c = 0$

or $$\frac{a}{3} + \frac{b}{2} + c = 0 \quad \text{.... (i)}$$

Let $f'(x) = ax^2 + bc + c$

On integrating both sides, we get

$$f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + k$$

Now, $f(1) = \frac{a}{3} + \frac{b}{2} + c + k$ [From (i)]

$= 0 + k = k$

and $f(0) = 0 + 0 + 0 + k = k$

Since *f* (x) is a polynomial of three degree, it is continuous and differentiable and *f* (0) = *f* (1), then by Rolle's theorem $f'(x) = 0$ i.e., $ax^2 + bx + c = 0$ has at least one real root between 0 and 1.

Example – 6

Verify Rolle's theorem for $f(x) = x(x+3)e^{-x/2}$ in [–3, 0].

Sol. We have $f(x) = x(x+3)e^{-x/2}$

∴ $$f'(x) = (2x+3)e^{-x/2} + x(x+3)e^{-x/2}(-1/2)$$

$$= -\frac{1}{2}(x^2 - x - 6)e^{-x/2}$$

which exists for every value of x in the interval [–3, 0].

Hence *f* (x) is differentiable and so also continuous in the interval [–3, 0]. Also *f*(–3) = *f*(0) = 0.

Thus all the three conditions of Rolle's theorem are satisfied. So $f'(x) = 0$ for at least one value of x lying in the open interval (–3, 0).

For $f'(x) \Rightarrow -\frac{1}{2}(x^2 - x - 6)e^{-x/2} = 0$

$\Rightarrow e^{-x/2} \neq 0, \therefore x^2 - x - 6 = 0$

$\Rightarrow (x-3)(x+2) = 0 \quad \Rightarrow x = -2, 3$

Since the value x = – 2 lies in the open interval (– 3, 0), the Rolle's theorem is verified.

Example – 7

If $f(x) = (x-1)(x-2)(x-3)$ and a = 0, b = 4., find 'c' using Lagrange's mean value theorem.

Sol. We have $f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$

∴ $f(a) = f(0) = (0-1)(0-2)(x-3) = -6$

and $f(b) = f(4) = (4-1)(4-2)(4-3) = 6$

∴ $$\frac{f(b) - f(a)}{b - a} = \frac{6-(-6)}{4-0} = \frac{12}{4} = 3 \quad \text{.... (1)}$$

Also $f'(x) = 3x^2 - 12x + 11$

gives $f'(c) = 3c^2 - 12c + 11$

From LMVT, $$\frac{f(b) - f(a)}{b-a} = f'(c) \quad \text{.... (2)}$$

$\Rightarrow \quad 3 = 3c^2 - 12c + 11$ {From (1) and (2)}

$\Rightarrow \quad 3c^2 - 12c + 8 = 0$

∴ $$c = \frac{12 \pm \sqrt{144 - 96}}{6} = 2 \pm \frac{2\sqrt{3}}{3}$$

As both of these values of c lie in the open interval (0, 4). Hence both of these are the required values of c.

Example – 8

Find the equation of the tangent to $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$ at the point (x_0, y_0).

Sol. $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$ Differentiating wrt x,

$\Rightarrow \frac{mx^{m-1}}{a^m}+\frac{my^{m-1}}{b^m}\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=-\frac{b^m}{a^m}\left(\frac{x}{y}\right)^{m-1}$

$\Rightarrow$ at the given point (x_0, y_0), slope of tangent is

$$\left.\frac{dy}{dx}\right|_{(x_0,y_0)}=-\left(\frac{b}{a}\right)^m\left(\frac{x_0}{y_0}\right)^{m-1}$$

$\Rightarrow$ the equation of tangent is

$$y-y_0=-\left(\frac{b}{a}\right)^m\left(\frac{x_0}{y_0}\right)^{m-1}(x-x_0)$$

$$a^m yy_0^{m-1}-a^m y_0^m=-b^m x\,x_0^{m-1}+b^m x_0^m$$

$$a^m yy_0^{m-1}+b^m x\,x_0^{m-1}=a^m y_0^m+b^m x_0^m$$

using the equation of given curve, the right side can be replaced by $a^m b^m$.

$\therefore a^m yy_0^{m-1}+b^m x\,x_0^{m-1}=a^m b^m$

$\Rightarrow$ the equation of tangent is

$$\frac{x}{a}\left(\frac{x_0}{a}\right)^{m-1}+\frac{y}{b}\left(\frac{y_0}{b}\right)^{m-1}=1$$

Example – 9

Find the equation of tangent to the curve $x^{2/3}+y^{2/3}=a^{2/3}$ at (x_0, y_0). Hence prove that the length of the portion of tangent intercepted between the axes is constant.

Sol. Method 1 :

$x^{2/3}+y^{2/3}=a^{2/3}$ Differentiating wrt x,

$$\frac{2}{3}x^{\frac{-1}{3}}+\frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx}=0$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(x_0,y_0)}=-\left(\frac{x_0}{y_0}\right)^{-\frac{1}{3}}$$

$\Rightarrow$ equation is $y-y_0=-\left(\frac{y_0}{x_0}\right)^{\frac{1}{3}}(x-x_0)$

$\Rightarrow x_0^{1/3}y-y_0x_0^{1/3}=-xy_0^{1/3}+x_0y_0^{1/3}$

$\Rightarrow x\,y_0^{1/3}+yx_0^{1/3}=x_0y_0^{1/3}+y_0x_0^{1/3}$

$\Rightarrow \frac{x\,y_0^{1/3}}{x_0^{1/3}\,y_0^{1/3}}+\frac{y\,x_0^{1/3}}{x_0^{1/3}\,y_0^{1/3}}=x_0^{2/3}+y_0^{2/3}$

$\Rightarrow$ equation of tangent is : $\frac{x}{x_0^{1/3}}+\frac{y}{y_0^{1/3}}=a^{2/3}$

Length intercepted between the axes :

length $=\sqrt{(x\text{ intercept})^2+(y\text{ intercept})^2}$

$=\sqrt{\left(x_0^{1/3}\,a^{2/3}\right)^2+\left(x_0^{1/3}\,a^{2/3}\right)^2}$

$=\sqrt{x_0^{2/3}a^{4/3}+y_0^{2/3}a^{4/3}}$

$=a^{2/3}\sqrt{x_0^{2/3}+y_0^{2/3}}$

$=a^{2/3}\sqrt{a^{2/3}}$

= a i.e. constant.

Method 2 :

Express the equation in parametric form

$x=a\sin^3 t,\quad y=a\cos^3 t$

Equation of tangent is :

$$(y-a\cos^3 t)=\frac{-3a\cos^2 t\sin t}{3a\sin^2 t\cos t}(x-a\sin^3 t)$$

$\Rightarrow y\sin t-a\sin t\cos^3 t=-x\cos t+a\sin^3 t\cos t$

$\Rightarrow x\cos t+y\sin t=a\sin t\cos t$

$\Rightarrow \frac{x}{\sin t}+\frac{y}{\cos t}=a$

in terms of (x_0, y_0) equation is :

$$\frac{x}{(x_0/a)^{1/3}}+\frac{y}{(y_0/a)^{1/3}}=a$$

Length of tangent intercepted between axes

$=\sqrt{(x_{int})^2+(y_{int})^2}$

$=\sqrt{a^2\sin^2 t+a^2\cos^2 t}=a$

Note..

1. The parametric form is very useful in these type of problems.
2. Equation of tangent can also be obtained by substituting b = a and m = 2/3 in the result $\frac{x}{a}\left(\frac{x_0}{a}\right)^{m-1}+\frac{y}{b}\left(\frac{y_0}{b}\right)^{m-1}=1.$

Example – 10

For the curve $xy = c^2$, prove that

(i) the intercept between the axes on the tangent at any point is bisected at the point of contact.

(ii) the tangent at any point makes with the co-ordinate axes a triangle of constant area.

Sol. Let the equation of the curve in parametric form be $x = ct$, $y = c/t$

Let the point of contact be (ct, c/t)

Equation of tangent is :

$$y - c/t = \frac{-c/t^2}{c}(x - ct)$$

$\Rightarrow$ $t^2y - ct = -x + ct$

$\Rightarrow$ $x + t^2y = 2\,ct$ ....(i)

(i) Let the tangent cut the x and y axes at A and B respectively.

Writing the equations as : $\frac{x}{2ct}+\frac{y}{2c/t}=1$

$\Rightarrow$ $x_{intercept} = 2\,ct$, $y_{intercept} = 2\,c/t$

$\Rightarrow$ $A \equiv (2ct, 0)$ and $B \equiv \left(0, \frac{2c}{t}\right)$

mid point of $AB \equiv \left(\frac{2ct+0}{2}, \frac{0+2c/t}{2}\right)$

$\equiv (ct, c/t)$

Hence, the point of contact bisects AB.

(ii) If O is the origin,

Area of triangle Δ OAB = 1/2 (OA) (OB)

$$=\frac{1}{2}(2ct)\frac{(2c)}{t}$$

$= 2\,c^2$

i.e. constant for all tangents because it is independent of t.

Example – 11

Find the equation of the tangent to $x^3 = ay^2$ at the point A (at^2, at^3). Find also the Point where this tangent meets the curve again.

Sol. Equation of tangent to : $x = at^2$, $y = at^3$ is

$$y - at^3 = \frac{3at^2}{2at}\left(x - at^2\right)$$

$\Rightarrow$ $2y - 2at^3 = 3tx - 3at^3$

i.e. $3\,tx - 2y - at^3 = 0$

Let B $\left(at_1^2, at_1^3\right)$ be the point where it again meets the curve.

$\Rightarrow$ slope of tangent at A = slope of AB

$$\frac{3at^2}{2at}=\frac{a\left(t^3-t_1^3\right)}{a\left(t^2-t_1^2\right)} \Rightarrow \frac{3t}{2}=\frac{t^2+t_1^2+t\,t_1}{t+t_1}$$

$\Rightarrow$ $3t^2 + 3\,tt_1 = 2t^2 + 2t_1^{\,2} + 2\,t\,t_1$

$\Rightarrow$ $2t_1^{\,2} - t\,t_1 - t^2 = 0$

$\Rightarrow$ $(t_1 - t)(2t_1 + t) = 0$

$\Rightarrow$ $t_1 = t$ or $t_1 = -t/2$

The relevant value is $t_1 = -t/2$

Hence the meeting point B is

$$=\left[a\left(\frac{-t}{2}\right)^2, a\left(\frac{-t}{2}\right)^3\right]=\left[\frac{at^2}{4}, \frac{-at^3}{8}\right]$$

Example – 12

Find the interval in which

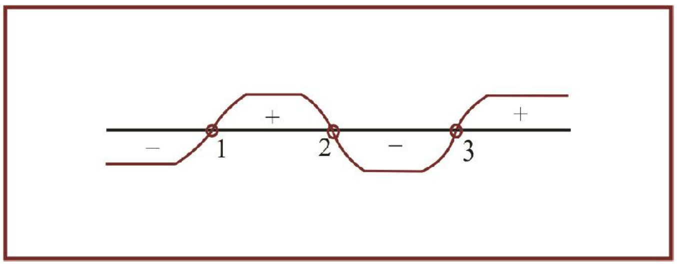
$f(x) = x^4 - 8x^3 + 22\,x^2 - 24x + 5$ is increasing.

Sol. Given $f(x) = x^4 - 8x^3 + 22\,x^2 - 24x + 5$

$\therefore$ $f'(x) = 4x^3 - 24\,x^2 + 44x - 24$

$= 4\,(x^3 - 6x^2 + 11\,x - 6)$

$= 4\,(x-1)(x-2)(x-3)$



For increasing function $f'(x) > 0$

or $4\,(x-1)(x-2)(x-3) > 0$

or $(x-1)(x-2)(x-3) > 0$

$\therefore$ $x \in (1, 2) \cup (3, \infty)$

Example – 13

Find the interval in which $f(x) = x - 2\sin x,\ 0 \le x \le 2\pi$ is increasing

Sol. Given $f(x) = x - 2\sin x$

$\therefore f'(x) = 1 - 2\cos x$

$f'(x) > 0$ or $1 - 2\cos x > 0 \qquad \therefore \cos x < \frac{1}{2}$

or $\quad -\cos x > -\frac{1}{2}$

or $\quad \cos(\pi + x) > \cos\frac{2\pi}{3}$

or $\quad 2n\pi - \frac{2\pi}{3} < \pi + x < 2n\pi + \frac{2\pi}{3},\ n \in I$

or $\quad 2n\pi - \frac{5\pi}{3} < x < 2n\pi - \frac{\pi}{3}$

For $\quad n = 1,\ \frac{\pi}{3} < x < \frac{5\pi}{3}$ which is true $(\because 0 \le x \le 2\pi)$

Hence, $x \in \left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$

Example – 14

For $0 < x \le \frac{\pi}{2}$, show that $x - \frac{x^3}{6} < \sin x < x$.

Sol. Let $f(x) = \sin x - x$

$f'(x) = \cos x - 1 = -(1 - \cos x) = -2\sin^2 x/2 < 0$

$\therefore \quad f(x)$ is a decreasing function

$x > 0$

$\therefore \quad f(x) < f(0) \Rightarrow \sin x - x < 0\ (\because f(0) = 0)$

$\Rightarrow \quad \sin x < x \qquad$ ......(1)

Now let $g(x) = x - \frac{x^3}{6} - \sin x \quad \therefore g'(x) = 1 - \frac{x^2}{2} - \cos x$

To find sign of $g'(x)$ we consider $\phi(x) = 1 - \frac{x^2}{2} - \cos x$

$\therefore \quad \phi'(x) = -x + \sin x < 0 \qquad$ [From (1)]

$\therefore \quad \phi(x)$ is a decreasing function $\quad \Rightarrow g'(x) < 0$

$\Rightarrow \quad g(x)$ is a decreasing function $\quad \because x > 0$

$\Rightarrow \quad g(x) < g(0)$

$\Rightarrow \quad x - \frac{x^3}{6} - \sin x < 0 \qquad (\because g(0) = 0)$

$\Rightarrow \quad x - \frac{x^3}{6} < \sin x \qquad$ ......(2)

Combining (1) and (2) we get $x - \frac{x^3}{6} < \sin x < x$.

Example – 15

Find the intervals of monotonicity of the function $f(x) = \frac{|x-1|}{x^2}$.

Sol. The given function $f(x)$ can be written as :

$$f(x) = \frac{|x-1|}{x^2} = \begin{cases} \frac{1-x}{x^2} & ; \quad x < 1, x \ne 0 \\ \frac{x-1}{x^2} & ; \quad x \ge 1 \end{cases}$$

Consider x < 1

$$f'(x) = \frac{-2}{x^3} + \frac{1}{x^2} = \frac{x-2}{x^3}$$

For increasing, $f'(x) > 0 \quad \Rightarrow \frac{x-2}{x^3} > 0$

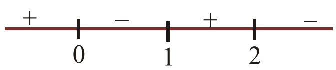
$\Rightarrow \quad x(x-2) > 0 \quad$ [as x^2 is positive]

$\Rightarrow \quad x \in (-\infty, 0) \cup (2, \infty)$.

Combining with $x < 1$, we get $f(x)$ is increasing in $x < 0$ and decreasing in $x \in (0, 1)$ ... (i)

Consider x ≥ 1

$$f'(x) = \frac{-1}{x^2} + \frac{2}{x^3} = \frac{2-x}{x^3}$$



For increasing $f'(x) > 0$

$\Rightarrow \quad (2 - x) > 0 \qquad$ [as x^3 is positive]

$\Rightarrow \quad (x - 2) < 0$.

$\Rightarrow \quad x < 2$.

Combining with $x > 1$, $f(x)$ is increasing in $x \in (1, 2)$ and decreasing in $x \in (2, \infty)$ ... (ii)

Combining (i) and (ii), we get :

$f(x)$ is strictly increasing on $x \in (-\infty, 0) \cup (1, 2)$ and strictly decreasing on $x \in (0, 1) \cup (2, \infty)$.

Example – 16

The function $f(x) = \log (x-2)^2 - x^2 + 4x + 1$ increases on the interval

(a) (1, 2) (b) (2, 3)

(c) (5/2, 3) (d) (2, 4)

Sol. (b), (c)

$f(x) = 2\log(x-2) - x^2 + 4x + 1$

$\Rightarrow \quad f'(x) = \dfrac{2}{x-2} - 2x + 4$

$\Rightarrow \quad f'(x) = 2\left[\dfrac{1-(x-2)^2}{x-2}\right] = -2\dfrac{(x-1)(x-3)}{x-2}$

$\Rightarrow \quad f'(x) = -\dfrac{2(x-1)(x-3)(x-2)}{(x-2)^2}$

$\therefore \quad f'(x) > 0 \Rightarrow -2(x-1)(x-3)(x-2) > 0$

$\Rightarrow \quad (x-1)(x-2)(x-3) < 0$

$\Rightarrow \quad x \in (-\infty, 1) \cup (2, 3).$

$\quad - \quad + \quad - \quad +$

$-\infty \quad 1 \quad 2 \quad 3 \quad \infty$

Example – 17

Show that $x/(1+x) < \log(1+x) < x$ for $x > 0$.

Sol. Let $f(x) = \log(1+x) - \dfrac{x}{1+x}$

$f'(x) = \dfrac{1}{1+x} - \dfrac{(1+x)-x}{(1+x)^2}$

$f'(x) = \dfrac{x}{(1+x)^2} > 0$ for > 0

$\Rightarrow \quad f(x)$ is increasing.

Hence $x > 0 \Rightarrow f(x) > f(0)$ by the definition of the increasing function.

$\Rightarrow \quad \log(1+x) - \dfrac{x}{1+x} > \log(1+0) - \dfrac{0}{1+0}$

$\Rightarrow \quad \log(1+x) - \dfrac{x}{1+x} > 0$

$\Rightarrow \quad \log(1+x) > \dfrac{x}{1+x} \quad \text{... (i)}$

Now, let $g(x) = x - \log(1+x)$

$g'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x} > 0$ for $x > 0$

$\Rightarrow \quad g(x)$ is increasing.

Hence $x > 0 \Rightarrow g(x) > g(0)$

$\Rightarrow \quad x - \log(1+x) > 0 - \log(1+0)$

$\Rightarrow \quad x - \log(1+x) > 0$

$\Rightarrow \quad x > \log(1+x) \quad \text{... (ii)}$

Combining (i) and (ii), we get :

$\dfrac{x}{1+x} < \log(1+x) < x$

Example – 18

Find critical points of $f(x) = x^{2/3}(2x-1)$.

Sol. $f(x) = 2x^{5/3} - x^{2/3}$

Differentiate w.r.t. x to get,

$f'(x) = \dfrac{10}{3}x^{2/3} - \dfrac{2}{3}x^{-1/3} = \dfrac{2}{3}\dfrac{(5x-1)}{x^{1/3}}.$

For critical points,

$f'(x) = 0$ or $f'(x)$ is not defined.

Put $f'(x) = 0$ to get $x = \dfrac{1}{5}.$

$f'(x)$ is not defined when denominator = 0.

$\Rightarrow \quad x^{1/3} = 0 \quad \Rightarrow \quad x = 0$

Now we can say that $x = 0$ and $x = \dfrac{1}{5}$ are critical points as $f(x)$ exists at both $x = 0$ and $x = \dfrac{1}{5}$.

$\Rightarrow \quad$ Critical points of $f(x)$ are $x = 0,\ x = \dfrac{1}{5}.$

Example – 19

Discuss concavity and convexity and find points of inflexion of $y = x^2 e^{-x}$.

Sol. Let $f(x) = x^2 e^{-x}$.

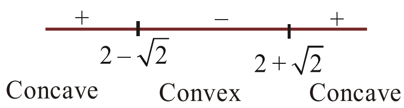
Differentiate w.r.t.x to get :

$$f'(x) = e^{-x}(2x) + (-e^{-x})\,x^2$$
$$= x\,e^{-x}[2 - x]$$

Differentiate again w.r.t. x to get :

$$f''(x) = (2 - 2x)\,e^{-x} + (2x - x^2)(-e^{-x})$$
$$= e^{-x}(2 - 2x - 2x + x^2)$$
$$= e^{-x}(x^2 - 4x + 2)$$
$$= e^{-x}(x - (2 - \sqrt{2}))\,(x - (2 + \sqrt{2}))$$

See the figure and observe how the sign of $f''(x)$ changes.



Sign of $f''(x)$ is changing at $x = 2 \pm \sqrt{2}$.

Therefore points of inflexion of $f(x)$ are $x = 2 \pm \sqrt{2}$.

$$f''(x) \geq 0\ \forall\ x \in [-\infty, 2 - \sqrt{2}] \cup [2 + \sqrt{2}, \infty]$$

Therefore $f(x)$ is "Concave upward"

$$\forall\ x \in (-\infty, 2 - \sqrt{2}] \cup [2 + \sqrt{2}, \infty)$$

Similarly we can observe

$$f''(x) \leq 0\ \forall\ x \in [2 - \sqrt{2}, 2 + \sqrt{2}]$$

Therefore $f(x)$ is "Convex upwards"

$$\forall\ x \in [2 - \sqrt{2}, 2 + \sqrt{2}]$$

Example – 20

Find points of local maximum and local minimum of $f(x) = x^{2/3}(2x - 1)$.

Sol. Let $f(x) = 2x^{5/3} - x^{2/3}$

Differentiate w.r.t. x to get :

$$f'(x) = 2\left(\frac{5}{3}\right)x^{2/3} - \frac{2}{3}x^{-1/3} = \frac{2}{3}\frac{(5x-1)}{x^{1/3}}$$

Critical points of $f(x)$ are $x = \frac{1}{5}$ and $x = 0$.

Using the following figure, we can determine how sign of $f'(x)$ is changing at $x = 0$ and $x = \frac{1}{5}$.

from figure,

Sign of $f'(x)$ in various intervals

$x = 0$ is point of local maximum as sign of $f'(x)$ changes from positive to negative and $x = \frac{1}{5}$ is point of local minimum as sign of $f'(x)$ is changing from negative to positive.

Example – 21

Determine the absolute extrema for the following function and interval.

$$g(t) = 2t^3 + 3t^2 - 12t + 4 \text{ on } [0, 2]$$

Sol. Differentiate w.r.t. t

$$g'(t) = 6t^2 + 6t - 12 = 6(t + 2)(t - 1)$$

Note that this problem is almost identical to the first problem. The only difference is the interval that we were working on.

The first step is to again find the critical points. From the first example we know these are $t = -2$ and $t = 1$. At this point it's important to recall that we only want the critical points that actually fall in the interval in question. This means that we only want $t = 1$ since $t = -2$ falls outside the interval so reject it.

Now for absolute maxima

We have,

$$\text{Max}\ \{g(1), g(0), g(2)\}$$

i.e., $\text{Max}\ \{-3, 4, 8\}$

On comparing all these values we get $g(t)$ has absolute max. as 8 at $t = 2$ and similarly absolute minimum of $g(t)$ is -3 at $t = 1$.

Example 22

Find the local maximum and local minimum values of the function $y = x^x$.

Sol. Let $f(x) = y = x^x$

$\Rightarrow \log y = x \log x$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \log x$

$\Rightarrow \frac{dy}{dx} = x^x (1 + \log x)$

$f'(x) = 0 \Rightarrow x^x(1 + \log x) = 0$

$\Rightarrow \log x = -1 \Rightarrow x = e^{-1} = 1/e.$

Method 1 : (First Derivative Test)

$f'(x) = x^x(1 + \log x)$

$f'(x) = x^x \log x$

$x < 1/e$ $\quad ex < 1$

$\Rightarrow f'(x) < 0$

$x > 1/e$ $\quad ex > 1$

$\Rightarrow f'(x) > 0$

The sign of $f'(x)$ changes from – ve to + ve around $x = 1/e$.

In other words, $f(x)$ changes from decreasing to increasing at $x = 1/e$.

Hence $x = 1/e$ is a point of local minimum.

Local minimum value $= (1/e)^{1/e} = e^{-1/e}$.

Method II : (Second Derivative Test)

$$f''(x) = (1 + \log x)\frac{d}{dx}x^x + x^x\left(\frac{1}{x}\right)$$

$$= x^x(1 + \log x)^2 + x^{x-1}$$

$$f''(1/e) = 0 + (e)^{(e-1)/e} > 0.$$

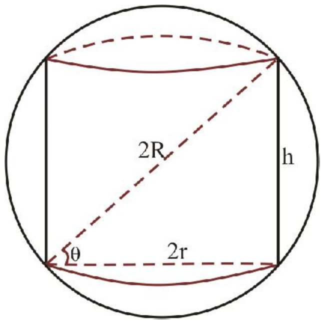
Hence $x = 1/e$ is a point of local minimum.

Local minimum value is $(1/e)^{1/e} = e^{-1/e}$.

Example – 23

Find the maximum surface area of a cylinder that can be inscribed in a given sphere of radius R.

Sol.



Let r be the radius and *h* be the height of cylinder. Consider the right triangle shown in the figure.

$2r = 2R\cos\theta$ and $\quad h = 2R\sin\theta$

Surface area of the cylinder $= 2\pi rh + 2\pi r^2$

$\Rightarrow S(\theta) = 4\pi R^2 \sin\theta\cos\theta + 2\pi R^2\cos^2\theta$

$\Rightarrow S(\theta) = 2\pi R^2\sin 2\theta + 2\pi R^2\cos^2\theta$

$\Rightarrow S'(\theta) = 4\pi R^2\cos 2\theta - 2\pi R^2\sin 2\theta$

$S'(\theta) = 0 \Rightarrow 2\cos 2\theta - \sin 2\theta = 0$

$\Rightarrow \tan 2\theta = 2 \Rightarrow \theta = \theta_0 = 1/2\tan^{-1}2$

$S''(\theta_0) = -8\pi R^2\sin 2\theta - 4\pi R^2\cos 2\theta$

$$S''(\theta) = -8\pi R^2\left(\frac{2}{\sqrt{5}}\right) - 4\pi R^2\left(\frac{1}{\sqrt{5}}\right) < 0$$

Hence surface area is maximum for $\theta = \theta_0 = 1/2\tan^{-1}2$

$S_{max} = 2\pi R^2\sin 2\theta_0 + 2\pi R^2\cos^2\theta_0$

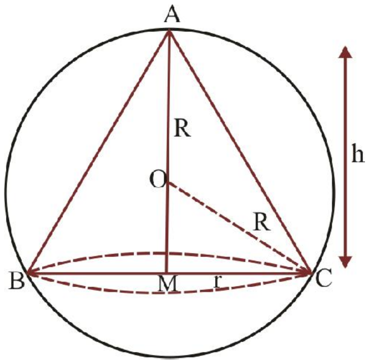
$$\Rightarrow S_{max} = 2\pi R^2\left(\frac{2}{\sqrt{5}}\right) + 2\pi R^2\left(\frac{1 + 1/\sqrt{5}}{2}\right)$$

$$\Rightarrow S_{max} = \pi R^2(1 + \sqrt{5})$$

Example – 24

Find the semi-vertical angle of the cone of maximum curved surface area that can be inscribed in a given sphere of radius R.

Sol.



Let h be the height of cone and r be the radius of the cone.

Consider the right ΔOMC where O is the centre of sphere and AM is perpendicular to the base BC of cone.

$OM = h - R,\ OC = R,\ MC = r$

$R^2 = (h-R)^2 + r^2$ ... (i)

and $r^2 + h^2 = l^2$ ... (ii)

where l is the slant height of cone.

Curve surface area = $C = \pi r l$

Using (i) and (ii), express C in terms of h only.

$$C = \pi r\sqrt{r^2+h^2} \Rightarrow C = \pi\sqrt{2hR-h^2}\ \sqrt{2hR}$$

We will maximise C^2.

Let $C^2 = f(h) = 2\pi^2 h R(2hR - h^2)$

$\Rightarrow$ $f'(h) = 2\pi^2 R(4hR - 3h^2)$

$f'(h) = 0 \Rightarrow 4hR - 3h^2 = 0$

$\Rightarrow$ $h = 4R/3$.

$f''(h) = 2\pi^2 R(4R - 6h)$

$$f''\left(\frac{4R}{3}\right) = 2\pi R^2(4R - 8R) < 0$$

Hence curved surface area is maximum for $h = \dfrac{4R}{3}$

Using (i), we get :

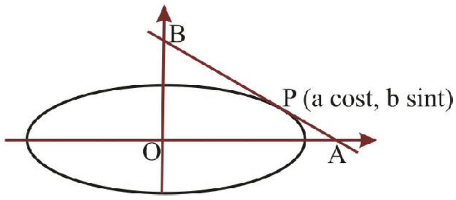
$$r^2 = 2hR - h^2 = \frac{8R^2}{9} \Rightarrow r = \frac{2\sqrt{2}}{3}R$$

Semi–vertical angle = $\theta = \tan^{-1} r/h = \tan^{-1} 1/\sqrt{2}$.

Example – 25

Prove that the minimum intercept made by axes on the tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is a + b. Also find the ratio in which the point of contact divides this intercept.

Sol.



Intercept made by the axes on the tangent is the length of the portion of the tangent intercepted between the axes.

Consider a point P on the ellipse whose coordinates are

$x = a\cos t,\ y = b\sin t$ (where t is the parameter)

The equation of the tangent is :

$$y - b\sin t = \frac{b\cos t}{-a\sin t}(x - a\cos t)$$

$$\Rightarrow \quad \frac{x}{a}\cos t + \frac{y}{b}\sin t = 1$$

$$\Rightarrow \quad OA = \frac{a}{\cos t},\ OB = \frac{b}{\sin t}$$

Length of intercept = l = AB = $\sqrt{\dfrac{a^2}{\cos^2 t} + \dfrac{b^2}{\sin^2 t}}$

We will minimise l^2.

Let $l^2 = f(t) = a^2\sec^2 t + \operatorname{cosec}^2 t$

$\Rightarrow$ $f'(t) = 2a^2\sec^2 t\tan t - 2b^2\operatorname{cosec}^2 t\cot t$

$f'(t) = 0 \Rightarrow a^2\sin^4 t = b^2\cos^4 t$

$\Rightarrow$ $t = \tan^{-1}\sqrt{b/a}$

$f''(t) = 2a^2(\sec^4 t + 2\tan^2 t\sec^2 t)$
$+ 2b^2(\operatorname{cosec}^4 t + 2\operatorname{cosec}^2 t\cot^2 t)$, which is positive.

Hence $f(t)$ is minimum for $\tan t = \sqrt{\dfrac{b}{a}}$.

$\Rightarrow$ $l_{min} = \sqrt{a^2(1 + b/a) + b^2(1 + a/b)}$

$\Rightarrow$ $l_{min} = a + b$

$$PA^2 = \left(a\cos t - \frac{a}{\cos t}\right)^2 + b^2\sin^2 t$$

$$= \frac{a^2\sin^4 t}{\cos^2 t} + b^2\sin^2 t$$

$$= (a^2\tan^2 t + b^2)\sin^2 t$$

$$= (ab + b^2)\frac{b}{a+b} = b^2$$

Hence $\dfrac{PA}{PB} = \dfrac{b}{a} \Rightarrow$ P divides AB in the ratio b : a

EXERCISE - 1 : BASIC OBJECTIVE QUESTIONS

Question Based on Rate of Change

1. The rate of change of the surface area of a sphere of radius r, when the radius is increasing at the rate of 2 cm/s is proportional to

(a) $\frac{1}{r}$ (b) $\frac{1}{r^2}$

(c) r (d) r^2

2. If the distance 's' metres traveresed by a particle in t seconds is given by $s = t^3 - 3t^2$, then the velocity of the particle when the acceleration is zero in m/s is

(a) 3 (b) –2

(c) –3 (d) 2

3. The sides of an equilateral triangle are increasing at the rate of 2 cm/s. The rate at which the area increases, when the side is 10 cm, is

(a) $\sqrt{3}\ cm^2/s$ (b) 10 cm^2/s

(c) $10\sqrt{3}\ cm^2/s$ (d) $\frac{10}{\sqrt{3}} cm^2/s$

4. Gas is being pumped into a spherical ballon at the rate of 30 ft^3/min. Then, the rate at which the radius increases when it reaches the value 15 ft, is

(a) $\frac{1}{30\pi}$ ft/min (b) $\frac{1}{15\pi}$ ft/min

(c) $\frac{1}{20}$ ft/min (d) $\frac{1}{15}$ ft/min

5. An object is moving in the clockwise direction around the unit circle $x^2 + y^2 = 1$. As it passes through the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, its y-coordinate is decreasing at the rate of 3 unit per second. The rate at which the x-coordinate changes at this point is (in unit per second)

(a) 2 (b) $3\sqrt{3}$

(c) $\sqrt{3}$ (d) $2\sqrt{3}$

6. The position of a point in time 't' is given by $x = a + bt - ct^2$, $y = at + bt^2$. Its acceleration at time 't' is

(a) b – c (b) b + c

(c) 2b – 2c (d) $2\sqrt{b^2 + c^2}$

7. If $V = \frac{4}{3}\pi r^3$, at what rate in cubic units is V increasing when r = 10 and $\frac{dr}{dt} = 0.01$?

(a) π (b) 4π

(c) 40π (d) 4π/3

8. Side of an equilateral triangle expands at the rate of 2 cm/s. The rate of increase of its area when each side is 10 cm, is

(a) $10\sqrt{2}\ cm^2/sec$ (b) $10\sqrt{3}\ cm^3/sec$

(c) 10 cm^2/sec (d) 5 cm^2/sec

9. The radius of a sphere is changing at the rate of 0.1 cm/s. The rate of change of its surface area when the radius is 200 cm, is

(a) 8π cm^2/sec (b) 12π cm^2/sec

(c) 160π cm^2/sec (d) 200π cm^2/sec

10. The distance moved by the particle in time t is given by $x = t^3 - 12t^2 + 6t + 8$. At the instant when its acceleration is zero, the velocity is

(a) 42 (b) –42

(c) 48 (d) –48

11. For what values of x is the rate of increase of $x^3 - 5x^2 + 5x + 8$ is twice the rate of increase of x ?

(a) $-3, -\frac{1}{3}$ (b) $-3, \frac{1}{3}$

(c) $3, -\frac{1}{3}$ (d) $3, \frac{1}{3}$

12. The radius of the base of a cone is increasing at the rate of 3 cm/minute and the altitude is decreasing at the rate of 4 cm/minute. The rate of change of lateral surface when the radius = 7 cm and altitude = 24 cm, is

(a) 54π cm^2/min (b) 7π cm^2/min

(c) 27π cm^2/min (d) none of these

13. The surface area of a sphere when its volume is increasing at the same rate as its radius, is

(a) 1 sq. units (b) $\frac{1}{2\sqrt{\pi}}$ sq. units

(c) 4π sq. units (d) $\frac{4\pi}{3}$ sq. units

14. The surface area of a cube is increasing at the rate of 2 cm^2/s. When its edge is 90 cm, the volume is increasing at the rate of

(a) 1620 cm^3/sec (b) 810 cm^3/sec

(c) 405 cm^3/sec (d) 45 cm^3/sec

15. A ladder 10 metres long rests with one end against a vertical wall, the other end on the floor. The lower end moves away from the wall at the rate of 2 metres/minute. The rate at which the upper end falls when its base is 6 metres away from the wall, is

(a) 3 metres/min (b) 2/3 metres/min

(c) 3/2 metres/min (d) none of these

16. If a particle moving along a line follows the law $s = \sqrt{1+t}$, then the acceleration is proportional to

(a) square of the velocity

(b) cube of the displacement

(c) cube of the velocity

(d) square of the displacement

17. If a particle is moving such that the velocity acquired is proportional to the square root of the distance covered, then its acceleration is

(a) a constant (b) $\propto s^2$

(c) $\propto \frac{1}{s^2}$ (d) $\propto \frac{1}{s}$

Errors & Approximation

18. The circumference of a circle is measured as 28 cm with an error of 0.01 cm. The percentage error in the area is

(a) $\frac{1}{14}$ (b) 0.01

(c) $\frac{1}{7}$ (d) none of these

19. If $y = x^n$, then the ratio of relative errors in y and x is

(a) 1 : 1 (b) 2 : 1

(c) 1 : n (d) n : 1

20. If the ratio of base radius and height of a cone is 1 : 2 and percentage error in radius is λ %, then the error in its volume is

(a) λ % (b) 2λ%

(c) 3λ% (d) none of these

21. The height of a cylinder is equal to the radius. If an error of α % is made in the height, then percentage error in its volume is

(a) α % (b) 2α%

(c) 3α% (d) none of these

22. If the percentage error in measuring the surface area of a sphere is α%, then the error in its volume is

(a) $\frac{3}{2}\alpha$% (b) $\frac{2}{3}\alpha$%

(c) 3α% (d) none of these

Roll's and L.M.V.T.

23. In the mean value theorem $\frac{f(b)-f(a)}{b-a} = f'(c)$, if a = 0, $b = \frac{1}{2}$ and $f(x) = x(x-1)(x-2)$, then value of c is

(a) $1-\frac{\sqrt{15}}{6}$ (b) $1+\sqrt{15}$

(c) $1-\frac{\sqrt{21}}{6}$ (d) $1+\sqrt{21}$

24. If the polynomial equation

$a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0 = 0$, n positive integer, has two different real roots α and β, then between α and β, the equation

$na_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + ... + a_1 = 0$ has

(a) exactly one root (b) atmost one root

(c) atleast one root (d) no root

25. The equation of $a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x = 0$ has a positive root $x = \alpha$, then the equation $n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + ... a_1 = 0$ has a positive root which is

(a) smaller than α (b) greater than α

(c) equal to α (d) greater than or equal to α

26. If $f(x) = \begin{vmatrix} \sin x & \sin a & \sin b \\ \cos x & \cos a & \cos b \\ \tan x & \tan a & \tan b \end{vmatrix}$, where $0 < a < b < \frac{\pi}{2}$, then the equation $f'(x) = 0$ has, in the interval (a, b)

(a) atleast one root (b) atmost one root

(c) no root (d) none of these

27. If $f(x)$ is differentiable in the interval [2, 5], where $f(2) = \frac{1}{5}$ and $f(5) = \frac{1}{2}$, then there exists a number c, $2 < c < 5$ for which $f'(c)$ is equal to

(a) 1/2 (b) 1/5

(c) 1/10 (d) none of these

Tangents and Normals

28. The tangent to the curve $5x^2 + y^2 = 1$ at $\left(\frac{1}{3}, -\frac{2}{3}\right)$ passes through the point

(a) (0, 0) (b) (1, –1)

(c) (–1, 1) (d) none of these

29. The tangent to the curve $x^2 + y^2 = 25$ is parallel to the line $3x - 4y = 7$ at the point

(a) (–3, –4) (b) (3, –4)

(c) (3, 4) (d) none of these

30. If the normal to the curve $y = f(x)$ at the point (3, 4) makes an angle $\frac{3\pi}{4}$ with the positive x-axis, then $f'(3)$ is equal to

(a) –1 (b) $-\frac{3}{4}$

(c) $\frac{4}{3}$ (d) 1

31. The equation of the tangent to the curve $y = \sqrt{9 - 2x^2}$ at the point where the ordinate and the abscissa are equal, is

(a) $2x + y - 3\sqrt{3} = 0$ (b) $2x + y + 3\sqrt{3} = 0$

(c) $2x - y - 3\sqrt{3} = 0$ (d) none of these

32. If the tangent at each point of the curve $y = \frac{2}{3}x^3 - 2ax^2 + 2x + 5$ makes an acute angle with the positive direction of x-axis, then

(a) $a \ge 1$ (b) $-1 \le a \le 1$

(c) $a \le -1$ (d) none of these

33. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1, 1) and the coordinate axes, lies in the first quadrant. If its area is 2, then the value of b is

(a) –1 (b) 3

(c) – 3 (d) 1

34. The equation of the tangent to the curve $(1 + x^2) y = 2 - x$, where it crosses the x-axis, is

(a) $x + 5y = 2$ (b) $x - 5y = 2$

(c) $5x - y = 2$ (d) $5x + y - 2 = 0$

35. Equation of the normal to the curve $y = -\sqrt{x} + 2$ at the point of its intersection with the curve $y = \tan(\tan^{-1} x)$ is

(a) $2x - y - 1 = 0$ (b) $2x - y + 1 = 0$

(c) $2x + y - 3 = 0$ (d) none of these

36. The curve $y - e^{xy} + x = 0$ has a vertical tangent at

(a) (1, 1) (b) (0, 1)

(c) (1, 0) (d) no point

37. If the line $ax + by + c = 0$ is a tangent to the curve $xy = 4$, then the possible answer is

(a) $a > 0, b > 0$ (b) $a > 0, b < 0$

(c) $a < 0, b > 0$ (d) none of these

Angle of Intersection

38. The angle between the curves $y = \sin x$ and $y = \cos x$ is

(a) $\tan^{-1}(2\sqrt{2})$ (b) $\tan^{-1}(3\sqrt{2})$

(c) $\tan^{-1}(3\sqrt{3})$ (d) $\tan^{-1}(5\sqrt{2})$

39. The angle between the tangents to the curve $y^2 = 2ax$ at the points where $x = \frac{a}{2}$, is

(a) $\pi/6$ (b) $\pi/4$

(c) $\pi/3$ (d) $\pi/2$

40. The angle between the tangents at those points on the curve $y = (x + 1)(x - 3)$ where it meets x-axis, is

(a) $\tan^{-1}\left(\frac{15}{8}\right)$ (b) $\tan^{-1}\left(\frac{8}{15}\right)$

(c) $\frac{\pi}{4}$ (d) none of these

41. The angle at which the curves $y = \sin x$ and $y = \cos x$ intersect in $[0, \pi]$, is

(a) $\tan^{-1} 2\sqrt{2}$ (b) $\tan^{-1}\sqrt{2}$

(c) $\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (d) none of these

42. The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$

(a) cut at right angles (b) touch each other

(c) cut at an angle $\frac{\pi}{3}$ (d) cut at an angle $\frac{\pi}{4}$

43. The two curves $y = 3^x$ and $y = 5^x$ intersect at an angle

(a) $\tan^{-1}\left(\frac{\log 5 - \log 3}{1 + \log 3.\log 5}\right)$ (b) $\tan^{-1}\left(\frac{\log 3 + \log 5}{1 - \log 3.\log 5}\right)$

(c) $\tan^{-1}\left(\frac{\log 3 + \log 5}{1 + \log 3.\log 5}\right)$ (d) none of these

44. The angle of intersection of the curve $y = x^2$ & $6y = 7 - x^3$ at (1, 1) is

(a) $\pi/5$ (b) $\pi/4$

(c) $\pi/3$ (d) $\pi/2$

45. The curves $x^3 + p\,xy^2 = -2$ and $3x^2y - y^3 = 2$ are orthogonal for

(a) $p = 3$ (b) $p = -3$

(c) no value of p (d) $p = \pm 3$

Length of Tangent Sub-tangent, Normal Sub normal

46. The length of subtangent to the curve $x^2 y^2 = a^4$ at the point $(-a, a)$ is

(a) 3a (b) 2a

(c) a (d) 4a

47. For the parabola $y^2 = 4ax$, the ratio of the sub-tangent to the abscissa is

(a) 1 : 1 (b) 2 : 1

(c) 1 : 2 (d) 3 : 1

48. The length of subtangent to the curve $x^2y^2 = a^4$ at the point $(-a, a)$ is

(a) 3a (b) 2a

(c) a (d) 4a

49. The product of the lengths of subtangent and subnormal at any point of a curve is

(a) square of the abscissae (b) square of the ordinate

(c) constant (d) None of these

Increasing and Decreasing

50. The function $f(x) = x + \cos x$ is

(a) always increasing

(b) always decreasing

(c) increasing for certain range of x

(d) None of the above

51. The function $f(x) = \tan^{-1} x + x$ increases in the interval

(a) $(1, \infty)$ (b) $(-1, \infty)$

(c) $(-\infty, \infty)$ (d) $(0, \infty)$

52. Which of the following statements is/are correct.

(a) x + sin x is increasing function

(b) sec x is neither increasing nor decreasing function

(c) x + sin x is decreasing function

(d) sec x is an increasing function

53. The function $f(x) = 2\log(x - 2) - x^2 + 4x + 1$ increases in the interval

(a) (1, 2) (b) (2, 3)

(c) [5/2, 3] (d) (2, 4)

54. The interval in which the function x^3 increases less rapidly than $6x^2 + 15x + 5$ is :

(a) $(-\infty, -1)$ (b) $(-5, 1)$

(c) $(-1, 5)$ (d) $(5, \infty)$

55. The function $y = \frac{2x^2 - 1}{x^4}$ is

(a) a decreasing function for all $x \in R - \{0\}$

(b) a increasing function for all $x \in R - \{0\}$

(c) increasing for $x > 0$

(d) none of these

56. The function $f(x) = \frac{\sin x}{x}$ is decreasing in the interval

(a) $\left(-\frac{\pi}{2}, 0\right)$ (b) $\left(0, \frac{\pi}{2}\right)$

(c) $(0, \pi)$ (d) none of these

57. If $f(x) = \frac{1}{x+1} - \log(1 + x), x > 0$, then f is

(a) an increasing function

(b) a decreasing function

(c) both increasing and decreasing function

(d) None of the above

58. Let $f(x) = \int_1^x e^x (x - 1)(x - 2)\,dx$. Then, f decreases in the interval

(a) $(-\infty, 2)$ (b) $(-2, -1)$

(c) $(1, 2)$ (d) $(2, \infty)$

59. Let the function $f(x) = \sin x + \cos x$, be defined in $[0, 2\pi]$, then $f(x)$

(a) increases in $(\pi/4, \pi/2)$

(b) decreases in $[\pi/4, 5\pi/4]$

(c) increases in $[0, \pi/4] \cup [5\pi/4, 2\pi]$

(d) decreases in $[0, \pi/4) \cup (\pi/2, 2\pi]$

60. If $f(x) = 2x + \cot^{-1} x + \log\left(\sqrt{1+x^2} - x\right)$, then $f(x)$:

(a) increases in $[0, \infty)$

(b) decreases in $[0, \infty)$

(c) neither increases nor decreases in $[0, \infty)$

(d) increases in $(-\infty, \infty)$

61. $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$ is an monotonically increasing function in the set of real numbers if a and b satisfy the condition.

(a) $a^2 - 3b - 15 > 0$ (b) $a^2 - 3b + 15 > 0$

(c) $a^2 - 3b + 15 < 0$ (d) $a > 0$ $b > 0$

62. The length of the longest interval, in which the function $3\sin x - 4\sin^3 x$ is increasing, is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{2}$ (d) π

63. The function $f(x) = 2x^2 - \log|x|$ monotonically decreases for

(a) $x \in (-\infty, -1/2] \cup (0, 1/2]$

(b) $x \in (-\infty, 1/2]$

(c) $x \in [-1/2, 0) \cup [1/2, \infty)$

(d) none of these

64. The function $f(x) = \frac{|x-1|}{x^2}$ is monotonically decreasing on :

(a) $(2, \infty)$ (b) $(0, 1)$

(c) $(0, 1) \cup (2, \infty)$ (d) $(-\infty, \infty)$

65. If $0 < x < \frac{\pi}{2}$, then

(a) $\frac{2}{\pi} > \frac{\sin x}{x}$ (b) $\frac{2}{\pi} < \frac{\sin x}{x} < 1$

(c) $\frac{\sin x}{x} < 1$ (d) $\frac{\sin x}{x} > 1$

66. $y = \log x$ satisfies for $x > 1$, the inequality

(a) $x - 1 > y$ (b) $x^2 - 1 > y$

(c) $y > x - 1$ (d) $(x - 1)/x < y$

Maxima and Minima

67. The function $f(x) = 2x^3 - 15x^2 + 36x + 4$ has local maxima at

(a) $x = 2$ (b) $x = 4$

(c) $x = 0$ (d) $x = 3$

68. Maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is

(a) 0 (b) 12

(c) 16 (d) 32

69. The function $f(x) = 2x^3 - 3x^2 - 12x + 4$ has

(a) no maxima and minima

(b) one maxima and one minima

(c) two maxima

(d) two minima

70. The minimum value of $2x + 3y$, when $xy = 6$ is $(x > 0)$

(a) 12 (b) 9

(c) 8 (d) 6

71. The greatest value of $f(x) = (x+1)^{1/3} - (x-1)^{1/3}$ on [0, 1] is :

(a) 1 (b) 2

(c) 3 (d) $2^{1/3}$

72. The function $f(x) = x^2 (x-2)^2$

(a) decreases on $(0, 1) \cup (2, \infty)$

(b) increase on $(-\infty, 0) \cup (1, 2)$

(c) has a local maximum value 0

(d) has a local maximum value 1

73. The maximum value of the function $y = x(x-1)^2, 0 \le x \le 2$ is

(a) 0 (b) 4/27

(c) –4 (d) none of these

74. Let $f(x) = (1+b^2)x^2 + 2bx + 1$ and m (b) the minimum value of $f(x)$ for a given b. As b varies, the range of m (b) is

(a) [0, 1] (b) (0, 1/2]

(c) $\left[\frac{1}{2}, 1\right]$ (d) (0, 1]

75. $f(x) = 1 + [\cos x]\, x$, in $0 \le x \le \frac{\pi}{2}$

(a) has a minimum value 0

(b) has a maximum value 2

(c) is continuous in $\left[0, \frac{\pi}{2}\right]$

(d) is not differentiable at $x = \frac{\pi}{2}$

76. The point in the interval $[0, 2\pi]$, where $f(x) = e^x \sin x$ has maximum slope, is

(a) 0 (b) $\frac{\pi}{2}$

(c) 2π (d) $\frac{3\pi}{2}$

77. The minimum value of x^x is attained (where x is positive real number) when x is equal to :

(a) e (b) e^{-1}

(c) 1 (d) e^2

78. The minimum value of $2^{(x^2-3)^3+27}$, is

(a) 2^{27} (b) 2

(c) 1 (d) 4

79. The maximum value of xy subject to $x + y = 8$, is

(a) 8 (b) 16

(c) 20 (d) 24

80. The maximum value of $x^3 - 3x$ in the intveral [0, 2], is

(a) –2 (b) 0

(c) 2 (d) None of these

81. The maximum area of the rectangle that can be inscribed in a circle of radius r, is

(a) πr^2 (b) r^2

(c) $\pi r^2/4$ (d) $2r^2$

82. If $A > 0, B > 0$ and $A + B = \frac{\pi}{3}$, then the maximum value of tan A tan B is

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{3}$

(c) 3 (d) $\sqrt{3}$

83. In a ΔABC, $B = 90°$ and $a + b = 4$. The area of the triangle is maximum when C, is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$

(c) $\frac{\pi}{3}$ (d) none of these

84. The point (0, 3) is nearest to the curve $x^2 = 2y$ at

(a) $\left(2\sqrt{2},0\right)$ (b) (0, 0)

(c) (2, 2) (d) none of these

85. The function $f(x) = \int_{-1}^{x} t\left(e^t-1\right)(t-1)(t-2)^3(t-3)^5\,dt$ local minimum at x =

(a) 0 (b) 1

(c) 2 (d) 3

86. The length of the smallest intercept made by the coordinate axes on any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$ is

(a) a + b (b) $\frac{a+b}{2}$

(c) $\frac{a+b}{4}$ (d) None of these

87. An isosceles triangle of vertical angle 2θ is inscribed in a circle of radius a. Then area of the triangle is maximum when θ =

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

88. The maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is

(a) 0 (b) 12

(c) 16 (d) 32

89. The function $f(x)=\int_{1}^{x}\left\{2(t-1)(t-2)^3+3(t-1)^2(t-2)^2\right\}dt$ attains its local maximum value at x =

(a) 1 (b) 2

(c) 3 (d) 4

90. If $f(x) = x^3 + 4x^2 + \lambda x + 1$ is a strictly decreasing function of x in the largest possible interval [–2, –2/3] then

(a) $\lambda = 4$ (b) $\lambda = 2$

(c) $\lambda = -1$ (d) λ has no real value

EXERCISE - 2 : PREVIOUS YEAR JEE MAINS QUESTIONS

1. The maximum distance from origin of a point on the curve

$x = a\sin t - b\sin\left(\frac{at}{b}\right)$

$y = a\cos t - b\cos\left(\frac{at}{b}\right)$, both a, b > 0, is **(2002)**

(a) $a - b$ (b) $a + b$

(c) $\sqrt{a^2 + b^2}$ (d) $\sqrt{a^2 - b^2}$

2. If $2a + 3b + 6c = 0$ (a, b, c, ∈ R), then the quadratic equation $ax^2 + bx + c = 0$ has **(2002)**

(a) at least one root in (0, 1) (b) at least one root in [2, 3]

(c) at least one root in [4, 5] (d) none of the above

3. The greatest value of $f(x) = (x+1)^{1/3} - (x-1)^{1/3}$ on [0, 1] is **(2002)**

(a) 1 (b) 2

(c) 3 (d) $\frac{1}{3}$

4. The function $f(x) = \cot^{-1}x + x$ increases in the interval **(2002)**

(a) $(1, \infty)$ (b) $(-1, \infty)$

(c) $(-\infty, \infty)$ (d) $(0, \infty)$

5. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where a > 0, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a equals **(2003)**

(a) 1 (b) 2

(c) $\frac{1}{2}$ (d) 3

6. The real number x when added to its inverse gives the minimum value of the sum at x equal to **(2003)**

(a) 1 (b) – 1

(c) – 2 (d) 2

7. A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa, is **(2004)**

(a) (2, 4) (b) (2, –4)

(c) $\left(-\frac{9}{8}, \frac{9}{2}\right)$ (d) $\left(\frac{9}{8}, \frac{9}{2}\right)$

8. A function $y = f(x)$ has a second order derivative $f'' = 6(x-1)$. If its graph passes through the point (2, 1) and at that point the tangent to the graph is $y = 3x - 5$, then the function is **(2004)**

(a) $(x-1)^2$ (b) $(x-1)^3$

(c) $(x+1)^3$ (d) $(x+1)^2$

9. The normal to the curve $x = a(1 + \cos\theta)$, $y = a\sin\theta$ at θ always passes through the fixed point **(2004)**

(a) (a, 0) (b) (0, a)

(c) (0, 0) (d) (a, a)

10. Area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is **(2005)**

(a) ab (b) 2 ab

(c) a/b (d) $\sqrt{ab}$

11. The normal to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any point θ is such that **(2005)**

(a) it makes angle $\frac{\pi}{2} + \theta$ with x–axis

(b) it passes through the origin

(c) it is a constant distance from the origin

(d) it passes through $\left(a\frac{\pi}{2}, -a\right)$

12. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3/min. When the thickness of ice is 5 cm, then the rate at which the thickness of ice decreases, is **(2005)**

(a) $\frac{1}{18\pi}$ cm/min (b) $\frac{1}{36\pi}$ cm/min

(c) $\frac{5}{6\pi}$ cm/min (d) $\frac{1}{54\pi}$ cm/min

13. Let f be differentiable for all x.
If $f(1) = -2$ and $f'(x) \geq 2$ for $x \in [1, 6]$, then **(2005)**

(a) $f(6) < 8$ (b) $f(6) \geq 8$

(c) $f(6) = 5$ (d) $f(6) < 5$

14. A function is matched below against an interval where it is supposed to be increasing. Which of the following pairs is incorrectly matched ? **(2005)**

	Interval	**Function**
(a)	$(-\infty, -4)$	$x^3 + 6x^2 + 6$
(b)	$\left(-\infty, \frac{1}{3}\right]$	$3x^2 - 2x + 1$
(c)	$[2, \infty)$	$2x^3 - 3x^2 - 12x + 6$
(d)	$(-\infty, \infty)$	$x^3 - 3x^2 + 3x + 3$

15. If x is real, the maximum value of $\frac{3x^2+9x+17}{3x^2+9x+7}$ is **(2006)**

(a) $\frac{1}{4}$ (b) 41

(c) 1 (d) $\frac{17}{7}$

16. The function $g(x) = \frac{x}{2} + \frac{2}{x}$ has a local minimum at **(2006)**

(a) $x = 2$ (b) $x = -2$

(c) $x = 0$ (d) $x = 1$

17. A triangular park is enclosed on two sides by a fence and on the third side by a straight river bank. The two sides having fence are of same length x. The maximum area enclosed by the park is **(2006)**

(a) $\frac{3}{2}x^2$ (b) $\sqrt{\frac{x^3}{8}}$

(c) $\frac{1}{2}x^2$ (d) πx^2

18. Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) is **(2006)**

(a) $\pi/2$ (b) $\pi/3$

(c) $\pi/6$ (d) $\pi/4$

19. A value of c for which conclusion of Mean Value Theorem holds for the function $f(x) = \log_e x$ on the interval [1, 3], is **(2007)**

(a) $\log_3 e$ (b) $\log_e 3$

(c) $2 \log_3 e$ (d) $\frac{1}{2}\log_e 3$

20. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in **(2007)**

(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(c) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$

21. If p and q are positive real numbers such that $p^2 + q^2 = 1$, then the maximum value of $(p + q)$ is **(2007)**

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$

(c) $\sqrt{2}$ (d) 2

22. How many real solutions does the equation $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$ have ? **(2008)**

(a) 5 (b) 7

(c) 1 (d) 3

23. Suppose the cubic $x^3 - px + q$ has three distinct real roots where $p > 0$ and $q > 0$. Then which one of the following holds? **(2008)**

(a) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

(b) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$

(c) The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$

(d) The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

24. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval $[-1, 1]$ **(2009)**

(a) $P(-1)$ is the minimum and $P(1)$ is the maximum of P

(b) $P(-1)$ is not minimum but $P(1)$ is the maximum of P

(c) $P(-1)$ is the minimum and $P(1)$ is not the maximum of P

(d) neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P

25. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis, is **(2010)**

(a) $y = 0$ (b) $y = 1$

(c) $y = 2$ (d) $y = 3$

26. Let $f : R \to R$ be defined by

$$f(x) = \begin{cases} k - 2x, & \text{if } x \le -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$$

If f has a local minimum at $x = -1$, then a possible value of k is **(2010)**

(a) 1 (b) 0

(c) $-\frac{1}{2}$ (d) -1

27. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t}\sin t\,dt$. Then, f has **(2011)**

(a) local minimum at π and 2π

(b) local minimum at π and local maximum at 2π

(c) local maximum at π and local minimum at 2π

(d) local maximum at π and 2π

28. Let f be a function defined by

$$f(x) = \begin{cases} \frac{\tan x}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$$

Statement I $x = 0$ is point of minima of f.

Statement II $f'(0) = 0$ **(2011)**

(a) Statement I is false, Statement II is true.

(b) Statement I is true, Statement II is true;

Statement II is correct explanation for Statement I.

(c) Statement I is true, Statement II is true;

Statement II is not a correct explanation for Statement I.

(d) Statement I is true, Statement II is false.

29. A spherical balloon is filled with 4500π cu m of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cu m/ min, then the rate (in m/min) at which the radius of the balloon decreases 49 min after the leakage began is **(2012)**

(a) $\frac{9}{7}$ (b) $\frac{7}{9}$

(c) $\frac{2}{9}$ (d) $\frac{9}{2}$

30. Let a, b $\in$ R be such that the function f given by $f(x) = \log|x| + bx^2 + ax$, $x \neq 0$ has extreme values at $x = -1$ and $x = 2$.

Statement I f has local maximum at $x = -1$ and $x = 2$.

Statement II $a = \frac{1}{2}$ and $b = \frac{-1}{4}$. **(2012)**

(a) Statement I is false, Statement II is true.

(b) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(d) Statement I is true, Statement II is false.

31. A line is drawn through the point (1, 2) to meet the coordinate axes at P and Q such that it forms a Δ OPQ, where O is the origin, if the area of the Δ OPQ is least, then the slope of the line PQ is **(2012)**

(a) $-\frac{1}{4}$ (b) -4

(c) -2 (d) $-\frac{1}{2}$

32. The intercepts on x-axis made by tangents to the curve, $y = \int_0^x |t|\,dt$, $x \in R$, which are parallel to the line $y = 2x$, are equal to **(2013)**

(a) ± 1 (b) ± 2

(c) ± 3 (d) ± 4

33. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$ then : **(2014)**

(a) $\alpha = 2, \beta = \frac{1}{2}$ (b) $\alpha = -6, \beta = \frac{1}{2}$

(c) $\alpha = -6, \beta = -\frac{1}{2}$ (d) $\alpha = 2, \beta = -\frac{1}{2}$

34. If f and g are differentiable functions in [0, 1] satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in]0, 1[$: **(2014)**

(a) $f'(c) = 2g'(c)$ (b) $2f'(c) = g'(c)$

(c) $2f'(c) = 3g'(c)$ (d) $f'(c) = g'(c)$

35. If the Rolle's theorem holds for the function $f(x) = 2x^3 + ax^2 + bx$ in the interval [-1, 1] for the point $c = \frac{1}{2}$, then the value of $2a + b$ is **(2014/Online Set–1)**

(a) 1 (b) -1

(c) 2 (d) -2

36. For the curve $y = 3\sin\theta\cos\theta$, $x = e^{\theta}\sin\theta, 0 \le \theta \le \pi$, the tangent is parallel to x-axis when θ is: **(2014/Online Set–2)**

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

37. Two ships A and B are sailing straight away from a fixed point O along routes such that $\angle AOB$ is always 120°. At a certain instance, OA = 8 km, OB = 6 km and the ship A is sailing at the rate of 20 km/hr while the ship B sailing at the rate of 30 km/hr. Then the distance between A and B is changing at the rate (in km/hr): **(2014/Online Set–2)**

(a) $\frac{260}{\sqrt{37}}$ (b) $\frac{260}{37}$

(c) $\frac{80}{\sqrt{37}}$ (d) $\frac{80}{37}$

38. The volume of the largest possible right circular cylinder that can be inscribed in a sphere of radius $= \sqrt{3}$ is: **(2014/Online Set–2)**

(a) $\frac{4}{3}\sqrt{3}\pi$ (b) $\frac{8}{3}\sqrt{3}\pi$

(c) 4π (d) 2π

39. If $f(\theta)=\begin{vmatrix}1 & \cos\theta & 1\\ -\sin\theta & 1 & -\cos\theta\\ -1 & \sin\theta & 1\end{vmatrix}\begin{vmatrix}1 & 2 & x\\ 3 & -1 & 2\end{vmatrix}$ and A and B are respectively the maximum and the minimum values of $f(\theta)$, then (A, B) is equal to: **(2014/Online Set–3)**

(a) (3, -1) (b) $\left(4, 2-\sqrt{2}\right)$

(c) $\left(2+\sqrt{2}, 2-\sqrt{2}\right)$ (d) $\left(2+\sqrt{2}, -1\right)$

40. If the volume of a spherical ball is increasing at the rate of 4π cc/sec, then the rateof increase of its radius (in cm/sec), when the volume is 288 πcc, is:

(2014/Online Set–4)

(a) $\frac{1}{6}$ (b) $\frac{1}{9}$

(c) $\frac{1}{36}$ (d) $\frac{1}{24}$

41. The normal to the curve, $x^2 + 2xy - 3y^2 = 0$, at (1, 1): **(2015)**

(a) meets the curve again in the third quadrant.

(b) meets the curve again in the fourth quadrant.

(c) does not meet the curve again.

(d) meets the curve again in the second quadrant.

42. Let f(x) be a polynomial of degree four having extreme value at x = 1 and x = 2. If $\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right]=3$, then f(2) is equal to: **(2015)**

(a) 0 (b) 4

(c) –8 (d) –4

43. If Roll's theorem holds for the function $f(x) = 2x^3 + bx^2 + cx$, $x \in [-1, 1]$, at the point $x = \frac{1}{2}$, then 2b + c equals :

(2015/Online Set–1)

(a) 1 (b) 2

(c) –1 (d) –3

44. Let k and K be the minimum and the maximum values of the function $f(x)=\frac{(1+x)^{0.6}}{1+x^{0.6}}$ in [0, 1] respectively, then the ordered pair (k, K) is equal to:

(2015/Online Set–2)

(a) $(2^{-0.4}, 1)$ (b) $(2^{-0.4}, 2^{0.6})$

(c) $(2^{-0.6}, 1)$ (d) $(1, 2^{0.6})$

45. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side =x unit and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then :

(2016)

(a) $(4-\pi)x = \pi r$ (b) $x = 2r$

(c) $2x = r$ (d) $2x = (\pi + 4)r$

46. Consider

$f(x)=\tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$, $x \in \left(0, \frac{\pi}{2}\right)$, A normal to y = f (x) at $x=\frac{\pi}{6}$ also passes through the point :

(2016)

(a) $\left(0, \frac{2\pi}{3}\right)$ (b) $\left(\frac{\pi}{6}, 0\right)$

(c) $\left(\frac{\pi}{4}, 0\right)$ (d) (0, 0)

47. If the tangent at a point P, with parameter t, on the curve $x = 4t^2 + 3$, $y = 8t^3 - 1$, $t \in R$, meets the curve again at a point Q, then the coordinates of Q are :

(2016/Online Set–1)

(a) $(t^2 + 3, -t^3 - 1)$ (b) $(4t^2 + 3, -8t^3 - 1)$

(c) $(t^2 + 3, t^3 - 1)$ (d) $(16t^2 + 3, -64t^3 - 1)$

48. The minimum distance of a point on the curve $y = x^2 - 4$ from the origin is : **(2016/Online Set–1)**

(a) $\frac{\sqrt{19}}{2}$ (b) $\sqrt{\frac{15}{2}}$

(c) $\frac{\sqrt{15}}{2}$ (d) $\sqrt{\frac{19}{2}}$

49. The normal to the curve $y(x-2)(x-3) = x + 6$ at the point where the curve intersects the y-axis passes through the point: **(2017)**

(a) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, \frac{1}{2}\right)$

(c) $\left(\frac{1}{2}, -\frac{1}{3}\right)$ (d) $\left(\frac{1}{2}, \frac{1}{3}\right)$

50. Twenty meters of wire is available for fencing off a flower-bed in the form of a circular sector. Then the maximum area (in sq. m) of the flower-bed, is: **(2017)**

(a) 12.5 (b) 10

(c) 25 (d) 30

51. The tangent at the point (2, “2) to the curve, $x^2y^2 - 2x = 4(1 - y)$ does not pass through the point : **(2017/Online Set–1)**

(a) $\left(4, \frac{1}{3}\right)$ (b) (8, 5)

(c) (–4, –9) (d) (–2, – 7)

52. The function f defined by $f(x) = x^3 - 3x^2 + 5x + 7$, is **(2017/Online Set–2)**

(a) increasing in R.

(b) decreasing in R.

(c) decreasing in $(0, \infty)$ and increasing in $(-\infty, 0)$.

(d) increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$.

53. If the curves $y^2 = 6x, 9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is **(2018)**

(a) $\frac{9}{2}$ (b) 6

(c) $\frac{7}{2}$ (d) 4

54. Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$,

$x \in R - \{-1, 0, 1\}$. If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of h(x) is : **(2018)**

(a) $2\sqrt{2}$ (b) 3

(c) -3 (d) $-2\sqrt{2}$

55. If a right circular cone, having maximum volume, is inscribed in a sphere of radius 3 cm, then the curved surface area (in cm^2) of this cone is : **(2018/Online Set–1)**

(a) $6\sqrt{2}\pi$ (b) $6\sqrt{3}\pi$

(c) $8\sqrt{2}\pi$ (d) $8\sqrt{3}\pi$

56. Let f(x) be a polynomial of degree 4 having extreme values at x = 1 and x = 2.

If $\lim_{x \to 0}\left(\frac{f(x)}{x^2} + 1\right) = 3$ then f (-1) is equal to: **(2018/Online Set–2)**

(a) $\frac{9}{2}$ (b) $\frac{5}{2}$

(c) $\frac{3}{2}$ (d) $\frac{1}{2}$

57. Let M and m be respectively the absolute maximum and the absolute minimum values of the function, $f(x) = 2x^3 - 9x^2 + 12x + 5$ in the interval [0, 3]. Then M-m is equal to : **(2018/Online Set–3)**

(a) 5 (b) 9

(c) 4 (d) 1

EXERCISE - 3 : ADVANCED OBJECTIVE QUESTIONS

APPLICATION OF DERIVATIVE–I

Single Type Questions

1. The altitude of a cone is 20 cm and its semi-vertical angle is 30°. If the semi-vertical angle is increasing at the rate of 2° per second, then the radius of the base is increasing at the rate of

(a) $\frac{2\pi}{27}$ cm/sec (b) $\frac{8\pi}{27}$ cm/sec

(c) $\frac{16\pi}{27}$ cm/sec (d) $\frac{4\pi}{27}$ cm/sec

2. A particle's velocity v at time t is given by $v = 2e^{2t}\cos\frac{\pi t}{3}$.

The least value of time at which the acceleration becomes zero, is

(a) 0 (b) $\frac{3}{2}$

(c) $\frac{3}{\pi}\tan^{-1}\left(\frac{6}{\pi}\right)$ (d) $\frac{3}{\pi}\cot^{-1}\left(\frac{6}{\pi}\right)$

3. A variable triangle is inscribed in a circle of radius R. If the rate of change of a side is R times the rate of change of the opposite angle, then that angle is

(a) π/6 (b) π/4

(c) π/3 (d) π/2

4. A particle moving on a curve has the position at time t given by $x = f'(t)\sin t + f''(t)\cos t$, $y = f'(t)\cos t - f''(t)\sin t$, where f is a thrice differentiable function. Then the velocity of the particle at time t is

(a) $f'(t) + f''(t)$ (b) $f'(t) + f'''(t)$

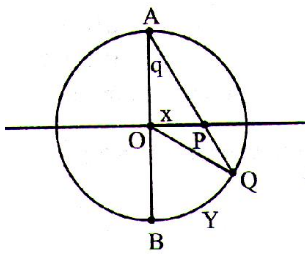
(c) $f'(t) + f'''(t)$ (d) none of these

5. The radius of a right circular cylinder increases at a constant rate. Its altitude is a linear function of the radius and increases three times as fast as radius. When the radius is 1 cm the altitude is 6 cm. When the radius is 6 cm, the volume is increasing at the rate of 1 cu cm/sec. When the radius is 36 cm, the volume is increasing at a rate of n cu cm/sec. The value of 'n' is equal to

(a) 12 (b) 22

(c) 30 (d) 33

6. The circle shown in figure has radius of 10 cm. If point P moves towards right with a speed of 5 cm/sec, then the rate at which the length of arc BQ is increasing at; the instant when ∠BAQ is 45°



(a) 10 cm/sec (b) $\frac{5}{\sqrt{2}}$ cm/sec

(c) $\frac{10}{\sqrt{2}}$ cm/sec (d) 5 cm/sec

7. l_1 and l_2 are the side lengths of two variable squares S_1 and S_2 respectively. If $l_1 = l_2 + l_2^3 + 6$ then rate of change of the area of S_2 with respect to rate of change of the area of S_1 and $l_2 = 1$ is equal to

(a) 3/4 (b) 4/3

(c) 3/2 (d) None

8. Let x be the length of one of the equal sides of an isosceles triangle, and let θ be the angle between them. If x is increasing at the rate (1/12)m/h, and θ is increasing at the rate of π/180 radians/h, then the rate in m^2/hr at which the area of the triangle is increasing when x = 12m and θ = π/4, is

(a) $2^{1/2}\left(1+\frac{2\pi}{5}\right)$ (b) $\frac{73}{2}\sqrt{2}$

(c) $\frac{3^{1/2}}{2}+\frac{\pi}{5}$ (d) $2^{1/2}\left(\frac{1}{2}+\frac{\pi}{5}\right)$

9. A flu epidemic hits Pune. Health officers estimate that the number of persons sick with the flu at time t (measured in days from the beginning of the epidemic) is estimated by $P(t) = 60t^2 - t^3, 0 \le t \le 40$. At what time t is the flu spreading at the rate of 900 people per day ?

(a) t = 10 and 30 (b) t = 10 only
(c) t = 38 (d) t = 25 and 32

10. In a ΔABC if sides a and b remain constant such that α is the error in C, then relative error in its area is

(a) $\alpha \cot C$ (b) $\alpha \sin C$
(c) $\alpha \tan C$ (d) $\alpha \cos C$

11. In a ΔABC the sides b and c are given. If there is an error ΔA in measuring angle A, then the error Δa in side a is given by (where S area of triangle)

(a) $\frac{S}{2a}\Delta A$ (b) $\frac{2S}{a}\Delta A$
(c) $bc \sin A\, \Delta A$ (d) none of these

12. The value of $(127)^{1/3}$ to four decimal places is

(a) 5.0267 (b) 5.4267
(c) 5.5267 (d) 5.001

13. If m be the slope of a tangent to the curve $e^{2y} = 1 + 4x^2$, then

(a) $m < 1$ (b) $|m| \le 1$
(c) $|m| > 1$ (d) none of these

14. Tangents are drawn from the origin to the curve $y = \sin x$. Their points of contact lie on the curve

(a) $x^2y^2 = x^2 + y^2$ (b) $x^2y^2 = x^2 - y^2$
(c) $x^2y^2 = y^2 - x^2$ (d) none of these

15. The equation of the tangent to the curve

$$y = \begin{cases} x^2 \sin\frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$ at the origin is

(a) $x = 0$ (b) $x = y$
(c) $y = 0$ (d) none of these

16. Number of possible tangents to the curve $y = \cos(x + y), -3\pi \le x \le 3\pi$, that are parallel to the line $x + 2y = 0$, is

(a) 1 (b) 2
(c) 3 (d) 4

17. The sum of the intercepts made on the axes of coordinates by any tangent to the curve $\sqrt{x} + \sqrt{y} = 2$ is equal to

(a) 4 (b) 2
(c) 8 (d) none of these

18. The number of values of c such that the straight line $3x + 4y = c$ touches the curve $\frac{x^4}{2} = x + y$ is

(a) 0 (b) 1
(c) 2 (d) 4

19. Equation of normal drawn to the graph of the function defined as $f(x) = \frac{\sin x^2}{x}, x \ne 0$ and $f(0) = 0$ at the origin is

(a) $x + y = 0$ (b) $x - y = 0$
(c) $y = 0$ (d) $x = 0$

20. The tangent to the curve $3xy^2 - 2x^2y = 1$ at (1, 1) meets the curve again at the point

(a) $\left(-\frac{16}{5}, -\frac{1}{20}\right)$ (b) $\left(\frac{16}{5}, \frac{1}{20}\right)$
(c) $\left(\frac{1}{20}, \frac{16}{5}\right)$ (d) $\left(-\frac{1}{20}, \frac{16}{5}\right)$

21. A curve is represented by the equations, $x = \sec^2 t$ and $y = \cot t$ where t is a parameter. If the tangent at the point P on the curve where $t = \pi/4$ meets the curve again at the point Q then |PQ| is equal to

(a) $\frac{5\sqrt{3}}{2}$ (b) $\frac{5\sqrt{5}}{2}$
(c) $\frac{2\sqrt{5}}{3}$ (d) $\frac{3\sqrt{5}}{2}$

22. At any two points of the curve represented parametrically by $x = a(2\cos t - \cos 2t)$; $y = a(2\sin t - \sin 2t)$ the tangents are parallel to the axis of x corresponding to the values of the parameter t differing from each other by

(a) $2\pi/3$ (b) $3\pi/4$
(c) $\pi/2$ (d) $\pi/3$

23. The ordinate of $y = (a/2)(e^{x/a} + e^{-x/a})$ is the geometric mean of the length of the normal and the quantity

(a) a/2 (b) a
(c) e (d) none of these

24. The two tangents to the curve $ax^2 + 2hxy + by^2 = 1, a > 0$ at the points where it crosses x-axis, are

(a) parallel (b) perpendicular

(c) inclined at an angle $\frac{\pi}{4}$ (d) none of these

25. The two curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ at the point (1,2)

(a) intersect orthogonally (b) intersect at an angle $\frac{\pi}{3}$

(c) touch each other (d) none of these

26. The lines $y = -\frac{3}{2}x$ and $y = -\frac{2}{5}x$ intersect the curve $3x^2 + 4xy + 5y^2 - 4 = 0$ at the points P and Q respectively. The tangents drawn to the curve at P and Q

(a) intersect each other at angle of 45°

(b) are parallel to each other

(c) are perpendicular to each other

(d) none of these

27. The sub-normal at any point of the curve $x^2y^2 = a^2(x^2 - a^2)$ varies as

(a) $(\text{abscissa})^{-3}$ (b) $(\text{abscissa})^3$

(c) $(\text{ordinate})^{-3}$ (d) none of these

28. The sub-tangent at any point of the curve $x^m y^n = a^{m+n}$ varies as

(a) $(\text{abscissa})^2$ (b) $(\text{abscissa})^3$

(c) abscissa (d) ordinate

29. If at any point on a curve the sub-tangent and sub-normal are equal, then the length of the normal is equal to

(a) $\sqrt{2}$ ordinate (b) ordinate

(c) $\sqrt{2 \text{ ordinate}}$ (d) none of these

30. The length of the perpendicular from the origin to the normal of curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any point θ is

(a) a (b) a/2

(c) a/3 (d) none of these

31. The sum of tangent and sub-tangent at any point of the curve $y = a\log(x^2 - a^2)$ varies as

(a) abscissa

(b) product of the coordinates

(c) ordinate

(d) none of these

32. For the curve $x^{m+n} = a^{m-n} y^{2n}$, where a is a positive constant and m, n are positive integers

(a) $(\text{sub-tangent})^m \propto (\text{sub-normal})^n$

(b) $(\text{sub-normal})^m \propto (\text{sub-tangent})^n$

(c) the ratio of subtangent and subnormal is constant

(d) none of the above

33. $|\sin 2x| - |x| - a = 0$ does not have solution if a lies in

(a) $\left(\frac{3\sqrt{3}-\pi}{6}, \infty\right)$ (b) $\left(\frac{3\sqrt{3}+\pi}{6}, \infty\right)$

(c) $(1, \infty)$ (d) None of these

34. A curve is represented parametrically by the equation $x = t + e^{at}$ and $y = -t + e^{at}$ when $t \in R$ and $a > 0$. If the curve touches the axis of x at the point A, then the coordinates of the point A are

(a) (1, 0) (b) (1/e, 0)

(c) (e, 0) (d) (2e, 0)

35. Let f be a differentiable function with $f(2) = 3$ and $f'(2) = 5$, and let g be the function defined by $g(x) = x f(x)$. y-intercept of the tangent line to the graph of 'g' at point with abscissa 2, is

(a) 20 (b) 8

(c) – 20 (d) – 18

36. The number of points with integral coordinates where tangent exists in the curve $y = \sin^{-1} 2x\sqrt{1-x^2}$ is

(a) 0 (b) 1

(c) 3 (d) None

37. Let $f(x) = \begin{cases} -x^2, & \text{for } x<0 \\ x^2+8, & \text{for } x\ge0 \end{cases}$. Then the x-intercept of the line that is tangent to both portions of the graph of $y = f(x)$ is

(a) zero (b) –1

(c) –3 (d) –4

38. If $px^2 + qx + r = 0$, p, q, r $\in$ R has no real zero and the line $y + 2 = 0$ is tangent to $f(x) = px^2 + qx + r$ then

(a) $p + q + r > 0$ (b) $p - q + r > 0$

(c) $r < 0$ (d) None of these

39. Tangent of acute angle between the curves $y = |x^2 - 1|$ and $y = \sqrt{7 - x^2}$ at their points of intersection is

(a) $\frac{5\sqrt{3}}{2}$ (b) $\frac{3\sqrt{5}}{2}$

(c) $\frac{5\sqrt{3}}{4}$ (d) $\frac{3\sqrt{5}}{4}$

40. If t, n, t′, n′ are the lengths of tangent, normal, subtangent & subnormal at a point P (x_1, y_1) on any curve $y = f(x)$ then

(a) $t^2 + n^2 = t'n'$ (b) $\frac{1}{t^2} + \frac{1}{n^2} = \frac{1}{t'n'}$

(c) $t'n' = tn$ (d) $nt' = n't$

41. Find the shortest distance between $xy = 9$ and $x^2+y^2 = 1$.

(a) $3\sqrt{2} + 1$ (b) 2

(c) 4 (d) $3\sqrt{2} - 1$

42. A curve passes through the point (2, 0) and the slope of the tangent at any point (x, y) is $x^2 - 2x$ for all values of x then $3y_{local\ max}$ is equal to

(a) 4 (b) 3

(c) 1 (d) 2

Multiple Type Questions

43. The abscissa of a point on the curve $xy = (a + x)^2$, the normal at which cuts off equal intercepts on the coordinate axes is

(a) $-a/\sqrt{2}$ (b) $\sqrt{2}\,a$

(c) $\sqrt{2}\,a/2$ (d) $-\sqrt{2}\,a$

44. For function $f(x) = \frac{\ell n\, x}{x}$, which of the following statements are true.

(a) $f(x)$ has horizontal tangent at $x = e$

(b) $f(x)$ cuts the x–axis only at one point

(c) $f(x)$ is many – one function

(d) $f(x)$ has one vertical tangent

45. Which of the following pair (s) of curves is/are orthogonal.

(a) $y^2 = 4ax$; $y = e^{-x/2a}$

(b) $y^2 = 4ax$; $x^2 = 4ay$ at (0, 0)

(c) $xy = a^2$; $x^2 - y^2 = b^2$

(d) $y = ax$; $x^2 + y^2 = c^2$

46. The length of the perpendicular from the origin to the normal of curve $x = a(\cos\theta + \theta \sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at a point θ is 'a', if θ =

(a) π/4 (b) π/3

(c) π/2 (d) π/6

47. A nursery sells plants after 6 year of growth. Two seedlings A and B are planted each of height 5 cm whose growth rates are $\frac{dh_A}{dt} = 0.5t + 2$ and $\frac{dh_B}{dt} = t + 1$ where heights h_A and h_B are in cms and t is the time in years. Then

(a) the height of the plants are equal at t = 3 (in years)

(b) the height of the plants are equal at t = 4 (in years)

(c) when the plants are sold, their heights are 26 cms and 29 cms.

(d) none of these

48. The angle between the tangent at any point P and the line joining P to the origin, where P is a point on the curve $ln(x^2 + y^2) = c\tan^{-1}\frac{y}{x}$, c is a constant, is

(a) independent of x and y

(b) dependent on c

(c) independent of c but dependent on x

(d) none of these

49. Let the parabolas $y = x(c - x)$ and $y = -x^2 - ax + b$ touch each other at the point (1, 0), then

(a) $a + b + c = 0$ (b) $a + b = 2$

(c) $b + c = 1$ (d) $a - c = -2$

50. The point on the curve $xy^2 = 1$, which is nearest to the origin is

(a) $(2^{1/3}, 2^{1/6})$ (b) $(2^{-1/3}, 2^{1/6})$

(c) $(2^{-1/3}, -2^{1/6})$ (d) $(-2^{-1/3}, 2^{1/6})$

51. The value of parameter a so that the line $(3-a)x + ay + (a^2-1) = 0$ is normal to the curve $xy = 1$, may lie in the interval

(a) $(-\infty, 0)$ (b) $(1, 3)$

(c) $(0, 3)$ (d) $(3, \infty)$

52. The points on the curve $y = x\sqrt{1-x^2}$, $-1 \le x \le 1$ at which the tangent line is vertical are

(a) $(-1, 0)$ (b) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$

(c) $(1, 0)$ (d) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Assertion Reason Type Questions

53. **Assertion :** The tangent at $x = 1$ to the curve $y = x^3 - x^2 - x + 2$ again meets the curve at $x = -2$.

Reason : When a equation of a tangent solved with the curve, repeated roots are obtained at point of tangency.

(a) A (b) B

(c) C (d) D

54. **Assertion :** The ratio of length of tangent to length of normal is directly proportional to the ordinate of the point of tangency at the curve $y^2 = 4ax$.

Reason : Length of normal & tangent to a curve

$y = f(x)$ is $\left|y\sqrt{1+m^2}\right|$ and $\left|\frac{y\sqrt{1+m^2}}{m}\right|$, where $m = \frac{dy}{dx}$.

(a) A (b) B

(c) C (d) D

55. **Assertion :** Tangent drawn at the point (0, 1) to the curve $y = x^3 - 3x + 1$ meets the curve thrice at one point only.

Reason : Tangent drawn at the point $(1, -1)$ to the curve $y = x^3 - 3x + 1$ meets the curve at 1 point only.

(a) A (b) B

(c) C (d) D

56. **Assertion :** Shortest distance between

$|x| + |y| = 2$ & $x^2 + y^2 = 16$ is $4 - \sqrt{2}$

Reason : Shortest distance between the two non intersecting differentiable curves lies along the common normal.

(a) A (b) B

(c) C (d) D

Paragraph Type Questions

Using the following passage, solve Q.57 to Q.59

Passage

Let $y = a\sqrt{x} + bx$ be curve, $(2x - y) + \lambda(2x + y - 4) = 0$ be family of lines.

57. If curve has slope $-\frac{1}{2}$ at (9, 0) then a tangent belonging to family of lines is

(a) $x + 2y - 5 = 0$ (b) $x - 2y + 3 = 0$

(c) $3x - y - 1 = 0$ (d) $3x + y - 5 = 0$

58. A line of the family cutting positive intercepts on axes and forming triangle with coordinate axes, then minimum length of the line segment between axes is

(a) $(2^{2/3} - 1)^{3/2}$ (b) $(2^{2/3} + 1)^{3/2}$

(c) $7^{3/2}$ (d) 27

59. Two perpendicular chords of curve $y^2 - 4x - 4y + 4 = 0$ belonging to family of lines form diagonals of a quadrilateral. Minimum area of quadrilateral is

(a) 16 (b) 32

(c) 64 (d) 50

Using the following passage, solve Q.60 to Q.62

Passage

If $y = \int_{u(x)}^{v(x)} f(t)\,dt$, let us define $\frac{dy}{dx}$ in a different manner as $\frac{dy}{dx} = v'(x) f^2(v(x)) - u'(x) f^2(u(x))$ and the equation of the tangent at (a, b) as

$$y - b = \left(\frac{dy}{dx}\right)_{(a,b)} (x - a)$$

60. If $y = \int_{x}^{x^2} t^2\,dt$, then equation of tangent at x = 1 is

(a) $y = x + 1$ (b) $x + y = 1$

(c) $y = x - 1$ (d) $y = x$

61. If $F(x) = \int_{1}^{x} e^{t^2/2}(1 - t^2)\,dt$, then $\frac{d}{dx}F(x)$ at x = 1 is

(a) 0 (b) 1

(c) 2 (d) –1

62. If $y = \int_{x^3}^{x^4} \ell nt\,dt$, then $\lim_{x \to 0^+} \frac{dy}{dx}$ is

(a) 0 (b) 1

(c) 2 (d) –1

Using the following passage, solve Q.63 to Q.65

Passage

If $y = f(x)$ is a curve and if there exists two points $A(x_1, f(x_1))$ and $B(x_2, f(x_2))$ on it such that $f'(x_1) = -\frac{1}{f'(x_2)} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$, then the tangent at x_1 is normal at x_2 for that curve.

63. Number of such lines on the curve y = sinx is

(a) 1 (b) 0

(c) 2 (d) infinite

64. Number of such lines on the curve $y = |\ell n\,x|$ is

(a) 1 (b) 2

(c) 0 (d) infinite

65. Number of such line on the curve $y^2 = x^3$ is

(a) 1 (b) 2

(c) 3 (d) 0

Match The Column

66.

	Column–I	Column–II	
(A)	If portion of the tangent at any point on the curve $x = at^3$, $y = at^4$ between the axes is divided by the abscissa of the point of contact in the ratio m : n externally, then |n + m| is equal to (m and n are coprime)	(P)	0
(B)	The area of triangle formed by normal at the point (1, 0) on the curve $x = e^{\sin y}$ with axes is	(Q)	1/2
(C)	If the angle between curves $x^2y = 1$ and $y = e^{2(1-x)}$ at the point (1, 1) is θ then tan θ is equal to	(R)	7
(D)	The length of sub-tangent at any point on the curve $y = be^{x/3}$ is equal to	(S)	3

67.

	Column–I	Column–II	
(A)	Circular plate is expanded by heat from radius 5 cm to 5.06 cm. Approximate increase in area is	(P)	4
(B)	If an edge of a cube increases by 1% then percentage increase in volume is	(Q)	0.6π
(C)	If the rate of decrease of $\frac{x^2}{2} - 2x + 5$ is twice the rate of decrease of x, then x is equal to (rate of decrease is non-zero)	(R)	3
(D)	Rate of increase in area of equilateral triangle of side 15cm, when each side is increasing at the rate of 0.1 cm/sec; is	(S)	$3\sqrt{3}/4$

Subjective Type Questions

68. If the length of the interval of 'a' such that the inequality $3 - x^2 > |x - a|$ has atleast one negative solution is k then find 4k.

69. Let α be the angle in radians between $\frac{x^2}{36} + \frac{y^2}{4} = 1$ and the circle $x^2 + y^2 = 12$ at their points of intersection. If $\alpha = \tan^{-1}\frac{k}{2\sqrt{3}}$, then find the value of k^2.

70. If A is the area of the triangle formed by positive x-axis and the normal and the tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ then $A/\sqrt{3}$ is equal to

APPLICATION OF DERIVATIVE–II

Single Correct

1. If $f(x)$ and $g(x)$ are differentiable functions for $0 \le x \le 1$ such that $f(0) = 2$, $g(0) = 0$, $f(1) = 6$, $g(1) = 2$, then in the interval (0, 1)

(a) $f'(x) = 0$, for all x

(b) $f'(x) = 2g'(x)$, for atleast one x

(c) $f'(x) = 2g'(x)$, for atmost one x

(d) none of these

2. If $a + b + c = 0$, then the equation $3ax^2 + 2bx + c = 0$ has, in the interval (0, 1)

(a) atleast one root (b) atmost one root

(c) no root (d) none of these

3. Between any two real roots of the equation $e^x \sin x = 1$, the equation $e^x \cos x = -1$ has

(a) atleast one root (b) exactly one root

(c) atmost one root (d) no root

4. If $a < 0$, $f(x) = e^{ax} + e^{-ax}$ and $S = \{x : f(x)$ is monotonically decreasing$\}$, then S is equals

(a) $\{x : x > 0\}$ (b) $\{x : x < 0\}$

(c) $\{x : x > 1\}$ (d) $\{x : x < 1\}$

5. Let $f(x)$ and $g(x)$ be defined and differentiable for $x \ge x_0$ and $f(x_0) = g(x_0)$, $f'(x) > g'(x)$ for $x > x_0$, then

(a) $f(x) < g(x), x > x_0$ (b) $f(x) = g(x), x > x_0$

(c) $f(x) > g(x), x > x_0$ (d) none of these

6. The function $f(x) = \dfrac{\ell n(\pi + x)}{\ell n(e + x)}$ is

(a) increasing on $(0, \infty)$

(b) deceasing on $(0, \infty)$

(c) increasing on $(0, \pi/e)$, decreasing on $(\pi/e, \infty)$

(d) decreasing on $(0, \pi/e)$, increasing on $(\pi/e, \infty)$

7. Let $f(x) = \cot^{-1}[g(x)]$, where $g(x)$ is an increasing function for $0 < x < \pi$. Then $f(x)$ is

(a) increasing in $(0, \pi)$

(b) decreasing in $(0, \pi)$

(c) increasing in $\left(0, \frac{\pi}{2}\right)$ and decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(d) none of these

8. Let $f'(x) > 0$ and $g'(x) < 0$ for all $x \in R$. Then,

(a) $f[g(x)] > f[g(x+1)]$

(b) $f[g(x)] > f[g(x-1)]$

(c) $g[f(x)] > g[f(x-1)]$

(d) none of these

9. Consider the following statements S and R :

S : Both sin x and cos x are decreasing functions in the interval $\left(\frac{\pi}{2}, \pi\right)$

R : If a differentiable function decreases in an interval (a, b), then its derivative also decreases in (a, b). Which of the following is true ?

(a) Both S and R are wrong.

(b) Both S and R are correct, but R is not the correct explanation for S.

(c) S is correct and R is the correct explanation for S.

(d) S is correct and R is wrong.

10. If the function f (x) increases in the interval (a, b) then the function $\phi(x) = [f(x)]^2$.

(a) Increases in (a, b)

(b) decreases in (a, b)

(c) we cannot say that $\phi(x)$ increases or decreases in (a, b)

(d) none of these

11. If $f(x)=\frac{x^2}{2-2\cos x}$; $g(x)=\frac{x^2}{6x-6\sin x}$ where $0<x<1$, then :

(a) both 'f' and 'g' are increasing functions

(b) 'f' is decreasing and 'g' is increasing function

(c) 'f' is increasing and 'g' is decreasing function

(d) both 'f' and 'g' are decreasing function

12. For $x\in\left(0, \tan^{-1}\sqrt{\frac{5}{2}}\right)$, the function

$$f(x)=\cot^{-1}\left(\frac{\sqrt{2}\sin x+\sqrt{5}\cos x}{\sqrt{7}}\right)$$

(a) increases in $\left(0, \tan^{-1}\sqrt{\frac{5}{2}}\right)$

(b) decreases in $\left(0, \tan^{-1}\sqrt{\frac{5}{2}}\right)$

(c) increases in $\left(0, \tan^{-1}\sqrt{\frac{2}{5}}\right)$ and decreases in $\left(\tan^{-1}\sqrt{\frac{2}{5}}, \tan^{-1}\sqrt{\frac{5}{2}}\right)$

(d) increases in $\left(\tan^{-1}\sqrt{\frac{2}{5}}, \tan^{-1}\sqrt{\frac{5}{2}}\right)$ and decreases in $\left(0, \tan^{-1}\sqrt{\frac{2}{5}}\right)$

13. If $f(x)=a^{\{a^{|x|}\operatorname{sgn} x\}}$; $g(x)=a^{[a^{|x|}\operatorname{sgn} x]}$ for $a>1$ and $x\in R$, where { } & [ ] denote the fractional part and integral part functions respectively, then which of the following statements hold good for the function h (x), where $(\ln a)\, h(x)=(\ln f(x)+\ln g(x))$.

(a) 'h' is even and increasing

(b) 'h' is odd and decreasing

(b) 'h' is even and decreasing

(d) 'h' is odd and increasing

14. If $\phi(x)=f(x)+f(2a-x)$ and $f''(x)>0$, $a>0$, $0\le x\le 2a$ then

(a) $\phi(x)$ increases in [a, 2a]

(b) $\phi(x)$ increases in [0, a]

(c) $\phi(x)$ decreases in [0, a]

(d) $\phi(x)$ decreases in [a, 2a]

15. If f is an even function then

(a) f^2 increases on (a, b)

(b) f cannot be monotonic

(c) f^2 need not increases on (a, b)

(d) f has inverse

16. Let $f(x)=x^{m/n}$ for $x\in R$ where m and n are integers, m even and n odd and $0<m<n$. Then

(a) $f(x)$ decreases on $(-\infty, 0]$

(b) $f(x)$ increases on $[0, \infty)$

(c) $f(x)$ increases on $(-\infty, 0]$

(d) $f(x)$ decreases on $[0, \infty)$

17. The greatest and the least value of the function,

$f(x)=\sqrt{1-2x+x^2}-\sqrt{1+2x+x^2}$, $x\in(-\infty,\infty)$ are

(a) 2, –2 (b) 2, –1

(c) 2, 0 (d) none

18. The difference between the greatest and least values of the function, $f(x)=\cos x+\frac{1}{2}\cos 2x-\frac{1}{3}\cos 3x$ is :

(a) 4/3 (b) 1

(c) 9/4 (d) 1/6

19. The function 'f' is defined by $f(x)=x^p(1-x)^q$ for all $x\in R$, where p,q are positive integers, has a local maximum value, for x equal to :

(a) $\frac{pq}{p+q}$ (b) 1

(c) 0 (d) $\frac{p}{p+q}$

20. Let $f(x) = \begin{cases} x^3 - x^2 + 10x - 5, & x \le 1 \\ -2x + \log_2(b^2 - 2), & x > 1 \end{cases}$ the set of values of b for which $f(x)$ have greatest value at x = 1 is given by :

(a) $1 \le b \le 2$

(b) $b = \{1, 2\}$

(c) $b \in (-\infty, -1)$

(d) $\left[-\sqrt{130}, -\sqrt{2}\right) \cup \left(\sqrt{2}, \sqrt{130}\right]$

21. The triangle formed by the tangent to the parabola $y = x^2$ at the point with abscissa x_1, the y-axis and the straight line $y = x_1^2$ has the greatest area where $x_1 \in [1, 3]$. Then x_1 equals:

(a) 3 (b) 2

(c) 1 (d) none

22. The least area of a circle circumscribing any right triangle of area S is :

(a) πS (b) $2\pi S$

(c) $\sqrt{2}\,\pi S$ (d) $4\pi S$

23. The largest area of a rectangle which has one side on the x-axis and the two vertices on the curve $y = e^{-x^2}$ is

(a) $\sqrt{2}\,e^{-1/2}$ (b) $2\,e^{-1/2}$

(c) $e^{-1/2}$ (d) none

24. A tangent to the curve $y = 1 - x^2$ is drawn so that the abscissa x_0 of the point of tangency belongs to the interval [0, 1]. The tangent at x_0 meets the x-axis and y-axis at A & B respectively. The minimum area of the triangle OAB, where O is the origin is

(a) $\frac{2\sqrt{3}}{9}$ (b) $\frac{4\sqrt{3}}{9}$

(c) $\frac{2\sqrt{2}}{9}$ (d) none

25. The minimum value of $a\tan^2 x + b\cot^2 x$ equals the maximum value of $a\sin^2\theta + b\cos^2\theta$ where $a > b > 0$, when

(a) a = b (b) a = 2b

(c) a = 3b (d) a = 4b

26. The greatest value of $f(x) = (x+1)^{1/3} - (x-1)^{1/3}$ on [0, 1] is

(a) 1 (b) 2

(c) 3 (d) 1/3

27. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its local maximum and local minimum at p and q respectively such that $p^2 = q$, then a equals.

(a) 3 (b) 1

(c) 2 (d) 1/2

28. If $ax^2 + \frac{b}{x} \ge c$ for all positive x, where a, b > 0, then

(a) $27\,ab^2 \ge 4c^3$ (b) $27\,ab^2 < 4\,c^3$

(c) $4\,ab^2 \ge 27c^3$ (d) none of these

29. If $ax + \frac{b}{x} \ge c$ for all positive x, where a, b, c > 0, then

(a) $ab < \frac{c^2}{4}$ (b) $ab \ge \frac{c^2}{4}$

(c) $ab \ge \frac{c}{4}$ (d) none of these

30. If $P(x) = a_0 + a_1x^2 + a_2x^4 + ... + a_nx^{2n}$ be a polynomial in $x \in R$ with $0 < a_1 < a_2 < ... < a_n$, then P(x) has

(a) no point of minima

(b) only one point of minima

(c) only two points of minima

(d) none of these

31. If $(x - a)^{2n}(x - b)^{2m+1}$, where m and n are positive integers and a > b, is the derivative of a function f, then

(a) x = a gives neither a maximum nor a minimum

(b) x = a gives a maximum

(c) x = b gives neither a maximum nor a minimum

(d) none of these

32. Let (h, k) be a fixed point, where h > 0, k > 0. A straight line passing through this point cuts the positive direction of the coordinate axes at the points P and Q. The minimum area of the Δ OPQ, O being the origin, is

(a) 2 *kh* (b) *kh*

(c) 4*kh* (d) none of these

33. A function f such that $f'(a) = f''(a) = ... f^{2n}(a) = 0$ and f has a local maximum value b at x = a, if $f(x)$ is

(a) $(x-a)^{2n+2}$ (b) $b-1-(x+1-a)^{2n+1}$

(c) $b-(x-a)^{2n+2}$ (d) $(x-a)^{2n+2}-b$.

34. For a function y = f(x), f $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ at a point x = a, then

(a) y must be maximum at x = a

(b) y must be minimum at x = a

(c) y may not have a maximum or minimum at x = a

(d) it is a constant function

35. The sum of the legs of a right triangle is 9cm. When the triangle rotates about one of the legs, a cone results which has the maximum volume. Then :

(a) slant height of such a cone is $3\sqrt{5}$

(b) maximum volume of the cone is 32 π

(c) curved surface of the cone is $18\sqrt{5}\,\pi$

(d) semi vertical angle of cone is $\tan^{-1}\sqrt{2}$

36. Moving along the x-axis there are two points with $x = 10 + 6t$, $x = 3 + t^2$. The speed with which they are moving away from each other at the time of encounter is (x is in cm and t is seconds)

(a) 16 cm/s (b) 20 cm/s

(c) 8 cm/s (d) 12 cm/s

37. Least value of the function $f(x) = 2^{x^2} - 1 + \frac{2}{2^{x^2}+1}$ is :

(a) 0 (b) 3/2

(c) 2/3 (d) 1

38. The set of values of 'a' for which the function $f(x) = \frac{ax^3}{3} + (a+2)x^2 + (a-1)x + 2$ possess a negative point of inflection.

(a) $(-\infty, -2) \cup (0, \infty)$ (b) $\{-4/5\}$

(c) $(-2, 0)$ (d) empty set

39. Number of solution (s) satisfying the equation, $3x^2 - 2x^3 = \log_2(x^2+1) - \log_2 x$ is :

(a) 1 (b) 2

(c) 3 (d) none

40. Let $f(x) = \cos 2\pi x + x - [x]$ ([.] denotes the greatest integer function). Then number of points in [0, 10] at which $f(x)$ assumes its local maximum value, is

(a) 0 (b) 10

(c) 9 (d) infinite

41. If x and y are real numbers satisfying the relation $x^2 + y^2 - 6x + 8y + 24 = 0$ then minimum value of $f(x) = \log_2(x^2+y^2)$ is

(a) 1 (b) 2

(c) 3 (d) 4

42. If $f(x) = 2x^3 - 3(a+1)x^2 + 6ax - 12$ has local maximum at x_1 and local minimum at x_2 and if $2x_1 = x_2$ then value of a can be:

(a) 1 (b) $\frac{1}{2}$

(c) – 1 (d) 2

43. Let $f(x)$ be defined as

$$f(x) = \begin{cases} \tan^{-1}\alpha - 5x^2, & 0 < x < 1 \\ -6x, & x \ge 1 \end{cases}$$

$f(x)$ can have a local maximum at x = 1 if value of α is

(a) 0 (b) – 1

(c) – tan 1 (d) – 2

44. A truck is to be driven 300 km on a highway at a constant speed of x kmph. Speed rules of the highway required that $30 \le x \le 60$. The fuel costs Rs. 10 per litre and is consumed at the rate of $2 + \frac{x^2}{600}$ liters per hour. The wages of the driver are Rs. 200 per hour. The most economical speed to drive the truck, in kmph, is

(a) 30 (b) 60

(c) $30\sqrt{3.3}$ (d) $20\sqrt{3.3}$

45. The curve $y = \frac{2x}{1+x^2}$ has

(a) exactly three points of inflection separated by a point of maximum and a point of minimum

(b) exactly two points of inflection with a point of maximum lying between them

(c) exactly two points of inflection with a point of minimum lying between them

(d) exactly three points of inflection separated by two points of maximum

46. Slope of tangent to the curve

$y = 2e^x \sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)$, where $0 \le x \le 2\pi$ is minimum at x =

(a) 0 (b) π
(c) 2π (d) none of these

47. If the angle made by the tangent drawn at any point (x, y) of a curve with positive x–axis is $\tan^{-1}(x^2 - 2x)$, $\forall\, x \in R$, then number of critical points of the curve is

(a) 0 (b) 1
(c) 2 (d) none of these

48. The total number of values of x, where $f(x) = 2^{-|x|}(\cos x + \cos\sqrt{3}\,x)$ attains its maximum value is

(a) 1 (b) 2
(c) 4 (d) None

49. Let $f(x) = \begin{cases} |x^3 + x^2 + 3x + \sin x|(3 + \sin 1/x), & x \ne 0 \\ 0, & x = 0 \end{cases}$ then number of points (where $f(x)$ attains its minimum value) is

(a) 1 (b) 2
(c) 3 (d) infinite many

50. The set of all values of the parameters a for which the points of local minimum of the function $y = 1 + a^2 x - x^3$ satisfy the inequality $\frac{x^2+x+2}{x^2+5x+6} \le 0$ is

(a) an empty set
(b) $\left(-3\sqrt{3}, -2\sqrt{3}\right)$
(c) $\left(2\sqrt{3}, 3\sqrt{3}\right)$
(d) $\left(-3\sqrt{3}, -2\sqrt{3}\right) \cup \left(2\sqrt{3}, 3\sqrt{3}\right)$

More than One correct Answer Type

51. The function $y = \frac{2x-1}{x-2}$ $(x \ne 2)$ with codomain = R – {2}

(a) is its own inverse
(b) decreases at all values of x in the domain
(c) has a graph entirely above x–axis
(d) is bound for all x.

52. Let $g'(x) > 0$ and $f'(x) < 0$, $\forall\, x \in R$, then

(a) $g(f(x+1)) > g(f(x-1))$
(b) $f(g(x-1)) > f(g(x+1))$
(c) $g(f(x+1)) < g(f(x-1))$
(d) $g(g(x+1)) < g(g(x-1))$

53. If $f(x) = x^3 - x^2 + 100x + 1001$, then

(a) $f(2000) > f(2001)$
(b) $f\left(\frac{1}{1999}\right) > f\left(\frac{1}{2000}\right)$
(c) $f(x+1) > f(x-1)$
(d) $f(3x-5) > f(3x)$

54. An extremum of the function,

$f(x) = \frac{2-x}{\pi}\cos \pi(x+3) + \frac{1}{\pi^2}\sin \pi(x+3)$ $0 < x < 4$

occurs at :

(a) x = 1 (b) x = 2
(c) x = 3 (d) x = π

55. If $f(x) = \frac{x}{1 + x\tan x}$, $x \in \left(0, \frac{\pi}{2}\right)$, then

(a) $f(x)$ has exactly one point of minimum
(b) $f(x)$ has exactly one point of maximum
(c) $f(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$
(d) maximum occurs at x_0 where $x_0 = \cos x_0$

56. Let $f(x) = (x-1)^4(x-2)^n$, $n \in N$. then $f(x)$ has

(a) local minimum at x = 2 if n is even
(b) local minimum at x = 1 if n is odd
(c) local maximum at x = 1 if n is odd
(d) local minimum at x = 1 if n is even

57. Let $g(x) = -\frac{f(-1)}{2}x^2(x-1) - f(0)(x^2-1) + \frac{f(1)}{2}x^2(x+1) - f'(0)\,x(x-1)(x+1)$ where f is a thrice differentiable function. Then the correct statements are

(a) there exists $x \in (-1, 0)$ such that $f'(x) = g'(x)$
(b) there exists $x \in (0, 1)$ such that $f''(x) = g''(x)$
(c) there exists $x \in (-1, 1)$ such that $f'''(x) = g'''(x)$
(d) there exists $x \in (-1, 1)$ such that $f'''(x) = 3f(1) - 3f(-1) - 6f'(0)$

58. If $f(x)$ is a differentiable function and $\phi(x)$ is twice differentiable function and α and β are roots of the equation $f(x) = 0$ and $\phi'(x) = 0$ respectively, then which of the following statement is true ? ($\alpha < \beta$).

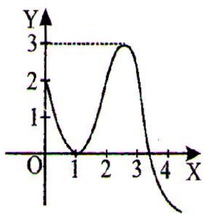
(a) there exists exactly one root of the equation $\phi'(x).f'(x) + \phi''(x).f(x) = 0$ and (α, β)

(b) there exists at least one root of the equation $\phi'(x).f'(x) + \phi''(x).f(x) = 0$ and (α, β)

(c) there exists odd number of roots of the equation $\phi'(x).f'(x) + \phi''(x).f(x) = 0$ and (α, β)

(d) None of these

59. The diagram shows the graph of the derivative of a function $f(x)$ for $0 \le x \le 4$ with $f(0) = 0$. Which of the following could be correct statements for $y = f(x)$?



(a) Tangent line to $y = f(x)$ at $x = 0$ makes an angle of $\sec^{-1}\sqrt{5}$ with the x-axis.

(b) f is strictly increasing in (0, 3)

(c) $x = 1$ is both an inflection point as well as point of local extremum.

(d) Number of critical point on $y = f(x)$ is two.

60. If $f : [-1, 1] \to R$ is a continuously differentiable function such that $f(1) > f(-1)$ and $|f'(y)| \le 1$ for all $y \in [-1, 1]$ then

(a) there exists an $x \in [-1, 1]$ such that $f'(x) > 0$

(b) there exists an $x \in [-1, 1]$ such that $f'(x) < 0$

(c) $f(1) \le f(-1) + 2$

(d) $f(-1) . f(1) < 0$

61. In a triangle ABC

(a) $\sin A \sin B \sin C \le \frac{3\sqrt{3}}{8}$

(b) $\sin^2 A + \sin^2 B + \sin^2 C \le \frac{9}{4}$

(c) sin A sin B sin C is always positive

(d) $\sin^2 A + \sin^2 B = 1 + \cos C$

Assertion Reason Type Questions

62. **Assertion :** If $g(x)$ is a differentiable function $g(1) \ne 0$, $g(-1) \ne 0$ and Rolles theorem is not applicable to $f(x) = \frac{x^2 - 1}{g(x)}$ in [–1, 1], then $g(x)$ has atleast one root in (–1, 1)

Reason : If $f(a) = f(b)$, then Rolles theorem is applicable for $x \in (a, b)$

(a) A (b) B

(c) C (d) D

63. **Assertion :** The equation $3x^2 + 4ax + b = 0$ has at least one root in (0, 1), if $3 + 4a = 0$.

Reason : $f(x) = 3x^2 + 4ax + b$ is continuous and differentiable in the interval (0, 1).

(a) A (b) B

(c) C (d) D

64. **Assertion :** The greatest of the numbers $1, 2^{1/2}, 3^{1/3}, 4^{1/4}, 5^{1/5}, 6^{1/6}, 7^{1/7}$ is $3^{1/3}$.

Reason : $x^{1/x}$ is increasing for $x < e$ and decreasing for $x > e$.

(a) A (b) B

(c) C (d) D

65. **Assertion :** Let $f : [0, \infty) \to [0, \infty)$ and $g : [0, \infty) \to [0, \infty)$ be non-increasing and non-decreasing functions respectively and $h(x) = g(f(x))$. If f and g are differentiable for all points in their respective domains and $h(0) = 0$ then $h(x)$ is constant function.

Reason : $g(x) \in [0, \infty) \Rightarrow h(x) \ge 0$ and $h'(x) \le 0$.

(a) A (b) B

(c) C (d) D

66. **Assertion :** If $f(x)$ is increasing function with concavity upwards, then concavity of $f^{-1}(x)$ is also upwards.

Reason : If $f(x)$ is decreasing function with concavity upwards, then concavity of $f^{-1}(x)$ is also upwards.

(a) A (b) B

(c) C (d) D

67. **Assertion :** Let $f(x) = 5 - 4(x - 2)^{2/3}$, then at $x = 2$ the function $f(x)$ attains neither least value nor greatest value.

Reason : $x = 2$ is the only critical point of $f(x)$.

(a) A (b) B

(c) C (d) D

68. **Assertion :** The largest term in the sequence

$a_n = \frac{n^2}{n^3+200}, n \in N$ is $\frac{(400)^{2/3}}{600}$.

Reason : $f(x) = \frac{x^2}{x^3+200}, x > 0$, then at $x = (400)^{1/3}$, $f(x)$ is maximum.

(a) A (b) B

(c) C (d) D

69. **Assertion :** for any triangle ABC

$$\sin\left(\frac{A+B+C}{3}\right) \geq \frac{\sin A + \sin B + \sin C}{3}$$

Reason : $y = \sin x$ is concave downward for $x \in (0, \pi]$.

(a) A (b) B

(c) C (d) D

70. **Assertion :** Among all the rectangles of given perimeter, the square has the largest area. Also among all the rectangles of given area, the square has the least perimeter.

Reason : For $x > 0$, $y > 0$, if $x + y$ = const, then xy will be maximum for $y = x$ and if xy = const, then $x + y$ will be minimum for $y = x$.

(a) A (b) B

(c) C (d) D

71. **Assertion :** The minimum distance of the fixed point $(0, y_0)$, where $0 \leq y_0 \leq \frac{1}{2}$, from the curve $y = x^2$ is y_0.

Reason : Maxima and minima of a function is always a root of the equation $f'(x) = 0$.

(a) A (b) B

(c) C (d) D

Paragraph Type Questions

Using the following passage, solve Q.72 to Q.74

Passage

Consider a function $f(x) = \left(\alpha - \frac{1}{\alpha} - x\right)(4 - 3x^2)$ where 'α' is a positive parameter

72. Number of points of extrema of $f(x)$ for a given value of α is

(a) 0 (b) 1

(c) 2 (d) 3

73. Absolute difference between local maximum and local minimum values of $f(x)$ in terms of α is

(a) $\frac{4}{9}\left(\alpha + \frac{1}{\alpha}\right)^3$ (b) $\frac{2}{9}\left(\alpha + \frac{1}{\alpha}\right)^3$

(c) $\left(\alpha + \frac{1}{\alpha}\right)^3$ (d) independent of α

74. Least possible value of the absolute difference between local maximum and local minimum values of $f(x)$ is

(a) $\frac{32}{9}$ (b) $\frac{16}{9}$

(c) $\frac{8}{9}$ (d) $\frac{1}{9}$

Using the following passage, solve Q.75 to Q.77

Passage

Let $f'(\sin x) < 0$ and $f''(\sin x) > 0 \ \forall x \in \left(0, \frac{\pi}{2}\right)$

Now consider a function $g(x) = f(\sin x) + f(\cos x)$

75. $g(x)$ decreases if x belongs to

(a) $\left(0, \frac{\pi}{4}\right)$ (b) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

(c) $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ (d) none of these

76. $g(x)$ increase if x belongs to

(a) $\left(0, \frac{\pi}{4}\right)$ (b) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

(c) $\left(\frac{\pi}{8}, \frac{\pi}{3}\right)$ (d) $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$

77. The set of critical points of $g(x)$ is

(a) $\left\{\frac{\pi}{8}, \frac{\pi}{6}\right\}$ (b) $\left\{\frac{\pi}{8}, \frac{\pi}{6}, \frac{\pi}{3}\right\}$

(c) $\left\{\frac{\pi}{8}, \frac{\pi}{6}, \frac{\pi}{4}\right\}$ (d) none of these

Using the following passage, solve Q.78 to Q.80

Passage

Consider the function $f(x) = \max\{x^2, (1-x)^2, 2x(1-x)\}$ where $0 \le x \le 1$.

78. The interval in which $f(x)$ is increasing is

(a) $\left(\frac{1}{3}, \frac{2}{3}\right)$ (b) $\left(\frac{1}{3}, \frac{1}{2}\right)$

(c) $\left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \frac{2}{3}\right)$ (d) $\left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{2}{3}, 1\right)$

79. The interval in which $f(x)$ is decreasing is

(a) $\left(\frac{1}{3}, \frac{2}{3}\right)$ (b) $\left(\frac{1}{3}, \frac{1}{2}\right)$

(c) $\left(0, \frac{1}{3}\right) \cup \left(\frac{1}{2}, \frac{2}{3}\right)$ (d) $\left(0, \frac{1}{2}\right) \cup \left(\frac{2}{3}, 1\right)$

80. Let RMVT is applicable for $f(x)$ on (a, b) then a + b + c is (where c is point such that $f'(c) = 0$)

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$

(c) $\frac{1}{2}$ (d) $\frac{3}{2}$

Using the following passage, solve Q.81 to Q.83

Passage

Consider f, g and h be three real valued differentiable functions defined on R.

Let $g(x) = x^3 + g''(1)x^2 + (3g'(1) - g''(1) - 1)x + 3g'(1)$,

$f(x) = x\,g(x) - 12x + 1$ and $f(x) = (h(x))^2$ where $h(0) = 1$.

81. The function y = f(x) has

(a) Exactly one local minima and no local maxima

(b) Exactly one local maxima and no local minima

(c) Exactly one local maxima and two local minima

(d) Exactly two local maxima and one local minima

82. Which of the following is/are true for the function y=g(x)?

(a) g (x) monotonically decreases in

$$\left(-\infty, 2 - \frac{1}{\sqrt{3}}\right) \text{and} \left(2 + \frac{1}{\sqrt{3}}, \infty\right)$$

(b) g (x) monotonically increases in

$$\left(2 - \frac{1}{\sqrt{3}}, 2 + \frac{1}{\sqrt{3}}\right)$$

(c) There exists exactly one tangent to y = g(x) which is parallel to the chord joining the points (1, g (1)) and (3, g(3))

(d) There exists exactly two distinct Lagrange's Mean Value in (0, 4) for the function y = g(x).

83. Which one of the following does not hold good for y = h(x) ?

(a) Exactly one critical point

(b) No point of inflection

(c) Exactly one real zero in (0, 3)

(d) Exactly one tangent parallel to x-axis

Match the Column

84.

	Column - I		**Column - II**
(A)	The equation $x \log x = 3 - x$ has at least one root in	(P)	(0, 1)
(B)	If $27a + 9b + 3c + d = 0$, then the equation $4ax^3 + 3bx^2 + 2cx + d = 0$ has at least one root in	(Q)	(1, 3)
(C)	If $c = \sqrt{3}$ & $f(x) = x + \frac{1}{x}$ then interval of x in which LMVT is applicable for $f(x)$, is	(R)	(0, 3)
(D)	If $c = \frac{1}{2}$ & $f(x) = 2x - x^2$, then interval of x in which LMVT is applicable for $f(x)$, is	(S)	(–1, 1)

85. **Column - I** **Column - II**

(A) If x is real, then the greatest and least value of the expression $\dfrac{x+2}{2x^2+3x+6}$ is — (P) 3

(B) If $a+b=1$; $a>0, b>0$, then the minimum value of $\sqrt{\left(1+\dfrac{1}{a}\right)\left(1+\dfrac{1}{b}\right)}$ is — (Q) $\dfrac{1}{3}$

(C) The maximum value attained by $y = 10 - |x-10|, -1 \le x \le 3$, is — (R) 5

(D) If $P(t^2, 2t), t \in [0, 2]$ is an arbitrary point on parabola $y^2=4x$. Q is foot of perpendicular from focus S on the tangent at P, then maximum area of triangle PQS is — (S) $-\dfrac{1}{13}$

86. **Column - I** **Column - II**

(A) The dimensions of the rectangle of perimeter 36 cm, which sweeps out the largest volume when revolved about one of its sides, are — (P) 6

(B) Let A (–1, 2) and B (2, 3) be two fixed points, A point P lying on $y = x$ such that perimeter of triangle PAB is minimum, then sum of the abscissa and ordinate of point P, is — (Q) 12

(C) If x_1 and x_2 are abscissae of two points on the curve $f(x) = x - x^2$ in the interval [0, 1], then maximum value of expression $(x_1+x_2) - (x_1^2 + x_2^2)$ is — (R) 4

(D) The number of non-zero integral values of 'a' for which the function $f(x) = x^4 + ax^3 + \dfrac{3x^2}{2} + 1$ is concave upward along the entire real line is — (S) 1/2

(T) 2

Subjective Type Questions

87. If $f(x)$ is a twice differentiable function such that $f(a) = 0, f(b) = 2, f(c) = -1, f(d) = 2, f(e) = 0$, where $a < b < c < d < e$, find the minimum number of zeroes of $g(x) = (f'(x))^2 + f''(x)f(x)$ in the interval [a, e].

88. From a given solid cone of height 'H', another inverted cone is carved such that its volume is maximum. Then find the ratio of height of the cone and height of the inscribed cone.

89. If α is an integer satisfying $|\alpha| \le 5 - |[x]|$, where x is a real number for which $2x \tan^{-1} x$ is greater than or equal to $\ln(1+x^2)$, then find the number of maximum possible values of α. (where [.] represents the greatest integer function)

90. The circle $x^2 + y^2 = 1$ cuts the x-axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis and the line segment PQ at S. If A is the maximum area of the triangle QSR then $3\sqrt{3}$ A is equal to _____.

91. A cylinderical vessel of volume $25\dfrac{1}{7}$ cu metres, open at the top is to be manufactured from a sheet of metal. (The value of π is taken as 22/7). If r and h are the radius and height of the vessel so that amount of metal is used in the least possible then rh is equal to

92. If k is a positive integer, such that

(i) $\cos^2 x \sin x > -\dfrac{7}{k}$, for all x

(ii) $\cos^2 x \sin x < -\dfrac{7}{k+1}$ for some x, then k must be equal to

EXERCISE - 4 : PREVIOUS YEAR JEE ADVANCED QUESTIONS

Objective Question I [Only one correct option]

1. If p, q, r are any real numbers, then **(1982)**

(a) max (p, q) < max (p, q, r)

(b) min (p, q) = $\frac{1}{2}$ {p + q – |p – q|}

(c) max (p, q) < min (p, q, r)

(d) None of the above

2. If y = a *l*og |x| + bx² + x has its extremum values at x = –1 and x = 2, then **(1983)**

(a) a = 2, b = –1 (b) a = 2, b = –1/2

(c) a = –2, b = 1/2 (d) None of these

3. The normal to the curve x = a (cos θ + θ sin θ), y = a (sin θ – θ cos θ) at any point 'θ' is such that **(1983)**

(a) it makes a constant angle with the x–axis

(b) it passes through the origin

(c) it is at a constant distance from the origin

(d) None of the above

4. If a + b + c = 0, then the quadratic equation $3ax^2 + 2bx + c = 0$ has **(1983)**

(a) at least one root in (0, 1)

(b) one root in (2, 3) and the other in (–2, –1)

(c) imaginary roots

(d) None of the above

5. Let $P(x) = a_0 + a_1x^2 + a_2x^4 + ... + a_nx^{2n}$ be a polynomial in a real variable x with $0 < a_0 < a_1 < a_2 < ... < a_n$. The function P (x) has **(1986)**

(a) neither a maxima nor a minima

(b) only one maxima

(c) only one minima

(d) only one maxima and only one minima

6. Let *f* and g be increasing and decreasing functions respectively from [0, ∞) to [0,∞). Let h (x) = *f* (g(x)). If h (0) = 0, then h (x) – h (1) is **(1987)**

(a) always negative (b) always positive

(c) strictly increasing (d) None of these

7. The function $f(x) = \frac{\log(\pi+x)}{\log(e+x)}$ is **(1995)**

(a) increasing on (0, ∞)

(b) decreasing on (0, ∞)

(c) increasing on (0, π/e), decreasing on (π/e, ∞)

(d) decreasing on (0, π/e), increasing on (π/e, ∞)

8. The slope of tangent to a curve y = *f*(x) at [x, *f*(x)] is 2x + 1. If the curve passes through the point (1, 2), then the area bounded by the curve, the x–axis and the line x = 1, is **(1995)**

(a) 5/6 (b) 6/5

(c) 1/6 (d) 6

9. On the inverval [0, 1] the function $x^{25}(1-x)^{75}$ takes its maximum value at the point. **(1995)**

(a) 0 (b) 1/4

(c) 1/2 (d) 1/3

10. If $f(x) = \frac{x}{\sin x}$ and $g(x) = \frac{x}{\tan x}$, where $0 < x \leq 1$, then in this interval : **(1997)**

(a) both *f* (x) and *g*(x) are increasing functions

(b) both *f* (x) and *g*(x) are decreasing functions

(c) *f* (x) is an increasing function

(d) *g* (x) is an increasing function

11. The number of values of x, where the function $f(x) = \cos x + \cos\left(\sqrt{2}x\right)$ attains its maximum, is **(1998)**

(a) 0 (b) 1

(c) 2 (d) infinite

12. If $f(x) = \frac{x^2-1}{x^2+1}$, for every real number x, then the minimum value of *f* **(1998)**

(a) does not exist because *f* is unbounded.

(b) is not attained even though *f* is bounded

(c) is equal to 1 (d) is equal to –1

13. The function $f(x) = \sin^4 x + \cos^4 x$ increases, if **(1999)**

(a) $0 < x < \frac{\pi}{8}$ (b) $\frac{\pi}{4} < x < \frac{3\pi}{8}$

(c) $\frac{3\pi}{8} < x < \frac{5\pi}{8}$ (d) $\frac{5\pi}{8} < x < \frac{3\pi}{4}$

14. For all x ∈ (0, 1) **(2000)**

(a) $e^x < 1 + x$ (b) $\log_e(1 + x) < x$

(c) sin x > x (d) $\log_e x > x$

15. Let $f(x) = \int e^x (x-1)(x-2)\,dx$. Then f decreases in the interval **(2000)**

(a) $(-\infty, -2)$ (b) $(-2, -1)$

(c) $(1, 2)$ (d) $(2, \infty)$

16. Let $f(x) = \begin{cases} |x|, & \text{for } 0<|x|\le 2 \\ 1, & \text{for } x=0 \end{cases}$ Then, at $x = 0$, f has **(2000)**

(a) a local maximum (b) no local maximum

(c) a local minimum (d) no extremum

17. If the normal to the curve, $y = f(x)$ at the point $(3, 4)$ makes an angle $3\pi/4$ with the positive x–axis, then $f'(3)$ is equal to **(2000)**

(a) -1 (b) $-3/4$

(c) $4/3$ (d) 1

18. If $f(x) = xe^{x(1-x)}$, then $f(x)$ is **(2001)**

(a) increasing in $\left[-\frac{1}{2}, 1\right]$ (b) decreasing in R

(c) increasing in R (d) decreasing in $\left[-\frac{1}{2}, 1\right]$

19. The maximum value of $(\cos\alpha_1).(\cos\alpha_2)\ldots(\cos\alpha_n)$, under the restrictions $0 \le \alpha_1, \alpha_2, \ldots \alpha_n \le \frac{\pi}{2}$ and $(\cot\alpha_1).(\cot\alpha_2)\ldots(\cot\alpha_n) = 1$ is **(2001)**

(a) $\frac{1}{2^{n/2}}$ (b) $\frac{1}{2^n}$

(c) $\frac{1}{2n}$ (d) 1

20. The length of a longest interval in which the function $3\sin x - 4\sin^3 x$ is increasing, is **(2002)**

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{2}$ (d) π

21. The point(s) on the curve $y^3 + 3x^2 = 12y$ where the tangent is vertical, is (are) **(2002)**

(a) $\left(\pm\frac{4}{\sqrt{3}}, -2\right)$ (b) $\left(\pm\sqrt{\frac{11}{3}}, 0\right)$

(c) $(0, 0)$ (d) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

22. The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ is **(2002)**

(a) $3y = 9x + 2$ (b) $y = 2x + 1$

(c) $2y = x + 8$ (d) $y = x + 2$

23. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ such that min $f(x) >$ max $g(x)$, then the relation between b and c, is – **(2003)**

(a) no real value of b & c (b) $0 < c < b\sqrt{2}$

(c) $|c| < |b|\sqrt{2}$ (d) $|c| > |b|\sqrt{2}$

24. If $f(x) = x^3 + bx^2 + cx + d$ and $0 < b^2 < c$, then in $(-\infty, \infty)$ **(2004)**

(a) $f(x)$ is strictly increasing function

(b) $f(x)$ has a local maxima

(c) $f(x)$ is strictly decreasing function

(d) $f(x)$ is bounded.

25. If $f(x)$ is differentiable and strictly increasing function, then the value of $\lim_{x\to 0}\frac{f(x^2)-f(x)}{f(x)-f(0)}$ is **(2004)**

(a) 1 (b) 0

(c) -1 (d) 2

26. Tangents are drawn to the ellipse $x^2 + 2y^2 = 2$, then the locus of the mid point of the intercept made by the tangents between the coordinate axes is **(2004)**

(a) $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$ (b) $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$

(c) $\frac{x^2}{2} + \frac{y^2}{4} = 1$ (d) $\frac{x^2}{4} + \frac{y^2}{2} = 1$

27. The angle between the tangents drawn from the point $(1, 4)$ to the parabola $y^2 = 4x$ is **(2004)**

(a) $\pi/6$ (b) $\pi/4$

(c) $\pi/3$ (d) $\pi/2$

28. The second degree polynomial $f(x)$, satisfying $f(0) = 0$, $f(1) = 1$, $f'(x) > 0$ for all $x \in (0, 1)$: **(2005)**

(a) $f(x) = \phi$

(b) $f(x) = ax + (1-a)x^2$; $\forall\, a \in (0, \infty)$

(c) $f(x) = ax + (1-a)x^2$; $a \in (0, 2)$

(d) No such polynomial

29. The tangent at $(1, 7)$ to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at **(2005)**

(a) $(6, 7)$ (b) $(-6, 7)$

(c) $(6, -7)$ (d) $(-6, -7)$

30. The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c - 1, e^{c-1})$ and $(c + 1, e^{c+1})$ **(2007)**

(a) on the left of $x = c$ (b) on the right of $x = c$

(c) at no point (d) at all points

31. Let the function $g : (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is **(2008)**

(a) even and is strictly increasing in $(0, \infty)$

(b) odd and is strictly decreasing in $(-\infty, \infty)$

(c) odd and is strictly increasing in $(-\infty, \infty)$

(d) neither even nor odd, but is strictly increasing in $(-\infty, \infty)$

32. The total number of local maxima and local minima of the function $f(x) = \begin{cases} (2+x)^3, & -3 < x \le -1 \\ x^{\frac{2}{3}}, & -1 < x < 2 \end{cases}$ is **(2008)**

(a) 0 (b) 1

(c) 2 (d) 3

33. Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$. If a, b and c denote respectively, the absolute maximum of f, g and h on [0, 1], then **(2010)**

(a) $a = b$ and $c \ne b$ (b) $a = c$ and $a \ne b$

(c) $a \ne b$ and $c \ne b$ (d) $a = b = c$

34. The number of points in $(-\infty, \infty)$, for which $x^2 - x\sin x - \cos x = 0$, is **(2013)**

(a) 6 (b) 4

(c) 2 (d) 0

35. A computer producing factory has only two plants T_1 and T_2. Plant T_1 produces 20% and plant T_2 produces 80% of the total computers produced. 7% of computers produced in the factory turn out to be defective. It is known that P (computer turns out to be defective given that it is produced in plant T_1) = 10P (computer turns out to be defective given that it is produced in plant T_2), where P(E) denotes the probability of an event E. A computer produced in the factory is randomly selected and it does not turn out to be defective. Then the probability that it is produced in plants T_2 is **(2016)**

(a) $\frac{36}{73}$ (b) $\frac{47}{79}$

(c) $\frac{78}{93}$ (d) $\frac{75}{83}$

Objective Questions II [One or more than one correct option]

36. If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then **(1986)**

(a) $a > 0, b > 0$ (b) $a > 0, b < 0$

(c) $a < 0, b > 0$ (d) $a < 0, b < 0$

37. If $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3 \end{cases}$, then **(1993)**

(a) $f(x)$ is increasing on $[-1, 2]$

(b) $f(x)$ is continuous on $[-1, 3]$

(c) $f'(2)$ does not exist

(d) $f(x)$ has the maximum value at $x = 2$.

38. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x. Then **(1998)**

(a) h is increasing whenever f is increasing.

(b) h is increasing whenever f is decreasing.

(c) h is decreasing whenever f is decreasing.

(d) nothing can be said in general

39. The function $f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5\,dt$ has local minimum at x equals to **(1999)**

(a) 0 (b) 1

(c) 2 (d) 3

40. On the ellipse $4x^2 + 9y^2 = 1$, the point at which the tangents are parallel to the line $8x = 9y$, are **(1999)**

(a) $\left(\frac{2}{5}, \frac{1}{5}\right)$ (b) $\left(-\frac{2}{5}, \frac{1}{5}\right)$

(c) $\left(-\frac{2}{5}, -\frac{1}{5}\right)$ (d) $\left(\frac{2}{5}, -\frac{1}{5}\right)$

41. If $f(x)$ is cubic polynomial which has local maximum at $x=-1$. If $f(2)=18, f(1)=-1$ and $f'(x)$ has local minimum at $x=0$, then **(2006)**

(a) the distance between $(-1, 2)$ and $(a, f(a))$ where $x=a$ is the point of local minima, is $2\sqrt{5}$.

(b) $f(x)$ is increasing for $x \in [1, 2\sqrt{5}]$

(c) $f(x)$ has local minima at $x=1$

(d) the value of $f(0)=5$

42. If $f(x)=\begin{cases} e^x, & 0 \le x \le 1 \\ 2-e^{x-1}, & 1 < x \le 2 \\ x-e, & 2 < x \le 3 \end{cases}$

and $g(x)=\int_0^x f(t)\,dt, x \in [1,3]$, then **(2006)**

(a) g (x) has local maxima at $x = 1 + \log_e 2$ and local minima at $x = e$

(b) $f(x)$ has local maxima at $x=1$ and local minima at $x=2$

(c) g (x) has no local minima

(d) $f(x)$ has no local maxima

43. For the function $f(x)=x\cos\frac{1}{x}, x \ge 1$. **(2009)**

(a) for at least one x in the interval $[1,\infty), f(x+2)-f(x)<2$

(b) $\lim_{x\to\infty} f'(x)=1$

(c) for all x in the interval $[1,\infty)$, $f(x+2)-f(x)>2$

(d) $f'(x)$ is strictly decreasing in the interval $[1,\infty)$

44. Let f be a real-valued function defined on the interval $(0,\infty)$, by $f(x) = \ln x + \int_0^x \sqrt{1+\sin t}\,dt$. Then which of the following statement(s) is (are) true ? **(2010)**

(a) $f''(x)$ exists for all $x \in (0,\infty)$

(b) $f'(x)$ exists for all $x \in (0,\infty)$ and f' is continuous on $(0,\infty)$, but not differentiable on $(0,\infty)$

(c) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha,\infty)$

(d) there exists $\beta > 0$ such that $|f(x)|+|f'(x)| \le \beta$ from all $x \in (0,\infty)$

45. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. The lengths of the sides of the rectangular sheet are **(2013)**

(a) 24 (b) 32

(c) 45 (d) 60

46. The function $f(x)=2|x|+|x+2|-||x+2|-2|x||$ has a local minimum or a local maximum at x is equal to **(2013)**

(a) -2 (b) $\frac{-2}{3}$

(c) 2 (d) $\frac{2}{3}$

47. Let $a \in R$ and let $f: R \to R$ be given by

$f(x)= x^5-5x+a$.

Then **(2014)**

(a) $f(x)$ has three real roots if $a>4$

(b) $f(x)$ has only one real root if $a > 4$

(c) $f(x)$ has three real roots if $a < -4$

(d) $f(x)$ has three real roots if $-4 < a < 4$

48. Let $f: \mathbb{R} \to (0,\infty)$ and $g: \mathbb{R} \to \mathbb{R}$, be twice differentiable functions such that f'' and g'' are continuous functions on . Suppose $f'(2)=g(2)=0$, $f''(2) \ne 0$ and $g'(2) \ne 0$. If then **(2016)**

(a) f has a local minimum at $x=2$

(b) f has a local maximum at $x=2$

(c) $f''(2)=f(2)$

(d) $f(x)-f''(x)=0$ for at least one $x \in$

Integer Type Questions

49. The maximum value of the function $f(x)=2x^3-15x^2+36x-48$ on the set $A=\{x \mid x^2+20 \le 9x\}$ is .... **(2009)**

50. The maximum value of the expression

$$\frac{1}{\sin^2\theta+3\sin\theta\cos\theta+5\cos^2\theta} \text{ is......}$$ **(2010)**

51. Let f be a function defined on R (the set of all real numbers) such that $f'(x)=2010(x-2009)(x-2010)^2(x-2011)^3(x-2012)^4$, for all $x \in R$. If g is a function defined on R with values in the interval $(0,\infty)$ such that $f(x)=\ln(g(x))$, for all $x \in R$, then the number of points in R at which g has a local maximum is ... **(2010)**

52. The number of distinct real roots of
$x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is .... **(2011)**

53. Let p (x) be a real polynomial of least degree which has a local maximum at x = 1 and a local minimum at x = 3. If p (1) = 6 and p (3) = 2, then p′ (0) is **(2012)**

54. Let $f : R \to R$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is **(2012)**

55. A vertical line passing through the point (h, 0) intersects the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the ΔPQR, $\Delta_1 = \max_{1/2 \le h \le 1} \Delta(h)$ and

$\Delta_2 = \min_{1/2 \le h \le 1} \Delta(h)$, then $\frac{8}{\sqrt{5}} \Delta_1 - 8\Delta_2$ is equal to **(2013)**

56. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point (1, 3) is **(2014)**

57. A cylindrical container is to be made from certain solid material with the following constraints : It has a fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container.

If the volume of material used to make the container is minimum when the inner radius of the container is 10 mm, then the value of $\frac{V}{250\pi}$ is **(2015)**

FILL IN THE BLANKS

58. The function $y = 2x^2 - \log|x|$ is monotonically increasing for values of $x(\neq 0)$, satisfying the inequalities... and monotonically decreasing for values of x satisfying the inequalities... **(1983)**

59. The set of all x for which $\log(1 + x) \le x$ is equal to... **(1987)**

60. If A > 0, B > 0 and A + B = π/3, then the maximum value of tan A tan B is..... **(1993)**

61. Let C be the curve $y^3 - 3xy + 2 = 0$. If H is the set of points on the curve C where the tangent is horizontal & V is the set of points on the curve C where the tangent is vertical, then H = ... and V = ... **(1994)**

Match the Columns

Match the conditions/expressions in Column I with statement in Column II.

62. Let the functions defined in Column I have domain (−π/2, π/2)

Column I	Column II
(A) x + sin x	(p) increasing
(B) sec x	(q) decreasing
	(r) neither increasing nor decreasing

(2008)

Passage Based Problem

Read the following passage and answer the questions.

Consider the function $f : (-\infty, \infty) \to (-\infty, \infty)$ defined by

$f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}$; $0 < a < 2$. **(2008)**

63. Which of the following is true ?

(a) $(2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 0$

(b) $(2 - a)^2 f''(1) - (2 + a)^2 f''(-1) = 0$

(c) $f'(1) f'(-1) = (2 - a)^2$

(d) $f'(1) f'(-1) = -(2 + a)^2$

64. Which of the following is true ?

(a) $f(x)$ is decreasing on (−1, 1) and has a local minimum at x = 1.

(b) $f(x)$ is increasing on (−1, 1) and has a local maximum at x = 1.

(c) $f(x)$ is increasing on (−1, 1) but has neither a local maximum nor a local minimum at x = 1.

(d) $f(x)$ is decreasing on (−1, 1) but has neither a local maximum nor a local minimum at x = 1.

65. Let $g(x) = \int_0^{e^x} \frac{f'(t)}{1 + t^2} dt$. Which of the following is true ?

(a) g′ (x) is positive on (−∞, 0) and negative on (0, ∞)

(b) g′ (x) is negative on (−∞, 0) and positive on (0, ∞)

(c) g′ (x) changes sign on both (−∞, 0) and (0, ∞)

(d) g′ (x) does not change sign (−∞, ∞)

Paragraph

Consider the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3$. Let s be the sum of all distinct real roots of f (x) and let t = |s| **(2010)**

66. The real numbers s lies in the interval

(a) $\left(-\frac{1}{4}, 0\right)$ (b) $\left(-11, -\frac{3}{4}\right)$

(c) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (d) $\left(0, \frac{1}{4}\right)$

67. The area bounded by the curve $y = f(x)$ and the lines $x = 0$, $y = 0$ and $x = t$, lies in the interval

(a) $\left(\frac{3}{4}, 3\right)$ (b) $\left(\frac{21}{64}, \frac{11}{16}\right)$

(c) $(9, 10)$ (d) $\left(0, \frac{21}{64}\right)$

68. The function $f'(x)$ is

(a) increasing in $\left(-t, -\frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$

(b) decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$

(c) increasing in $(-t, t)$

(d) decreasing in $(-t, t)$

Paragraph

Let $f(x) = (1-x)^2 \sin^2 x + x^2$ for all $x \in R$ and let

$$g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t\right) f(t)\,dt \text{ for all } x \in (1, \infty)$$

69. Which of the following is true ? **(2012)**

(a) g is increasing on $(1, \infty)$

(b) g is decreasing on $(1, \infty)$

(c) g is increasing on $(1, 2)$ and decreasing on $(2, \infty)$

(d) g is decreasing on $(1, 2)$ and increasing on $(2, \infty)$

70. Consider the statements

P : There exists some $x \in R$ such that

$f(x) + 2x = 2(1 + x^2)$

Q : There exists some $x \in R$ such that

$2f(x) + 1 = 2x(1 + x)$

Then, **(2012)**

(a) Both P and Q are true (b) P is true and Q is false

(c) P is false and Q is true (d) Both P and Q are false

Paragraph

Let $f : [0, 1] \to R$ (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x$, $x \in [0, 1]$.

71. Which of the following is true for $0 < x < 1$? **(2013)**

(a) $0 < f(x) < \infty$ (b) $-\frac{1}{2} < f(x) < \frac{1}{2}$

(c) $-\frac{1}{4} < f(x) < 1$ (d) $-\infty < f(x) < 0$

72. If the function $e^{-x} f(x)$ assumes its minimum in the interval $[0, 1]$ at $x = \frac{1}{4}$, which of the following is true ? **(2013)**

(a) $f'(x) < f(x), \frac{1}{4} < x < \frac{3}{4}$

(b) $f'(x) > f(x), 0 < x < \frac{1}{4}$

(c) $f'(x) < f(x), 0 < x < \frac{1}{4}$

(d) $f'(x) < f(x), \frac{3}{4} < x < 1$

Paragraph

Let $f(x) = x + \log_e x - x \log_e x,\ x \in (0, \infty)$.

Column 1 contains information about zeros of f(x), f'(x) and f''(x).

Column 2 contains information about the limiting behavior of f(x), f'(x) and f''(s) at infinity.

Column 3 contains information about increasing/decreasing nature of f(x) and f'(x).

Column 1	Column 2	Column 3
(I) $f(x) = 0$ for some $x \in (1, e^2)$	(i) $\lim_{x\to\infty} f(x) = 0$	(P) f is increasing in (0, 1)
(II) $f'(x) = 0$ for some $x \in (1, e)$	(ii) $\lim_{x\to\infty} f(x) = -\infty$	(Q) f is decreasing in (e, e^2)
(III) $f'(x) = 0$ for some $x \in (0, 1)$	(iii) $\lim_{x\to\infty} f'(x) = -\infty$	(R) f' is increasing in (0, 1)
(IV) $f''(x) = 0$ for some $x \in (1, e)$	(iv) $\lim_{x\to\infty} f''(x) = 0$	(S) f' is decreasing in (e, e^2)

(2017)

73. Which of the following options is the only CORRECT combination ?

(a) (I) (ii) (R) (b) (IV) (i) (S)

(c) (III) (iv) (P) (d) (II) (iii) (S)

74. Which of the following options is the only CORRECT combination ?

(a) (I) (i) (P) (b) (II) (ii) (Q)

(c) (III) (iii) (R) (d) (IV) (iv) (S)

75. Which of the following options is the only INCORRECT combination ?

(a) (II) (iii) (P) (b) (I) (iii) (P)

(c) (III) (i) (R) (d) (II) (iv) (Q)

True/False

76. For $0 < a < x$, the minimum value of function $\log_a x + \log_x a$ is 2. **(1984)**

Analytical and Descriptive Questions

77. Let x and y be two real variables such that $x > 0$ and $xy = 1$. Find the minimum value of $x + y$. **(1981)**

78. If $ax^2 + b / x \geq c$ for all positive x where $a > 0$ and $b > 0$, show that $27ab^2 \geq 4c^3$. **(1982)**

79. If $f(x)$ and $g(x)$ are differentiable function for $0 \leq x \leq 1$ such that $f(0) = 2$, $g(0) = 0$, $f(1) = 6$, $g(1) = 2$.

Then show that there exists c satisfying $0 < c < 1$ and $f'(c) = 2g'(c)$. **(1982)**

80. Show that

$1 + x \log (x + \sqrt{x^2 + 1}) \geq \sqrt{1 + x^2}$ for all $x \geq 0$. **(1983)**

81. A swimmer S is in the sea at a distance d km from the closest point A on a straight shore. The house of the swimmer is on the shore at a distance L km from A. He can swim at a speed of u km/h and walk at a speed of v km/h ($v > u$). At what point on the shore should he land so that he reaches his house in the shortest possible time? **(1983)**

82. Find the coordinates of the point on the curve $y = \frac{x}{1+x^2}$, where the tangent to the curve has the greatest slope. **(1984)**

83. Let $f(x) = \sin^3 x + \lambda \sin^2 x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the intervals in which λ should lie in the order that $f(x)$ has exactly one minima and exactly one maxima. **(1985)**

84. Find all the tangents to the curve $y = \cos(x + y)$, $-2\pi \leq x \leq 2\pi$, that are parallel to the line $x + 2y = 0$. **(1985)**

85. Let A $(p^2, -p)$ B (q^2, q), C $(r^2, -r)$ be the vertices of the triangle ABC. A parallelogram AFDE is drawn with vertices D, E and F on the line segments. BC, CA and AB respectively. Using calculus, show that maximum area of such a parallelogram is $\frac{1}{4}(p + q)(q + r)(p - r)$ **(1986)**

86. Find the point on the curve $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$ that is farthest from the point $(0, -2)$. **(1987)**

87. A point P is given on the circumference of a circle of radius r. Chord QR is parallel to the tangent at P. Determine the maximum possible area of the triangle PQR. **(1990)**

88. Show that

$2 \sin x + 2 \tan x \geq 3x$, where $0 \leq x < \frac{\pi}{2}$. **(1990)**

89. A window of fixed perimeter (including the base of the arch) is in the form of a rectangle surmounted by a semi-circle. The semi-circular portion is fitted with coloured glass while the rectangular part is fitted with clear glass. The clear glass transmits three times as much light per square meter as the coloured glass does.

What is the ratio for the sides of the rectangle so that the window transmits the maximum light ? **(1991)**

90. Three normals are drawn from the point (c, 0) to the curve $y^2 = x$. Show that c must be greater than $\frac{1}{2}$. One normal is always the x–axis. Find c for which the other two normals are perpendicular to each other. **(1991)**

91. What normal to the curve $y = x^2$ forms the shortest chord? **(1992)**

92. Find the equation of the normal to the curve

$y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at $x = 0$ **(1993)**

93. Tangent at a point P_1 {other than (0, 0)} on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3, and so on. Show that the abscissae of $P_1, P_2, P_3, \ldots, P_n$, form a GP. Also find the ratio

[area $(\Delta P_1P_2P_3)$]/[area $(\Delta P_2P_3P_4)$] **(1993)**

94. Let $f(x) = \begin{cases} -x^3 + \frac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)}, & 0 \leq x \leq 1 \\ 2x - 3, & 1 \leq x \leq 3 \end{cases}$

Find all possible real values of b such that $f(x)$ has the smallest value at $x = 1$. **(1993)**

95. The circle $x^2 + y^2 = 1$ cuts the x–axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x–axis and the line segment PQ at S. Find the maximum area of the triangle QSR. **(1994)**

96. The curve $y = ax^3 + bx^2 + cx + 5$, touches the x–axis at P (–2, 0) and cuts the y – axis at a point Q, where its gradient is 3. Find a, b, c. **(1994)**

97. Let (h, k) be a fixed point, where $h > 0, k > 0$. A straight line passing through this point cuts the positive directions of the coordinate axes at the points P and Q. Find the minimum area of the triangle OPQ, O being the origin. **(1995)**

98. Determine the points of maxima and minima of the function

$f(x) = \frac{1}{8} \ln x - bx + x^2, x > 0$ where $b \geq 0$ is a constant. **(1996)**

99. Let $f(x) = \begin{cases} xe^{ax}, & x \leq 0 \\ x + ax^2 - x^3, & x > 0 \end{cases}$

where a is a positive constant. Find the interval in which $f'(x)$ is increasing. **(1996)**

100. Suppose, $f(x)$ is a function satisfying the following conditions

(a) $f(0) = 2, f(1) = 1$

(b) has a minimum value at $x = \frac{5}{2}$ and

(c) $\forall x, f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

where, a, b are some constants. Determine the constant a, b and the function $f(x)$. **(1998)**

101. Let $-1 \leq p \leq 1$. Show that the equation $4x^3 - 3x - p = 0$ has a unique root in the interval $\left[\frac{1}{2}, 1\right]$ and identify it. **(2001)**

102. A straight line L with negative slope passes through the point (8, 2) and cuts the positive coordinate axes at points P and Q. Find the absolute minimum value of OP + OQ, as L varies, where O is the origin. **(2002)**

103. Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$, is minimum. **(2003)**

104. For the circle $x^2 + y^2 = r^2$, find the value of r for which the area enclosed by the tangents drawn from the point P (6, 8) to the circle and the chord of contact is maximum. **(2003)**

105. If $P(1) = 0$ and $\frac{dP(x)}{dx} > P(x)$ for all $x \geq 1$, then prove that $P(x) > 0$ for all $x > 1$. **(2003)**

106. Using the relation $2(1 - \cos x) < x^2, x \neq 0$ or otherwise, prove that $\sin(\tan x) \geq x, \ \forall x \in \left[0, \frac{\pi}{4}\right]$. **(2003)**

107. Prove that $\sin x + 2x \geq \frac{3x.(x+1)}{\pi} \forall x \in \left[0, \frac{\pi}{2}\right]$.

(Justify the inequality, if any used). **(2004)**

108. If $|f(x_1) - f(x_2)| \leq (x_1 - x_2)^2$, for all $x_1, x_2 \in R$. Find the equation of tangent to the curve $y = f(x)$ at the point (1, 2). **(2005)**

109. If $f(x)$ is twice differentiable function such that $f(a) = 0, f(b) = 2, f(c) = -1, f(d) = 2, f(e) = 0$, where $a < b < c < d < e$, then the minimum number of zeroes of $g(x) = \{f'(x)\}^2 + f''(x).f(x)$ in the interval [a, e] is ? **(2006)**

110. For all θ in [0, π/2], show that $\cos(\sin\theta) \geq \sin(\cos\theta)$.

Assertion Type Question

111. **(a)** For all $x \in (0,1)$: **(2000)**

(a) $e^x < 1 + x$ (b) $\log_e(1+x) < x$

(c) $\sin x > x$ (d) $\log_e x > x$

(b) Consider the folloiwng statement S and R :

S : Both sin x & cos x are decreasing functions in the interval (π/2, π).

R : If a differentiable function decreases in an interval (a, b), then its derivative also decreases in (a, b).

Which of the following is true ?

(a) both S and R are wrong

(b) both S and R are correct, but R is not the correct explanation for S.

(c) S is correct and R is the correct explanation for S

(d) S is correct and R is wrong.

ANSWER KEY

EXERCISE - 1 : BASIC OBJECTIVE QUESTIONS

1. (c)	2. (c)	3. (c)	4. (a)	5. (b)	6. (d)	7. (b)	8. (b)	9. (c)	10. (b)
11. (d)	12. (a)	13. (a)	14. (d)	15. (c)	16. (c)	17. (a)	18. (a)	19. (d)	20. (c)
21. (c)	22. (a)	23. (c)	24. (c)	25. (a)	26. (a)	27. (c)	28. (d)	29. (b)	30. (d)
31. (a)	32. (b)	33. (c)	34. (a)	35. (a)	36. (c)	37. (a)	38. (a)	39. (d)	40. (b)
41. (a)	42. (a)	43. (a)	44. (d)	45. (b)	46. (c)	47. (b)	48. (c)	49. (b)	50. (a)
51. (a,b,c,d)	52. (a,b)	53. (b,c)	54. (c)	55. (d)	56. (b,c)	57. (b)	58. (c)	59. (b,c)	60. (a,d)
61. (c)	62. (a)	63. (a)	64. (a,b,c)	65. (b,c)	66. (a,b,d)	67. (a)	68. (b)	69. (b)	70. (a)
71. (b)	72. (d)	73. (d)	74. (d)	75. (c)	76. (c)	77. (b)	78. (c)	79. (b)	80. (c)
81. (d)	82. (b)	83. (c)	84. (c)	85. (b, d)	86. (a)	87. (a)	88. (b)	89. (a)	90. (a)

EXERCISE - 2 : PREVIOUS YEAR JEE MAINS QUESTIONS

1. (b)	2. (a)	3. (b)	4. (c)	5. (b)	6. (a)	7. (d)	8. (b)	9. (a)	10. (b)
11. (a,c)	12. (a)	13. (b)	14. (b)	15. (b)	16. (a)	17. (c)	18. (a)	19. (c)	20. (d)
21. (c)	22. (c)	23. (b)	24. (b)	25. (d)	26. (d)	27. (c)	28. (c)	29. (c)	30. (c)
31. (c)	32. (a)	33. (a)	34. (a)	35. (b)	36. (c)	37. (a)	38. (c)	39. (c)	40. (d)
41. (b)	42. (a)	43. (c)	44. (a)	45. (b)	46. (a)	47. (a)	48. (c)	49. (b)	50. (c)
51. (d)	52. (a)	53. (a)	54. (a)	55. (d)	56. (a)	57. (b)			

EXERCISE - 3 : ADVANCED OBJECTIVE QUESTIONS

APPLICATION OF DERIVATIVE–I

1. (b)	2. (c)	3. (c)	4. (c)	5. (d)	6. (d)	7. (d)	8. (d)	9. (a)	10. (a)
11. (b)	12. (a)	13. (b)	14. (b)	15. (c)	16. (c)	17. (a)	18. (b)	19. (a)	20. (a)
21. (d)	22. (a)	23. (b)	24. (a)	25. (c)	26. (c)	27. (a)	28. (c)	29. (a)	30. (a)
31. (b)	32. (a)	33. (a,b,c)	34. (d)	35. (c)	36. (c)	37. (b)	38. (c)	39. (c)	40. (b)
41. (d)	42. (a)	43. (a,c)	44. (a,b,c)	45. (a,b,c,d)	46. (a,b,c,d)	47. (b, c)	48. (a,b)	49. (a,c,d)	50. (b,c)
51. (a,d)	52. (a, c)	53. (d)	54. (a)	55. (c)	56. (d)	57. (b)	58. (b)	59. (b)	60. (c)
61. (a)	62. (a)	63. (b)	64. (c)	65. (b)	66. (A)–(R), (B)–(Q), (C)–(P), (D)–(S)				

67. (A)–(Q), (B)–(R), (C)–(P), (D)–(S) 68. 0025 69. 0016 70. 0002

APPLICATION OF DERIVATIVE–II

1. (b)	2. (a)	3. (a)	4. (b)	5. (c)	6. (b)	7. (b)	8. (a)	9. (d)	10. (c)
11. (c)	12. (d)	13. (d)	14. (a,c)	15. (b,c)	16. (a, b)	17. (a)	18. (c)	19. (d)	20. (d)
21. (a)	22. (a)	23. (a)	24. (b)	25. (d)	26. (b)	27. (c)	28. (a)	29. (b)	30. (b)
31. (a)	32. (a)	33. (c)	34. (c)	35. (a,c)	36. (c)	37. (d)	38. (a)	39. (a)	40. (b)
41. (d)	42. (b,d)	43. (d)	44. (b)	45. (a)	46. (b)	47. (c)	48. (a)	49. (a)	50. (d)
51. (a,b)	52. (b,c)	53. (b,c)	54. (a,c)	55. (b,d)	56. (a,c,d)	57. (a, b, c, d)		58. (b)	59. (a, b, d)
60. (a,c)	61. (a,b,c)	62. (c)	63. (d)	64. (a)	65. (a)	66. (d)	67. (d)	68. (d)	69. (a)
70. (a)	71. (c)	72. (c)	73. (a)	74. (a)	75. (a)	76. (b)	77. (d)	78. (d)	79. (c)
80. (d)	81. (c)	82. (d)	83. (c)						

84. (A)–(Q,R), (B)–(R), (C)–(Q), (D)–(P) 85. (A)–(Q), (S), (B)–(P), (C)–(P), (D)–(R) 86. (A)–(P, Q), (B)–(R), (C)–(S), (D)–(T)

87. 6 88. 3:1 89. 0011 90. 4 91. 4 92. 0018

EXERCISE - 4 : PREVIOUS YEAR JEE ADVANCED QUESTIONS

1. (b)	2. (b)	3. (c)	4. (a)	5. (c)	6. (d)	7. (b)	8. (a)	9. (b)	10. (c)
11. (b)	12. (d)	13. (b)	14. (b)	15. (c)	16. (a)	17. (d)	18. (a)	19. (a)	20. (a)
21. (d)	22. (d)	23. (d)	24. (a)	25. (c)	26. (a)	27. (c)	28. (c)	29. (d)	30. (a)
31. (c)	32. (c)	33. (d)	34. (c)	35. (c)	36. (b, c)	37. (a,b,c,d)	38. (a, c)	39. (b, d)	40. (b, d)
41. (b, c)	42. (a, b)	43. (b,c,d)	44. (b, c)	45. (a,c)	46. (a,b)	47. (b,d)	48. (a,d)	49. 7	50. 2
51. 9	52. 1	53. 2	54. 9	55. 5	56. (8)	57. (4)	58. 9		

59. $x\in\left[-\frac{1}{2},0\right)\cup\left[\frac{1}{2},\infty\right), x\in\left(-\infty,-\frac{1}{2}\right]\cup\left(0,\frac{1}{2}\right]$ 60. $x>-1$ 61. $\frac{1}{3}$ 62. $H=\phi, V=\{1,1\}$.

63. $(A\to p; B\to r)$ 64. (a) 65. (a) 66. (c) 67. (a) 68. (b) 69. (b) 70. (c) 71. (d)

72. (c) 73. (d) 74. (b) 75. (c) 76. False 77. 2 81. $\frac{ud}{\sqrt{v^2-u^2}}$ 82. $x=0, y=0$

83. $\lambda\in\left(-\frac{3}{2},\frac{3}{2}\right)$-{0} 84. $x+2y=\frac{\pi}{2}$ and $x+2y=-\frac{3\pi}{2}$ 86. (0, 2) 87. $\frac{3\sqrt{3}}{4}r^2$ sq. unit 89. $6:6+\pi$ 90. $c=\frac{3}{4}$

91. $\sqrt{2}x-2y+2=0, \sqrt{2}x+2y-2=0$ 92. $y+x-1=0$ 93. 1 : 16 94. $b\in(-2,-1)\cup[1,\infty]$ 95. $\frac{4\sqrt{3}}{9}$

96. $a=-\frac{1}{2}, b=-\frac{3}{4}, c=3$ 97. 2h k 98. Maxima at $x=\frac{\left(b-\sqrt{b^2-1}\right)}{4}$ and minima at $x=\frac{1}{4}\left(b+\sqrt{b^2-1}\right)$

99. $\left[-\frac{2}{a},\frac{a}{3}\right]$ 100. $a=\frac{1}{4}, b=-\frac{5}{4}; f(x)=\frac{1}{4}x^2-\frac{5}{4}x+2$ 101. $\cos\left(\frac{1}{3}\cos^{-1}p\right)$ 102. 18 103. (2, 1)

104. 5 unit 108. $y-2=0$ 109. 6 111. a – b; b – d

Dream on !!