NOTES

M A T H E M A T I C S

QUADRATIC EQUATION

INTRODUCTION TO QUADRATIC EQUATIONS

What you already know

- Number theory
- Solving inequalities
- Wavy curve method

What you will learn

- Solving quadratic equation
- Relation between roots and coefficient
- Nature of roots

General expression of a polynomial of degree 'n'

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + a_4x^{n-4} + ... + a_n$ where $a_0, a_1, a_2 ... \in R$ and $n \in W$ $a_0 \neq 0$

For a real polynomial,

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + a_4x^{n-4} + ... + a_n$

Coefficients should be real numbers and powers of x should be whole numbers.

Quick Query 1

Identify which of the following is/are a polynomial.

(a) $\sqrt{5}x^3 + 2x - 7$

(b) $3x^2 - 6x^{-1} + 5$

(c) $\frac{2}{7}x^2 - 6\sqrt{3}x + 7$

(d) $3x^3 + \frac{2}{x^{\frac{2}{3}}} + 4x$

General expression of a quadratic polynomial

A polynomial of degree 2 is known as a quadratic polynomial For example:
$ax^2 + bx + c$, a, b, c $\in$ R, and $a \neq 0$

For example:

$x^2 - 2x + 5 = 1x^2 + (-2)x + 5$ (a = 1, b = −2, c = 5), i.e. $ax^2 + bx + c$

→ Degree = 2
→ a, b, c $\in$ R
→ $a = 1 \neq 0$

} **Quadratic polynomial**

General form of a quadratic equation: $ax^2 + bx + c = 0$, a, b, c $\in$ R, and a $\neq$ 0

For example:

$2x^2 - 7x + 10 = 0$ $\qquad$ $-x^2 + 5x + 6 = 0$

$2x^2 + 1 = 2x$ $\qquad$ $3x^2 - 4x + 4 = (x - 1)^2$

Methods to find roots of a Quadratic Equation

$ax^2 + bx + c = 0$, a,b,c $\in$ R, and a $\neq$ 0

- Completing the square method
- Quadratic formula
- Factorisation method

Solve $4x^2 + 4\sqrt{3}x + 3 = 0$

Step 1:

We have $4x^2 + 4\sqrt{3}x + 3 = 0 \Rightarrow 4(x^2 + \sqrt{3}x + \frac{3}{4}) = 0 \Rightarrow x^2 + \sqrt{3}x + \frac{3}{4} = 0$

Step 2:

$x^2 + \sqrt{3}x + \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{3}{4} = 0 \Rightarrow \left(x + \frac{\sqrt{3}}{2}\right)^2 - \frac{3}{4} + \frac{3}{4} = 0 \Rightarrow \left(x + \frac{\sqrt{3}}{2}\right)^2 = 0 \Rightarrow x = -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}$

Concept Check 1

Find the roots of the equation: $x^2 + 2x - 11 = 0$

Relation between Roots and Coefficients

$ax^2 + bx + c$: a, b, c $\in$ R, and a $\neq$ 0

Let p and q be its roots.

S = Sum of the roots = $p + q = -\frac{b}{a}$

P = Product of the roots = $pq = \frac{c}{a}$

D = Difference of the roots = $|p - q| = \frac{\sqrt{b^2 - 4ac}}{|a|} = \frac{\sqrt{D}}{|a|}$

Proof

Consider,

$(p + q)^2 = p^2 + q^2 + 2pq$

$\Rightarrow \left(-\frac{b}{a}\right)^2 = p^2 + q^2 + 2\frac{c}{a} \Rightarrow \frac{b^2}{a^2} - 2\frac{c}{a} = p^2 + q^2$

$\Rightarrow \frac{b^2 - 2ac}{a^2} = p^2 + q^2 \longrightarrow (1)$

Now,

$(p - q)^2 = p^2 + q^2 - 2pq$

$= \frac{b^2 - 2ac}{a^2} - 2\frac{c}{a}$

$= \frac{b^2 - 2ac - 2ac}{a^2}$

$\Rightarrow (p - q)^2 = \frac{b^2 - 4ac}{a^2}$

$\Rightarrow |p - q| = \frac{\sqrt{b^2 - 4ac}}{|a|} = \frac{\sqrt{D}}{|a|}$

Hence proved.

Find the sum, product, and the difference of the roots of the equation $x^2 + 6x + 9 = 0$.

$x^2 + 6x + 9 = 0 \Rightarrow x^2 + 3x + 3x + 9 = 0$

$(x + 3)(x + 3) = 0 \Rightarrow (x + 3)^2 = 0$

$x = -3, -3$ are the roots of the equation.

Sum of the roots $= -3 + (-3) = -6$

Product of the roots $= (-3)(-3) = 9$

Difference of the roots $= -3 - (-3) = 0$

Here $a = 1$, $b = 6$, $c = 9$

$D = b^2 - 4ac = 36 - 4 \times 1 \times 9 = 36 - 36 = 0$

Sum of the roots $= \frac{-b}{a} = \frac{-6}{1} = -6$

Product of the roots $= \frac{c}{a} = \frac{9}{1} = 9$

Difference of the roots $= \frac{\sqrt{D}}{|a|} = \frac{0}{1} = 0$

The quadratic equation whose roots are p, q is given by $x^2 - (p + q)x + pq = 0$.
If S = sum of roots = p + q and P = product of roots = pq, then we write the equation as $x^2 - Sx + P = 0$.

Quick Query 2

Let -2 and $\frac{6}{5}$ be the roots of a quadratic equation. Find the quadratic equation.

Find the sum of all the real roots of x satisfying the equation $3^{(x - 1)(x^2 + 5x - 50)} = 1$.

Step 1:

If $a^p = 1 \Rightarrow p = 0$

$(x - 1)(x^2 + 5x - 50) = 0 \Rightarrow (x - 1)(x^2 + 10x - 5x - 50) = 0 \Rightarrow (x - 1)(x + 10)(x - 5) = 0 \Rightarrow x = -10, 1, 5$

Step 2:

Sum of the real roots $= (-10 + 1 + 5) = -4$

Concept Check 2

The sum of all the real roots of x satisfying the equation

$(x^2 - 5x + 5)^{(x^2 + 4x - 60)} = 1$

(a) 6 (b) 5 (c) 3 (d) -4

Concept Check 3

JEE MAINS 2018

Let α, β be the roots of $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$,

then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is

(a) 3 (b) -3 (c) 6 (d) -6

Nature of Roots

$ax^2 + bx + c = 0$: a,b,c ∈ R, and a ≠ 0 — Two roots:

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

D = Discriminant = $\Delta = b^2 - 4ac$

- $\Delta = 0$ or $b^2 - 4ac = 0$ → Real and equal roots
- $\Delta > 0$ or $b^2 - 4ac > 0$ → Real but unequal roots
- $\Delta < 0$ or $b^2 - 4ac < 0$ → Non-real roots

If one root of the equation $x^2 + px + 12 = 0$ is 4, while the equation $x^2 + px + q = 0$ has equal roots, then q =

(a) 4 (b) 12 (c) 3 (d) $\frac{49}{4}$

Step 1:

Solving equation 1: $x^2 + px + 12 = 0$. Let α = 4 and β be the roots of the equation.

Sum of the roots = $\alpha + \beta = \frac{-b}{a} \Rightarrow 4 + \beta = -p$

Product of the roots = $\alpha \times \beta = \frac{c}{a} \Rightarrow 4 \times \beta = 12 \Rightarrow \beta = 3 \Rightarrow p = -7$

Step 2:

Solving equation 2: $x^2 + px + q = 0$. For equal roots, D = 0

$D = b^2 - 4ac = 0 \Rightarrow p^2 - 4q = 0 \Rightarrow (-7)^2 - 4q = 0 \Rightarrow q = \frac{49}{4}$

Concept check 4

If both roots of the equation $2x^2 + px + 8 = 0$ are equal, then p =

(a) 8 (b) 4 (c) -8 (d) -4

Concept Check 5

The number of integral values of k for which the equation $(k - 2)x^2 + 8x + k + 4 = 0$ has real roots is

(a) 9 (b) 10 (c) 11 (d) 12

Summary sheet

For quadratic equation, $ax^2 + bx + c = 0$, $a, b, c \in R$, and $a \neq 0$, having α and β as its roots:

Key takeaways

If $\Delta = b^2 - 4ac = 0$, then equation will have real and equal roots.

If $\Delta = b^2 - 4ac > 0$, then equation will have real and unequal roots.

If $\Delta = b^2 - 4ac < 0$, then equation will have non-real roots.

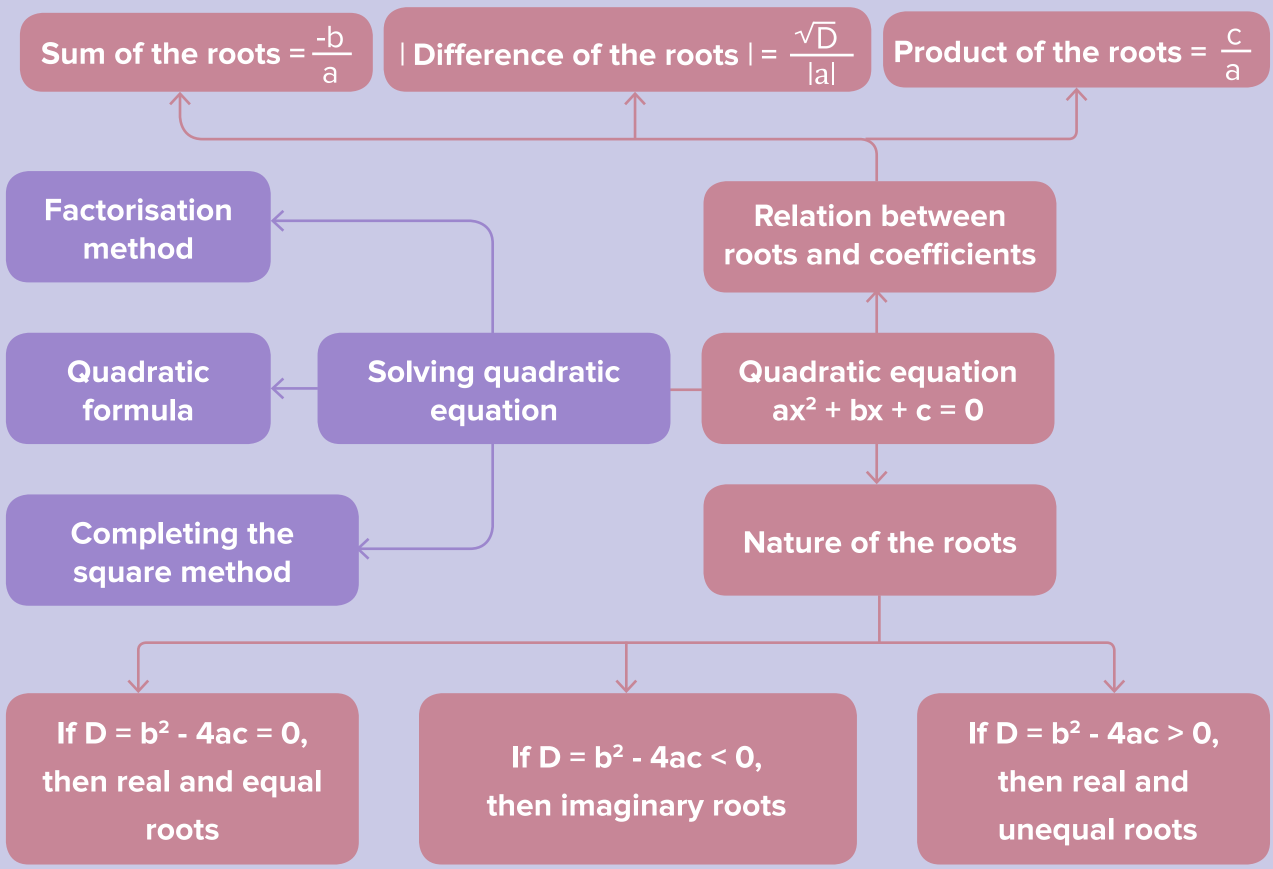
Key formulae

Sum of the roots = $\alpha + \beta = \frac{-b}{a}$

Product of the roots = $\alpha\beta = \frac{c}{a}$

Difference of the roots = $|\alpha - \beta| = \frac{\sqrt{D}}{|a|}$, where $D = b^2 - 4ac$

Mind map



Self-Assessment

1. Find the values of a for which one root of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as the other.
2. If the difference between roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then find the possible values of a.
3. If $f(x) = ax^2 + bx + c$ and $g(x) = -ax^2 + bx + c = 0$ and $ac \neq 0$, then prove that f(x)g(x) have at least two real roots.

Answers

Quick Query:

1.

(a) $\sqrt{5}x^3 + 2x - 7$: This is a polynomial expression as coefficients are real numbers and exponents are whole numbers.

(b) $3x^2 - 6x^{-1} + 5$: This is not a polynomial expression as exponent -1 is not a whole number.

(c) $\frac{2}{7}x^2 - 6\sqrt{3}x + 7$: This is a polynomial as coefficients are real numbers and exponents are whole numbers.

(d) $3x^3 + \frac{2}{x^{\frac{2}{3}}} + 4x$: This is not a polynomial expression as exponent is $\frac{-2}{3}$ which is not a whole number

---

2. Given, -2 and 6/5 are the roots of a quadratic equation.

Sum of the roots = $S = -2 + \frac{6}{5} = -\frac{4}{5}$

Product of the roots = $P = -2 \times \frac{6}{5} = -\frac{12}{5}$

The equation will be $x^2 - Sx + P = 0 \Rightarrow x^2 + \frac{4x}{5} - \frac{12}{5} = 0$

Concept Check 1

1. $x^2 + 2x - 11 = 0$

Given quadratic equation, $x^2 + 2x - 11 = 0$

Here a = 1, b = 2, and c = -11

Then, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm 4\sqrt{3}}{2} = -1 \pm 2\sqrt{3}$

Concept Check 2

Step 1:

Solving case 1: $a^{\text{exponent}} = 1$, when a = 1

$x^2 - 5x + 5 = 1 \Rightarrow x^2 - 5x + 4 = 0 \Rightarrow x^2 - 4x - x + 4 = 0 \Rightarrow (x - 4)(x - 1) = 0 \Rightarrow x = 1, 4$

Step 2:

Solving case 2: $a^{\text{exponent}} = 1, \Rightarrow$ exponent = 0

$x^2 + 4x - 60 = 0 \Rightarrow x^2 + 10x - 6x - 60 = 0 \Rightarrow (x + 10)(x - 6) = 0 \Rightarrow x = -10, 6$

Step 3:

Solving case 3: $a^{\text{exponent}} = 1$, when a = -1 and exponent is even

$x^2 - 5x + 5 = -1 \Rightarrow x^2 - 5x + 6 = 0 \Rightarrow x^2 - 3x - 2x + 6 = 0 \Rightarrow (x - 3)(x - 2) = 0 \Rightarrow x = 2, 3$

Checking exponent = $x^2 + 4x - 60$ for x = 2, 3

$x = 2 \Rightarrow x^2 + 4x - 60 = 2^2 + 4 \cdot 2 - 60 = -48$, which is even

$x = 3 \Rightarrow x^2 + 4x - 60 = 3^2 + 4 \cdot 3 - 60 = -39$, which is odd, $\therefore$ x = 3 is rejected.

Step 4:

All possible real roots are: □, □, □, −□□, □

Sum of all the roots = □ + □ + □ + (−□□) + □ = □

Concept Check 3

Solution

We have, $a_n = \alpha^n - \beta^n$ for $n \geq 1$

α and β be the roots of $x^2 - 6x - 2 = 0$

$\Rightarrow \alpha^2 - 6\alpha - 2 = 0 \Rightarrow \alpha^2 = 6\alpha + 2 \Rightarrow \alpha^{10} = 6\alpha^9 + 2\alpha^8 \longrightarrow$ (1)

$\Rightarrow \beta^2 - 6\beta - 2 = 0 \Rightarrow \beta^2 = 6\beta + 2 \Rightarrow \beta^{10} = 6\beta^9 + 2\beta^8 \longrightarrow$ (2)

Subtracting (2) from (1)

$\alpha^{10} - \beta^{10} = 6(\alpha^9 - \beta^9) + 2(\alpha^8 - \beta^8)$

$\Rightarrow a_{10} = 6a_9 + 2a_8$

$\Rightarrow a_{10} - 2a_8 = 6a_9 \Rightarrow \frac{a_{10} - 2a_8}{2a_9} = 3$

Concept Check 4

Given, the roots of the equation $2x^2 + px + 8 = 0$ are equal.

Roots are equal if $D = 0 \Rightarrow b^2 - 4ac = 0 \Rightarrow p^2 - 4 \times 2 \times 8 = 0 \Rightarrow p^2 = 64 \Rightarrow p = -8, 8$.

Therefore (a) and (c) are the correct answers.

Concept Check 5

Step 1:

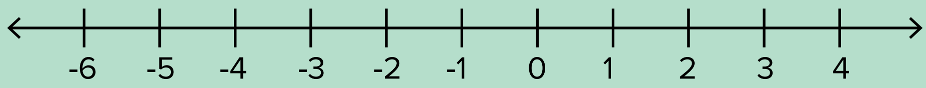
For real roots,

$D \geq 0 \Rightarrow b^2 - 4ac \geq 0 \Rightarrow 8^2 - 4(k - 2)(k + 4) \geq 0 \Rightarrow 16 - (k - 2)(k + 4) \geq 0$

$\Rightarrow 16 - k^2 - 2k + 8 \geq 0 \Rightarrow -k^2 - 2k + 24 \geq 0 \Rightarrow k^2 + 6k - 4k - 24 \leq 0 \Rightarrow (k + 6)(k - 4) \leq 0$

Step 2:

⇒ k ∈ [−□,□]



Therefore, for 11 integral values of k, the equation will have real roots.

Self-Assessment

1. Let α and 2α be the two roots of $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$

Sum of the roots $= \alpha + 2\alpha = 3\alpha = \dfrac{-b}{a} = \dfrac{1 - 3a}{a^2 - 5a + 3}$

Product of the roots $= \alpha \times 2\alpha = 2\alpha^2 = \dfrac{c}{a} = \dfrac{2}{a^2 - 5a + 3}$

$\Rightarrow 2\left[\dfrac{1}{9}\dfrac{(1 - 3a)^2}{(a^2 - 5a + 3)^2}\right] = \dfrac{2}{a^2 - 5a + 3}$

$\Rightarrow \dfrac{(1 - 3a)^2}{a^2 - 5a + 3} = 9 \Rightarrow 9a^2 - 6a + 1 = 9a^2 - 45a + 27$

$\Rightarrow 39a = 26 \Rightarrow a = \dfrac{2}{3}$

2. If the difference between roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then find the possible values of a.

   Let α, β be the roots of $x^2 + ax + 1 = 0$

   $\Rightarrow |\alpha - \beta| < \sqrt{5} \Rightarrow a^2 - 4 < 5$ $\qquad |\alpha - \beta| = \dfrac{\sqrt{D}}{|a|} = \dfrac{\sqrt{a^2 - 4}}{1}$

   $\Rightarrow a^2 < 9 \Rightarrow |a| < 3 \Rightarrow a \in (-3, 3)$

3. Let D_1 and D_2 be the discriminants of $ax^2 + bx + c = 0$ and $-ax^2 + bx + c = 0$ respectively, then

   $D_1 = b^2 - 4ac$ and $D_2 = b^2 + 4ac$

   Now $ac \neq 0$, either $ac > 0$ or $ac < 0$

   If $ac > 0$, then $D_2 > 0$ and $g(x) = -ax^2 + bx + c = 0$ have real and unequal roots.

   If $ac < 0$, then $D_1 > 0$ and $f(x) = ax^2 + bx + c = 0$ have real and unequal roots.

   Thus f(x)g(x) have at least two roots.

QUADRATIC EQUATIONS

NATURE OF ROOTS AND COMMON ROOTS OF QUADRATIC EQUATIONS

NOTES

What you already know

- General expression of quadratic equations
- Solving quadratic equations

What you will learn

- Nature of the roots
- Common roots
- Roots under particular conditions

Nature of the Roots

If we have a quadratic equation with roots α and β, $ax^2 + bx + c = 0$; $a, b, c \in R$ and $a \neq 0$

$$\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

D = Discriminant = $\Delta = b^2 - 4ac$

Case I :

If $a, b, c \in R$, and $\Delta = 0$

$ax^2 + bx + c = 0$

↓

Roots are real and equal

↓

$$\alpha, \beta = \frac{-b}{2a}$$

Case II :

If $a, b, c \in R$, and $\Delta < 0$

$ax^2 + bx + c = 0$

↓

Roots are non-real and unequal

↓

α, β occurs in conjugate pairs

↓

$\alpha = p + iq$ and $\beta = p - iq$ $\quad i = \sqrt{-1}$

Case III :

If $a, b, c \in R$, and $\Delta > 0$

$ax^2 + bx + c = 0$

↓

⇒ $a, b, c \in Q$ and Δ is a perfect square of a rational number.

↓

⇒ Roots are rational and unequal.

Example:

For $3x^2 + 5x - 2 = 0$

$a = 3$, $b = 5$, $c = -2$

$\Delta = b^2 - 4ac = 5^2 - 4(3)(-2) = 25 + 24 = 49$

$\Delta > 0$ and Δ is a perfect square of a rational number.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \frac{-5 \pm \sqrt{49}}{6} = -2, \frac{1}{3}$$

Roots are rational and unequal.

Case IV :

If $a, b, c \in R$, and $\Delta > 0$
$ax^2 + bx + c = 0$

↓

$a, b, c \in Q$, and Δ is not a perfect square of a rational number.

↓

Roots are irrational and unequal which always exists as conjugate pairs i.e., if $\boldsymbol{p} + \sqrt{\boldsymbol{q}}$ is one root, then $\boldsymbol{p} - \sqrt{\boldsymbol{q}}$ is the other root.

Example:

For $x^2 - 4x + 1 = 0$

$a = 1, b = -4, c = 1$

$\Delta = b^2 - 4ac = -4^2 - 4(1)(1) = 16 - 4 = 12$

$\Delta > 0$ and Δ is not a perfect square of a rational number.

$\therefore x^2 - 4x + 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

Roots are irrational and unequal.

Case V :

If $a, b, c \in R$, and $\Delta > 0$
$ax^2 + bx + c = 0$

↓

$a = 1$, $b, c \in Z$, and Δ is a perfect square of an integer.

↓

Roots are distinct integers.

Proof:

We have, $x^2 + bx + c = 0$ $(a = 1)$

$\Delta = b^2 - 4ac > 0$ and Δ is a perfect square of an integer

$$\text{Now, } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$$

$$\text{Roots } \alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$$

$\Delta > 0$ and Δ is a perfect square of an integer.

Case I : When b is an even integer.
Then, $-b$, $b^2 - 4ac$ are even integers.

$$\text{So, } \frac{-b \pm \sqrt{b^2 - 4ac}}{2} \text{ will be integers.}$$

Roots α & β are distinct integers.

Case II : When b is an odd integer.
Then, $-b$, b^2 are odd integers and $4c$ is an even integer.

$$\text{So, } \frac{-b \pm \sqrt{b^2 - 4ac}}{2} \text{ are integers.}$$

Roots α & β are distinct integers.

If the two roots of $(p - 1)(x^2 + x + 1) + (p + 1)(x^2 + x + 1)^2 = 0$ are real and distinct, then what is the set of all values of 'p'?

(a) $(-1, 1)$ (b) $\left(\frac{-1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$ (c) $\left(-1, \frac{-1}{7}\right)$ (d) $(-\infty, -2) \cup (2, \infty)$

Step 1:

Simplify $(p - 1)(x^2 + x + 1) + (p + 1)(x^2 + x + 1)^2 = 0 \Rightarrow (x^2 + x + 1)((p - 1) + (p + 1)(x^2 + x + 1)) = 0$

But $(x^2 + x + 1) \neq 0$, as $\Delta = 1^2 - 4.\,1.\,1 = -3 < 0 \Rightarrow (p - 1) + (p + 1)(x^2 + x + 1) = 0$ have real and distinct roots

Step 2:

$(p - 1) + (p + 1)(x^2 + x + 1) = 0 \Rightarrow (p + 1)x^2 + (p + 1)x + 2p = 0 \Rightarrow a = p + 1,\ b = p + 1$ and $c = 2p$

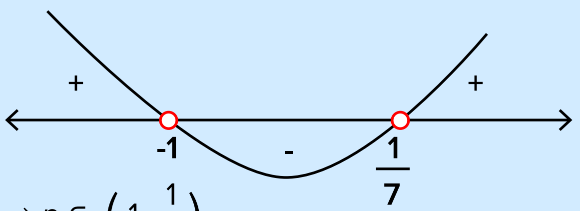
For real and distinct roots $\Delta = b^2 - 4ac > 0 \Rightarrow (p + 1)^2 - 4(p + 1)(2p) > 0 \Rightarrow (p + 1)(p + 1 - 8p) > 0$

$(p + 1)(1 - 7p) > 0 \Rightarrow (p + 1)7\left(p - \frac{1}{7}\right) < 0$

Step 3:

Using wavy curve method

$(p + 1)(1 - 7p) > 0 \Rightarrow (p + 1)7\left(p - \frac{1}{7}\right) < 0$



$\Rightarrow p \in \left(-1, \frac{1}{7}\right)$

Concept Check 1

If p and $q (p \neq q)$ are the roots of the equation $x^2 + px + q = 0$, then

(a) $p = 1, q = -2$

(b) $p = 0, q = 1$

(c) $p = -2, q = 0$

(d) $p = -2, q = 1$

Common Roots

Consider two quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$

Case 1:

Only one common root, say α

α will satisfy both equations

$\Rightarrow a_1\alpha^2 + b_1\alpha + c_1 = 0$ and

$a_2\alpha^2 + b_2\alpha + c_2 = 0$

By cross multiplication

$$\underbrace{\frac{\alpha^2}{b_1c_2 - b_2c_1}}_{①} = \underbrace{\frac{\alpha}{c_1a_2 - c_2a_1}}_{②} = \underbrace{\frac{1}{a_1b_2 - a_2b_1}}_{③}$$

$$① = ② \Rightarrow \alpha = \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$$

$$② = ③ \Rightarrow \alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

$$\Rightarrow \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1} = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

$\Rightarrow (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1) = (c_1a_2 - c_2a_1)^2$

This is the required condition for one root of two quadratic equations to be common.

If one of the roots of the equations $x^2 + 2x + 3k = 0$ and $2x^2 + 3x + 5k = 0$ is common, then what is the value of k?

(a) 0 (b) -2 (c) 2 (d) -1

Solution : $1x^2 + 2x + 3k = 0$ and $2x^2 + 3x + 5k = 0$

$a_1 = 1$, $b_1 = 2$, $c_1 = 3k$, $a_2 = 2$, $b_2 = 3$, $c_2 = 5k$

Let α be the common root

$$\Rightarrow \alpha = \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1} = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \qquad \Rightarrow \alpha = \frac{3k \times 2 - 5k \times 1}{1 \times 3 - 2 \times 2} \Rightarrow \alpha = -k$$

Also, $(c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$

$\Rightarrow (3k \times 2 - 5k \times 1)^2 = (2 \times 5k - 3 \times 3k)(1 \times 3 - 2 \times 2)$

$\Rightarrow k^2 = -k$

$\Rightarrow k^2 + k = 0$

$\Rightarrow k(k + 1) = 0$

$\therefore k = 0, k = -1$

Case 2:

Both roots are common, say α, β

Two quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$

Conditions :

Sum of the roots of the both the equation should be equal.

$$\alpha + \beta = \frac{-b_1}{a_1} \qquad \Rightarrow \frac{b_1}{a_1} = \frac{b_2}{a_2} \qquad \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} \qquad \alpha + \beta = \frac{-b_2}{a_2}$$

Product of the roots of the both the equations should be equal.

$$\alpha \times \beta = \frac{c_1}{a_1} \qquad \Rightarrow \frac{c_1}{a_1} = \frac{c_2}{a_2} \qquad \Rightarrow \frac{a_1}{a_1} = \frac{c_1}{c_2} \qquad \alpha \times \beta = \frac{c_2}{a_2}$$

So required condition is $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

If $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$ have a common root, then what is a : b : c ?

(a) 1 : 2 : 3 (b) 3 : 2 : 1 (c) 1 : 3 : 2 (d) 3 : 1 : 2

Step 1 :

Observe equation 1: $x^2 + 2x + 3 = 0$

$\Delta = b^2 - 4ac = 2^2 - 4(1)(3) = -8 < 0$

⇒ Equation 1 have imaginary roots.

Step 2 :

Recall, imaginary roots always occur in pairs, so equation 1: $x^2 + 2x + 3 = 0$ and equation 2: $ax^2 + bx + c = 0$ have both common roots.

$$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow \frac{1}{a} = \frac{2}{b} = \frac{3}{c}$$

$\Rightarrow a : b : c = 1 : 2 : 3$

Non-real roots occur in conjugate pairs. Therefore, quadratic equations with non-real common roots will always have both roots common.

Concept Check 2

If the equation $x^2 + bx - 1 = 0$, $b \neq 1$ and $x^2 + x - b = 0$ have a common root, then what is $|b|$?

(a) 0 (b) 3 (c) $\sqrt{3}$ (d) $\sqrt{2}$

Roots under particular conditions

$ax^2 + bx + c = 0$ where $a, b, c \in R$, $a \neq 0$

Case I : With constant term $c = 0$

$\Rightarrow ax^2 + bx + 0 = 0 \Rightarrow x(ax + b) = 0 \Rightarrow x = 0, \frac{-b}{a}$

$c = 0 \Rightarrow$ Roots are $0, \frac{-b}{a}$

Case II : With b = 0

$\Rightarrow ax^2 + 0x + c = 0 \Rightarrow ax^2 = -c \Rightarrow x = \pm\sqrt{\frac{-c}{a}}$

For real roots $\frac{c}{a} < 0$

$b = 0 \Rightarrow$ Roots are $\pm\sqrt{\frac{-c}{a}}$

Case III : With $a = c$

$\Rightarrow ax^2 + bx + a = 0$

Example:

$x^2 - 4x + 1 = 0$

Here, $a = c = 1$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$

Let $\alpha = 2 + \sqrt{3}$

$\frac{1}{\alpha} = \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3}$

If $a = c$, then the roots are reciprocal to each other i.e., roots are α & $\frac{1}{\alpha}$, where $\alpha \neq 0$

Case IV : With $b = c = 0$

$\Rightarrow ax^2 + 0x + 0 = 0 \Rightarrow ax^2 = 0 \Rightarrow x = 0, 0$

$a = c = \mathbf{0} \Rightarrow x = 0, 0$

Case V : a, b, c are of same sign

$x^2 + 5x + 6 = 0 \Rightarrow (x + 2)(x + 3) \Rightarrow x = -2, -3$

a, b, c are of same sign ⇒ Roots are negative.

Case VI : $a + b + c = 0$

$2x^2 - x - 1 = 0$

Here, $a = 2$, $b = -1$, $c = -1$; $a + b + c = 0$

$2x^2 - x - 1 = 0 = (2x + 1)(x - 1) = 0$; $x = \frac{-1}{2}, 1$

$a + b + c = 0 \Rightarrow$ Roots are $1, \frac{c}{a}$

Case VII : $a - b + c = 0$

$-3x^2 - x + 2 = 0$

$\Rightarrow (x + 1)(-3x + 2) = 0 \Rightarrow x = -1, \frac{2}{3}$

$a - b + c = 0 =$ Roots are $-1, \frac{-c}{a}$

Case VIII : $a{\cdot}c > 0$

$x^2 + 5x + 6 = 0 \Rightarrow (x + 2)(x + 3) = 0 \Rightarrow x = -2, -3$

Roots are of same sign.

Case IX : $a{\cdot}c < 0$

$-3x^2 - x + 2 = 0 \Rightarrow (x + 1)(-3x + 2) = 0 \Rightarrow x = -1, \frac{2}{3}$

Roots will be of opposite sign.

Concept Check 3

The quadratic equations $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ have one root in common. The other roots are integers in the ratio 4 : 3. Then the common root is _____.

Summary Sheet

Key Takeaways

For quadratic equation, $ax^2 + bx + c = 0$, $a, b, c \in$ R and $a \neq 0$

1. If $\Delta = 0$, roots α, β are real & equal.
2. If $\Delta < 0$, roots α, β are non-real & unequal. α & β are conjugates of each other.
3. If $\Delta > 0$ & Δ is a perfect square of a rational number, roots are rational & unequal.
4. If $\Delta > 0$ & $a, b, c \in Q$ & Δ is not a perfect square of a rational number, roots are irrational and conjugates.
5. If $a = 1$, $b, c \in Z$ & Δ is perfect square of an integer, roots are distinct Integers.

For quadratic equation, $ax^2 + bx + c = 0$, $a, b, c \in$ R and $a \neq 0$

1. For constant term $c = 0$, roots are $0, \frac{-b}{a}$.
2. For $b = 0$, roots are $\pm\sqrt{\frac{-c}{a}}$
3. For $a = c$, roots are α and $\frac{1}{\alpha}$
4. For $b = c = 0$, roots are 0, 0
5. If a, b, c are of the same sign, roots are negative.
6. If $a + b + c = 0$, roots are $1, \frac{c}{a}$
7. If $a - b + c = 0$, roots are $-1, \frac{-c}{a}$
8. If $a.c > 0$, roots are of the same sign
9. If $a.c < 0$ roots will be of opposite sign

Key Results

Two quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$

For one common root

$$\Rightarrow \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1} = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

$$\Rightarrow (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1) = (c_1a_2 - c_2a_1)^2$$

For both common roots

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Mind Map

Quadratic equation $ax^2 + bx + c = 0$

- **$\Delta = b^2 - 4ac$**
  - If $\Delta = 0 \Rightarrow$ Roots are real and equal
  - If $\Delta < 0 \Rightarrow$ Roots are non-real and conjugate
  - If $\Delta > 0$ and Δ is a perfect square of a rational number. Roots are rational and unequal.
  - If $\Delta > 0$ and Δ is not a perfect square. Roots are irrational & conjugate.
  - $a = 1$, $b, c \in Z$ and Δ is a perfect square of an integer. Roots are distinct integers.
- **Common Root**
  - **One common root**: $(b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1) = (c_1a_2 - c_2a_1)^2$
  - **For both common roots**: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
- For constant term $c = 0$, roots are $0, \frac{-b}{a}$
- For $b = 0$, roots are $\pm\sqrt{\frac{-c}{a}}$
- For $a = c$, roots are α and $\frac{1}{\alpha}$
- For $b = c = 0$, roots are 0, 0.
- If a, b, c are of the same sign, roots are negative.
- If $a + b + c = 0$, roots are $1, \frac{c}{a}$
- If $a - b + c = 0$, roots are $-1, \frac{-c}{a}$
- If $a.c < 0$ roots will be of opposite sign
- If $a.c > 0$, roots are of the same sign

Self-assessment

1. If a, b, $c \in R$ and $2b = a + c$, then check nature of roots of equation $ax^2 + 2bx + c = 0$.
2. If $ax^2 + bx + a = 0$ and $x^3 - 2x^2 + 2x - 1 = 0$ have two roots common, then find the value of $a + b$.
3. If the equation $4x^2 - x - 1 = 0$ and $3x^2 + (\lambda + \mu)x + \lambda - \mu = 0$ have a common root, then what are the rational values of λ and μ?

Answers

Concept Check 1 : If p and q are the roots of the equation $x^2 + px + q = 0$, then

Step 1: $x^2 + px + q = 0 \Rightarrow$ Sum of the roots $= \frac{-b}{a} = -p$; Product of the roots $= \frac{c}{a} = q$
Given, p and q are roots $\Rightarrow p + q = -p \Rightarrow q = -2p$ and $pq = q \Rightarrow p(-2p) = -2p$
$\Rightarrow -p^2 = -p \Rightarrow p = 0, 1$

Step 2: Given $p \neq q$, rejecting $p = 0$ since for $p = 0$, $q = 0$
Therefore, $p = 1$ and $q = -2p = -2$
Hence, $(p, q) = (1, -2)$

Concept Check 2: If the equations $x^2 + bx - 1 = 0$, $b \neq 1$ and $x^2 + x - b = 0$ have a common root, then $|b|$ is

Step 1: Let α be the common root of the equation 1: $x^2 + bx - 1 = 0$ and equation 2: $x^2 + x - b = 0$
$\Rightarrow \alpha^2 + b\alpha - 1 = 0$ ...1 and $\alpha^2 + \alpha - b = 0$ ...2

Step 2: Subtracting equation 1 from equation 2
$\alpha^2 + b\alpha - 1 - (\alpha^2 + \alpha - b) = 0 \Rightarrow \alpha(b - 1) - 1 + b = 0 \Rightarrow \alpha = -1$ and $b = 0$
Therefore, $|b| = 0$

Concept Check 3 :

Step 1: Let their common root be α. Other roots are in ratio 4 : 3 $\Rightarrow$ Other two roots are 4β, 3β
For equation 1: $x^2 - 6x + a = 0$, Product of the roots $= \alpha.4\beta = a$ 1
Similarly for equation 2: $x^2 - cx + 6 = 0$, Product of the roots $= \alpha.3\beta = 6$ 2

Step 2: From 1 and 2, $a = 8 \Rightarrow x^2 - 6x + a = 0 \Rightarrow x^2 - 6x + 8 = 0 \Rightarrow (x - 2)(x - 4) \Rightarrow x = 2, 4$

Step 3: For $x = 2, 4$; $\alpha = 2, 4$ respectively
But for $\alpha = 4 \Rightarrow \beta = \frac{1}{2}$ other roots 4β, 3β are 2, $\frac{3}{2}$ resulting in non-integral roots.
Therefore, $\alpha \neq 4$ and $\alpha = 2$. Hence, the common root is 2

Self-assessment :

1. Given equation $ax^2 + 2bx + c = 0$ and $2b = a + c$. Hence,

   $D = 4b^2 - 4ac = (a + c)^2 - 4ac = (a - c)^2 \geq 0$

   Thus, the roots are real.

2. By observation, $x = 1$ is the root of the equation . Thus we have,

   $(x - 1)(x^2 - x + 1) = 0$

   Now roots of $x^2 - x + 1 = 0$ are non-real

   Then equation $ax^2 + bx + a = 0$ has both roots common with $x^2 - x + 1 = 0$

   Hence we have,

   $\frac{a}{1} = \frac{b}{-1} = \frac{a}{1} \Rightarrow a + b = 0$

3. Roots of $4x^2 - x - 1 = 0$ are irrational. So, one root common implies both roots are common.

   Therefore,

   $\frac{4}{3} = \frac{-1}{\lambda + \mu} = \frac{-1}{\lambda - \mu} \Rightarrow \lambda = \frac{-3}{4}$ and $\mu = 0$

MATHEMATICS

QUADRATIC EQUATION

GRAPHS OF QUADRATIC POLYNOMIAL

What you already know

- Solving quadratic equations
- Nature of roots

What you will learn

- Transforming quadratic equations
- Visualisation of a quadratic equation

Transformation of a Quadratic Equation

Changing the sign of the roots of f(x) = 0

Given: $f(x) = ax^2 + bx + c = 0$ with roots α, β

Required: Transformed quadratic equation with roots -α, -β

⇒ S = Sum of the roots $= -\alpha - \beta = -(\alpha + \beta) = \frac{b}{a}$

⇒ P = Product of the roots $= (-\alpha)(-\beta) = \frac{c}{a}$

Recall, a quadratic equation with S as the sum of the roots and P as the product of the roots is $x^2 - Sx + P$

⇒ Required transformed equation is $ax^2 - bx + c = 0$

$f(-x) = ax^2 - bx + c = 0$

Conclusion: Replace x with -x

Adding constants to the roots of f(x) = 0

Given: $f(x) = ax^2 + bx + c = 0$ with roots α, β

Required: Transformed quadratic equation with roots α + k, β + k, where k is some constant.

⇒ If x is the root of given equation, and y is the root of transformed equation, then $\mathbf{y = x + k}$

⇒ $x = y - k$

⇒ $f(y - k) = a(y - k)^2 + b(y - k) + c = 0$

Example: Given $f(x) = x^2 + 5x + 6 = (x + 2)(x + 3) = 0$ with roots -2, -3

Required transformed equation with roots α + 1, β + 1 i.e. -1, -2

Here S = Sum of the roots = -1 + (-2) = -3 and P = Product of roots = 2

⇒ $x^2 - Sx + P = x^2 + 3x + 2 = 0$

For $f(x) = x^2 + 5x + 6 = 0$

$f(x - 1) = (x - 1)^2 + 5(x - 1) + 6 = x^2 + 3x + 2 = 0$ = Required transformed equation

Conclusion: Replace x with x - k

Multiplying the roots of f(x) = 0 by a constant

Given: $f(x) = ax^2 + bx + c = 0$ with roots α and β

Required: Transformed quadratic equation with roots $k\alpha$ and $k\beta$, where k is some non zero constant

$\Rightarrow$ If x is the root of given equation, and y is root of transformed equation, then $\mathbf{y = kx}$

$\Rightarrow x = \frac{y}{k}$

$\Rightarrow f(\frac{y}{k}) = a(\frac{y}{k})^2 + b(\frac{y}{k}) + c = 0$

Conclusion: Replace x with $\frac{x}{k}$

Using linear combination of roots of f(x) = 0

Given: $f(x) = ax^2 + bx + c = 0$ with roots α and β

Required: Transformed quadratic equation with roots $p\alpha + q$ and $p\beta + q$

$\Rightarrow$ If x is the root of given equation, and y is root of transformed equation, then $y = px + q$

$\Rightarrow x = \frac{y - q}{p} \Rightarrow f\left(\frac{y-q}{p}\right) = a\left(\frac{y-q}{p}\right)^2 + b\left(\frac{y-q}{p}\right) + c = 0$

Example: Given $f(x) = x^2 - 9x + 20 = 0$ with roots $\alpha = 4$ and $\beta = 5$

Required transformed equation with root $2\alpha + 1 = 9$ and $2\beta + 1 = 11$

$f(\frac{x-1}{2}) = (\frac{x-1}{2})^2 - 9(\frac{x-1}{2}) + 20 = 0$

$\Rightarrow \frac{x^2 - 2x + 1}{4} - 9\frac{x-1}{2} + 20 = 0 \Rightarrow \frac{x^2 - 2x + 1 - 18x + 18 + 80}{4} = 0$

$\Rightarrow \frac{x^2 - 20x + 99}{4} = 0$

$\Rightarrow x^2 - 20x + 99 = 0$

Conclusion: Replace x with $\frac{x-q}{p}$, $p \neq 0$

$x^2 - Sx + P = 0$

$S = 9 + 11 = 20$; $P = (9)(11) = 99$

$\Rightarrow x^2 - 20x + 99 = 0$

Reciprocating the roots of f(x) = 0

Given: $f(x) = ax^2 + bx + c = 0$ with roots α and β

Required: Transformed quadratic equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}$

$\Rightarrow$ If x is the root of given equation, and y is root of transformed equation, then $\mathbf{y = \frac{1}{x}}$

$\Rightarrow x = \frac{1}{y} \Rightarrow f(\frac{1}{y}) = a(\frac{1}{y})^2 + b(\frac{1}{y}) + c = 0 \Rightarrow cy^2 + by + a = 0$

Conclusion: Replace x with $\frac{1}{x}$

Squaring the roots of f(x) = 0

Given: $f(x) = ax^2 + bx + c = 0$ with roots α, β

Required: Transformed quadratic equation with roots α^2, β^2

$\Rightarrow$ If x is the root of given equation, and y is root of transformed equation, then $\mathbf{y = x^2}$

$\Rightarrow x = \sqrt{y} \Rightarrow f(\sqrt{y}) = a(\sqrt{y})^2 + b(\sqrt{y}) + c = 0$

Conclusion: Replace x with $\sqrt{x}$

Exponentiating the roots of f(x) = 0 to n

Given: $f(x) = ax^2 + bx + c = 0$ with roots α, β

Required: Transformed quadratic equation with roots α^n, β^n

$\Rightarrow$ If x is the root of given equation, and y is root of transformed equation, then $\mathbf{y = x^n}$

$\Rightarrow x = y^{\frac{1}{n}} \Rightarrow f(y^{\frac{1}{n}}) = a(y^{\frac{1}{n}})^2 + b(y^{\frac{1}{n}}) + c = 0$

Conclusion: Replace x with $x^{\frac{1}{n}}$

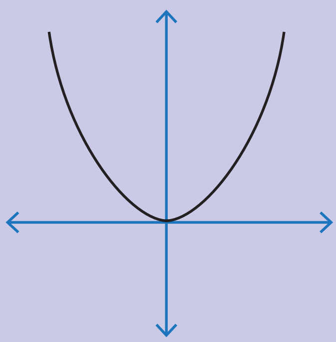
Concept Check: 1

If α, β are non-zero roots of $f(x) = px^2 + qx + r = 0$, then the quadratic equation with roots $\frac{1}{p\alpha + q}, \frac{1}{p\beta + q}$ is

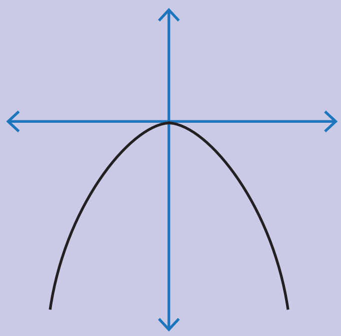
(a) $f(-rx) = 0$ (b) $f(\frac{p}{x - q})$ (c) $f(\frac{1}{px + q})$ (d) None of the these

Graph of a Quadratic Polynomial

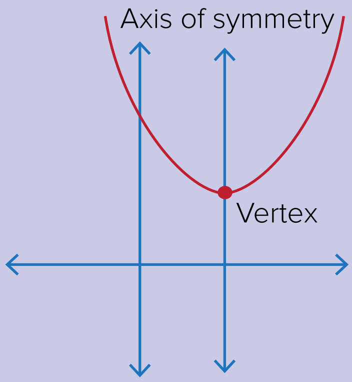
$y = f(x) = x^2$



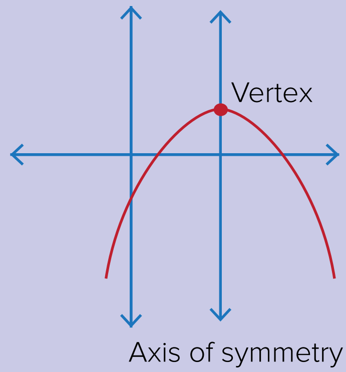
$y = f(x) = -x^2$



Graph of a quadratic expression is always a parabola



Upward-opening parabola



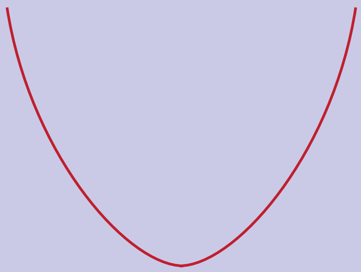
Downward-opening Parabola

How to plot the graph?

3 pieces of information are required to plot a graph of a quadratic expression

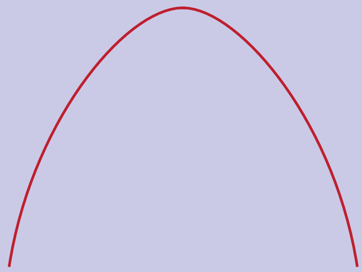
I. **Sign of the leading coefficient** ($a > 0$ upward opening, $a < 0$ downward opening)

Upward-opening parabola



$a > 0$

Downward-opening parabola



$a < 0$

II. **Vertex** = $\mathbf{\frac{-b}{2a}}$, where the min value of f(x) is achieved (upward-opening)

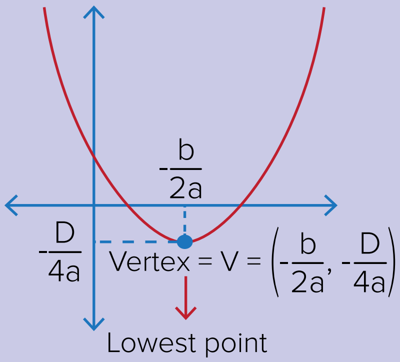
$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right) = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right]$$

$$y + \frac{D}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

When $a > 0$,

$$y + \frac{D}{4a} = a\left(x + \frac{b}{2a}\right)^2 \geq 0 \Rightarrow \text{Minimum} = 0$$

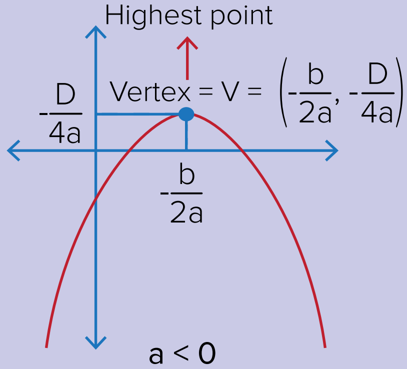
$$\left(x + \frac{b}{2a}\right)^2 = 0 \Rightarrow x = -\frac{b}{2a} \Rightarrow y = -\frac{D}{4a}$$



When $a < 0$,

$$y + \frac{D}{4a} = a\left(x + \frac{b}{2a}\right)^2 \leq 0 \Rightarrow \text{Maximum} = 0$$

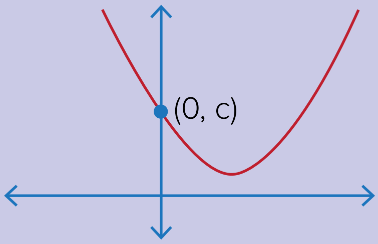
$$\left(x + \frac{b}{2a}\right)^2 = 0 \Rightarrow x = -\frac{b}{2a} \Rightarrow y = -\frac{D}{4a}$$

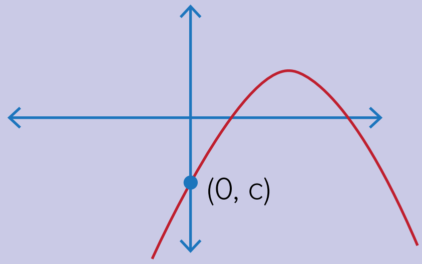


III. **Y-intercept**

Y-intercept is the y-coordinate of a point where a graph of a quadratic expression, i.e., the parabola, intersects the y-axis.

The coordinates of the point where parabola cuts the y–axis, i.e. (0, c).





Example: $y = x^2 - 3x + 2$

Here, $a = 1$, $b = -3$, $c = 2$ and $D = b^2 - 4ac = 1$

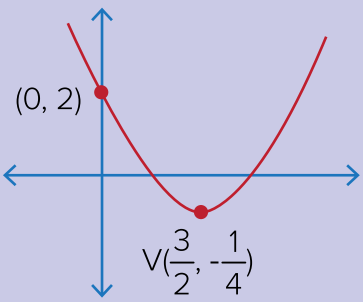
Three pieces of information

1. Leading coefficient $a = 1 > 0$

   $\Rightarrow$ Upward-opening parabola

2. Vertex = $V = \left(-\frac{b}{2a}, -\frac{D}{4a}\right) = \left(\frac{3}{2}, \frac{-1}{4}\right)$

3. Intercept (0, c) = (0, 2)



Example: $y = -x^2 + 2x - 1$

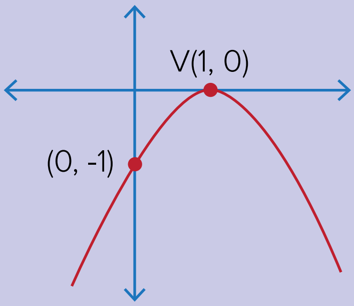
Here $a = -1$, $b = 2$, $c = -1$ and $D = b^2 - 4ac = 0$

Three pieces of information

1. $a = -1 < 0 \Rightarrow$ Downward-opening parabola

2. Vertex = $V = \left(-\frac{b}{2a}, -\frac{D}{4a}\right) = (1, 0)$

3. Intercept (0, c) = (0, -1)



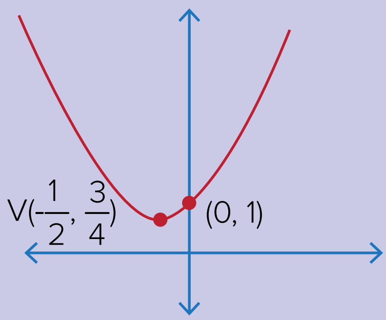
Example: $y = x^2 + x + 1$

Here, $a = 1$, $b = 1$, $c = 1$ and $D = b^2 - 4ac = -3$

1. $a = 1 > 0 \Rightarrow$ Upward-opening parabola

2. Vertex = $V = \left(-\frac{b}{2a}, -\frac{D}{4a}\right) = \left(\frac{-1}{2}, \frac{3}{4}\right)$

3. Intercept (0, c) = (0, 1)



$a > 0 \Rightarrow y = f(x)$ is an upward-opening parabola.

Case 1

The vertex of parabola = $V = \left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ is below x-axis.

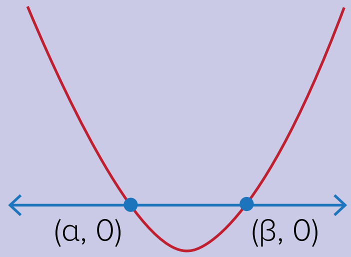
$\Rightarrow -\frac{D}{4a} < 0 \Rightarrow D > 0 \Rightarrow$ Real and unequal roots α, β

For $\alpha < \beta$

Observe $f(x) > 0 \ \forall \ x \in (-\infty, \alpha) \cup (\beta, \infty)$

$f(x) < 0 \ \forall \ x \in (\alpha, \beta)$

$f(x) = 0 \Rightarrow x \in \{\alpha, \beta\}$



Intersects the x-axis at two distinct points.

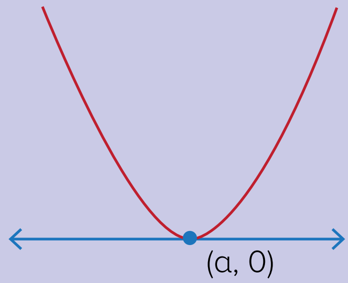
Case 2

The vertex of parabola = $V = \left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ touches x-axis.

$\Rightarrow -\frac{D}{4a} = 0 \Rightarrow D = 0 \Rightarrow$ Real and equal roots $\alpha = \beta$

Observe $f(x) > 0 \ \ \forall \ x \in R - \{\alpha\}$

$f(x) = 0 \Rightarrow x \in \{\alpha\}$



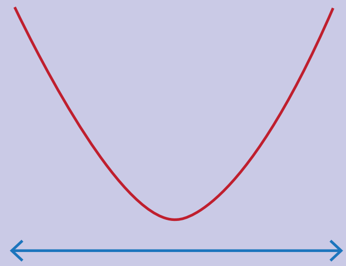
Touches the x-axis

Case 3

The vertex of parabola = $V = \left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ is above x-axis.

$\Rightarrow -\frac{D}{4a} > 0 \Rightarrow D < 0 \Rightarrow$ Non real roots

Observe $f(x) > 0 \ \ \forall \ x \in R$

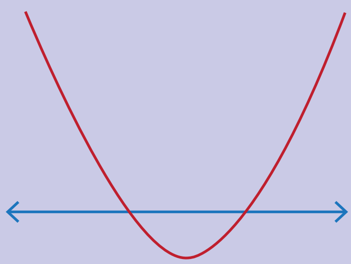


Does not intersect the x-axis.

To summarise

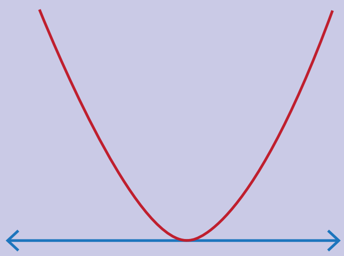
$a > 0$

$D > 0$



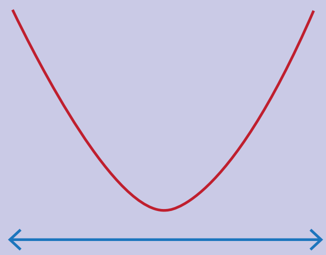
Intersects the x-axis at two distinct points.

$D = 0$



Touches the x-axis

$D < 0$



Does not intersect the x-axis.

$a < 0$ and for $a > 0 \Rightarrow y = f(x)$ is upward-opening parabola.

Case 1

The vertex of parabola = $V = \left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ is above x-axis.

$\Rightarrow -\frac{D}{4a} > 0$

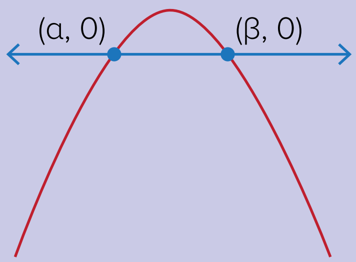
$\Rightarrow D > 0$ as $a < 0 \Rightarrow$ Real and unequal roots α, β

For $\alpha < \beta$

Observe $f(x) < 0 \ \forall \ x \in (-\infty, \alpha) \cup (\beta, \infty)$

$f(x) > 0 \ \forall \ x \in (\alpha, \beta)$

$f(x) = 0 \Rightarrow x \in \{\alpha, \beta\}$



Intersects the x-axis at two distinct points.

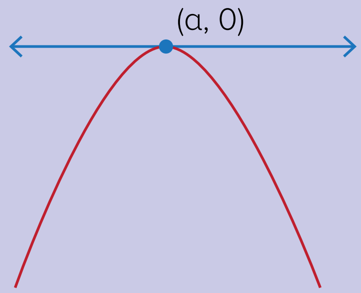
Case 2

The vertex of parabola = $V = \left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ touches x-axis.

$\Rightarrow -\frac{D}{4a} = 0 \Rightarrow D = 0 \Rightarrow$ Real and equal roots $\alpha = \beta$

Observe $f(x) < 0 \ \ \forall \ x \in R - \{\alpha\}$

$f(x) = 0 \Rightarrow x \in \{\alpha\}$



Touches the x-axis

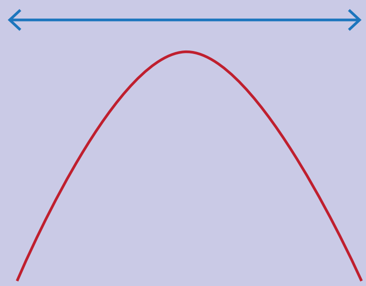
Case 3

The vertex of parabola = $V = \left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ is below x-axis.

$\Rightarrow -\frac{D}{4a} < 0 \Rightarrow D < 0$ as $a < 0 \Rightarrow$ Non real roots

No real roots $D < 0$

Observe $f(x) < 0 \ \ \forall \ x \in R$

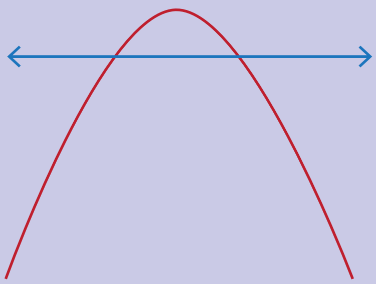


Does not intersect the x-axis.

To summarise

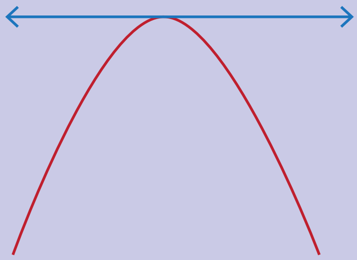
$a < 0$

$D > 0$



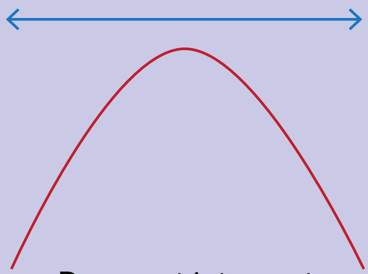
Intersects the x-axis at two distinct points.

$D = 0$



Touches the x-axis

$D < 0$



Does not intersect the x-axis.

Concept Check 2

If roots of $ax^2 + bx + 1 = 0$ are imaginary, then:

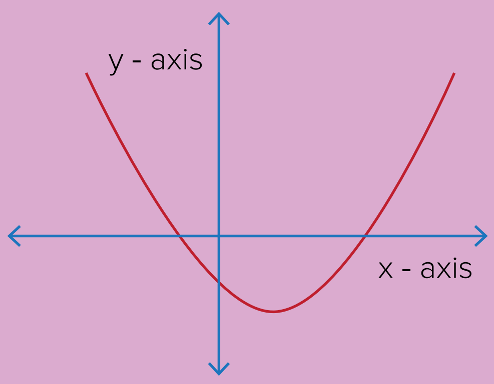
(a) $\frac{a}{9} + \frac{b}{3} + 1 < 0$ (b) $4a - 2b + 1 > 0$ (c) $\frac{a}{9} + \frac{b}{3} + 1 > 0$ (d) $4a - 2b + 1 < 0$

Concept Check 3

The graph of $ax^2 + bx + c$ is given. Then,

(a) $b^2 - 4ac > 0$ (b) $b < 0$

(c) $a > 0$ (d) None of these



Summary

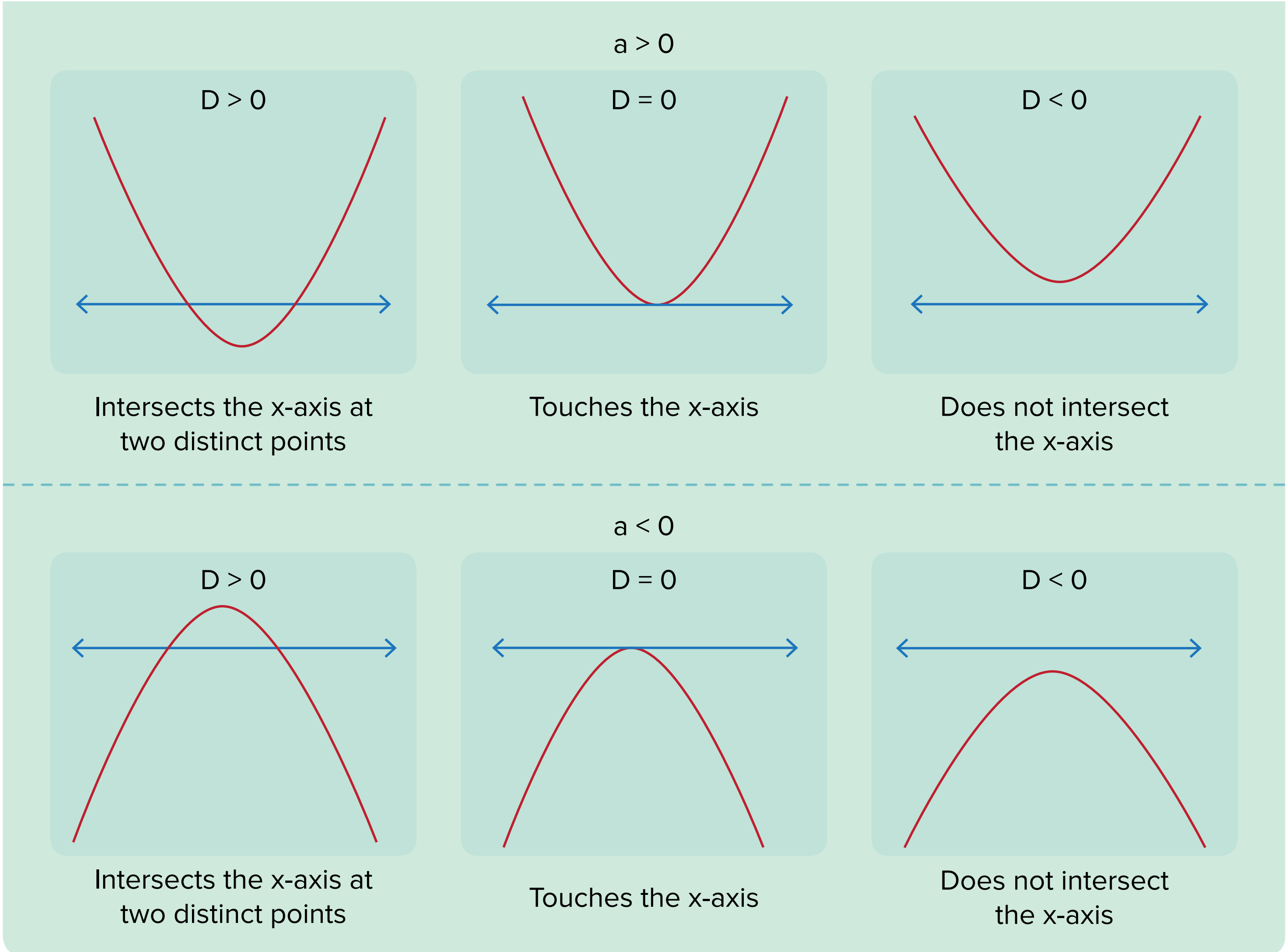
Key Takeaways

If $f(x) = ax^2 + bx + c = 0$ is a quadratic equation with roots α, β, then

1. $f(-x)$ is a transformed quadratic equation with roots $-\alpha, -\beta$
2. $f(x - k)$ is a transformed quadratic equation with roots $\alpha + k, \beta + k$
3. $f(\frac{x}{k})$ is a transformed quadratic equation with roots $k\alpha, k\beta$
4. $f(\frac{x - q}{p})$ is a transformed quadratic equation with roots $p\alpha + q$ and $p\beta + q$
5. $f(\frac{1}{x})$ is a transformed quadratic equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}$
6. $f(\sqrt{x})$ is a transformed quadratic equation with roots α^2, β^2
7. $f(x^{\frac{1}{n}})$ is a transformed quadratic equation with roots α^n, β^n

Key Graphs

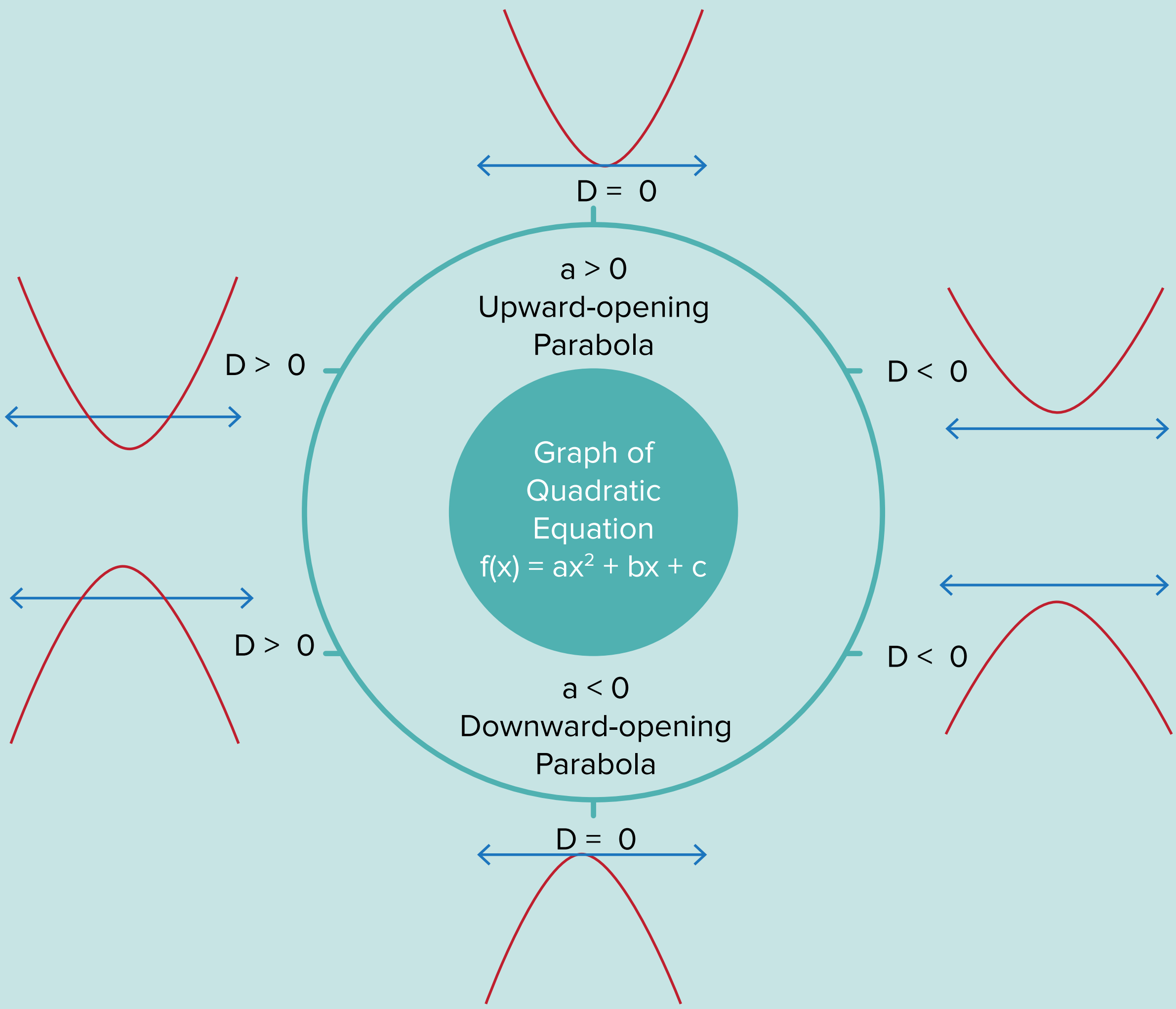
For parabola, graph of quadratic equation $f(x) = ax^2 + bx + c$



Mind map

$f(x) = ax^2 + bx + c = 0$
Roots α, β

- For roots $p\alpha + q$, $p\beta + q$
  Replace x with $\frac{x - q}{p}$
- For roots $-\alpha$, $-\beta$
  Replace x with $-x$
- For roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$
  Replace x with $\frac{1}{x}$
- For roots $\alpha + k$, $\beta + k$
  Replace x with $x - k$
- For roots α^2, β^2
  Replace x with $\sqrt{x}$
- For roots $k\alpha$, $k\beta$
  Replace x with $\frac{x}{k}$
- For roots α^n, β^n
  Replace x with $x^{\frac{1}{n}}$



Self-Assessment

1. If the roots of the equation $ax^2 + bx + c = 0$ are reciprocal of the roots of the equation $px^2 + qx + r = 0$, then prove $ap = rc$.
2. If α, β are roots of the equation $ax^2 + bx + c = 0$ then, find the quadratic equation whose roots are $\frac{1}{(a\alpha + b)^2}, \frac{1}{(a\beta + b)^2}$.
3. If the vertex of the curve $y = -2x^2 - 4px - k$ is $(-2, 7)$, then find the value of k.

Answers

Concept Check 1

If x is the root of the given equation $f(x) = px^2 + qx + r = 0 \Rightarrow x(px + q) = -r \Rightarrow px + q = \frac{-r}{x}$

Let y be the root of the transformed equation $\Rightarrow y = \frac{1}{px + q} = \frac{-x}{r} \Rightarrow x = -ry$

Substituting $x = -ry$ in f(x)

$f(-ry) = p(-ry)^2 + q(-ry) + r = 0 \Rightarrow pr^2y^2 - qry + r = 0$

Therefore, the transformed quadratic equation is $pr^2x^2 - qrx + r = 0 = f(-rx)$.

Concept Check 2

We have, roots of $f(x) = ax^2 + bx + 1 = 0$ are imaginary $\Rightarrow D = b^2 - 4ac = b^2 - 4a < 0$

Also, $f(0) = 1 > 0 \Rightarrow f(x) > 0 \ \forall \ x \in R$, as $f(x) = ax^2 + bx + 1$ is upward-opening parabola with $a > 1$

$f(-2) = 4a - 2b + 1 > 0$

$f(\frac{1}{3}) = \frac{a}{9} + \frac{b}{3} + 1 > 0$

Options (b) and (c) represent correct answers.

Concept Check 3

We can observe $a > 0$, as the graph is upward-opening

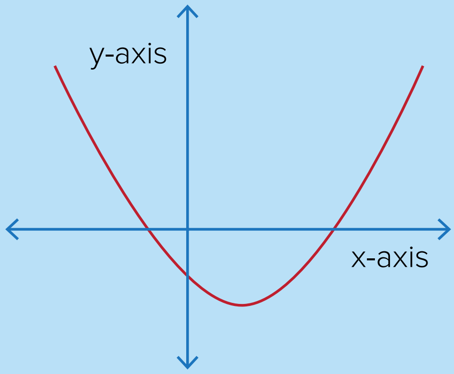
Clearly, $f(x) = 0$ has two distinct real roots $\Rightarrow D = b^2 - 4ac > 0$

Vertex = $V = (-\frac{b}{2a}, -\frac{D}{4a})$ is in fourth quadrant, i.e., $x > 0$ and $y < 0$

$\Rightarrow x = \frac{-b}{2a} > 0 \Rightarrow b < 0$

Intercept = $(0, c)$ is negative $\Rightarrow c < 0$

Hence, options (a), (b), (c) are the correct answers.



Self-Assessment

1. Let α, β be the roots of equation $f(x) = ax^2 + bx + c = 0$, then roots of the equation

   $g(x) = px^2 + qx + r = 0$, will be $\frac{1}{\alpha}, \frac{1}{\beta}$

   Observe

   $g(\frac{1}{x}) = p\frac{1}{x^2} + q\frac{1}{x} + r = 0$ will have roots $\alpha, \beta \Rightarrow rx^2 + qx + p = 0$ and $ax^2 + bx + c = 0$ have both roots common, i.e., α, β.

   $\Rightarrow \frac{a}{r} = \frac{b}{q} = \frac{c}{p} = k \Rightarrow ap = rc.$

   Hence proved.

2.

We know that

α, β are the roots of

$ax^2 + bx + c + 0$...(1)

Now,

$a\alpha^2 + b\alpha + c = 0$

$\Rightarrow (a\alpha + b) = -\frac{c}{\alpha}, (a\beta + b) = -\frac{c}{\beta}$

Therefore,

$\frac{1}{(a\alpha + b)^2} = \frac{\alpha^2}{c^2}$

$\frac{1}{(a\beta + b)^2} = \frac{\beta^2}{c^2}$

The quadratic equation whose roots are $\frac{\alpha^2}{c^2}, \frac{\beta^2}{c^2}$ is

Let $y = \frac{x^2}{c^2}$, where $x = \alpha, \beta$

$c^2 y = x^2$

$\Rightarrow x = c\sqrt{y}$

Putting this in the equation (1),

$a(c\sqrt{y})^2 + b(c\sqrt{y}) + c = 0$

$\Rightarrow ac^2y + bc\sqrt{y} + c = 0$

$\Rightarrow (ac^2y + c)^2 = (-bc\sqrt{y})^2$

$\Rightarrow a^2c^4y^2 + (2ac^3 - b^2c^2)y + c^2 = 0$

$\Rightarrow (a^2c^2)y^2 + (2ac - b^2)y + 1 = 0$

Hence, the required quadratic equation is

$(a^2c^2)x^2 + (2ac - b^2)x + 1 = 0.$

3.

Comparing $y = -2x^2 - 4px - k = 0$ with $y = ax^2 + bx + c = 0$

We get, a = -2, b = -4p, c = -k

So, the coordinates of the vertex is

$\left(\frac{-b}{2a}, \frac{-D}{4a}\right) = (-2, 7)$

$\Rightarrow \frac{-(-4p)}{-4} = -2 \; , \; \frac{-(16p^2 - 8k)}{-8} = 7$

$\Rightarrow p = 2$

$\Rightarrow \frac{-(16p^2 - 8k)}{-8} = 7 \Rightarrow 16 \times 4 - 8k = 56 \Rightarrow k = 1$

M A T H E M A T I C S

QUADRATIC EQUATIONS

RANGE AND LOCATION OF THE ROOTS

NOTES

What you already know

- Solving Quadratic Equation
- Plotting of graph

What you will learn

- Range of a quadratic equation
- Range under restricted domain
- Location of roots

Range of Quadratic Polynomial Function

$f(x) = ax^2 + bx + c,\ a \neq 0,\ a, b, c \in R$

We will find the range of quadratic polynomial functions with the help of observations from graphs. The values obtained from the graph on the y-axis, is the range of the function. A quadratic polynomial function can have two types of graphs:

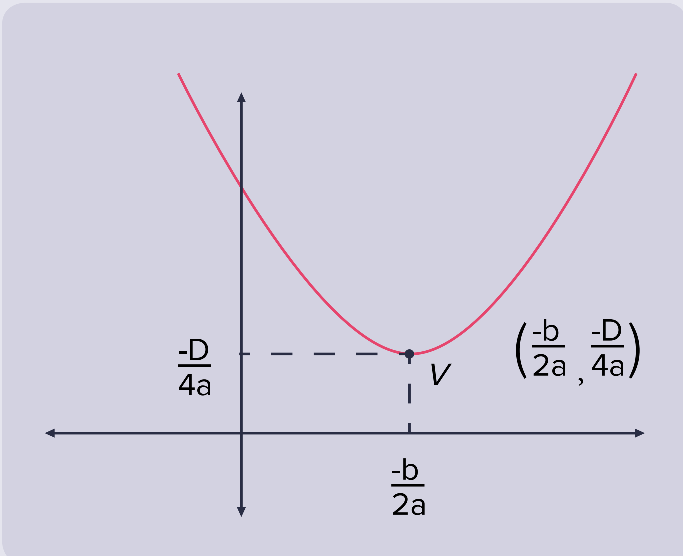
- Upward opening parabola, for $a > 0$
- Downward opening parabola, for $a < 0$

Case 1: a > 0

Observations from the graph:

- It's an upward opening parabola.
- As the domain of f(x): $x \in R$, the graph covers all values of $x \in R$.
- The graph sweeps all the values greater than or equal to vertex = y_{min}.

$\Rightarrow$ Range of $f(x) = [\ y_{min}, \infty\ )$



Vertex V of parabola = $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$

Therefore, for $a > 0$, the function attains its minimum value, $y_{min} = \frac{-D}{4a}$ at $x = \frac{-b}{2a}$

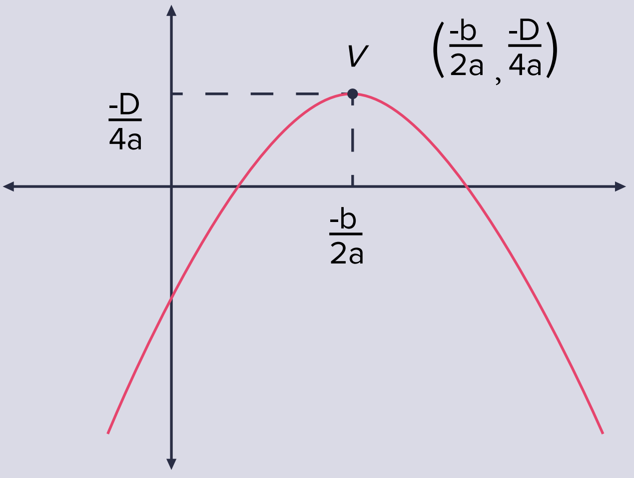
Case 2: a < 0

- It is a downward opening parabola.
- As the domain of f(x): $x \in R$, the graph covers all values of $x \in R$.
- The graph sweeps all the values less than or equal to vertex = y_{max}.

$\Rightarrow$ Range of $f(x) = (-\infty, y_{max}]$

Vertex V of parabola = $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$

Therefore, for a < 0, the function attains its maximum value $y_{max} = \frac{-D}{4a}$ at $x = \frac{-b}{2a}$



If $f(x) = x^2 - 2x - 3$, then find the range of f(x), for the following range of values of x:

(a) $x \in R$ **(b) $x \in [0, 3]$** **(c) $x \in [-2, 0]$**

Solution :

Given, $f(x) = x^2 - 2x - 3$
Here a = 1, b = -2, c = -3
$D = b^2 - 4ac = 4 + 12 = 16$

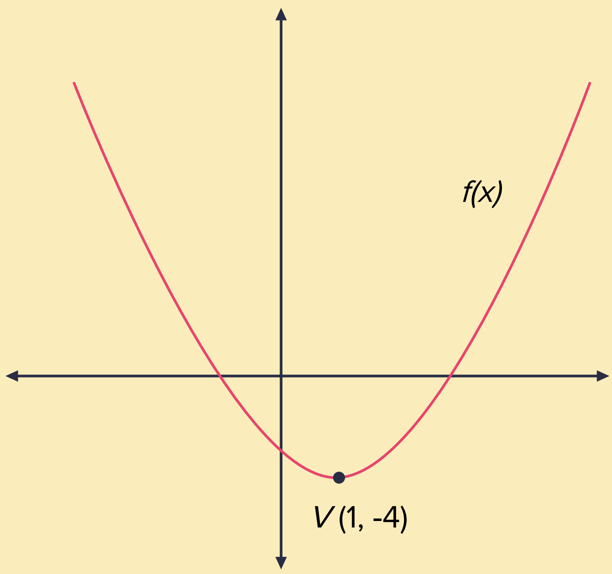
Observe:

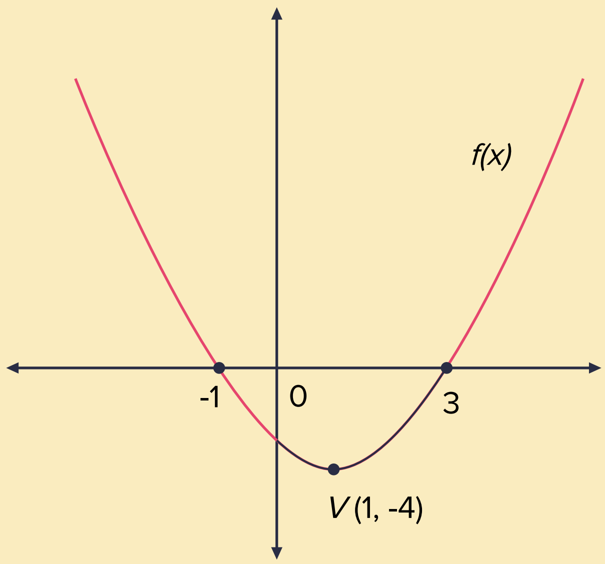
- Upward opening parabola (Since a > 0)
- Vertex V = $\left(\frac{-b}{2a}, \frac{-D}{4a}\right) = (1, -4)$
- Intercept = (0, c) = (0, -3)

(a) $x \in R$

For $x \in R$, we can observe from the graph, that the parabola attains all the values greater than or equal to $y_{min} = \frac{-D}{4a} = -4$.

Hence, range $R(f) = [-4, \infty)$, for $x \in R$



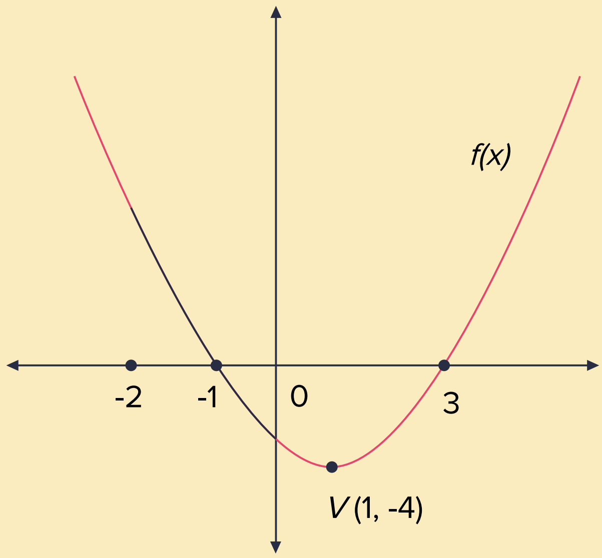


(b) $x \in [0, 3]$

Restricted part of domain = [0, 3]
As x is moving from 0 to 3, we see that f(x) first decreases till minima, then starts to increase till 3.

So, sweeped values of y = $[y_{min}, f(3)]$
$y_{min} = -4$, and $f(3) = 0$.

So range is [-4,0], for $x \in [0, 3]$



(c) $x \in [-2, 0]$

We can see from the graph that when x is increasing from -2 to 0, f(x) is decreasing in the interval [f(-2), f(0)].

$f(-2) = 5$, $f(0) = -3$
$R(f) = [-3, 5]$, for $x \in [-2, 0]$

Location of Roots

Given

A quadratic polynomial function $f(x) = ax^2 + bx + c$; $a \neq 0$, $a, b, c \in R$. Let α and β be the roots of quadratic equation $f(x) = 0$.

Required

To find conditions on coefficients of a quadratic polynomial, such that α, β lie in specified intervals.

Example: Conditions on a, b, c such that roots of $f(x) = 0$, $\alpha, \beta \in \left(\frac{1}{2}, \infty\right)$.

Approach

Graph of the quadratic function $y = f(x)$ is an upward opening parabola (for $a > 0$) or a downward opening parabola (for $a < 0$). When we divide f(x) by a, we end up with function $\frac{f(x)}{a} = x^2 + \frac{bx}{a} + \frac{c}{a}$, that has 1 as the leading coefficient.

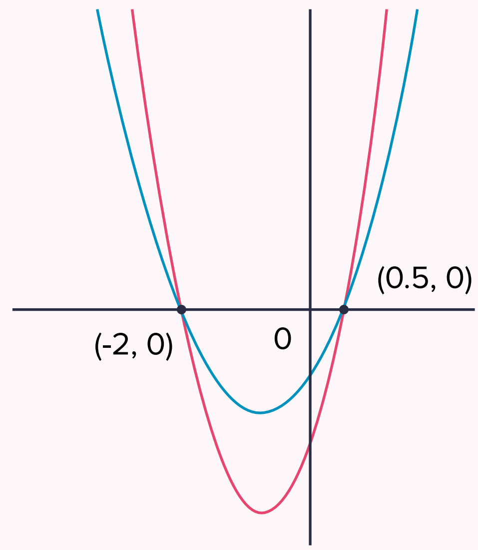
Note that both $f(x) = 0$ and $\frac{f(x)}{a} = 0$ have α and β as roots. So, to find the condition on coefficients (based on the location of the roots), we will use $\frac{f(x)}{a}$ instead of f(x).

All discussions are done using the graphs of $\frac{f(x)}{a}$.

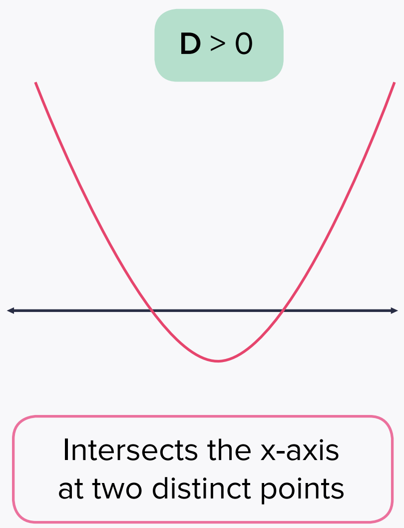
For example, consider the quadratic function,
$f(x) = 2x^2 + 3x - 2$
(the red line in the given graph)

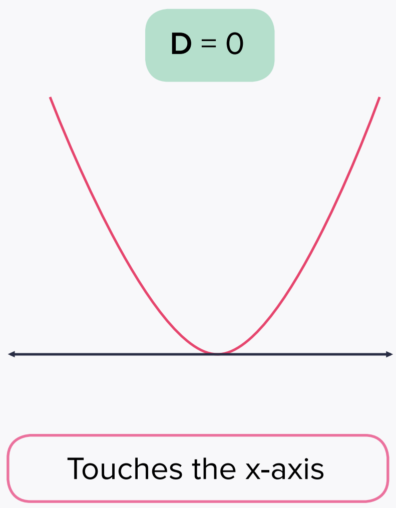
We divide it by a = 2 and get,
$\frac{f(x)}{2} = x^2 + \left(\frac{3}{2}\right) x - 1$
(the blue line in the given graph)

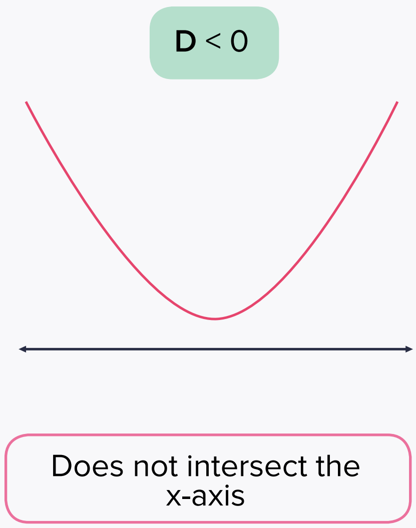
From the graph, it is clear that both these graphs are different, but have the same roots.



The graph of a quadratic function (a > 0) can be plotted in 3 ways.



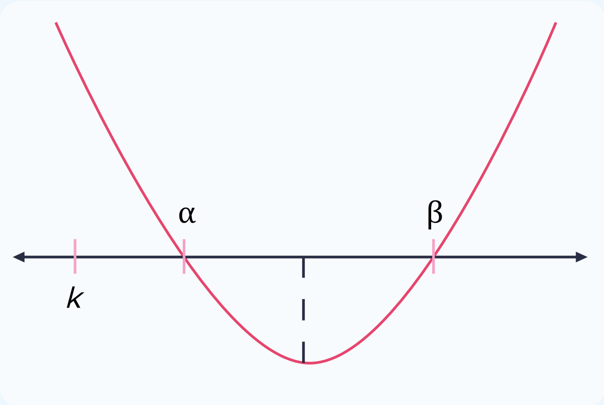




These are the three questions we will be solving.

- I. Both roots are greater than k → i.e. $\alpha, \beta > k$
- II. Both roots are less than k → i.e. $\alpha, \beta < k$
- III. k lies between the two roots → i.e. $\alpha < k < \beta$

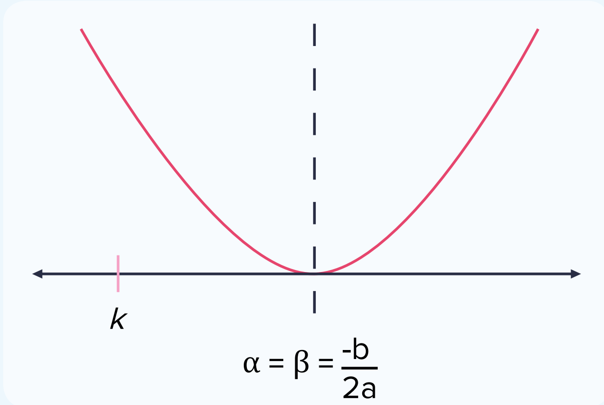
I. Both roots are greater than *k*



Case 1: Distinct roots; $k < \alpha < \beta$

Observations from the graph

- The graph intersects the x-axis at two distinct points, α and β
  $\Rightarrow D > 0$
- The graph lies above x-axis at k.
  $\Rightarrow f(k) > 0$
- The vertex or midpoint of roots α and β lie ahead of the k on the x-axis.
  $\Rightarrow$ Midpoint of α and β
  $\Rightarrow \frac{\alpha + \beta}{2} = \frac{-b}{2a}$ (Since, $\alpha + \beta = \frac{-b}{a}$)
  $\Rightarrow k < \frac{-b}{2a}$



Case 2: Equal Roots; $k < \alpha = \beta$

Observations from the graph

- The graph touches the x-axis at $x = \alpha = \beta$.
  $\Rightarrow D = 0$
- The graph lies above x-axis at k.
  $\Rightarrow f(k) > 0$
- The vertex or roots $\alpha = \beta$ lie ahead of k on the x-axis.
  $\Rightarrow k < \frac{-b}{2a}$

Combined condition:

$D \geq 0$

$f(k) > 0$

$k < \frac{-b}{2a}$

II. Both roots are less than *k*

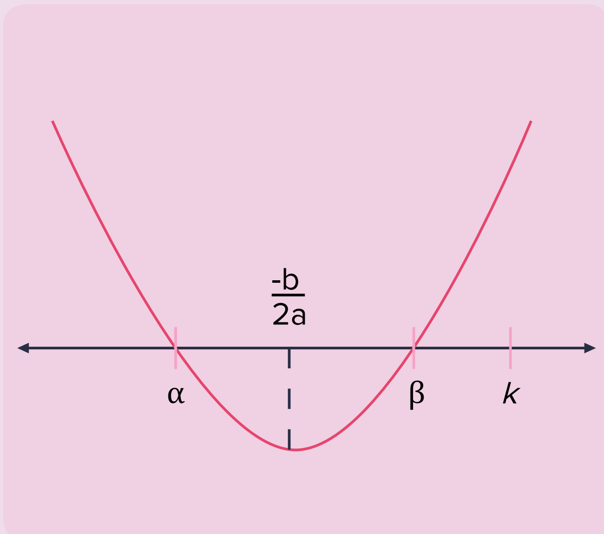
$\alpha, \beta < k$

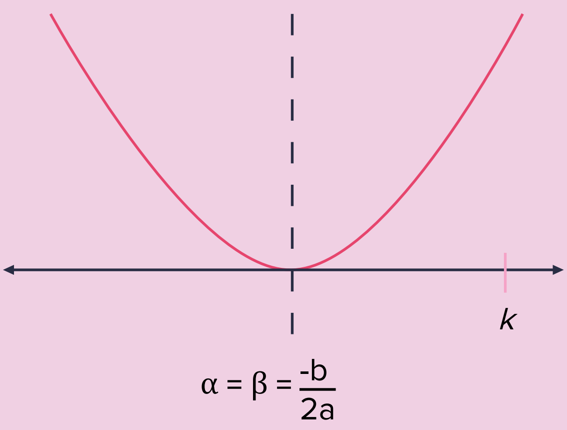
Again, there will be two cases.

Case 1: Distinct roots; $\alpha < \beta < k$

Observations from the graph:

- The graph intersects the x-axis at two distinct points α and β
  $\Rightarrow D > 0$
- The graph lies above x-axis at k
  $\Rightarrow f(k) > 0$
- The vertex or midpoint of roots α and β lies behind the k on the x-axis.
  $\Rightarrow k > \frac{-b}{2a}$





Case 2: Equal Roots; k > α = β

Observations from the graph:

- The graph touches the x-axis at x = α = β
  ⇒ D = 0
- The graph lies above x-axis at k
  ⇒ f(k) > 0
- The vertex or roots α = β lie behind k on the x-axis.
  $\Rightarrow k > \frac{-b}{2a}$

Combined condition:

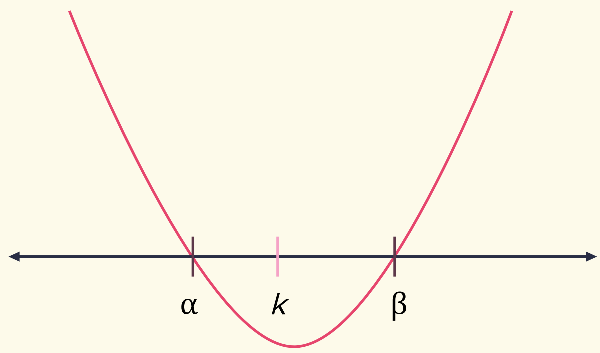
$D \geq 0$

$f(k) > 0$

$k > \frac{-b}{2a}$

III: k lies between the two roots ⇒ α < *k* < β

Solution :



Observations from the graph

- The graph intersects the x-axis at two distinct points, α and β.
  ⇒ D > 0
- The graph lies below the x-axis at k.
  ⇒ f(k) < 0

Recall that for an upward opening parabola, f(k) < 0 will automatically satisfy the condition of real and distinct roots, i.e. D > 0.

If both the roots of $x^2 - 2mx + m^2 + m - 5 = 0$ are less than 5, then m lies in which of the following interval?

(a) (5, 6] (b) $(6, \infty)$ (c) $(-\infty, 4)$ (d) [4, 5]

Solution :

Step 1:

Given, $x^2 - 2mx + m^2 + m - 5 = 0$ are having roots α and β such that, $\alpha, \beta < 5$, a = 1, b = -2m, $c = m^2 + m - 5$

Conditions applicable: $D \geq 0$

$f(k) > 0$

$k > \frac{-b}{2a}$

Step 2:

$D \geq 0 \Rightarrow (-2m)^2 - 4.1.(m^2 + m - 5) > 0 \Rightarrow 4m^2 - 4m^2 - 4m + 20 \geq 0 \Rightarrow 5 - m > 0 \Rightarrow m \in (-\infty, 5]$

Step 3:

$f(5) > 0 \Rightarrow 5^2 - 10m + m^2 + m - 5 > 0 \Rightarrow m^2 - 9m + 20 > 0 \Rightarrow (m-4)(m-5) > 0 \Rightarrow m \in (-\infty, 4) \cup (5, \infty)$

Step 4:

$5 > \frac{-b}{2a} \Rightarrow 5 > \frac{2m}{2} \Rightarrow m \in (-\infty, 5)$

Step 5:

From step 2, 3, and 4, we get $m \in (-\infty, 4)$

Concept Check 1

The values of 'a' for which one root of the equation $x^2 - (a + 1)x + a^2 + a - 8 = 0$ greater than 2 and the other root is less than 2, lie in which of the following intervals?

(a) (-2, 3) (b) (3, 10)

(c) $\left(\frac{-11}{3}, 3\right)$ (d) $\left(\frac{-11}{3}, -2\right)$

Concept Check 2

If the root(s) of $(m-2)x^2 + 8x + m + 4 = 0$ is/are greater than 0, then m lies in which of the following intervals?

(a) [-6, -4) (b) (-6, -4) (c) (2, 4) (d) [2, 4)

Summary Sheet

For quadratic polynomial $ax^2 + bx + c$

Key Takeaways

(a) Range of an upward opening parabola ($a > 0$) is $\left[\frac{-D}{4a}, \infty\right)$

(b) Range of a downward opening parabola ($a < 0$) is $\left(-\infty, \frac{-D}{4a}\right]$

Key Conditions

(a) Both roots are greater than k $\Rightarrow k < \alpha, \beta$

$D \geq 0$

$f(k) > 0$

$k < \frac{-b}{2a}$

(b) Both roots are less than k $\Rightarrow \alpha, \beta < k$

$D \geq 0$

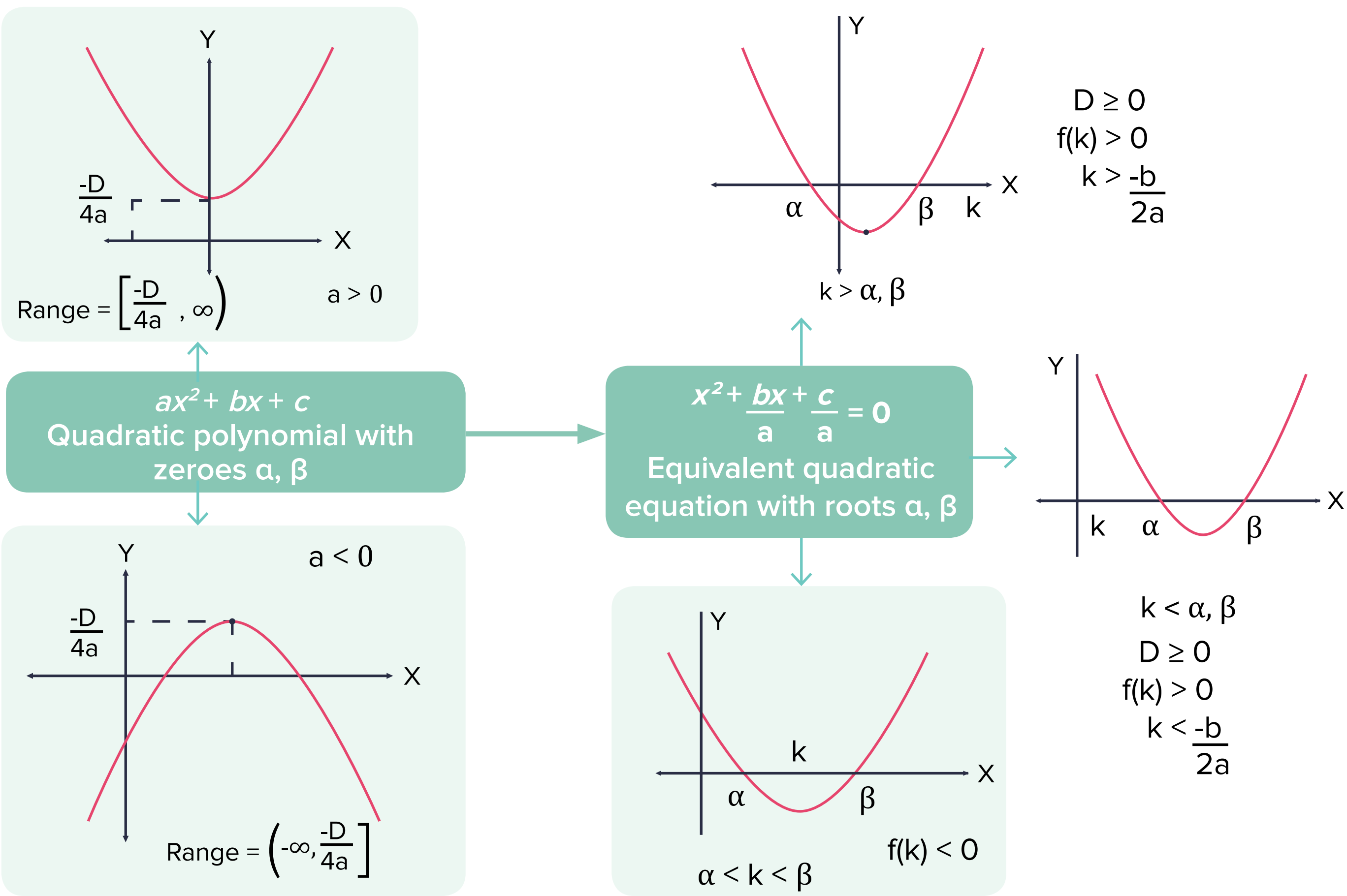
$f(k) > 0$

$k > \frac{-b}{2a}$

(c) k lies between the two roots $\Rightarrow \alpha < k < \beta$

$D > 0, f(k) < 0$

Mind map



Self-Assessment

(1) If the expression $(n - 2)x^2 + 8x + (n + 4)$ is negative $\forall x \in R$, then specify the range of n.

(2) If the roots of $x^2 - 6kx + (2 - 2k + 9k^2) = 0$ are greater than 3, then find the range of k.

(3) What is the set of values of 'a' for which both roots of the quadratic polynomial $f(x) = ax^2 + (a - 3)x + 1$ lie only on one side of the y–axis?

Answers

Concept Check 1

Step 1

Given, $x^2 - (a + 1)x + a^2 + a - 8 = 0$ with roots α, β, such that, $\alpha < 2 < \beta$
$a = 1, b = -(a + 1), c = a^2 + a - 8, k = 2$
Condition applicable: $f(k) < 0$

Step 2

$f(2) < 0 \Rightarrow 2^2 - (a + 1)2 + a^2 + a - 8 < 0 \Rightarrow a^2 - a - 6 < 0 \Rightarrow (a + 2)(a - 3) < 0 \Rightarrow a \in (-2, 3)$
Final answer: $a \in (-2, 3)$

Concept Check 2

Step 1

Given, $(m - 2)x^2 + 8x + m + 4 = 0$ with roots α, β such that $\alpha, \beta > 0$.
$a = m - 2, b = 8, c = m + 4, k = 0$
There are two cases: Case 1: $m \neq 2$, Case 2: $m = 2$

Step 2

As $m \neq 2$, dividing the equation by (m-2), $x^2 + \frac{8}{m-2}x + \frac{m+4}{m-2} = 0$
As, $\alpha, \beta > 0 = k$
Conditions applicable: I. $D \geq 0$ II. $f(k) > 0$ III. $k < \frac{-b}{2a}$

$$D \geq 0 \Rightarrow \frac{64}{(m-2)^2} - 4\frac{m+4}{m-2} \geq 0 \Rightarrow \frac{16 - (m+4)(m-2)}{(m-2)^2} \geq 0 \Rightarrow \frac{(m+6)(m-4)}{(m-2)^2} \leq 0$$

$\Rightarrow m \in [-6, 4] - \{2\}$ ....A

Step 3

$f(0) > 0 \Rightarrow \frac{(m+4)}{(m-2)} > 0 \quad m \in (-\infty, -4) \cup (2, \infty)$ ...B

Step 4

$0 < \frac{-b}{2a} \Rightarrow 0 < \frac{\frac{-8}{m-2}}{4} \Rightarrow m - 2 < 0 \Rightarrow m \in (-\infty, 2)$...C
For Case 1: From A, B, and C, solution set = [-6, -4).

Step 5

Solving for case 2, when m = 2
$(m-2)x^2 + 8x + m + 4 = 0 \Rightarrow 8x + 6 = 0 \Rightarrow x = \frac{-3}{4}$. But we require $x > 0$,
so no solution set for Case 2.

Step 6

From Case 1 and Case 2, final solution set = [-6, -4).

Self-Assessment

(1) Given: $f(x) = (n-2)x^2 + 8x + (n+4) < 0 \ \forall\ x \in R$

$\Rightarrow$ Downward opening parabola lying entirely below the x-axis.

$\Rightarrow a < 0$ and $D < 0$

Step 1 $a < 0 \Rightarrow n - 2 < 0 \Rightarrow n < 2 \Rightarrow n \in (-\infty, 2) = A$

Step 2 $D < 0 \Rightarrow b^2 - 4ac < 0 \Rightarrow 8^2 - 4(n+4)(n-2) < 0 \Rightarrow 16 - n^2 - 2n + 8 < 0$

$\Rightarrow n^2 + 2n - 24 > 0$

$\Rightarrow (n+6)(n-4) > 0 \Rightarrow n \in (-\infty, -6) \cup (4, \infty) = B$

Step 3 Final solution $A \cap B = (-\infty, 2) \cap ((-\infty, -6) \cup (4, \infty)) = (-\infty, -6)$

(2) Given: $f(x) = x^2 - 6kx + (2 - 2k + 9k^2) = 0$ have roots greater than 3. So, three conditions are:

$D \ge 0$

$f(k) > 0$, here $k = 3$

$k < \frac{-b}{2a}$

Step 1 $D \ge 0 \Rightarrow (-6k)^2 - 4(1)(2 - 2k + 9k^2) \ge 0 \Rightarrow 36k^2 - (8 - 8k + 36k^2) \ge 0 \Rightarrow k \ge 1 \Rightarrow k \in [1, \infty) = A$

Step 2 $f(3) > 0 \Rightarrow 9 - 18k + 2 - 2k + 9k^2 > 0 \Rightarrow 9k^2 - 20k + 11 > 0 \Rightarrow (9k - 11)(k - 1) > 0$

$\Rightarrow k < 1$ or $k > \frac{11}{9} = B$

Step 3 $3 < \frac{-b}{2a} \Rightarrow 3 < \frac{-(-6k)}{2} \Rightarrow 3 < 3k \Rightarrow k > 1 = C$

Step 4 From A, B and C

Final solution: $k > \frac{11}{9} \Rightarrow k \in \left(\frac{11}{9}, \infty\right)$

(3)

Step 1 Given: Roots of $f(x) = ax^2 + (a-3)x + 1 = 0$, $a \neq 0$ lie only on one side of the y-axis.

$\Rightarrow$ Either both roots are negative or both roots are positive

Also, $D \ge 0 \Rightarrow (a-3)^2 - 4a \ge 0 \Rightarrow a^2 - 6a + 9 - 4a \ge 0 \Rightarrow a^2 - 10a + 9 \ge 0$

$\Rightarrow a \in (-\infty, 1] \cup [9, \infty) - \{0\}$ ....A

Step 2 Case 1: Both roots negative

$\Rightarrow$ Sum of roots negative and product positive

$\Rightarrow \frac{-b}{a} < 0$ and $\frac{c}{a} > 0 \Rightarrow \frac{-(a-3)}{a} < 0$ and $\frac{1}{a} > 0$

As the second inequality demands $a > 0$, we reject the possibility of $a < 0$ in the first inequality too.

$\Rightarrow a > 0$ and $a - 3 > 0 \Rightarrow a > 3 \Rightarrow a \in (3, \infty)$ ..B

Step 3 Case 1: Both roots positive

$\Rightarrow$ Sum of roots positive and product positive

$\Rightarrow \frac{-b}{a} > 0$ and $\frac{c}{a} > 0 \Rightarrow \frac{-(a-3)}{a} > 0$ and $\frac{1}{a} > 0 \Rightarrow a > 0$ and $a - 3 < 0 \Rightarrow a \in (0, 3)$ ...C

Step 4 From A, B, and C, we can observe final solution will be $A \cap (B \cup C)$.

Final solution: $a \in (0, 1] \cup [9, \infty)$

MATHEMATICS

QUADRATIC EQUATIONS

MORE ON LOCATION OF ROOTS AND HIGH ORDER EQUATIONS

NOTES

What you already know

- Location of roots:
  - Both roots lie between k_1 and $k_2 \Rightarrow k_1 < \alpha, \beta < k_2$
  - k_1 and k_2 lie between both the roots $\Rightarrow \alpha < k_1, k_2 < \beta$
  - Exactly one root lies between k_1 and k_2
    $\Rightarrow k_1 < \alpha < k_2$ or $k_1 < \beta < k_2$
  - At least one root is greater than k
    $\Rightarrow \beta < k < \alpha$ or $\alpha < k < \beta$ or $k < \alpha, \beta$
- Identity Equations

What you will learn

- Plotting of graph
- Location of roots:
  - Both roots are greater than k
  - Both roots are less than k
  - k lies in between the two roots.

Location of Roots

Given: Quadratic function $f(x) = ax^2 + bx + c$; $a \neq 0$, a, b, c $\in R$
Let α and β are the roots of quadratic equation $f(x) = 0$

Required: To find conditions on coefficients of quadratic polynomial, such that α, β lie in specified intervals. For example: Conditions on a, b, c such that roots of $f(x) = 0$, $\alpha, \beta \in (\frac{1}{2}, \infty)$

Approach: Graph of the quadratic function $y = f(x)$ is an upward opening parabola (for $a > 0$) or is a downward opening parabola (for $a < 0$). When we divide f(x) by a, we end up with function $\frac{f(x)}{a} = x^2 + \frac{b}{a} \times \frac{c}{a}$, that has 1 as the leading coefficient.

Note that both $f(x) = 0$ and $\frac{f(x)}{a} = 0$ have α and β as roots. so to find the condition on coefficients (based on the location of the roots), we will conveniently use $\frac{f(x)}{a}$ instead of f(x). All discussions are done using the graphs of $\frac{f(x)}{a}$.

$$y = f(x) = x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \rightarrow \alpha, \beta$$

Que IV : Both roots lie between k_1 & k_2 — i.e. $k_1 < \alpha, \beta < k_2$

Que V : k_1 & k_2 lie between both the roots — i.e. $\alpha < k_1, k_2 < \beta$

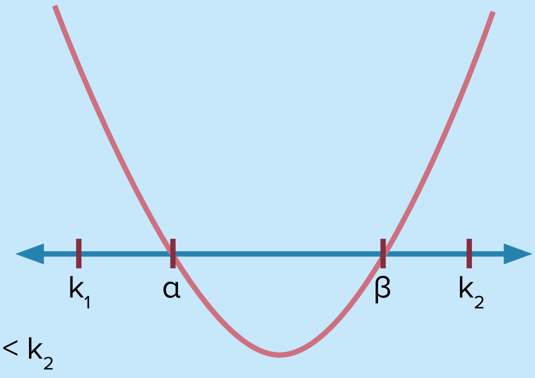
Que VI : Exactly one root lies between k_1 & k_2 — i.e. $k_1 < \alpha < k_2$ or $k_1 < \beta < k_2$

Que VII : Atleast one root is greater than k — i.e. $\beta < k < \alpha$ or $\alpha < k < \beta$ or α & $\beta > k$

Both roots lie between k_1 and $k_2 \Rightarrow k_1 < \alpha, \beta < k_2$

There will be 2 cases.

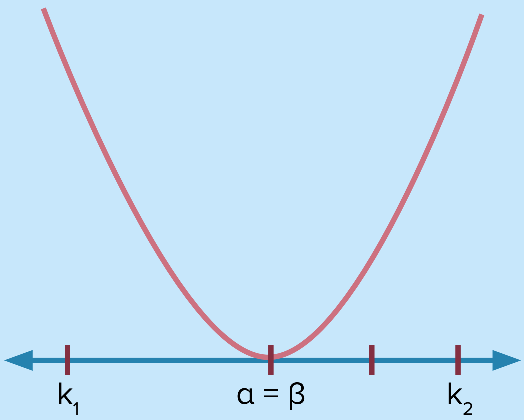
Case 1: $\alpha \neq \beta$



Observations from graph:

1. The graph intersects the x-axis at two distinct points
   $\Rightarrow D > 0$
2. The graph lies above x-axis at k_1 and k_2
   $\Rightarrow f(k_1) > 0$
   $\Rightarrow f(k_2) > 0$
3. Vertex $V = \left(\frac{-b}{2a}, \frac{D}{4a}\right)$ lies between k_1 and $k_2 \Rightarrow k_1 < \frac{-b}{2a} < k_2$

Case 2: $\alpha = \beta$



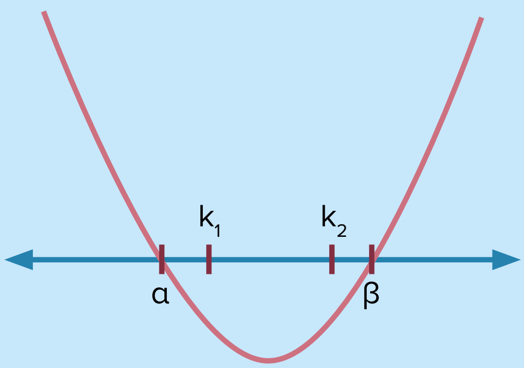
Observations from graph:

1. The graph touches the x-axis at $\alpha = \beta$
   $\Rightarrow D = 0$
2. The graph lies above x-axis at k_1 and k_2
   $\Rightarrow f(k_1) > 0$
   $\Rightarrow f(k_2) > 0$
3. Vertex $V = \left(\frac{-b}{2a}, \frac{D}{4a}\right)$ lies between k_1 and k_2
   $\Rightarrow k_1 < \frac{-b}{2a} < k_2$

Clubbing both the cases, we get

$$D \geq 0$$
$$f(k_1) > 0$$
$$f(k_2) > 0$$
$$k_1 < \frac{-b}{2a} < k_2$$

k_1 and k_2 lie between both the roots i.e $\alpha < k_1, k_2 < \beta$



Observations from graph:

1. The graph touches the x-axis at α and β
   $\Rightarrow D > 0$
2. The graph lies below x-axis at k_1 and k_2
   $\Rightarrow f(k_1) < 0$
   $\Rightarrow f(k_2) < 0$

$f(k_1) < 0$ or $f(k_2) < 0$ will automatically satisfy the condition of real roots, i.e $D > 0$, as for upward opening parabola $D < 0 \Rightarrow f(x) > 0 \ \forall \ x \in R$ and cannot attain negative values at k_1 and k_2

Therefore, there will be 2 conditions for k_1 and k_2 to lie between both the roots i.e $\alpha < k_1, k_2 < \beta$
$f(k_1) < 0$ and $f(k_2) < 0$

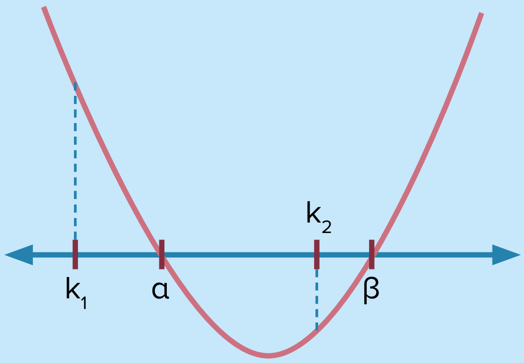
Exactly one root lies between k_1 and k_2

There will be two cases i.e $k_1 < \alpha < k_2$ or $k_1 < \beta < k_2$

Case 1: $k_1 < \alpha < k_2$

Observations from graph:

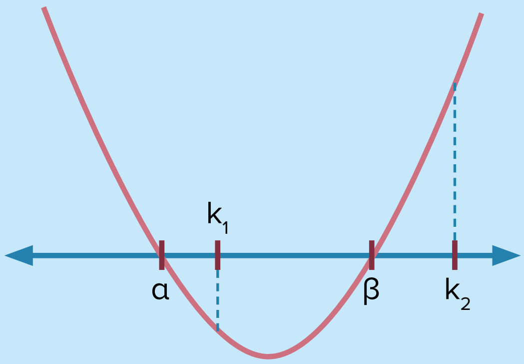
1. The graph intersects the x-axis at two distinct points
   $\Rightarrow D > 0$
2. The graph lies above x-axis at k_1
   $\Rightarrow f(k_1) > 0$
3. The graph lies below x-axis at k_2
   $\Rightarrow f(k_2) < 0$



Case 2: $k_1 < \beta < k_2$

Observations from graph:

1. The graph intersects the x-axis at two distinct points
   $\Rightarrow D > 0$
2. The graph lies below x-axis at k_1
   $\Rightarrow f(k_1) < 0$
3. The graph lies above x-axis at k_2
   $\Rightarrow f(k_2) > 0$



As α and β are two different roots, therefore $D > 0$ is implicit in the problem statement.
From the above two cases, combined condition will be
$f(k_1) \times f(k_2) < 0$

At least one root is greater than k

There will be three cases.
$\alpha < k < \beta$ or
$k < \alpha, \beta$ or
$k < \alpha = \beta$

Case 1: $\alpha < k < \beta$

Observations from graph:

1. The graph intersects the x-axis at α and β

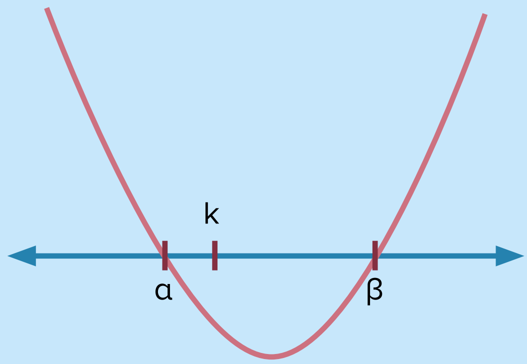
   $\Rightarrow D > 0$

2. The graph lies below x-axis at k

   $\Rightarrow f(k) < 0$

3. The bigger root β lies ahead of the k

   $\Rightarrow \dfrac{-b + \sqrt{b^2 - 4ac}}{2a} > k$



Case 2: $k < \alpha, \beta$

Observations from graph:

1. The graph intersects the x-axis at α and β

   $\Rightarrow D > 0$

2. The graph lies above x-axis at k

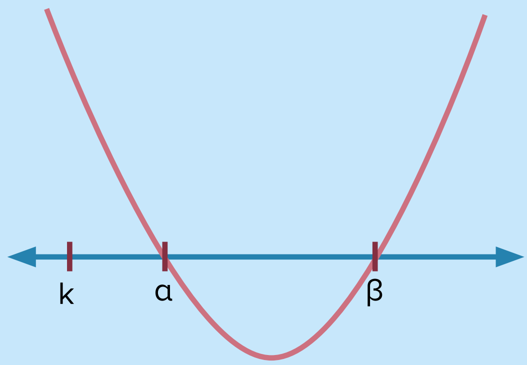
   $\Rightarrow f(k) > 0$

4. Midpoint of α and β lies ahead of k and

   $\Rightarrow k < \dfrac{-b}{2a}$

3. The bigger root β lies ahead of the k

   $\Rightarrow \dfrac{-b + \sqrt{b^2 - 4ac}}{2a} > k$



Case 3: $k < \alpha = \beta$

Observations from graph:

1. The graph intersects the x-axis at $\alpha = \beta$

   $\Rightarrow D = 0$

2. The graph lies above x-axis at k

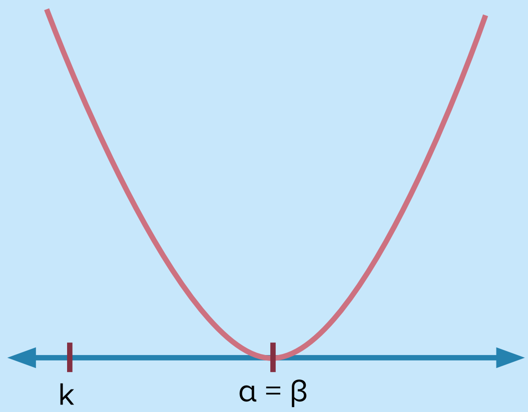
   $\Rightarrow f(k) > 0$

4. The vertex of parabola lies ahead of k

   $\Rightarrow k < \dfrac{-b}{2a}$

3. The bigger root β lies ahead of the k

   $\Rightarrow \dfrac{-b + \sqrt{b^2 - 4ac}}{2a} > k$



So, from Case 1, 2 and 3 we can club the common condition

1. $D \geq 0$
2. $k < \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

Polynomial equation of degree n, with real coefficients
$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + a_4x^{n-4} + \dots\dots\dots + a_n = 0$; $a \neq 0$ and $a_0, a_1, \dots a_n \in R$ has n roots.

The values of m for which the equation $x^4 + (1 - 2m)x^2 + (m^2 - 1) = 0$ has

(i) four real distinct solutions

(a) $m \in (1, 2)$ (b) $m \in (1, \frac{5}{4})$ (c) $m \in [0, 2)$ (d) $m \in \Phi$

Solution

We have $f(x) = x^4 + (1 - 2m)x^2 + m^2 - 1 = 0$

Step - 1:

Let $x^2 = t \Rightarrow f(x) = x^4 + (1 - 2m)x^2 + m^2 - 1 = 0 \Rightarrow f(t) = t^2 + (1 - 2m)t + m^2 - 1 = 0$

The given biquadratic equation in x $f(x) = 0$ with roots x_1, x_2, x_3 and x_4 has been transformed into a quadratic equation in t, $f(t) = 0$ with t_1 and t_2 as the roots.

Step - 2:

As $x^2 = t \Rightarrow x = \pm\sqrt{t}$, Four roots of $f(x) = 0$ will be

$x_1 = +\sqrt{t_1}$, $x_2 = -\sqrt{t_1}$, $x_3 = +\sqrt{t_2}$, $x_4 = -\sqrt{t_2}$

For $f(x) = 0$ with four real roots x_1, x_2, x_3 and x_4, $t_1, t_2 > 0$, $t_1 \neq t_2$

Step - 3:

Both roots t_1, t_2 of $f(t) = 0$ are greater than 0, i.e $t_1, t_2 > 0$, $t_1 \neq t_2$, $k = 0$

A. $D > 0 \Rightarrow (1 - 2m)^2 - 4(m^2 - 1) > 0$

$\Rightarrow 4m^2 - 4m + 1 - 4m^2 + 4 > 0 \Rightarrow m < \frac{5}{4} \Rightarrow m \in (-\infty, \frac{5}{4}) = A$

B. $\frac{-b}{2a} > 0 \Rightarrow \frac{-(1 - 2m)}{2} > 0 \Rightarrow 1 - 2m < 0 \Rightarrow m > \frac{1}{2} \Rightarrow m \in (\frac{1}{2}, \infty) = B$

C. $f(0) > 0 \Rightarrow m^2 - 1 > 0 \Rightarrow (m+1)(m-1) > 0 \Rightarrow m \in (-\infty, -1) \cup (1, \infty) = C$

1. **$D \geq 0$ and**
2. **$\frac{-b}{2a}$ and**
3. **$f(k) > 0$**

Final Solution:

$A \cap B \cap C \Rightarrow m \in (1, \frac{5}{4})$

Concept Check 1

The values of m for which the equation $x^4 + (1 - 2m)x^2 + m^2 - 1 = 0$ has three real distinct solutions.

(a) $m \in (-1, 2)$ (b) $m \in (1, \frac{5}{4})$ (c) $m \in [0, 2)$ (d) $m \in \{1\}$

Concept Check 2

The values of m for which the equation $x^4 + (1 - 2m)x^2 + m^2 - 1 = 0$ has two real distinct solutions.

(a) $m \in (-1, 1) \cup \{\frac{5}{4}\}$ (b) $m \in (-1, \frac{5}{4})$ (c) $m \in [0, 2)$ (d) $m \in \Phi$

Identity

$ax^2 + bx + c = 0$ is said to be an identity in x if it is satisfied by all $x \in R$

Or

Number of roots > 2

Or

$a = b = c = 0$

For example:

$x^2 - 4x + 4 = (x - 2)^2$ is satisfied $\forall$ $x \in R$, hence an identity

$x^2 - 4x + 4 = (x - 2)^2$

$x^2 - 4x + 4 = x^2 - 4x + 4$

$0.x^2 + 0.x + 0 = 0$

Coefficient of x^2 = Coefficient of x = Constant term = 0

Similarly,

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + a_4x^{n-2} + ... + a_n = 0$ is an identity in x if it is satisfied by all $x \in R$

Or

Number of roots > n

Or

$a_0 = a_1 = a_2 = a_3 = ... = a_n = 0$

Concept Check 2

The values of 'p' for which the equation $(p^2 - 3p + 2)x^2 - (p^2 - 5p + 4)x + p - p^2 = 0$ possess more than two roots is/are

(a) 0 (b) 1 (c) 2 (d) 3

Summary

For quadratic equation $ax^2 + bx + c = 0$

Key Conditions

1. Both roots lie between k_1 and $k_2 \Rightarrow k_1 < \alpha, \beta < k_2$

$$D \geq 0, f(k_1) > 0, f(k_2) > 0 \text{ and } k_1 < \frac{-b}{2a} < k_2$$

2. Both roots are less than $k \Rightarrow \alpha, \beta < k$

$$D \geq 0, f(k) > 0 \text{ and } k > \frac{-b}{2a}$$

3. k_1 and k_2 lie between both the roots i.e $\alpha < k_1, k_2 < \beta$

$$f(k_1) < 0 \text{ and } f(k_2) < 0$$

4. Exactly one root lie between k_1 and k_2

$$f(k_1) \times f(k_2) < 0$$

5. At least one root is greater than k

$$D \geq 0 \text{ and } k < \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Key Takeaways

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + a_4x^{n-2} + ... + a_n = 0$ is an identity in x if it is satisfied by all $x \in R$

Mind Map

$ax^2 + bx + c = 0$

Quadratic Polynomial

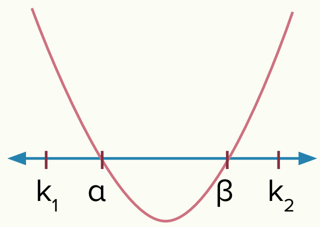
→ Identity in x iff $a = b = c = 0$

↓

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Equivalent Quadratic Equation

$k_1 < \alpha, \beta < k_2$



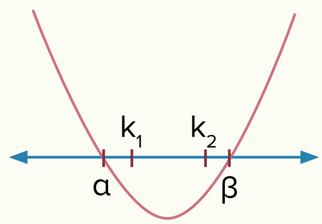
$D \geq 0$

$f(k_1) > 0$

$f(k_2) > 0$

$k_1 < \frac{-b}{2a} < k_2$

$\alpha < k_1, k_2 < \beta$

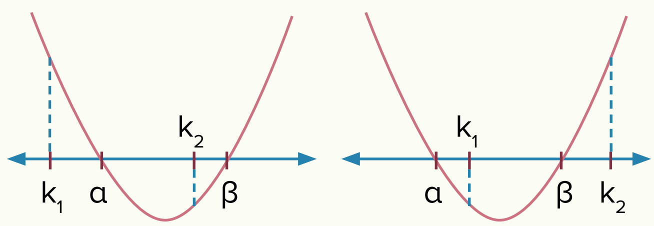


$D > 0$

$f(k_1) < 0$

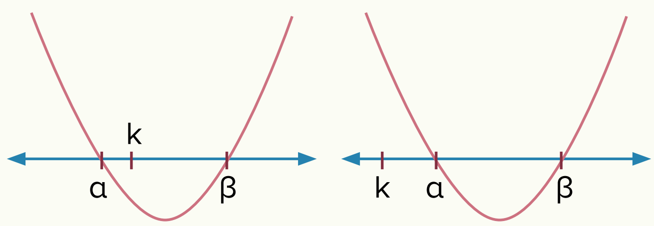
$f(k_2) < 0$

$k_1 < \alpha < k_2$ or $k_1 < \beta < k$



$D > 0$ $f(k_1) \times f(k_2) < 0$

$\alpha < k < \beta$ or $k < \alpha, \beta$ or $k < \alpha = \beta$



$D \geq 0$

$k < \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

Self- Assessment

1. Let $f(x) = 4x^2 - 4(a - 2)x + a - 2 = 0$, **a** $\in$ R be a quadratic equation, the range of values of **a** for which at least one root lies in $(0, \frac{1}{2})$

(a) $[3,\infty)$ (b) $(-\infty, 2) \cup (3,\infty)$ (c) $(-\infty,2)$ (d) Φ

2. Find the values of '**p**' for which the roots of $(p - 2)x^2 + 2px + p + 3 = 0$ lies in $(-2, 1)$

(a) $(-\infty, = \frac{-1}{4}) \cup [5, 7]$ (b) $(-\infty, 1) \cup [5, 7]$ (c) $(-\infty, \frac{-1}{4}) \cup (5, 6]$ (d) Φ

Answers

Concept Check 1

Step 1: Let $x^2 = t \Rightarrow f(x) = x^4 + (1 - 2m)x^2 + m^2 - 1 = 0 \Rightarrow f(t) = t^2 + (1 - 2m)t + m^2 - 1 = 0$

The given biquadratic equation in x $f(x) = 0$ with roots x_1, x_2, x_3 and x_4 has been transformed into a quadratic equation in t, $f(t) = 0$ with t_1 and t_2 as the roots.

Step 2: As $x^2 = t \Rightarrow x = \pm\sqrt{t}$ Four roots of $f(x) = 0$ will be

$x_1 = +\sqrt{t_1}, x_2 = -\sqrt{t_1}, x_3 = +\sqrt{t_2}, x_4 = -\sqrt{t_2}$

Three distinct real roots only if $t_1 > 0$ and $t_2 = 0$

Step 3:

A. $D > 0 \Rightarrow b^2 - 4ac > 0 \Rightarrow (1 - 2m)^2 - 4m^2 + 4 > 0 \Rightarrow -4m + 5 > 0 \Rightarrow m \in \left(-\infty, \frac{5}{4}\right)$

B. For $t_2 = 0 \Rightarrow f(0) = 0 \Rightarrow f(0) = m^2 - 1 = 0 \Rightarrow m = \{-1, 1\}$

C. For $t_1 > 0$, $\frac{-b}{2a} > 0 \Rightarrow \frac{-(1 - 2m)}{2} > 0 \Rightarrow 1 - 2m < 0 \Rightarrow m > \frac{1}{2} \Rightarrow m \in \left(\frac{1}{2}, \infty\right)$

For final solution, A, B and C have to be simultaneously satisfied

Hence final solution $A \cap B \cap C = \left(-\infty, \frac{5}{4}\right) \cap \{-1, 1\} \cap \left(\frac{1}{2}, \infty\right) = \{1\}$

Concept Check 2

Step 1: Let $x^2 = t \Rightarrow f(x) = x^4 + (1 - 2m)x^2 + m^2 - 1 = 0 \Rightarrow f(t) = t^2 + (1 - 2m)t + m^2 - 1 = 0$

The given biquadratic equation in x $f(x) = 0$ with roots x_1, x_2, x_3 and x_4 has been transformed into a quadratic equation in t, $f(t) = 0$ with t_1 and t_2 as the roots.

Step 2: As $x^2 = t \Rightarrow x = \pm\sqrt{t}$ Four roots of $f(x) = 0$ will be

$x_1 = +\sqrt{t_1}, x_2 = -\sqrt{t_1}, x_3 = +\sqrt{t_2}, x_4 = -\sqrt{t_2}$

For $f(x) = 0$ will have two real roots when $t_1 > 0$ and $t_2 < 0 \Rightarrow 0$ lies in between the roots t_1 and t_2 of $f(t) = 0$ Or $t_1 = t_2 > 0$

Step 3:

Case 1: $t_1 < 0 < t_2$. For 0 to sit in between the roots $f(0) < 0$

$\Rightarrow f(0) = m^2 - 1 < 0 \Rightarrow (m - 1)(m + 1) < 0 \Rightarrow m \in (-1, 1)$

Step 4:

Case 2: $t_1 = t_2 > 0$

$\Rightarrow D = 0 \Rightarrow b^2 - 4ac = 0 \Rightarrow (1 - 2m)^2 - 4m^2 + 4 = 0 \Rightarrow m = \frac{5}{4}$

But for $m = \frac{5}{4}$, $t_1 = t_2 = \frac{3}{4} > 0 \Rightarrow$ Two real solutions $x_1 = \frac{\sqrt{3}}{2}$, $x_2 = \frac{-\sqrt{3}}{2}$

Hence final solution = Case 1 $\cup$ Case 2 $= (-1, 1) \cup \{\frac{5}{4}\}$

Concept Check 3

Step 1: Given $f(x) = (p^2 - 3p + 2)x^2 - (p^2 - 5p + 4)x + p - p^2 = 0$ have more than two roots

$\Rightarrow f(x) = 0$ is an identity in x

$\Rightarrow$ Coefficient of x^2 = Coefficient of x = Constant term = 0

Step 2: $(p^2 - 3p + 2) = 0$ and $(p^2 - 5p + 4) = 0$ and $p - p^2 = 0$

$\Rightarrow (p^2 - 3p + 2) = 0 \Rightarrow p = \{1, 2\}$

$\Rightarrow (p^2 - 5p + 4) = 0 \Rightarrow p = \{1, 4\}$

$\Rightarrow p - p^2 = 0 \Rightarrow p = \{0, 1\}$

Hence p = 1 is the correct answer.

Self- Assessment 1

Step 1: We have, $f(x) = 4x^2 - 4(a - 2)x + a - 2 = 0$

Dividing both sides by 4, we get : $f(x) = x^2 - (a - 2)x + \frac{a-2}{4} = 0$. Let α, β be its roots

Given : At least one root lies in $(0, \frac{1}{2})$

Step 2: Case 1: Exactly one root lies in $(0, \frac{1}{2})$

$\Rightarrow f(0). f(\frac{1}{2}) < 0$

$\Rightarrow \frac{a-2}{4}(\frac{1}{4} - (\frac{a-2}{2}) + \frac{a-2}{4}) < 0$

$\Rightarrow (a - 2)(a - 3) > 0 \; a \in (-\infty, 2) \cup (3, \infty)$

Step 3: Case 2: Both root lies in $(0, \frac{1}{2})$

1. $D \geq 0 \Rightarrow b^2 - 4ac \geq 0 \Rightarrow (a - 2)^2 - 4(\frac{a-2}{4}) \geq 0 \Rightarrow (a - 2)(a - 3) \geq 0 \Rightarrow a \in (-\infty, 2] \cup [3, \infty) = A$

2. $f(k_1) > 0 \Rightarrow f(0) > 0 \Rightarrow \frac{a-2}{4} > 0 \Rightarrow a \in (2, \infty) = B$

3. $f(k_2) > 0 \Rightarrow f(\frac{1}{2}) > 0 \Rightarrow (\frac{1}{4} - \frac{a-2}{2} + \frac{a-2}{4}) > 0 \Rightarrow (a - 3) < 0 \Rightarrow a \in (-\infty, 3) = C$

4. $k_1 < \frac{-b}{2a} < k_2 \Rightarrow 0 < \frac{a-2}{2} < \frac{1}{2} \Rightarrow 2 < a < 3 \Rightarrow a \in (2, 3) = D$

Case 2 Solution Set = $A \cap B \cap C \cap D = \phi$

Step 3: Final Solution Set = = Case 1 ∪ Case 2 = $((-\infty,2) \cup ((3,\infty)) \cup \phi = (-\infty,2) \cup (3,\infty)$

Self- Assessment 2

Step 1: We have $(p - 2)x^2 + 2px + p + 3 = 0$

Case 1: $p - 2 \neq 0$

$\therefore$ dividing both sides by **p - 2**, we get, $x^2 + \frac{2p}{p-2}x + \frac{p+3}{p-2} = 0$. Let α, β be its roots.

Given : $\alpha, \beta \in (\mathbf{-2}, \mathbf{1})$

Solving Case 1

Step 2: D ≥ 0

$$\Rightarrow b^2 - 4ac \geq 0 \Rightarrow \left(\frac{2p}{p-2}\right)^2 - 4(1)\left(\frac{p+3}{p-2}\right) \geq 0 \Rightarrow \frac{p-6}{(p-2)^2} \leq 0$$

$$\Rightarrow p \in (-\infty, 6] - \{2\} = A$$

Step 3: $f(k_1) > 0 \Rightarrow f(-2) > 0 \Rightarrow (-2)^2 + \left(\frac{2p}{p-2}\right)(-2) + \frac{p+3}{p-2} > 0$

$\Rightarrow 4 - \frac{4p}{p-2} + \frac{p+3}{p-2} > 0 \Rightarrow \frac{p-5}{p-2} > 0 \Rightarrow p \in (-\infty, 2) \cup (5, \infty) = B$

Step 4: $f(k_2) > 0 \Rightarrow f(1) > 0 \Rightarrow 1 + \frac{2p}{p-2} + \frac{p+3}{p-2} > 0$

$\Rightarrow \mathrm{p} < -\frac{1}{4}$ *or* $p > 2 \Rightarrow p \in \left(-\infty - \frac{1}{4}\right) \cup (2, \infty) = C$

Step 5: $k_1 < -\frac{b}{2a} < \mathrm{k}_2 \Rightarrow -2 < -\frac{2p}{2(p-2)} < 1$

$\Rightarrow p \in (-\infty, 1) \cup (4, \infty) = D$

Step 6: Solving Case 2 : p - 2 = 0 ⇒ p = 2

$\Rightarrow f(x) = (p-2)x^2 + 2px + (p+3) = 0$ becomes

$2px + p + 3 = 0 \Rightarrow x = \frac{-p-3}{2p} = -\frac{5}{4}$

$x = -\frac{5}{4}$ is an acceptable solution as roots $\in (-2, 1)$

Step 7: Final solution (Case 1) U (Case 2)

Final solution set $p \in \left(-\infty, -\frac{1}{4}\right) \cup (5, 6]$

M A T H E M A T I C S

QUADRATIC EQUATIONS

EQUATIONS REDUCIBLE TO QUADRATIC EQUATIONS

NOTES

What you already know

- Plotting quadratic polynomials,
- Solving quadratic equations
- Nature of roots of quadratic equations

What you will learn

- Solving biquadratic equations of special types

The value(s) of k for which the equation $x^4 + (1-2k)x^2 + (k^2-1) = 0$ has only one real solution is/are

Solution

Let $f(x = x^4 + 1-2k \ x^2 + k^2 - 1 = 0$

Now, let $x^2 = t \Rightarrow t \geq 0$

Thus the equation becomes

$f(t = t^2 + 1-2k \ t + k^2 - 1 = 0$

Let's say this equation will have roots t_1 and t_2

$\Rightarrow x^2 = t_1, x^2 = t_2$

$\Rightarrow x_1 = \sqrt{t_1}, x_2 = -\sqrt{t_1}, x_3 = \sqrt{t_2}, x_4 = -\sqrt{t_2}$

For only one real root

Case 1: $t_1 = t_2 = 0$

$\Rightarrow x_1 = \sqrt{0}, x_2 = -\sqrt{0}, x_3 = \sqrt{0}, x_4 = -\sqrt{0}$

$\Rightarrow x_1 = x_2 = x_3 = x_4 = 0$

a. Therefore, sum of roots $= x_1 + x_2 + x_3 + x_4 = 0$

$\Rightarrow -\dfrac{b}{2a} = 0$

$\Rightarrow (1-2k) = 0$

$\Rightarrow k = \dfrac{1}{2}$

b. Now, product of roots $= x_1x_2x_3x_4 = 0$

$\Rightarrow \dfrac{c}{a} = 0$

$\Rightarrow k^2 - 1 = 0$

$\Rightarrow (k-1)(k+1) = 0$

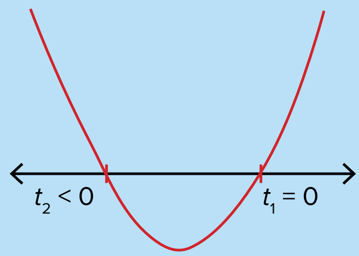
$\Rightarrow k \in \{-1, 1\}$

Taking intersection of (a) and (b), we get, $k \in \Phi$

Case 2: $t_1 = 0, t_2 < 0$

Thus for this to happen, the necessary conditions are:

1. $D > 0$
2. $-\frac{b}{2a} < 0$
3. $f(0) = 0$



Now,

1. $D > 0$

$\Rightarrow (1-2k)^2 - 4(k^2-1) > 0$

$\Rightarrow 1 + 4k^2 - 4k - 4k^2 + 4 > 0$

$\Rightarrow -4k + 5 > 0$

$\Rightarrow -4k > -5$

$\Rightarrow k < \frac{5}{4}$

$\Rightarrow k \in \left(-\infty, \frac{5}{4}\right)$

2. $-\frac{b}{2a} < 0$

$\Rightarrow -\frac{(1-2k)}{2} < 0$

$\Rightarrow 1 - 2k > 0$

$\Rightarrow k < \frac{1}{2}$

$\Rightarrow k \in \left(-\infty, \frac{1}{2}\right)$

3. $f(0) = 0$

$\Rightarrow 0^2 + (1-2k)0 + (k^2 - 1) = 0$

$\Rightarrow k^2 - 1 = 0$

$\Rightarrow k^2 = 1$

$\Rightarrow k = \pm 1$

$\Rightarrow k \in \{-1, 1\}$

Taking intersection of (1), (2), and (3) we get,

Solution set: $k = -1$

Therefore, from Case 1 and 2, solution set for only one real root: $k = -1$

Concept Check

1. What is/are value(s) of k for which the equation $x^4 + (1-2k)x^2 + (k^2 - 1) = 0$ has only non real solutions?

The value of α for which the equation $(x^2 - 2x)^2 - (\alpha + 3)(x^2 - 2x) + (\alpha - 2) = 0$ has four real solutions are

Solution

Let $f(x) = (x^2 - 2x)^2 - (\alpha + 3)(x^2 - 2x) + (\alpha - 2) = 0$

Now, let $x^2 - 2x = t$

$\Rightarrow x^2 - 2x + 1 - 1 = t$

$\Rightarrow (x^2 - 2x + 1) - 1 = t$

$\Rightarrow (x-1)^2 - 1 = t$

Now, since the square of a real number is always positive,

$\Rightarrow (x-1)^2 \geq 0$

Adding -1 to both sides,

$\Rightarrow (x-1)^2 - 1 \geq -1$

Or $t \geq -1$

$\Rightarrow t \in [-1, \infty)$

Thus, substituting t, we get,

$f(x) = t^2 - (\alpha + 3)t + (\alpha - 2) = 0$

Also,

$\Rightarrow (x-1)^2 = t + 1$

$\Rightarrow x - 1 = \pm\sqrt{t+1}$

$\Rightarrow x_1 = 1 + \sqrt{t_1 + 1}, x_2 = 1 - \sqrt{t_1 + 1}$

Similarly,

$x_3 = 1 + \sqrt{t_2 + 1}, x_4 = 1 - \sqrt{t_2 + 1}$

For four real solutions we have,

$x_1, x_2, x_3,$ and x_4 are all real and distinct.

Therefore, t_1 and t_2 must be strictly greater than -1.

$\Rightarrow t_1 > -1$ and $t_2 > -1$

Necessary conditions:

1. $D > 0$
2. $-\frac{b}{2a} > -1$
3. $f(-1) > 0$

Now,

1.$D > 0$

$\Rightarrow (\alpha+3)^2 - 4(\alpha-2) > 0$

$\Rightarrow \alpha^2 + 6\alpha + 9 - 4\alpha + 8 > 0$

$\Rightarrow \alpha^2 + 2\alpha + 17 > 0$

Since, the discriminant (D) of this equation is negative, the equation will have positive values only and hence this equation holds true for all real values of α.

2.$-\frac{b}{2a} > -1$

$\Rightarrow -\left(-\frac{(\alpha+3)}{2}\right) > -1$

$\Rightarrow \alpha + 3 > -2$

$\Rightarrow \alpha > -5$

$\Rightarrow \alpha \in (-5, \infty)$

3.$f(-1) > 0$

$\Rightarrow (-1)^2 - (\alpha + 3)(-1) + (\alpha - 2) > 0$

$\Rightarrow 1 + \alpha + 3 + \alpha - 2 > 0$

$\Rightarrow 2\alpha + 2 > 0$

$\Rightarrow \alpha > -1$

$\Rightarrow \alpha \in (-1, \infty$

Taking intersection of (1), (2), and (3) we get,

Value of α for four real solutions:

$\alpha \in (-1, \infty$

Concept Check

2. What are the values of α for which the equation $(x^2 - 2x^{\,2} - (\alpha + 3)(x^2 - 2x) + (\alpha - 2) = 0$ has two real solutions?

Equations Reducible to Quadratic Forms

In the last session, we solved biquadratic equations by solving the equivalent quadratic equations. Similarly, special polynomial equations of higher degree can be solved.

Type 1: $ax^4 + bx^3 + cx^2 + bx + a = 0, a \neq 0$

As $a \neq 0$, $x = 0$ can not be the root of the given equation.

Divide the given equation by x^2, as $x \neq 0$,

we get : $a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0$

Let $x + \frac{1}{x} = t$

$\Rightarrow x^2 + \frac{1}{x^2} + 2 = t^2$

$\Rightarrow a\left(t^2 - 2\right) + bt + c = 0$

$\Rightarrow at^2 + bt + c - 2a = 0$

$\Rightarrow$ A quadratic equation in t

However, there is one key distinction between a general quadratic equation $ax^2 + bx + c = 0$ and $at^2 + bt + c - 2a = 0$,

i.e., x in $f(x)$ can attain all real values, but t in $f(t)$ is constrained.

Let $y = x + \frac{1}{x}$, $x \in R - \{0\}$

Range of y

$\Rightarrow x^2 + (-y)x + 1 = 0$ (This equation has real roots)

$\Rightarrow D \geq 0$

$\Rightarrow y^2 - 4 \geq 0$

$\Rightarrow (y-2)(y+2) \geq 0$

$\Rightarrow y \in (-\infty, -2] \cup [2, \infty)$

$\Rightarrow t \notin R$

but $t \in (-\infty, -2] \cup [2, \infty)$

Solve $2x^4 - 9x^3 + 14x^2 - 9x + 2 = 0$

Solution

Let $f(x) = 2x^4 - 9x^3 + 14x^2 - 9x + 2 = 0$

Comparing the equation with $ax^4 + bx^3 + cx^2 + dx + e = 0$

$a = e = 2$ and $b = d = -9$

Thus, $f(x)$ follows Type 1 conditions.

Hence, dividing the equation by x^2

$$\Rightarrow 2x^2 - 9x + 14 - \frac{9}{x} + \frac{2}{x^2} = 0$$

$$\Rightarrow 2x^2 + \frac{2}{x^2} - 9x - \frac{9}{x} + 14 = 0$$

$$\Rightarrow 2\left(x^2 + \frac{1}{x^2}\right) - 9\left(x + \frac{1}{x}\right) + 14 = 0 \qquad (1)$$

Now, put $x + \frac{1}{x} = t$ where $t \in (-\infty, -2] \cup [2, \infty)$

On squaring we get, $x^2 + \frac{1}{x^2} + 2 = t^2$

$$\Rightarrow x^2 + \frac{1}{x^2} = t^2 - 2$$

Substituting in equation(1),

$$\Rightarrow 2(t^2 - 2) - 9t + 14 = 0$$

$$\Rightarrow 2t^2 - 9t + 10 = 0$$

Now, splitting the middle term

$$\Rightarrow 2t^2 - 5t - 4t + 10 = 0$$

$$\Rightarrow t(2t - 5) - 2(2t - 5) = 0$$

$$\Rightarrow (2t - 5)(t - 2) = 0$$

$$\Rightarrow t = \frac{5}{2} \text{ or } t = 2$$

Since, both the values lie in the range of t, hence both are acceptable.
Now,

Case 1: $t = \frac{5}{2}$

$$\Rightarrow x + \frac{1}{x} = \frac{5}{2}$$

$$\Rightarrow \frac{x^2 + 1}{x} = \frac{5}{2}$$

$$\Rightarrow 2(x^2 + 1) = 5x$$

$$\Rightarrow 2x^2 - 5x + 2 = 0$$

Splitting the middle term,

$$\Rightarrow 2x^2 - 4x - x + 2 = 0$$

$$\Rightarrow 2x(x - 2) - 1(x - 2) = 0$$

$$\Rightarrow (2x - 1)(x - 2) = 0$$

$$\Rightarrow x = 2, \frac{1}{2}$$

Case 2: $t = 2$

$$\Rightarrow x + \frac{1}{x} = 2$$

$$\Rightarrow \frac{x^2 + 1}{x} = 2$$

$$\Rightarrow x^2 + 1 = 2x$$

$$\Rightarrow x^2 - 2x + 1 = 0$$

Splitting the middle term,

$$\Rightarrow x^2 - x - x + 1 = 0$$

$$\Rightarrow x(x - 1) - 1(x - 1) = 0$$

$$\Rightarrow (x - 1)(x - 1) = 0$$

$$\Rightarrow (x - 1)^2 = 0$$

$$\Rightarrow x = 1$$

Therefore, the roots of the equation are $1, 1, 2, \frac{1}{2}$

Type 2: $x-a)(x-b)(x-c)(x-d)=Ax^2$ where $ab = cd$

$(x^2 + ab - (a + b)x)\ (x^2 + cd - (c + d)x) = Ax^2$

Dividing the general equation by x^2, we get,

$$\Rightarrow \left(x+\frac{ab}{x}-(a+b)\right)\left(x+\frac{cd}{x}-(c+d)\right) = A$$

$$\Rightarrow \left(x+\frac{ab}{x}-(a+b)\right)\left(x+\frac{ab}{x}-(c+d)\right) = A \qquad (As,\ ab = cd)$$

Put $x+\frac{ab}{x} = t$, the equation transforms to

$$\Rightarrow (t-(a+b))(t-(c+d)) = A$$

Solve $(x+2)(x+3)(x+8)(x+12) = 4x^2$

Step 1: Given : $(x+2)(x+3)(x+8)(x+12) = 4x^2$ or $(x+2)(x+12)(x+3)(x+8) = 4x^2$

Comparing the equation with $(x-a)(x-b)(x-c)(x-d) = Ax^2$ and $ab = cd$

$a = -2,\ b = -12,\ c = -3$, and $d = -8$. Here $ab = cd = 24$

Thus the equation follows Type 2 condition.

Step 2 : $(x + 2)(x + 12)(x + 3)(x + 8) = 4x^2$

$$\Rightarrow (x^2 + 14x + 24)(x^2 + 11x + 24) = 4x^2$$

$$\Rightarrow \left(x + \frac{24}{x} + 14\right)\left(x + \frac{24}{x} + 11\right) = 4$$

Let $x + \frac{24}{x} = t \Rightarrow\ t + 14)(t+11) = 4 \Rightarrow t^2 + 25t + 150 = 0$

Here $t \in \left(-\infty, -2\sqrt{24}\right] \cup \left[2\sqrt{24}, \infty\right)$

$\Rightarrow (t+10)(t+15) = 0$

$\Rightarrow t = -10, -15$ (Both acceptable solutions)

Step 3 : Case I : $t = -10$

$\Rightarrow x + \frac{24}{x} = -10$

$\Rightarrow x^2 + 10x + 24 = 0$

$\Rightarrow (x + 4)(x+6) = 0$

$\Rightarrow x = -4,\ -6$

Step 4 : Case II : $t = -15$

$\Rightarrow x + \frac{24}{x} = -15$

$\Rightarrow x^2 + 15x + 24 = 0$

$\Rightarrow x = \frac{-15 \pm \sqrt{15^2 - 4\times 24}}{2} = \frac{-15 \pm \sqrt{129}}{2}$

Therefore the roots are $\frac{-15 \pm \sqrt{129}}{2}, -6,$ and -4.

Type 3 : $(x-a)(x-b)(x-c)(x-d)=A,$

where $a+b=c+d$

$\Rightarrow \left(x^2-(a+b)x+ab\right)\left(x^2-(c+d)x+cd\right)=A$

Let, $x^2-(a+b)x = x^2-(c+d)x = t$

$\Rightarrow (t+ab)(t+cd)=A$

Solve $x+9)(x-3)(x-7)(x+5)=385$

Given, $(x+9)(x-3)(x-7)(x+5)=385$

Now, the equation can be rearranged as:

$x+9)(x-7)(x+5)(x-3)=385$

Comparing this equation with

$(x-a)(x-b)(x-c)(x-d)=A$

Here,

$a=-9,\ b=7,\ c=-5,$ and $d=3$

$\Rightarrow a+b = c+d = -2$

Therefore, the equation follows Type 3 conditions.

Now,

$x+9)(x-7)(x+5)(x-3)=385$

Multiplying the first two and the last two terms,

$\Rightarrow \left(x^2+2x-63\right)\left(x^2+2x-15\right)=385$

Now, put $x^2+2x=t$ (where $t\in[-1,\infty)$)

$\Rightarrow (t-63)(t-15)=385$

$\Rightarrow t^2-15t-63t+945=385$

$\Rightarrow t^2-78t+560=0$

Splitting the middle term,

$\Rightarrow t^2-70t-8t+560=0$

$\Rightarrow t\ t-70)-8(t-70)=0$

$\Rightarrow (t-8)(t-70)=0$

$\Rightarrow t=8\ or\ t=70$

Since both the values lies in the range of *t*, hence both are acceptable.
Now,

Case 1: $t = 70$

$\Rightarrow x^2 + 2x = 70$

$\Rightarrow x^2 + 2x - 70 = 0$

Using Quadratic formula,

$$\Rightarrow x = \frac{-2 \pm \sqrt{4 + 4 \times 70}}{2}$$

$$\Rightarrow x = \frac{-2 \pm \sqrt{284}}{2}$$

$\Rightarrow x = -1 \pm \sqrt{71}$

Case 2: $t = 8$

$\Rightarrow x^2 + 2x = 8$

$\Rightarrow x^2 + 2x - 8 = 0$

Splitting the middle term,

$\Rightarrow x^2 + 4x - 2x - 8 = 0$

$\Rightarrow x \ \ x + 4) - 2(x + 4) = 0$

$\Rightarrow (x - 2)(x + 4) = 0$

$\Rightarrow x = 2, -4$

Therefore, the roots are $2,\ -4,\ -1 \pm \sqrt{71}$

Concept Check

3. Solve $x^4 + 4x^2 - 21 = 0$

Summary sheet

Biquadratic equations reducible to quadratic equations can be classified into various types

Key Takeaways

1. Type 1: When leading coefficient and constant term are same and coefficient of x^3 and x are same;

   $ax^4 + bx^3 + cx^2 + bx + a = 0$

   Put $x + \frac{1}{x} = t$

2. **Type 2:** $(x-a)(x-b)(x-c)(x-d) = Ax^2$, where $ab = cd$

   Put $x + \frac{ab}{x} = t$

3. **Type 3:** When a + b = c +d

   $x-a)(x-b)(x-c)(x-d) = A$

   where $a + b = c + d$

   Put $x^2 - (a+b)x = t$

Mind map

$ax^4 + bx^3 + cx^2 + dx + e = 0$

General biquadratic equation,

Type 1:

$ax^4 + bx^3 + cx^2 + bx + a = 0$

Put $x + \frac{1}{x} = t$

Type 2:

$(x-a)(x-b)(x-c)(x-d) = Ax^2$

where $ab = cd$; Put $x + \frac{ab}{x} = t$

Type 3:

When $a + b = c + d$

$x-a)(x-b)(x-c)(x-d) = A$

Put $x^2 - (a+b)x = t$

Self-Assessment

1. The roots of $2x^4 + x^3 - 11x^2 + x + 2 = 0$ are ?

2. Solve $\frac{5x}{x^2+3x+1} + \frac{7x}{x^2+5x+1} = 2$

Answer

Concept Check 1

Let $f(x) = x^4 + (1 - 2k)x^2 + (k^2 - 1) = 0$

Now, let $x^2 = t \Rightarrow t \geq 0$

Thus the equation becomes

$f(t) = t^2 + (1 - 2k)t + (k^2 - 1) = 0$

Let's say this equation will have roots t_1 and t_2

$\Rightarrow x^2 = t_1, x^2 = t_2,$

For no real solutions

Case 1: $t_1 < 0$ and $t_2 < 0$

Therefore, the necessary conditions are:

$1. D \geq 0$

$2. -\frac{b}{2a} < 0$

$3. f\ 0) > 0$

$1. D \geq 0$

$\Rightarrow (1-2k)^2 - 4(k^2-1) \geq 0$

$\Rightarrow 1 + 4k^2 - 4k - 4k^2 + 4 \geq 0$

$\Rightarrow -4k + 5 \geq 0$

$\Rightarrow -4k \geq -5$

$\Rightarrow k \leq \frac{5}{4}$

$\Rightarrow k \in \left(-\infty, \frac{5}{4}\right]$

$2. -\frac{b}{2a} < 0$

$\Rightarrow -\frac{(1-2k)}{2} < 0$

$\Rightarrow (1 - 2k) > 0$

$\Rightarrow k < \frac{1}{2}$

$\Rightarrow k \in \left(-\infty, \frac{1}{2}\right)$

$3. f(0) > 0$

$\Rightarrow 0^2 + (1-2k)0 + (k^2-1) > 0$

$\Rightarrow k^2 - 1 > 0$

$\Rightarrow (k-1)(k+1) > 0$

$\Rightarrow k \in (-\infty, -1) \cup (1, \infty)$

Thus, from (1), (2), and (3), we get the solution set:

$k \in (-\infty, -1)$

Case 2: Both t_1 and t_2 are imaginary.

Necessary conditions:

D < 0

$\Rightarrow 1-2k)^2 - 4(k^2-1) < 0$

$\Rightarrow 1 + 4k^2 - 4k - 4k^2 + 4 < 0$

$\Rightarrow -4k + 5 < 0$

$\Rightarrow k > \frac{5}{4}$

$\Rightarrow k \in \left(\frac{5}{4}, \infty\right)$

Therefore, from Case 1 and 2, we get the solution set for no solution:

$k \in (-\infty, -1) \cup \left(\frac{5}{4}, \infty\right)$

Concept Check 2

Let $f(x) = (x^2 - 2x)^2 - (a + 3)(x^2 - 2x) + (a - 2) = 0$

Now, let $x^2 - 2x = t$

$\Rightarrow x^2 - 2x + 1 - 1 = t$

$\Rightarrow (x^2 - 2x + 1) - 1 = t$

$\Rightarrow (x - 1)^2 - 1 = t$

Observe, $(x - 1)^2 - 1 = t$

$\Rightarrow (x-1)^2 = t+1 \Rightarrow x-1 = \pm\sqrt{t+1} \Rightarrow x_1 = 1+\sqrt{t_1+1},\ x_2 = 1-\sqrt{t_1+1}$

Similarly, $x_3 = 1+\sqrt{t_2+1},\ x_4 = 1-\sqrt{t_2+1}$

Case 1 : $t_1 = t_2 > -1$

Therefore, the necessary conditions are:

$1.\ D = 0$

$2.\ -\dfrac{b}{2a} > -1$

$3.\ f(-1) > 0$

Now,

1. D = 0

$\Rightarrow (\alpha + 3)^2 - 4(\alpha - 2) = 0$

$\Rightarrow \alpha^2 + 6\alpha + 9 - 4\alpha + 8 = 0$

$\Rightarrow \alpha^2 + 2\alpha + 17 = 0$ (i)

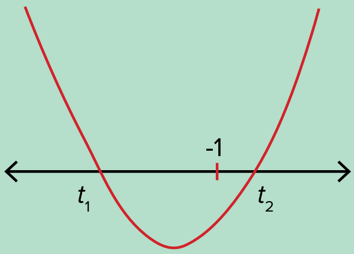
Since discriminant of (i) is negative

(D = 4 - 68 = - 64).

Hence, equation will have no real roots.

$\Rightarrow \alpha = \Phi$

Case 2 : $t_1 > -1$ and $t_2 < -1$



Necessary conditions:

$\Rightarrow f(-1) < 0$

$\Rightarrow (-1)^2 - (\alpha+3)(-1) + (\alpha-2) < 0$

$\Rightarrow 1 + \alpha + 3 + \alpha - 2 < 0$

$\Rightarrow 2\alpha + 2 < 0$

$\Rightarrow \alpha < -2$

$\Rightarrow \alpha \in (-\infty, -1)$

$2.\ -\dfrac{b}{2a} > -1$

$\Rightarrow -\left(-\dfrac{(\alpha+3)}{2}\right) > -1$

$\Rightarrow \alpha + 3 > -2$

$\Rightarrow \alpha > -5$

$\Rightarrow \alpha \in (-5, \infty)$

$3.\ f(-1) > 0$

$\Rightarrow (-1)^2 - (\alpha+3)(-1) + (\alpha-2) > 0$

$\Rightarrow 1 + \alpha + 3 + \alpha - 2 > 0$

$\Rightarrow 2\alpha + 2 > 0$

$\Rightarrow \alpha > -1$

$\Rightarrow \alpha \in (-1, \infty)$

Taking intersection of (1), (2), and (3) we get,

Solution set: Φ

Therefore, from Case 1 and 2 we get,

The values of for two real solutions: $\alpha \in (-\infty,\ -1)$

Concept Check 3

We have, $x^4+4x^2-21=0$

Put $x^2=t$ where $t\in[0,\infty)$

$\Rightarrow t^2+4t-21=0$

$\Rightarrow (t-3)(t+7)=0$

$\Rightarrow t=3$ *or* $t=-7$

Since, $t=-7$ is not in the range of t, we will reject this value

$\Rightarrow t=3$

Or $x^2=3$

$\Rightarrow x=\pm\sqrt{3}$

Self-assessment

1.

Step 1: Given equation $2x^4+x^3-11x^2+x+2=0$

Dividing the equation by x^2, as $x=0$, is not the root of given equation.

$$\Rightarrow 2x^2+x-11+\frac{1}{x}+\frac{2}{x^2}=0 \Rightarrow 2\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)-11=0$$

$$\Rightarrow 2\left(\left(x+\frac{1}{x}\right)^2-2\right)+\left(x+\frac{1}{x}\right)-11=0 \Rightarrow 2\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)-15=0$$

Step 2 : Let $x+\frac{1}{x}=t$

$\Rightarrow 2t^2+t-15=0$

$\Rightarrow (t+3)(2t-5)=0$

$\Rightarrow t=-3, \frac{5}{2}$

Both are acceptable solution as $t\in(-\infty,-2]\cup[2,\ \infty)$

Step 3 : For $t=-3$

$\Rightarrow x+\frac{1}{x}=-3$

$\Rightarrow x^2+3x+1=0$

$$\Rightarrow x=\frac{-3\pm\sqrt{9-4}}{2}=\frac{-3\pm\sqrt{5}}{2}$$

Step 4: For $t=\frac{5}{2}$ $\Rightarrow x+\frac{1}{x}=\frac{5}{2}$ $\Rightarrow 2x^2-5x+2=0$ $\Rightarrow (2x-1)(x-2)=0$ $\Rightarrow x=\frac{1}{2},2$

Step 5 : Four roots of equation $2x^4+x^3-11x^2+x+2=0$ are $\frac{-3\pm\sqrt{5}}{2}, \frac{1}{2}$, and 2

2.

Step 1: Given $\frac{5x}{x^2+3x+1}+\frac{7x}{x^2+5x+1}=2$

Dividing numerator and denominator by x

$\Rightarrow \frac{5}{x+3+\frac{1}{x}}+\frac{7}{x+5+\frac{1}{x}}=2$

Let $x+\frac{1}{x}=t \quad \Rightarrow \frac{5}{t+3}+\frac{7}{t+5}=2$

Note $t\in(-\infty,-2]\cup[2,\infty)$

Step 2: $\frac{5}{t+3}+\frac{7}{t+5}=2$

$\Rightarrow t^2+2t-8=0$

$\Rightarrow (t+4)(t-2)=0$

$\Rightarrow t=-4,\ 2$ (Both acceptable solutions)

Step 3: Case I: $t=2$

$\Rightarrow x+\frac{1}{x}=2$

$\Rightarrow x^2-2x+1=0$

$\Rightarrow (x-1)^2=0$

$\Rightarrow x=1$

Step 4: Case II: $t=-4$

$\Rightarrow x+\frac{1}{x}=-4$

$\Rightarrow x^2+4x+1=0$

$\Rightarrow x=\frac{-4\pm\sqrt{16-4}}{2}=-2\pm\sqrt{3}$

Therefore, the roots are 1, $-2\pm\sqrt{3}$

MATHEMATICS

QUADRATIC EQUATIONS

THEORY OF EQUATIONS

NOTES

What you will learn

- Difference between a polynomial and an identity on the basis of roots
- Nature of Roots
- Factor Theorem
- Formulae of roots of polynomial with several degrees

What you already know

- Polynomial equations
- Identities
- Zeros of polynomial functions

Important Results

- A polynomial equation of degree n has n roots (real or imaginary).
- If the number of roots is greater than the degree of polynomial equation, then it becomes an identity.
- If all the coefficients are real, then the imaginary roots, if any, always occur in conjugate pairs i.e., the number of imaginary roots is always even.
- If the degree is even, then the number of real roots is also even.
- If the degree is odd, then the number of real roots is also odd. In fact, at least one root is always real.
- Factor theorem: $x = k$ is a root of $f(x) = 0 \Leftrightarrow (x - k)$ is a factor of $f(x)$

Relation between roots and coefficients of the polynomial

Given: $f(x) = ax^3 + bx^2 + cx + d = 0$ with roots α, β and γ

$$ax^3 + bx^2 + cx + d = a(x - \alpha)(x - \beta)(x - \gamma)$$

$$= a(x - \alpha)(x^2 - (\beta + \gamma)x + \beta\gamma) = a(x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma)$$

(i) Sum of the roots	$= \alpha + \beta + \gamma$	$= \frac{-b}{a} = \frac{-\text{Coeff. of } x^2}{\text{Coeff. of } x^3}$
(ii) Sum of the product of roots taken two at a time	$= \alpha\beta + \beta\gamma + \alpha\gamma$	$= \frac{c}{a} = \frac{\text{Coeff. of } x^2}{\text{Coeff. of } x^3}$
(iii) Product of the roots	$= \alpha\beta\gamma$	$= \frac{-d}{a} = \frac{-\text{Constant term}}{\text{Coeff. of } x^3}$

Given: $f(x) = ax^4 + bx^3 + cx^2 + dx + e = 0$ with roots ∝, β, γ and δ

$ax^4 + bx^3 + cx^2 + dx + e = a(x - \propto)(x - \beta)(x - \gamma)(x - \delta)$

$= a(x^4 - (\propto + \beta + \gamma + \delta)x^3 + (\propto\beta + \beta\gamma + \propto\gamma + \propto\delta + \beta\delta + \gamma\delta)x^2 - (\propto\beta\gamma + \beta\gamma\delta + \propto\gamma\delta + \propto\beta\delta)x + \propto\beta\gamma\delta)$

(i) Sum of the roots	$= \propto + \beta + \gamma + \delta$	$= -\frac{b}{a} = -\frac{\text{Coeff. of } x^3}{\text{Coeff. of } x^4}$
(ii) Sum of product of roots taken two at a time	$= \propto\beta + \beta\gamma + \gamma\delta + \delta\propto + \propto\gamma + \beta\delta$	$= \frac{c}{a} = \frac{\text{Coeff. of } x^2}{\text{Coeff. of } x^4}$
(iii) Sum of product of roots taken three at a time	$= \propto\beta\gamma + \beta\gamma\delta + \gamma\delta\propto + \delta\propto\beta$	$= -\frac{d}{a} = \frac{-\text{Coeff. of } x}{\text{Coeff. of } x^4}$
(iv) Product of roots	$= \propto\beta\gamma\delta$	$= \frac{e}{a} = \frac{\text{Constant term}}{\text{Coeff. of } x^4}$

Given: $f(x) = a_0x^n + a_1x^{n-1} + \ldots.. + a_{n-1}x^1 + a_n = 0$ with roots $\propto_1, \propto_2, \ldots \propto_{n-1}, \& \propto_n$

(i) Sum of the roots	$= \propto_1 + \propto_2 + \propto_3 + \ldots$	$= \frac{-a_1}{a_0} = \frac{-\text{Coeff. of } x^{n-1}}{\text{Coeff. of } x^n}$
(ii) Sum of the product of roots taken two at a time	$= \propto_1\propto_2 + \propto_1\propto_3 + \propto_1\propto_4 + \ldots$	$= \frac{a_2}{a_0} = \frac{\text{Coeff. of } x^{n-2}}{\text{Coeff. of } x^n}$
(iii) Sum of the product of roots taken three at a time	$= \propto_1\propto_2\propto_3 + \propto_1\propto_2\propto_4 + \propto_1\propto_2\propto_5 + \ldots$	$= \frac{-a_3}{a_0} = \frac{-\text{Coeff. of } x^{n-3}}{\text{Coeff. of } x^n}$
(iv) Product of roots	$= \propto_1\propto_2\propto_3\propto_4\ldots$	$= \frac{(-1)^n a_n}{a_0} = \frac{(-1)^n\text{Constant term}}{\text{Coeff. of } x^3}$

Let ∝, β, γ and δ be the real roots of the equation $x^4 - 9x^3 + 28x^2 - \lambda x + 12 = 0$, such that $\gamma\delta = 2$. Then the value of λ is given by:

(a) 34 (b) -34 (c) 4 (d) -4

Step 1: Given $\gamma\delta = 2$

(i) Sum of the roots	$= \propto + \beta + \gamma + \delta$	$= -\frac{b}{a} = -\frac{\text{Coeff. of } x^3}{\text{Coeff. of } x^4} = 9$
(ii) Sum of the product of roots taken two at a time	$= \propto\beta + \beta\gamma + \gamma\delta + \delta\propto + \propto\gamma + \beta\delta$	$= \frac{c}{a} = \frac{\text{Coeff. of } x^2}{\text{Coeff. of } x^4} = 28$
(iii) Sum of the product of roots taken three at a time	$= \propto\beta\gamma + \beta\gamma\delta + \gamma\delta\propto + \delta\propto\beta$	$= -\frac{d}{a} = \frac{-\text{Coeff. of } x}{\text{Coeff. of } x^4} = \lambda$
(iv) Product of roots	$= \propto\beta\gamma\delta$	$= \frac{e}{a} = \frac{\text{Constant term}}{\text{Coeff. of } x^4} = 12$

Step 2: Given $\gamma\delta = 2$

$\propto + \beta + \gamma + \delta = 9$

$\beta\gamma + \delta\propto + \propto\gamma + \beta\delta = 20$ (As $\gamma\delta = 2$ & $\propto\beta = 6$)

$(\propto + \beta)(\gamma + \delta) = 20$

∴ $(\propto + \beta)$ & $(\gamma + \delta)$ are roots of the equation

$x^2 - 9x + 20 = 0 \Rightarrow (x - 5)(x - 4) = 0$

$\Rightarrow (\propto + \beta) = 5$ & $(\gamma + \delta) = 4$

Step 3:

$\lambda = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta$

$= \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta)$

$\Rightarrow \lambda = 6(4) + 2(5)$

$= 34$

Concept Check 1

If α, β and γ are the roots of $x^3 + qx + r = 0$, then the equation whose roots are ∝ + β, β + γ and γ + α is

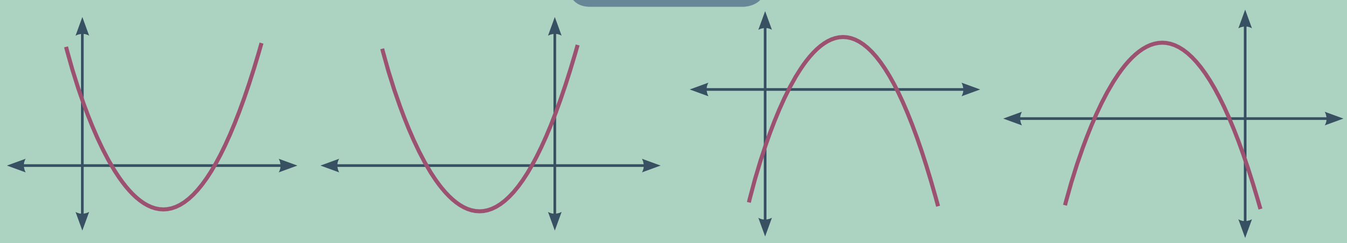
(a) $-2x^3 - qx + 2r = 0$ (b) $-x^3 + qx + r = 0$ (c) $x^3 + qx - r = 0$ (d) $-2x^3 + qx + 2r = 0$

If $f(x) = ax^2 + bx + c = 0$; $|b| > |a+c|$, then
(a) one root of $f(x) = 0$ is positive, the other root is negative
(b) exactly one of the roots of $f(x) = 0$ lies in (-1, 1)
(c) 1 lies between the roots of $f(x) = 0$
(d) both the roots of $f(x) = 0$ are less than 1

Step 1: Given $f(x) = ax^2 + bx + c = 0$; $|b| > |a + c|$
$\Rightarrow b^2 > (a + c)^2 \Rightarrow (a + c + b)(a + c - b) < 0$

Step 2: Note $f(1)f(-1) = (a + b + c)(a + b - c) < 0$ Recall, exactly one root lies in (k_1, k_2) *iff* $f(k_1)\ f(k_2) < 0$
$\Rightarrow$ There is exactly one root in (-1,1)

We have, $f(x) = ax^2 + bx + c = 0$ $f(-1)\ f(1) < 0$



Therefore, the final solution is option (b).

Concept Check 2

If $f(x) = ax^2 + bx + c = 0$; $a(a + b + c) < 0 < c(a + b + c)$, then
(a) one root of $f(x) = 0$ is less than 0, the other root is greater than 1
(b) exactly one of the roots of $f(x) = 0$ lies in (0,1)
(c) both the roots of $f(x) = 0$ lie in (0,1)
(d) atleast one of the roots of $f(x) = 0$ lies in (0,1)

The equation $\left(\frac{(x^2 + ax + b)}{(x^2 + x + 1)}\right) = c$ has three distinct real roots, then the value of $a + b + c$ will be equal to:

(a) 1 (b) 2 (c) 3 (d) 4

Step 1: Given $\frac{(x^2 + ax + b)}{(x^2 + x + 1)} = c \Rightarrow \frac{x^2 + ax + b}{x^2 + x + 1} - c = 0 \Rightarrow \frac{(1 - c)x^2 + (a - c)x + b - c}{x^2 + x + 1} = 0$

Step 2: $(1 - c)x^2 + (a - c)x + b - c = 0$ has more than 2 roots, therefore its an identity
$\Rightarrow 1 - c = a - c = b - c = 0 \Rightarrow a = b = c = 1 \Rightarrow a + b + c = 3$

Concept Check 3

If $ax^2 - 2bx + c = 0$ and $px^2 - 2qx + r = 0$ have a root in common and $b^2 = ac$, then which of the following option(s) holds true?

(a) $\frac{a}{p} = \frac{b}{q} = \frac{c}{r}$ (b) $q^2 = pr$ (c) $pc + ra = 2bq$ (d) $b^2pr = q^2ac$

The roots of the equation $6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$ are:

(a) $-3, -\frac{1}{3}, \frac{1}{2}, 1, 2$ (b) $-2, -\frac{1}{3}, \frac{1}{2}, 1, 2$ (C) $-3, -1, -\frac{1}{3}, \frac{1}{2}, 2$ (d) $3, -\frac{1}{2}, -\frac{1}{3}, 1, 2$

Step 1: Given $f(x) = 6x^5 - x^4 - 43x^3 + 43x^2 + x - 6$

Observe $f(1) = 0 \Rightarrow x = 1$ is a root of $f(x) \Rightarrow (x - 1)$ is a factor of $f(x) \Rightarrow f(x) = (x - 1)\, q(x)$

Step 2: Using Synthetic Division Method or Long Division Method $q(x) = 6x^4 + 5x^3 - 38x^2 + 5x + 6$ such that $f(x) = (x - 1)\, q(x)$

Dividing $q(x) = 0$ by x^2, we get $6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0$

Let $t = x + \frac{1}{x}$, where $t \in (-\infty, -2] \cup [2, \infty)$

$\Rightarrow 6(t^2 - 2) + 5t - 38 = 0 \Rightarrow 6t^2 + 5t - 50 = 0 \Rightarrow (3t + 10)(2t - 5) = 0$

$\Rightarrow t = -\frac{10}{3}, \frac{5}{2}$ (Both accepatable solutions)

Step 3: $t = -\frac{10}{3} = x + \frac{1}{x} \Rightarrow 3x^2 + 10x + 3 = 0$

$\Rightarrow (3x + 1)(x + 3) = 0 \Rightarrow x = -3, -\frac{1}{3}$

Step 4: $t = \frac{5}{2} = x + \frac{1}{x} \Rightarrow 2x^2 - 5x + 2 = 0$

$\Rightarrow (2x - 1)(x - 2) = 0 \Rightarrow x = \frac{1}{2}, 2$

Therefore the roots of $f(x) = 0$ are $-3, -\frac{1}{3}, \frac{1}{2}, 1, 2$

Summary

Key Takeaways

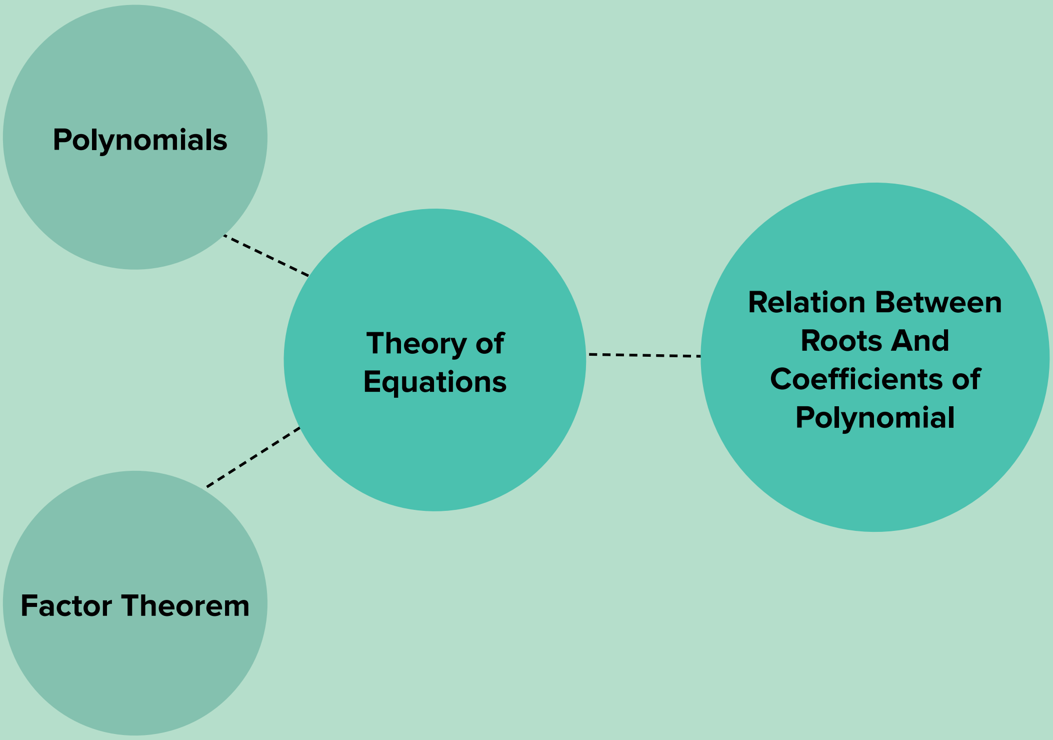
- A polynomial equation of degree n has n roots(real or imaginary).
- If the number of roots is greater than the degree of polynomial equation, then it becomes an identity.
- If all coefficients of polynomial equation are real, then the imaginary roots (if any) occur in conjugate pairs i.e., the number of imaginary roots is always even.
- If the degree of polynomial equation is even, and all coefficients are real , then the number of real roots is also even.
- If the degree of polynomial equation is odd and coefficients are real, then the number of real roots is also odd. In fact, at least one root is always real.
- Factor theorem: k is a root of $f(x) = 0 \Leftrightarrow (x - k)$ is a factor of $f(x)$

Key Results

Given: $f(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x^1 + a_n = 0$ with roots $\propto_1, \propto_2, ... \propto_{n-1}, \& \propto_n$

(i) Sum of the roots	$= \propto_1 + \propto_2 + \propto_3 + ...$	$= \frac{-a_1}{a_0} = \frac{\text{- Coeff. of } x^{n-1}}{\text{Coeff. of } x^n}$
(ii) Sum of the product of roots taken two at a time	$= \propto_1\propto_2 + \propto_1\propto_3 + \propto_1\propto_4 + ...$	$= \frac{a_2}{a_0} = \frac{\text{Coeff. of } x^{n-2}}{\text{Coeff. of } x^n}$
(iii) Sum of the product of roots taken three at a time	$= \propto_1\propto_2\propto_3 + \propto_1\propto_2\propto_4 + \propto_1\propto_2\propto_5 + ...$	$= \frac{-a_3}{a_0} = \frac{\text{- Coeff. of } x^{n-3}}{\text{Coeff. of } x^n}$
(iv) Product of roots	$= \propto_1\propto_2\propto_3\propto_4 ...$	$= \frac{(-1)^n a_n}{a_0} = \frac{(-1)^n \text{ Constant term}}{\text{Coeff. of } x^3}$

Mind Map



Self-Assessment

1. If p(x) is a polynomial of degree greater than 2 such that p(x) leaves remainder a and - a when divided by x + a and x - a respectively. If p(x) is divided by $x^2 - a^2$ then what is the remainder ?

2. If $x^3 + 5x^2 + px + q = 0$ and $x^3 + 7x^2 + px + r = 0$ have two roots in common and their third roots are γ_1 and γ_2 respectively, then what is the value of $|\gamma_1 + \gamma_2|$?

Answers

Concept check 1

Step 1: Given f(x) $= x^3 + qx + r = 0$ has roots α, β, and γ

$\alpha + \beta + \gamma = 0 \Rightarrow \alpha + \beta = -\gamma;\ \beta + \gamma = -\alpha;\ \alpha + \gamma = -\beta$

$\alpha\beta + \beta\gamma + \gamma\alpha = q\ \&\ \alpha\beta\gamma = -r$

∴ Equation with roots $(\alpha + \beta)$, $(\beta + \gamma)$ & $(\alpha + \gamma)$ is same as equation with roots $-\alpha$, $-\beta$ and $-\gamma$

Step 2: Required transformation of cubic equation with roots $-\alpha, -\beta, -\gamma$

$f(x) = 0$ has roots $\alpha, \beta, \gamma \Rightarrow f(-x) = 0$ will have roots $-\alpha, -\beta, -\gamma$

$f(-x) = -x^3 - qx + r = 0 \Rightarrow x^3 + qx - r = 0$

Concept check 2

Step 1: Given $f(x) = ax^2 + bx + c = 0$ & $a(a + b + c) < 0 < c(a + b + c)$

Observe $f(1) = a + b + c$ & $f(0) = c$

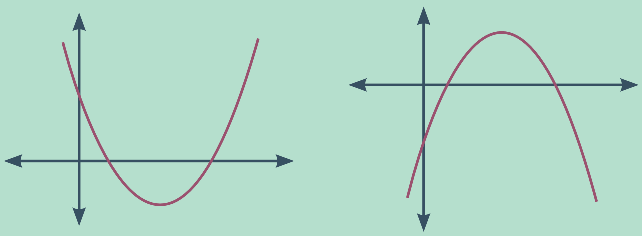
$\Rightarrow af(1) = a(a + b + c) < 0$ ...(I) & $f(0)\,f(1) = c(a + b + c) > 0$ ....(II)

Step 2: Condition I: $af(1) < 0 \Rightarrow a$ & $f(1)$ are of opposite signs

Case 1: $a > 0$ & $f(1) < 0$ Case 2: $a < 0$ & $f(1) > 0$

$\Rightarrow$ 1 lies between the roots of $f(x) = 0 \Rightarrow \alpha < 1 < \beta$

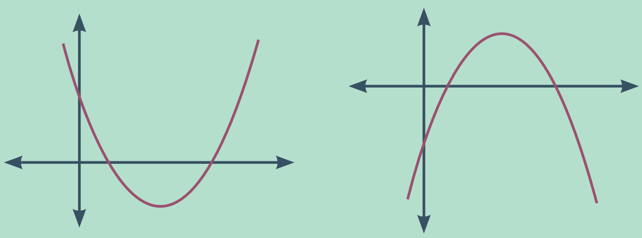
$af(1) < 0 \Rightarrow a$ & $f(1)$ are of opposite signs



Both roots of $f(x) = 0$ lie on opposite sides of 1 or 1 lies between the roots.

Step 3: Condition II: $f(0)f(1) > 0 \Rightarrow f(0)$ & $f(1)$ have the same sign. From condition 1, we know 1 lies between the roots of $f(x) = 0$. As $f(0)$ & $f(1)$ have same sign $\Rightarrow$ 0 also lies between the roots of $f(x) = 0 \Rightarrow \alpha < 0 < 1 < \beta$

$f(0)f(1) > 0 \Rightarrow f(0)$ & $f(1)$ are of the same signs



Therefore, one root of $f(x) = 0$ is less than 0 and other root is greater than 1

Concept check 3

Step 1: Let $f(x) = ax^2 - 2bx + c$; $g(x) = px^2 - 2qx + r$

Now, Discriminant D of $f(x) = (2b)^2 - 4ac = 0$ ($\because b^2 = ac$)

$\Rightarrow f(x) = 0$ has equal roots α, α.

Here $\alpha = \frac{b}{a}$

Step 2: As both the equations $f(x) = 0$ and $g(x) = 0$ have a common root, $\frac{b}{a}$ is a root of $g(x) = 0$

$\Rightarrow g\left(\frac{b}{a}\right) = p\left(\frac{b}{a}\right)^2 - 2q\left(\frac{b}{a}\right) + r = 0 \Rightarrow pb^2 - 2qab + ra^2 = 0$

$\Rightarrow pac + ra^2 = 2qab \Rightarrow pc + ra = 2bq$ ($\because b^2 = ac$ & $a \neq 0$)

Self- Assessment

1. **Step 1**:

   As the degree of remainder is always lesser than the degree of the divisor, we have taken remainder with degree less than 2 here.

   Let $ux + v$ be the remainder when $p(x)$ is divided by $x^2 - a^2$

   Using division algorithm, $p(x) = (x^2 - a^2)\, q(x) + (ux + v)$...(i) where $q(x)$ is the quotient.

   We know that, by remainder theorem, $p(-a) = a$, and $p(a) = -a$.

   Putting $x = -a$ and $x = a$ in the equation (i)

   $a = -ua + v$ and $-a = ua + v$. Solving them simultaneously, $v = 0$ and $u = -1$.

   Hence, the required remainder is: $ux + v = -x + 0 = -x$

2. **Step 1:**

   Let the roots of $x^3 + 5x^2 + px + q = 0$ be α, β, and γ_1 ....(i)

   Roots of $x^3 + 7x^2 + px + r = 0$ are α, β, and γ_2 .....(ii)

   Subtracting equation (ii) from (i),

   $\Rightarrow -2x^2 + q - r = 0$ ..(iii)

   As α and β are common roots, so the roots of the equation (iii) are α and β

   $\Rightarrow \alpha + \beta = 0$

   Step 2:

   From equation (i)

   $\alpha + \beta + \gamma_1 = -5 \Rightarrow \gamma_1 = -5$

   Similarly, from equation (ii)

   $\alpha + \beta + \gamma_2 = -7 \Rightarrow \gamma_2 = -7$

   Therefore, $\gamma_1 + \gamma_2 = -12 \Rightarrow |\gamma_1 + \gamma_2| = 12$