CHAPTER 12

Linear Programming

Summary

- **Linear Programming**

  Linear programming is a method which provides the optimization (maximization or minimization) of a linear function composed of certain variables subject to the number of constraints.
- **Applications of Linear Programming**
  - Used in finding highest margin, maximum profit, minimum cost etc.
  - Used in industry, commerce, management science etc.
- **Linear Programming Problem (LPP)**

  Linear Programming problem is a type of problem in which a linear function z is maximized or minimized on certain conditions that are determined by a set of linear inequalities with non-negative variables.
- **Mathematical Formulation of LPP**
  - **Optimal value:** Maximum or Minimum value of a linear function
  - **Objective Function:** The function which is to be optimized (maximized/minimized).
  - **Linear objective function:** $Z = ax + by$ is a linear function form, where a, b are constants, which has to be maximized or minimized is called a linear objective function.

    For example- $Z = 340x + 60y$, where variables x and y are called **decision variables**.
- **Constraints:** The limitations as disparities on the factors of a LPP are called constraints. The conditions $x \geq 0$, $y \geq 0$ are called **non-negative restrictions**.
- **'Linear'** states that all mathematical relations used in the problem are linear relations. **Programming** refers to the method of determining a particular program or plan of action.
- **Mathematical Formulation of the Problem**

  A general LPP can be stated as (Max/Min) $Z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$ subject to given constraints and the non-negative restrictions.
  - $x_1, x_2, \ldots\ldots, x_n \geq 0$ and all are variables.
  - $c_1, c_2, \ldots\ldots c_n$ are constants.
- **Graphical methods to solve a Linear Programming Problem**
- **Corner Point method:** This method is used to solve the LPP graphically by finding the corner points.

  Procedure-

  (i) Replace the signs of inequality by the equality and consider each constraint as an equation.

  (ii) Plotting each equation on the graph that will represent a straight line.

  (iii) The common region that satisfies all the constraints and the non-negative restrictions is known as the **feasible region**. It is a convex polygon.

  (iv) Determining the vertices of the convex polygon. These vertices of the polygon are also known as the extreme points or corners of the feasible region.

  (v) Finding the values of Objective function at each of the extreme points. Now, finding the point at which the value of the objective function is optimum as that is the optimal solution of the given LPP.
- **General features of a LPP**
  - The feasible region is always a convex region.
  - The maximum (or minimum) solution of the objective function occurs at the vertex (corner) of the feasible region.
  - If two corner points produce the same optimum (maximum or minimum) value of the objective function, then every point on the line segment joining these points will also give the same optimum (maximum or minimum) value.
- **Different Types of Linear Programming Problems**
  - **Manufacturing problems**

    In order to make maximum profit, determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed manpower, machine hours, warehouse space per unit of the output etc., in order to make maximum profit.
  - **Diet problems**

    Determining the minimum amount of different nutrients which should be included in a diet so as to minimize the cost of the diet.

- **Transportation problems**
  To find the cheapest way of transporting a product from factories situated at different locations to different markets.
- **Allocation problems**
  These problems are concerned with the allocation of a particular land/area of a company or any organization by choosing a certain number of employees and a certain amount of area to complete the assignment within the required deadline, given that a single person works on only one job within the assignment.

EXERCISE

1. For the constraint of a linear optimizing function $z = x_1 + x_2$, given by $x_1 + x_2 \leq 1$, $3x_1 + x_2 \geq 3$ and $x_1, x_2 \geq 0$

(*a*) There are two feasible regions
(*b*) There are infinite feasible regions
(*c*) There is no feasible regions
(*d*) None of these

2. Which of the following is not a vertex of the positive region bounded by the inequalities $2x + 3y \leq 6$, $5x + 3y \leq 15$ and $x, y \geq 0$

(*a*) (0, 2) (*b*) (0, 0)
(*c*) (3, 0) (*d*) None of these

3. The intermediate solutions of constraints must be checked by substituting them back into

(*a*) Objective function
(*b*) Constraint equations
(*c*) Not required
(*d*) None of these

4. A basic solution is called non-degenerate, if

(*a*) All the basic variables are zero
(*b*) None of the basic variables is zero
(*c*) At least one of the basic variables is zero
(*d*) None of these

5. If the number of available constraints is 3 and the number of parameters to be optimized is 4, then

(*a*) The objective function can be optimized
(*b*) The constraints are short in number
(*c*) The solution is problem oriented
(*d*) None of these

6. Objective function of a linear programming problem is

(*a*) A constraint
(*b*) A function to be optimized
(*c*) A relation between the variables
(*d*) None of these

7. If the constraints in a linear programming problem are changed

(*a*) The problem is to be re-evaluated
(*b*) Solution is not defined
(*c*) The objective function has to be modified
(*d*) The change in constraints is ignored

8. Which of the following statements is correct

(*a*) Every linear programming problem admits an optimal solution
(*b*) A linear programming problem admits a unique optimal solution
(*c*) If a linear programming problem admits two optimal solutions, it has an infinites number of optimal solution
(*d*) The set of all feasible solutions of a linear programming is not a convex set

9. The minimum value of linear objective function $c = 2x + 2y$ under linear constraints $3x + 2y \geq 12$, $x + 3y \geq 11$ and $x, y \geq 0$, is

(*a*) 10 (*b*) 12
(*c*) 6 (*d*) 5

10. The maximum value of $P = x + 3y$ such that $2x + y \leq 20$, $x + 2y \leq 20$, $x \geq 0$, $y \geq 0$, is

(*a*) 10 (*b*) 60
(*c*) 30 (*d*) None of these

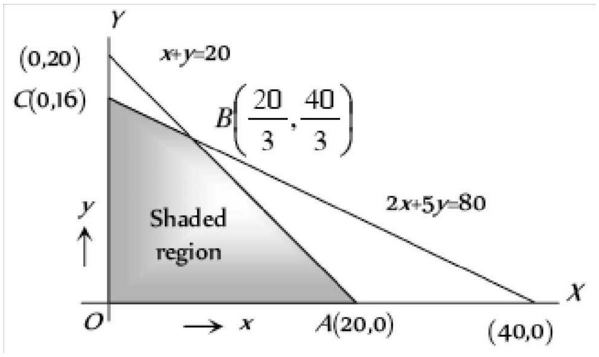
11. Inequations $3x - y \geq 3$ and $4x - y > 4$

(a) Have solution for positive x and y
(b) Have no solution for positive x and y
(c) Have solution for all x
(d) Have solution for all y

12. The constraints
$-x_1 + x_2 + \leq 1$; $-x_1 + 3x_2 \leq 9$; $x, x_2 \geq 0$ define

(a) Bounded feasible space
(b) Unbounded feasible space
(c) Both bounded and unbounded feasible space
(d) None of these

13.



Shaded region is represented by

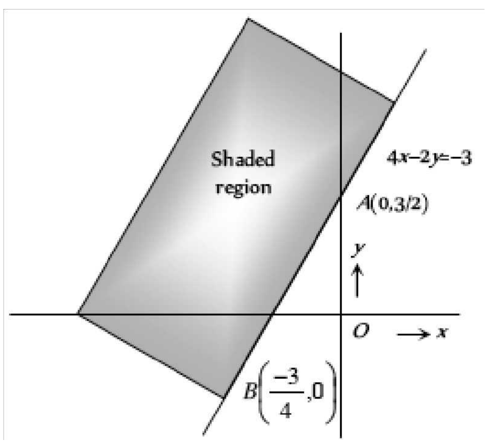
(a) $2x + 5y \geq 80, x + y \leq 20, x \geq 0, y \leq 0$

(b) $2x + 5y \geq 80, x + y \geq 20, x \geq 0, y \geq 0$

(c) $2x + 5y \leq 80, x + y \leq 20, x \geq 0, y \geq 0$

(d) $2x + 5y \leq 80, x + y \leq 20, x \leq 0, y \leq 0$

14.



Shaded region is represented by

(a) $4x - 2y \leq 3$ (b) $4x - 2y \leq -3$

(c) $4x - 2y \geq 3$ (d) $4x - 2y \geq -3$

15. A firm makes pants and shirt. A shirt takes 2 hour on machine and 3 hour of man labour while a pant takes 3 hour on machine and 2 hour of man labour. In a week there are 70 hour of machine and 75 hour of man labour available. If the firm determines to make x shirts and y pants per week, then for this the linear constraints are

(a) $x \geq 0, y \geq 0, 2x + 3y \geq 70, 3x + 2y \geq 75$

(b) $x \geq 0, y \geq 0, 2x + 3y \leq 70, 3x + 2y \geq 75$

(c) $x \geq 0, y \geq 0, 2x + 3y \geq 70, 3x + 2y + \leq 75$

(d) $x \geq 0, y \geq 0, 2x + 3y \leq 70, 3x + 2y \leq 75$

16. The minimum value of the objective function $Z = 2x + 10y$ for linear constraints $x - y \geq 0$, $x - 5y \leq -5$ and $x, y \geq 0$ is

(a) 10 (b) 15

(c) 12 (d) 8

17. Maximize $z = 3x + 2y$, subject to $x + y \geq 1$, $y - 5x \leq 0$, $x - y \geq -1$, $x + y \leq 6$, $x \leq 3$ and $x, y \geq 0$

(a) $x = 3$ (b) $y = 3$

(c) $z = 15$ (d) All the above

18. Maximum value of $4x + 5y$ subject to the constraints $x + y \leq 20$, $x + 2y \leq 35$, $x - 3y \leq 12$ is

(a) 84 (b) 95

(c) 100 (d) 96

19. The maximum value of $Z = 4x + 3y$ subjected to the constraints $3x + 2y \geq 160$, $5x + 2y \geq 200$, $x + 2y \geq 80$, $x, y \geq 0$ is

(a) 320 (b) 300

(c) 230 (d) None of these

20. The maximum value of $\mu = 3x + 4y$ subjected to the conditions $x + y \leq 40$, $x + 2y \leq 60$; $x, y \geq 0$ is

(a) 130 (b) 120

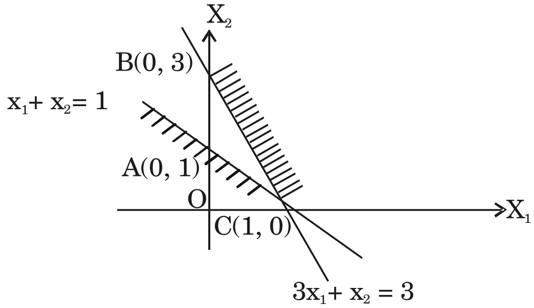
(c) 40 (d) 140

Answer Keys

1. (*c*)	**2.** (*d*)	**3.** (*b*)	**4.** (*b*)	**5.** (*b*)	**6.** (*b*)	**7.** (*a*)	**8.** (*c*)	**9.** (*a*)	**10.** (*c*)
11. (*a*)	**12.** (*b*)	**13.** (*c*)	**14.** (*b*)	**15.** (*d*)	**16.** (*b*)	**17.** (*d*)	**18.** (*b*)	**19.** (*d*)	**20.** (*d*)

Solutions

1. Clearly, from graph there is no feasible region.



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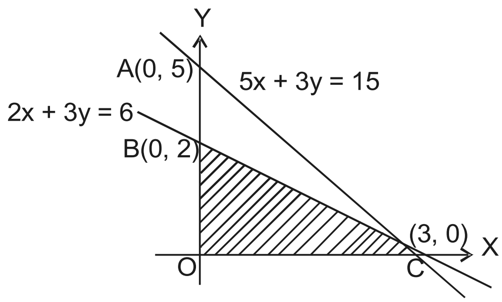


Here (0, 2), (0, 0) and (3, 0) all are vertices of feasible region.

3. The intermediate solutions of constraints must be checked by substituting them back into constraint equations.

4. A basic solution is called non-degenerate, if none of the basic variables is zero.

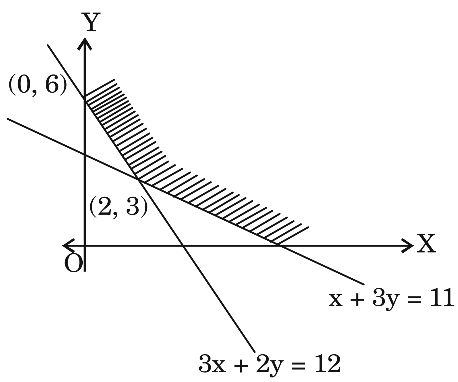
5. If the number of available constraints is 3 and the number of parameters to be optimized is 4, then the constraints are short in number.

6. Objective function of a linear programming problem is a function to be optimized.

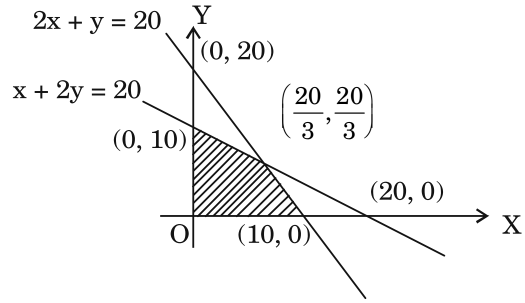
7. If the constraints in a linear programming problem are changed the problem is to be re-evaluated.

8. If a linear programming problem admits two optimal solutions, it has an infinite number of optimal solutions.

9. Minimum (z) = 2(2) + 2(3) $\Rightarrow$ c = 10

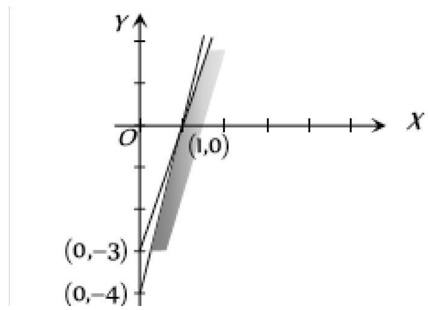


10. Obviously, P = x + 3y will be maximum at (0, 10)



$\therefore$ P = 0 + 3 × 10 = 30

11.



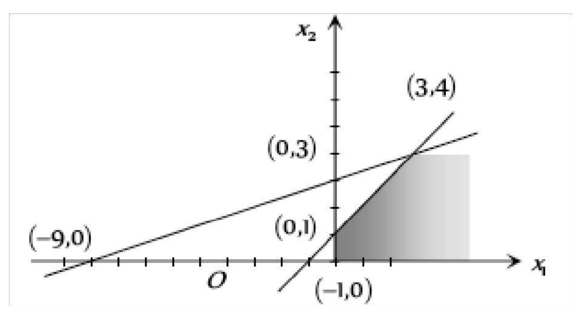
(a) Following figure will be obtained on drawing the graphs of given inequations :

From $3x - y \geq 3$, $\frac{x}{1} + \frac{y}{-3} = 1$

From $4x - y \geq 4$, $\frac{x}{1} + \frac{y}{-4} = 1$

Clearly the common region of both the inequations is true for positive value of (x, y). It is also true for positive values of x and negative values of y.

12.



It is clear from the graph, the constraints define an unbounded feasible space.

13. In all the given equations, the origin is present in shaded area. Answer (c) satisfy this condition.

14. Origin is not present in given shaded area.

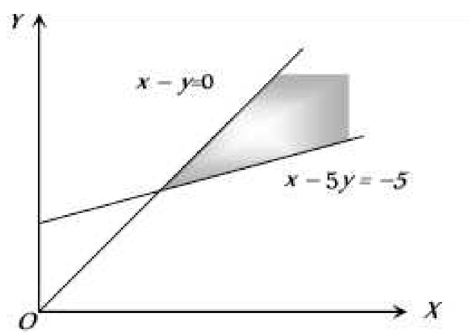
So $4x - 2y \le -3$ satisfy this condition.

15.

	Working time on machine	Man labour
Shirt(x)	2 hours	3 hours
Pant(y)	3 hours	2 hours
Availability	70 hours	75 hours

Linear constraints are $2x + 3y \le 70$, $3x + 2y \le 75$ and $x, y \ge 0$

16.



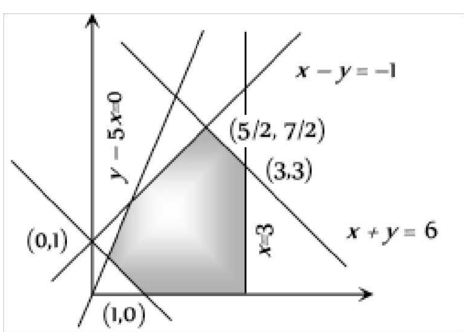
Required region is unbounded whose vertex is

$\left(\frac{5}{4}, \frac{5}{4}\right)$

Hence the minimum value of objective function is

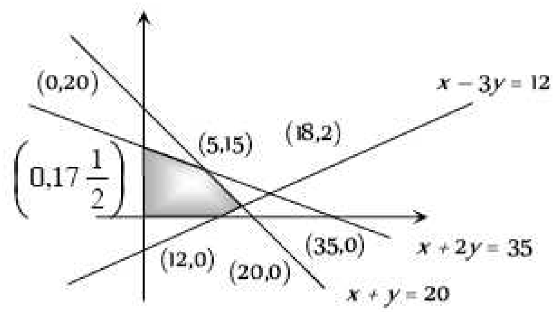
$$= 2\times\frac{5}{4} + 10\times\frac{5}{4} = 15.$$

17.



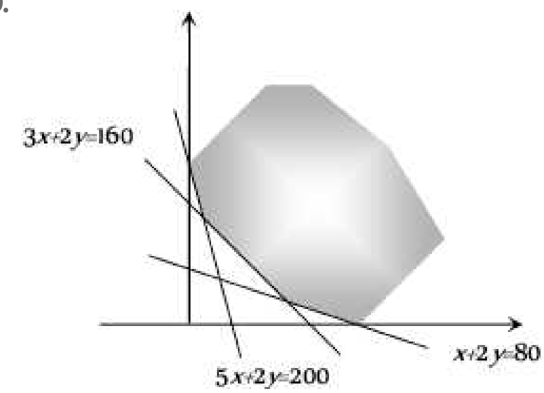
The shaded region represents the bounded region (3, 3) satisfies, so x = 3, y = 3 and z = 15.

18.



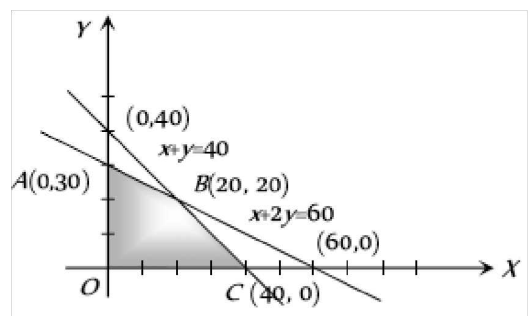
Obviously, max. $4x + 5y = 95$. It is at (5, 15).

19.



Obviously, it is unbounded. Therefore its maximum value does not exist.

20.



Obviously Max $\mu = 3x + 4y$ at (20, 20)

$\mu = 60 + 80 = 140$