Continuity and Differentiability

Question1

Consider the function. $f(x) = \begin{cases} \dfrac{a(7x-12-x^2)}{b|x^2-7x+12|} & , x < 3 \\ 2^{\frac{\sin(x-3)}{x-[x]}} & , x > 3 \\ b & , x = 3 \end{cases}$

Where [x] denotes the greatest integer less than or equal to x .
If S denotes the set of all ordered pairs (a,b) such that f(x) is continuous at x = 3 , then the number of elements in S is :

[27-Jan-2024 Shift 1]

Options:

A.

2

B.

Infinitely many

C.

4

D.

1

Answer: D

Solution:

$f(3^-) = \frac{a}{b}\frac{(7x-12-x^2)}{|x^2-7x+12|}$ (for $f(x)$ to be cont.)

$\Rightarrow f(3^-) = \frac{-a}{b}\frac{(x-3)(x-4)}{(x-3)(x-4)}; x < 3 \Rightarrow \frac{-a}{b}$

Hence $f(3^-) = \frac{-a}{b}$

Then $f(3^+) = 2^{\lim_{x\to 3^+}\left(\frac{\sin(x-3)}{x-3}\right)} = 2$ and $f(3) = b$.

Hence $f(3) = f(3^+) = f(3^-)$

$\Rightarrow b = 2 = -\frac{a}{b}$

$b = 2, a = -4$

Hence only 1 ordered pair $(-4, 2)$.

Question2

Consider the function $f : (0, 2) \to R$ defined by $f(x) = \frac{x}{2} + \frac{2}{x}$ and the function $g(x)$ defined by

$$g(x) = \begin{cases} \min\{f(t)\} & 0 < t \le x \text{ and } 0 < x \le 1 \\ \frac{3}{2} + x & 1 < x < 2 \end{cases}$$ .. Then

[27-Jan-2024 Shift 2]

Options:

A.

g is continuous but not differentiable at x = 1

B.

g is not continuous for all x∈ (0, 2)

C.

g is neither continuous nor differentiable at x = 1

D.

g is continuous and differentiable for all x ∈ (0, 2)

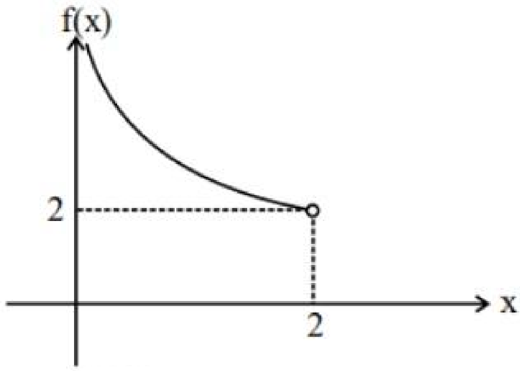
Answer: A

Solution:

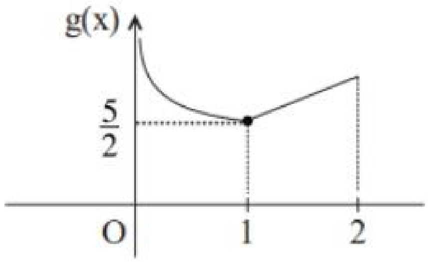
$f:(0,2)\longrightarrow R;\ f(x)=\frac{x}{2}+\frac{2}{x}$

$f'(x)=\frac{1}{2}-\frac{2}{x^2}$

$\therefore f(x)$ is decreasing in domain.



$$g(x)=\begin{cases}\frac{x}{2}+\frac{2}{x} & 0<x\le 1\\ \frac{3}{2}+x & 1<x<2\end{cases}$$



Question3

Let $f(x)=\sqrt{\lim_{r\to x}\left\{\frac{2r^2[(f(r))^2-f(x)f(r)]}{r^2-x^2}-r^3e^{\frac{f(r)}{r}}\right\}}$ be differentiable in $(-\infty,0)\cup(0,\infty)$ and $f(1)=1$. Then the value of ea, such that $f(a)=0$, is equal to______ *[29-Jan-2024 Shift 2]*

Answer: 2

Solution:

$f(1)=1, f(a)=0$

$$f^2(x)=\lim_{r\to x}\left(\frac{2r^2(f^2(r)-f(x)f(r))}{r^2-x^2}-r^3e^{\frac{f(r)}{r}}\right)$$

$$=\lim_{r\to x}\left(\frac{2r^2f(r)}{r+x}\frac{(f(r)-f(x))}{r-x}-r^3e^{\frac{f(r)}{r}}\right)$$

$$f^2(x)=\frac{2x^2f(x)}{2x}f'(x)-x^3e^{\frac{f(x)}{x}}$$

$$y^2=xy\frac{dy}{dx}-x^3e^{\frac{y}{x}}$$

$$\frac{y}{x}=\frac{dy}{dx}-\frac{x^2}{y}e^{\frac{y}{x}}$$

Put $y=vx\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$

$$v=v+x\frac{dv}{dx}-\frac{x}{v}e^{v}$$

$$\frac{dv}{dx}=\frac{e^v}{v}\Rightarrow e^{-v}v\,dv=dx$$

Integrating both side

$$e^v(x+c)+1+v=0$$

$f(1)=1\Rightarrow x=1, y=1$

$$\Rightarrow c=-1-\frac{2}{e}$$

$$e^v\left(-1-\frac{2}{e}+x\right)+1+v=0$$

$$e^{\frac{y}{x}}\left(-1-\frac{2}{e}+x\right)+1+\frac{y}{x}=0$$

$$x=a, y=0\Rightarrow a=\frac{2}{e}$$

$$ae=2$$

Question4

If the function $f(x)=\begin{cases}\dfrac{1}{|x|} & , |x|\geq 2\\ ax^2+2b & , |x|<2\end{cases}$ is differentiable on $\mathbb{R}$, then $48(a+b)$ is equal to____

[30-Jan-2024 Shift 1]

Answer: 15

Solution:

$$f(x)\begin{cases}\frac{1}{x};\ x\geq 2\\ ax^2+2b;\ -2<x<2\\ -\frac{1}{x};\ x\leq -2\end{cases}$$

Continuous at $x=2 \Rightarrow \frac{1}{2}=\frac{a}{4}+2b$

Continuous at $x=-2 \Rightarrow \frac{1}{2}=\frac{a}{4}+2b$

Since, it is differentiable at $x=2$

$-\frac{1}{x^2}=2ax$

Differentiable at $x=2 \Rightarrow \frac{-1}{4}=4a \Rightarrow a=\frac{-1}{16}, b=\frac{3}{8}$

Question5

Let $f:\mathbb{R}-\{0\}\rightarrow\mathbb{R}$ be a function satisfying $f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)}$ for all $x, y, f(y)\neq 0$. If $f'(1)=2024$, then

[30-Jan-2024 Shift 2]

Options:

A.

$xf'(x)-2024f(x)=0$

B.

$xf'(x)+2024f(x)=0$

C.

$xf'(x)+f(x)=2024$

D.

$xf'(x)-2023f(x)=0$

Answer: A

Solution:

$f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)}$

$f'(1)=2024$

$f(1)=1$

Partially differentiating w. r. t. x

$f'\left(\frac{x}{y}\right)\cdot\frac{1}{y}=\frac{1}{f(y)}f'(x)$

$y\longrightarrow x$

$f'(1)\cdot\frac{1}{x}=\frac{f'(x)}{f(x)}$

$2024f(x)=xf'(x)\Rightarrow xf'(x)-2024f(x)=0$

Question6

Let a and b be real constants such that the function f defined by $f(x)=\begin{cases}x^2+3x+a, & x\le 1\\ bx+2, & x>1.\end{cases}$ be differentiable on R. Then, the value of $\int_{-2}^{2}f(x)$ dx equals

[30-Jan-2024 Shift 2]

Options:

A.

15/6

B.

19/6

C.

21

D.

17

Answer: D

Solution:

f is continuous

$\therefore\ 4 + a = b + 2$

$a = b - 2$

$f'(x) = 2x + 3,\quad x < 1$

$\qquad b,\qquad x > 1$

f is differentiable

$\therefore\ b = 5$

$\therefore\ a = 3$

$$\int_{-2}^{1}(x^2 + 3x + 3)dx + \int_{1}^{2}(5x + 2)\,dx$$

$$= \left[\frac{x^3}{3} + \frac{3x^2}{2} + 3x\right]_{-2}^{1} + \left[\frac{5x^2}{2} + 2x\right]_{1}^{2}$$

$$= \left(\frac{1}{3} + \frac{3}{2} + 3\right) - \left(\frac{-8}{3} + 6 - 6\right) + \left(10 + 4 - \frac{5}{2} - 2\right)$$

$$= 6 + \frac{3}{2} + 12 - \frac{5}{2} = 17$$

Question7

Consider the function f : (0, ∞) ⟶ R defined by f(x) = $e^{-|\log_e x|}$. If m and n be respectively the number of points at which f is not continuous and f is not differentiable, then m+n is

[31-Jan-2024 Shift 2]

Options:

A.

0

B.

3

C.

1

D.

2

Answer: C

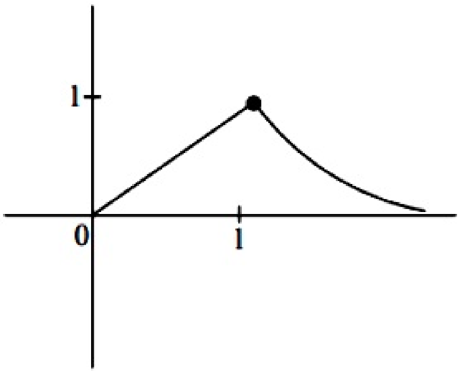
Solution:

$f:(0,\infty)\longrightarrow R$

$f(x)=e^{-|\log_e x|}$

$$f(x)=\frac{1}{e^{|\ln x|}}=\begin{cases}\frac{1}{e^{-\ln x}};\ 0<x<1\\ \frac{1}{e^{\ln x}};\ x\geq 1\end{cases}$$

$$\begin{cases}\frac{1}{\frac{1}{x}}=x;\ 0<x<1\\ \frac{1}{x},\ x\geq 1\end{cases}$$



m = 0 (No point at which function is not continuous)

n = 1 (Not differentiable)

$\therefore m+n=1$

Question8

Let $f:R\longrightarrow R$ be defined as

$$f(x)=\begin{cases}\dfrac{a-b\cos 2x}{x^2} & ; & x<0\\ x^2+cx+2 & ; & 0\leq x\leq 1\\ 2x+1 & ; & x>1\end{cases}$$

If f is continuous everywhere in R and m is the number of points where f is NOT differential then m + a + b + c equals :

[1-Feb-2024 Shift 1]

Options:

A.

1

B.

4

C.

3

D.

2

Answer: D

Solution:

At $x = 1$, $f(x)$ is continuous therefore,

$f(1) = f(1) = f(1^+)$

$f(1) = 3 + c$ ......(1)

$f(1^+) = \lim_{h \to 0} 2(1 + h) + 1$

$f(1^+) = \lim_{h \to 0} 3 + 2h = 3$ ......(2)

from (1) & (2)

$c = 0$

at $x = 0$, $f(x)$ is continuous therefore,

$f(0^-) = f(0) = f(0^+)$ ......(3)

$f(0) = f(0^+) = 2$ ......(4)

$f(0^-)$ has to be equal to 2

$$\lim_{h \to 0} \frac{a - b\cos(2h)}{h^2}$$

$$\lim_{h \to 0} \frac{a - b\left\{1 - \frac{4h^2}{2!} + \frac{16h^4}{4!} + \ldots\right\}}{h^2}$$

$$\lim_{h \to 0} \frac{a - b + b\left\{2h^2 - \frac{2}{3}h^4 \ldots\right\}}{h^2}$$

for limit to exist $a - b = 0$ and limit is $2b$ ....(5)

from (3), (4) & (5)

$a = b = 1$

checking differentiability at $x = 0$

$$\text{LHD}: \lim_{h \to 0} \frac{\frac{1 - \cos 2h}{h^2} - 2}{-h}$$

$$\lim_{h \to 0} \frac{1 - \left(1 - \frac{4h^2}{2!} + \frac{16h^4}{4!} \ldots\right) - 2h^2}{-h^3} = 0$$

$$\text{RHD}: \lim_{h \to 0} \frac{(0 + h)^2 + 2 - 2}{h} = 0$$

Function is differentiable at every point in its domain

$\therefore m = 0$

$m + a + b + c = 0 + 1 + 1 + 0 = 2$

Question9

Let f(x) = 2| x^2 + 5|x| −3|, x∈R. If m and n denote the number of points where f is not continuous and not differentiable respectively, then m + n is equal to :

[1-Feb-2024 Shift 2]

Options:

A.

5

B.

2

C.

0

D.

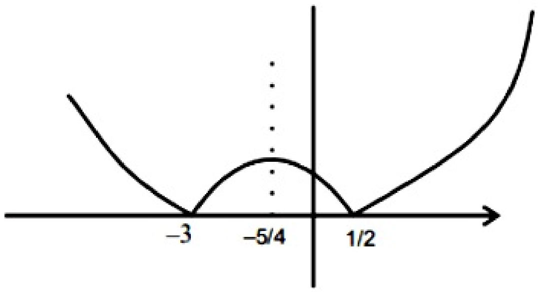
3

Answer: D

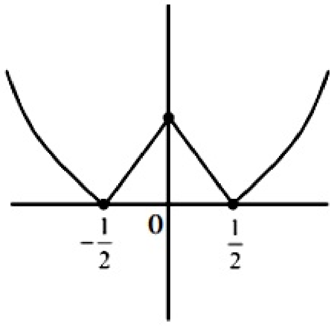
Solution:

$f(x) = 2x^2 + 5|x| - 3|$

Graph of $y = |2x^2 + 5x - 3|$



Graph of $f(x)$



Number of points of discontinuity $= 0 = m$

Number of points of non-differentiability $= 3 = n$

Question10

Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$.; Then at x = 0

[24-Jan-2023 Shift 1]

Options:

A. f is continuous but not differentiable

B. f is continuous but f' is not continuous

C. f and f ' both are continuous

D. f' is continuous but not differentiable

Answer: B

Solution:

Solution:

Continuity of $f(x) : f(0^+) = h^2 \cdot \sin \frac{1}{h} = 0$

$f(0^-) = (-h)^2 \cdot \sin\left(\frac{-1}{h}\right) = 0$

$f(0) = 0$

f (x) is continuous

$$f'(0^+) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \frac{h^2 \cdot \sin\left(\frac{1}{h}\right) - 0}{h} = 0$$

$$f'(0^-) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \frac{h^2 \cdot \sin\left(\frac{1}{-h}\right) - 0}{-h} = 0$$

f (x) is differentiable.

$$f'(x) = 2x \cdot \sin\left(\frac{1}{x}\right) + x^2 \cdot \cos\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2}$$

$$f'(x) = \begin{cases} 2x \cdot \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} .$$

(x) is not continuous (as $\cos\left(\frac{1}{x}\right)$ is highly oscillating at $x = 0$)

Question11

If the function

$$\mathbf{f(x) =} \begin{cases} (1 + |\cos x|)^{\frac{\lambda}{|\cos x|}} & , 0 < x < \frac{\pi}{2} \\ \mu & , x = \frac{\pi}{2} \\ e^{\frac{\cot 6x}{\cot 4x}} & , \frac{\pi}{2} < x < \pi \end{cases} .$$

is continuous at $x = \frac{\pi}{2}$, then $9\lambda + 6\log_e \mu + \mu^6 - e^{6\lambda}$ is equal to

[25-Jan-2023 Shift 2]

Options:

A. 11

B. 8

C. $2e^4 + 8$

D. 10

Answer: D

Solution:

Solution:

$\Rightarrow \lim_{x \to \frac{\pi}{2}^+} e^{\frac{\cot 6x}{\cot 4x}} = \lim_{x \to \frac{\pi}{2}^+} e^{\frac{\sin 4x \cdot \cos 6x}{\sin 6x \cdot \cos 4x}} = e^{2/3}$

$\Rightarrow \lim_{x \to \frac{\pi}{2}^-} \left(1 + |\cos x|\right)^{\frac{\lambda}{|\cos x|}} = e^{\lambda}.$

$\Rightarrow f(\pi / 2) = \mu$

For continuous function $\Rightarrow e^{2/3} = e^{\lambda} = \mu$

$\lambda = \frac{2}{3}, \mu = e^{2/3}$

Now, $9\lambda + 6\log_e \mu + \mu^6 - e^{6\lambda} = 10$

Question12

Let $a \in \mathbb{Z}$ and [t] be the greatest integer $\le t$. Then the number of points, where the function $f(x) = [a + 13 \sin x]$, $x \in (0, \pi)$ is not differentiable, is
[6-Apr-2023 shift 1]

Answer: 25

Solution:

Solution:

$f(x) = [a + 13\sin x] = a + [13\sin x]$ in $(0, \pi)$

$x \in (0, \pi)$

$\Rightarrow 0 < 13\sin x \le 13$

$\Rightarrow [13\sin x] = \{0, 1, 2, 3, \ldots 12, 13,\}$

Total point of N.D. $= 25$.

Question13

Let $f : (-2, 2) \to \mathbb{R}$ be defined by

$f(x) = \begin{cases} x[x] & -2 < x < 0 \\ (x-1)[x] & 0 \le x < 2 \end{cases}$.

where [x] denotes the greatest integer function. If m and n respectively are the number of points in $(-2, 2)$ at which $y = |f(x)|$ is not continuous and not differentiable, then $m + n$ is equal to ______.
[10-Apr-2023 shift 1]

Answer: 4

Solution:

$$f(x) = \begin{cases} -2x & -2 < x < -1 \\ -x & -1 \le x < 0 \\ 0 & 0 \le x < 1 \\ x-1 & 1 \le x < 2 \end{cases}.$$

Clearly $f(x)$ is discontinuous at $x = -1$ also non differentiable.
$\therefore m = 1$
Now for differentiability

$$f'(x) = \begin{cases} -2 & -2 < x < -1 \\ -1 & -1 < x < 0 \\ 0 & 0 < x < 1 \\ -1 & 1 < x < 2 \end{cases}.$$

Clearly $f(x)$ is non-differentiable at $x = -1, 0, 1$
Also, $|f(x)|$ remains same.
$\therefore n = 3$
$\therefore m + n = 4$

Question14

Let $f(x) = [x^2 - x] + |-x + [x]|$, where $x \in \mathbb{R}$ and $[t]$ denotes the greatest integer less than or equal to t. Then, f is :
[11-Apr-2023 shift 1]

Options:

A. not continuous at $x = 0$ and $x = 1$

B. continuous at $x = 0$ and $x = 1$

C. continuous at $x = 1$, but not continuous at $x = 0$

D. continuous at $x = 0$, but not continuous at $x = 1$

Answer: C

Solution:

Solution:
Here $f(x) = [x(x-1)] + \{x\}$
$f(0^+) = -1 + 0 = -1$ $f(1^+) = 0 + 0 = 0$
$f(0) = 0$ $f(1) = 0$
$f(1^-) = -1 + 1 = 0$
$\therefore f(x)$ is continuous at $x = 1$, discontinuous at $x = 0$

Question15

Let f and g be two functions defined by $f(x) = \begin{cases} x+1 & x<0 \\ |x-1|, & x \geq 0 \end{cases}$. and

$g(x) = \begin{cases} x+1 & x<0 \\ 1 & x \geq 0 \end{cases}$. Then (gof) (x) is

[11-Apr-2023 shift 2]

Options:

A. continuous everywhere but not differentiable at x = 1

B. continuous everywhere but not differentiable exactly at one point

C. differentiable everywhere

D. not continuous at $x = -1$

Answer: B

Solution:

Solution:

$$f(x) = \begin{cases} x+1 & x<0 \\ 1-x & 0 \leq x < 1 \\ x-1 & 1 \leq x \end{cases}.$$

$$g(x) = \begin{cases} x+1 & x<0 \\ 1 & x \geq 0 \end{cases}.$$

$$g(f(x)) = \begin{cases} x+2 & x<-1 \\ 1 & x \geq -1 \end{cases}.$$

∴g(f(x)) is continuous everywhere
g(f(x)) is not differentiable at $x = -1$
Differentiable everywhere else

Question16

Let [x] be the greatest integer ≤x. Then the number of points in the interval $(-2, 1)$, where the function $f(x) = |[x]| + \sqrt{x - [x]}$ is discontinuous, is _______.
[12-Apr-2023 shift 1]

Answer: 2

Solution:

Solution:
Need to check at doubtful points
discont at x I only
at $x = -1 \Rightarrow f(-1^{+}) = 1 + 0 = 1$

$\Rightarrow f(-1^-) = 2 + 1 = 3$
at $x = 0 \Rightarrow f(0^+) = 0 + 0 = 0$
$\Rightarrow f(0^-) = 1 + 1 = 2$
at $x = 1 \Rightarrow f(1^+) = 1 + 0 = 1$
$\Rightarrow f(1^-) = 0 + 1 = 1$
discont. at two points

Question17

Let [x] denote the greatest integer function and $f(x) = \max\{1 + x + [x], 2 + x, x + 2[x]\}$, $0 \le x \le 2$. Let m be the number of points in [0, 2], where f is not continuous and n be the number of points in (0, 2), where f is not differentiable. Then $(m + n)^2 + 2$ is equal to [15-Apr-2023 shift 1]

Options:

A. 6

B. 3

C. 2

D. 11

Answer: B

Solution:

Solution:

$$\text{Let } g(x) = 1 + x + [x] = \begin{cases} 1 + x; & x \in [0, 1) \\ 2 + x; & x \in [1, 2) \\ 5; & x = 2 \end{cases}.$$

$$\lambda(x) = x + 2[x] = \begin{cases} x; & x \in [0, 1) \\ x + 2; & x \in [1, 2) \\ 6; & x = 2 \end{cases}.$$

$r(x) = 2 + x$

$$f(x) = \begin{cases} 2 + x; & x \in [0, 2) \\ 6; & x = 2 \end{cases}.$$

f(x) is discontinuous only at $x = 2 \Rightarrow m = 1$
f(x) is differentiable in $(0, 2) \Rightarrow n = 0$
$(m + n)^2 + 2 = 3$

Question18

Let $f(x) = \begin{cases} \dfrac{\sin(x - [x])}{x - [x]} & x \in (-2, -1) \\ \max\{2x, 3[|x|]\} & |x| < 1 \\ 1 & \text{othewise} \end{cases}$. where [t] denotes greatest integer $\le t$. If m is the number of points where f is not continuous and n is the number of points where f is not differentiable, then the ordered pair (m, n) is :

[24-Jun-2022-Shift-2]

Options:

A. (3, 3)

B. (2, 4)

C. (2, 3)

D. (3, 4)

Answer: C

Solution:

Solution:

$$f(x) = \begin{cases} \frac{\sin(x-[x])}{x[x]} & x \in (-2,-1) \\ \max\{2x, 3[|x|]\} & |x| < 1 \\ 1 & \text{otherwise} \end{cases}.$$

$$f(x) = \begin{cases} \frac{\sin(x+2)}{x+2} & x \in (-2,-1) \\ 0 & x \in (-1,0] \\ 2x & x \in (0,1) \\ 1 & \text{otherwise} \end{cases}.$$

It clearly shows that $f(x)$ is discontinuous

At $x = -1, 1$ also non differentiable

and at $x = 0$, $L.H.D = \lim_{h \to 0} \frac{f(0-h)-f(0)}{-h} = 0$

R. H.D $= \lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = 2$

$\therefore$ f(x) is not differentiable at x = 0

$\therefore m = 2, n = 3$

Question19

Let f (x) = $[2x^2 + 1]$ and g(x) = $\begin{cases} 2x-3 & x < 0 \\ 2x+3 & x \geq 0. \end{cases}$, where [t] is the greatest integer ≤t. Then, in the open interval (−1, 1), the number of points where fog is discontinuous is equal to____

[25-Jun-2022-Shift-2]

Answer: 62

Solution:

Solution:

$f(g(x)) = [2g^2(x)] + 1$

$$= \begin{cases} [2(2x-3)^2]+1; & x<0 \\ [2(2x+3)^2]+1; & x \geq 0. \end{cases}$$

∴ fog is discontinuous whenever $2(2x-3)^2$ or $2(2x+3)^2$ belongs to integer except $x = 0$.
∴62 points of discontinuity.

Question20

Let f, g : R → R be two real valued functions defined as

$$f(x) = \begin{cases} -|x+3| & x<0 \\ e^x & x \geq 0 \end{cases}$$

$$\text{and } g(x) = \begin{cases} x^2+k_1x & x<0 \\ 4x+k_2 & x \geq 0 \end{cases}$$

where k_1 and k_2 are real constants. If (gof) is differentiable at x = 0, then (gof) (−4) + (gof)(4) is equal to :
[26-Jun-2022-Shift-1]

Options:

A. $4(e^4+1)$

B. $2(2e^4+1)$

C. $4e^4$

D. $2(2e^4-1)$

Answer: D

Solution:

Solution:

∵ gof is differentiable at $x = 0$

So R.H.D = L.H.D

$$\frac{d}{dx}(4e^x+k_2) = \frac{d}{dx}((-|x+3|)^2 - k_1|x+3|)$$

$\Rightarrow 4 = 6 - k_1 \Rightarrow k_1 = 2$

Also $f(f(0^+)) = g(f(0^-))$

$\Rightarrow 4 - k_2 = 9 - 3k_1 \Rightarrow k_2 = -1$

Now $g(f(-4)) + g(f(4))$

$= g(-1) - g(e^4) = (1-k_1) + (4e^4 + k_2)$

$= 4e^4 - 2$

$= 2(2e^4 - 1)$

Question21

Let f (x) = min{1, 1 + x sin x}, 0 ≤ x ≤ 2π. If m is the number of points,

where f is not differentiable and n is the number of points, where f is not continuous, then the ordered pair (m, n) is equal to
[26-Jun-2022-Shift-2]

Options:

A. (2, 0)

B. (1, 0)

C. (1, 1)

D. (2, 1)

Answer: B

Solution:

Solution:

$f(x) = \min\{1, 1 + x\sin x\}, 0 \le x \le x$

$f(x) = \begin{cases} 1 & 0 \le x < \pi \\ 1 + x\sin x & \pi \le x \le 2\pi. \end{cases}$

Now at $x = \pi$, $\lim_{x \to \pi^-} f(x) = 1 = \lim_{x \to \pi^-} f(x)$

$\therefore$ f(x) is continuous in $[0, 2\pi]$

Now, at $x = \pi L.H.D = \lim_{h \to 0} \frac{f(\pi - h) - f(\pi)}{-h} = 0$

$R.H.D = \lim_{h \to 0} \frac{f(\pi + h) - f(\pi)}{h} = 1 - \frac{(\pi + h)\sin h - 1}{h} = -\pi$

$\therefore$ f(x) is not differentiable at $x = \pi$

$\therefore (m, n) = (1, 0)$

Question22

Let f : R → R be defined as

$$\mathbf{f(x)} = \begin{cases} [e^x], & x < 0 \\ ae^x + [x - 1], & 0 \le x < 1 \\ b + [\sin(\pi x)], & 1 \le x < 2 \\ [e^{-x}] - c, & x \ge 2. \end{cases}$$

where a, b, c ∈ R and [t] denotes greatest integer less than or equal to t. Then, which of the following statements is true?
[28-Jun-2022-Shift-1]

Options:

A. There exists a, b, c ∈ R such that f is continuous on R.

B. If f is discontinuous at exactly one point, then a + b + c = 1

C. If f is discontinuous at exactly one point, then a + b + c ≠ 1

D. f is discontinuous at at least two points, for any values of a, b and c

Answer: C

Solution:

Solution:

$$f(x) = \begin{cases} 0 & x < 0 \\ ae^{x} - 1 & 0 \le x < 1 \\ b & x = 1 \\ b - 1 & 1 < x < 2 \\ -c & x \ge 2 \end{cases}.$$

To be continuous at $x = 0$
$a - 1 = 0$
to be continuous at $x = 1$
$ae - 1 = b = b - 1 \Rightarrow$ not possible
to be continuous at $x = 2$
$b - 1 = -c \Rightarrow b + c = 1$
If $a = 1$ and $b + c = 1$ then $f(x)$ is discontinuous at exactly one point.

Question23

Let f, g : R → R be functions defined by

$f(x) = \begin{cases} [x], & x < 0 \\ |1 - x|, & x \ge 0. \end{cases}$ and

$g(x) = \begin{cases} e^{x} - x, & x < 0 \\ (x - 1)^{2} - 1, & x \ge 0. \end{cases}$

where [x] denote the greatest integer less than or equal to x. Then, the function fog is discontinuous at exactly :
[28-Jun-2022-Shift-2]

Options:

A. one point

B. two points

C. three points

D. four points

Answer: B

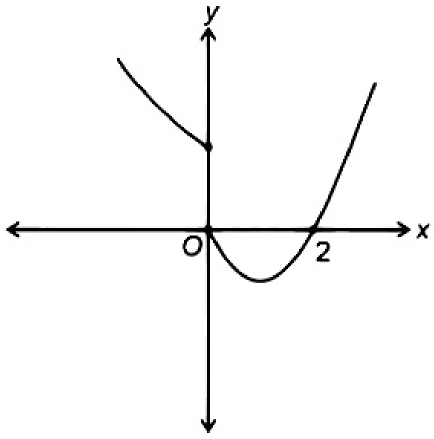
Solution:

Solution:

$f(x) = \begin{cases} [x], & x < 0 \\ |1 - x|, & x \ge 0 \end{cases}$

and $g(x) = \begin{cases} e^{x} - x, & x < 0 \\ (x - 1)^{2} - 1, & x \ge 0 \end{cases}$

$f \circ g(x) = \begin{cases} [g(x)], & g(x) < 0 \\ |1 - g(x)|, & g(x) \ge 0 \end{cases}.$



(graph of $y = g(x)$)

$$= \begin{cases} |1 + x - e^x|, & x < 0 \\ 1, & x = 0 \\ [(x-1)^2 - 1], & 0 < x < 2 \\ |2 - (x-1)^2|, & x \geq 2 \end{cases}$$

So, x = 0, 2 are the two points where fog is discontinuous.

Question24

Let f : R → R be a function defined by :

$$\mathbf{f(x)} = \begin{cases} \max\limits_{t \leq x}\{t^3 - 3t\}; & x \leq 2 \\ x^2 + 2x - 6; & 2 < x < 3 \\ [x-3] + 9; & 3 \leq x \leq 5 \\ 2x + 1; & x > 5. \end{cases}$$

where [t] is the greatest integer less than or equal to t . Let m be the number of points where f is not differentiable and I $= \int_{-2}^{2} f(x)dx$. Then the ordered pair (m, l) is equal to :
[29-Jun-2022-Shift-1]

Options:

A. $\left(3, \frac{27}{4}\right)$

B. $\left(3, \frac{23}{4}\right)$

C. $\left(4, \frac{27}{4}\right)$

D. $\left(4, \frac{23}{4}\right)$

Answer: C

Solution:

$$f(x) = \begin{cases} x^3 - 3x & x \le -1 \\ 2 & -1 < x < 2 \\ x^2 + 2x - 6 & 2 < x < 3 \\ 9 & 3 \le x < 4 \\ 10 & 4 \le x < 5 \\ 11 & x = 5 \\ 2x + 1 & x > 5 \end{cases}$$

Clearly f(x) is not differentiable at
$x = 2, 3, 4, 5 \Rightarrow m = 4$

$$I = \int_{-2}^{-1} (x^3 - 3x)\,dx + \int_{-1}^{2} 2 \cdot dx = \frac{27}{4}$$

Question25

The number of points where the function

$$\mathbf{f(x) =} \begin{cases} |2x^2 - 3x - 7| & \text{if} \quad x \le -1 \\ [4x^2 - 1] & \text{if} \quad -1 < x < 1 \\ |x + 1| + |x - 2| & \text{if} \quad x \ge 1. \end{cases}$$

[t] denotes the greatest integer ≤t , is discontinuous is [24-Jun-2022-Shift-1]

Answer: 7

Solution:

$\because f(-1) = 2$ and $f(1) = 3$

For $x \in (-1, 1)$, $(4x^2 - 1) \in [-1, 3)$

hence $f(x)$ will be discontinuous at $x = 1$ and also

whenever $4x^2 - 1 = 0, 1$ or 2

$\Rightarrow x = \pm \frac{1}{2}, \pm \frac{1}{\sqrt{2}}$ and $\pm \frac{\sqrt{3}}{2}$

So there are total 7 points of discontinuity.

Question26

Let $\mathbf{f(x) =} \begin{cases} |4x^2 - 8x + 5|, \text{if } 8x^2 - 6x + 1 \ge 0 \\ [4x^2 - 8x + 5], \text{if } 8x^2 - 6x + 1 < 0 \end{cases}$,

where [α] denotes the greatest integer less than or equal to α. Then the number of points in R where f is not differentiable is [25-Jul-2022-Shift-1]

Answer: 3

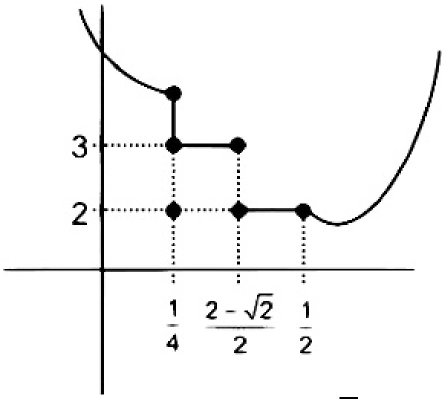
Solution:

Solution:

$$f(x) = \begin{cases} |4x^2 - 8x + 5| & \text{if } 8x^2 - 6x + 1 \geq 0 \\ [4x^2 - 8x + 5], & \text{if } 8x^2 - 6x + 1 < 0. \end{cases}$$

$$= \begin{cases} 4x^2 - 8x + 5 & \text{if } x \in \left[-\infty, \frac{1}{4}\right] \cup \left[\frac{1}{2}, \infty\right) \\ [4x^2 - 8x + 5] & \text{if } x \in \left(\frac{1}{4}, \frac{1}{2}\right). \end{cases}$$

$$f(x) = \begin{cases} 4x^2 - 8x + 5 & \text{if } x \in \left(-\infty, \frac{1}{4}\right] \cup \left[\frac{1}{2}, \infty\right) \\ 3 & x \in \left(\frac{1}{4}, \frac{2-\sqrt{2}}{2}\right) \\ 2 & x \in \left[\frac{2-\sqrt{2}}{2}, \frac{1}{2}\right). \end{cases}$$



$\therefore$ Non-diff at $x = \frac{1}{4}, \frac{2-\sqrt{2}}{2}, \frac{1}{2}$

Question27

If $f(x) = \begin{cases} x + a, & x \leq 0 \\ |x - 4|, & x > 0. \end{cases}$ and $g(x) = \begin{cases} x + 1, & x < 0 \\ (x - 4)^2 + b, & x \geq 0. \end{cases}$ are continuous on R, then $(gof)(2) + (fog)(-2)$ is equal to : [26-Jul-2022-Shift-1]

Options:

A. -10

B. 10

C. 8

D. -8

Answer: D

Solution:

Solution:

$\because f(x)$ and $g(x)$ are continuous on R

$\therefore a = 4$ and $b = 1 - 16 = -15$
then $(g \circ f)(2) + (f o g)(-2)$
$= g(2) + f(-1)$
$= -11 + 3 = -8$

Question28

If for $p \neq q \neq 0$, the function $f(x) = \frac{\sqrt[7]{p(729 + x)} - 3}{\sqrt[3]{729 + qx} - 9}$ is continuous at $x = 0$, then :
[27-Jul-2022-Shift-2]

Options:

A. $7pqf(0) - 1 = 0$

B. $63qf(0) - p^2 = 0$

C. $21qf(0) - p^2 = 0$

D. $7pqf(0) - 9 = 0$

Answer: B

Solution:

Solution:
$f(x) = \frac{\sqrt[7]{p(729 + x)} - 3}{\sqrt[3]{729 + qx} - 9}$
for continuity at $x = 0$, $\lim_{x \to 0} f(x) = f(0)$
Now, $\therefore \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sqrt[7]{p(729 + x)} - 3}{\sqrt[3]{729 + qx} - 9}$
$\Rightarrow p = 3$ (To make indeterminant form)
So, $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{(3^7 + 3x)^{\frac{1}{7}} - 3}{(729 + qx)^{\frac{1}{3}} - 9}$
$$= \lim_{x \to 0} \frac{3\left[\left(1 + \frac{x}{3^6}\right)^{\frac{1}{7}} - 1\right]}{9\left[\left(1 + \frac{q}{729}x\right)^{\frac{1}{3}} - 1\right]} = \frac{1}{3} \cdot \frac{\frac{1}{7} \cdot \frac{1}{3^6}}{\frac{1}{3} \cdot \frac{q}{729}}$$
$\therefore f(0) = \frac{1}{7q}$
$\therefore$ Option (B) is correct.

Question29

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \lim_{n \to \infty} \frac{\cos(2\pi x) - x^{2n} \sin(x - 1)}{1 + x^{2n + 1} - x^{2n}}$ is continuous for all x in :
[28-Jul-2022-Shift-2]

Options:

A. $R - \{-1\}$

B. $\mathbb{R} - \{-1, 1\}$

C. R − {1}

D. R − {0}

Answer: B

Solution:

Solution:

$f(x) = \lim_{n \to \infty} \frac{\cos(2\pi x) - x^{2n}\sin(x-1)}{1 + x^{2n+1} - x^{2n}}$

For $|x| < 1$, $f(x) = \cos 2\pi x$, continuous function

$|x| > 1, f(x) = \lim_{n \to \infty} \frac{\frac{1}{x^{2n}}\cos 2\pi x - \sin(x-1)}{\frac{1}{x^{2n}} + x - 1}$

$= \frac{-\sin(x-1)}{x-1}$, continuous

For $|x| = 1$, $f(x) = \begin{cases} 1 & \text{if } x = 1 \\ -(1 + \sin 2) & \text{if } x = -1. \end{cases}$

Now,

$\lim_{x \to 1^+} f(x) = -1, \lim_{x \to 1^-} f(x) = 1$, so discontinuous at $x = 1$

$\lim_{x \to 1^+} f(x) = 1, \lim_{x \to -1^-} f(x) = -\frac{\sin 2}{2}$, so discontinuous at $x = -1$

$\therefore f(x)$ is continuous for all $x \in R - \{-1, 1\}$

Question30

The number of points, where the function $f : R \to R, f(x) = |x-1|\cos|x-2|\sin|x-1| + (x-3)x^2 - 5x + 4|$, is NOT differentiable, is : [29-Jul-2022-Shift-1]

Options:

A. 1

B. 2

C. 3

D. 4

Answer: B

Solution:

Solution:

$f : R \to R$

$f(x) = |x-1|\cos|x-2| \sin|x-1| + (x-3)\left|x^2 - 5x + 4\right|$

$= |x-1|\cos|x-2| \sin|x-1| + (x-3)|x-1||x-4|$

$= |x-1|[\cos|x-2| \sin|x-1| + (x-3)|x-4|]$

Sharp edges at $x = 1$ and $x = 4$

$\therefore$ Non-differentiable at $x = 1$ and $x = 4$

Question31

Let the function $f(x) = \begin{cases} \dfrac{\log_e(1+5x) - \log_e(1+axx)}{x} & ; \text{ if } x \neq 0 \\ 10 & ; \text{ if } x = 0. \end{cases}$ **be continuous at** $x = 0$.
Then α is equal to
[29-Jul-2022-Shift-2]

Options:

A. 10

B. −10

C. 5

D. −5

Answer: D

Solution:

Solution:
$f(x)$ is continuous at $x = 0$
$\therefore f(0) = \lim_{x \to 0} f(x)$
$\Rightarrow 10 = \lim_{x \to 0} \dfrac{\log_e(1+5x) - \log_e(1+\alpha x)}{x}$
$= \lim_{x \to 0} \dfrac{\log(1+5x)}{5x} \times 5 - \dfrac{\log_e(1+\alpha x)}{\alpha x} \times \alpha$
$= 1 \times 5 - \alpha$
$\Rightarrow \alpha = 5 - 10 = -5$

Question32

If [t] denotes the greatest integer ≤t, then the number of points, at which the function $f(x) = 4\left|2x+3\right| + 9\left[x + \frac{1}{2}\right] - 12[x+20]$ **is not differentiable in the open interval (−20, 20), is ________.**
[29-Jul-2022-Shift-2]

Answer: 79

Solution:

$f(x) = 4\,|\,2x+3\,| + 9\left[x + \dfrac{1}{2}\right] - 12[x+20]$

$= 4\,|\,2x+3\,| + 9\left[x + \dfrac{1}{2}\right] - 12[x] - 240$

$f(x)$ is non differentiable at $x = -\dfrac{3}{2}$

and $f(x)$ is discontinuous at $\{-19, -18, \ldots, 18, 19\}$

as well as $\left\{-\dfrac{39}{2}, -\dfrac{37}{2}, \ldots, -\dfrac{3}{2}, -\dfrac{1}{2}, \dfrac{1}{2}, \ldots, \dfrac{39}{2}\right\}$,

at same point they are also non differentiable

$\therefore$ Total number of points of non differentiability
$= 39 + 40$
$= 79$

Question33

Let $f : R \rightarrow R$ be defined asIf $f(x)$ is continuous on R, then $a + b$ equals [2021, 26 Feb. Shift-11]

Options:

A. -3

B. -1

C. 3

D. 1

Answer: B

Solution:

Solution:
Given, $f(x)$ is continuous on R.
If $f(x)$ is continuous, then f is continuous at $x = 1$
$\Rightarrow \lim_{x \rightarrow 1^-} f(x) = f(1) = \lim_{x \rightarrow 1^+} f(x)$
$\Rightarrow |a + 1 + b| = \sin \pi = 0$
$\Rightarrow a + b = -1 \ldots$ (i)
Also, f is continuous at $x = -1$
$\Rightarrow \lim_{x \rightarrow -1^-} f(x) = f(-1) = \lim_{x \rightarrow -1^+} f(x)$
$\Rightarrow 2 \sin\left(\frac{-\pi}{2}(-1)\right) = \left|a - 1 + b\right|$
$\Rightarrow 2 = |a + b - 1| \ldots$ (ii)
Eq. (ii) is satisfied.
$\therefore a + b = -1$

Question34

Let f be any function defined on R and let it satisfy the condition $|f(x) - f(y)| \leq |(x - y)^2|, \forall (x, y) \in R$ If $f(0) = 1$, then [2021,26 Feb. Shift-1]

Options:

A. $f(x)$ can take any value in R

B. $f(x) < 0, \forall x \in R$

C. $f(x) = 0, \forall x \in R$

D. $f(x) > 0, \forall x \in R$

Answer: D

Solution:

Given, $|f(x)-f(y)| \le x-y|^2$

$\Rightarrow \frac{|f(x)-f(y)|}{|x-y|} \le x-y\Big|$

Now, taking the limit,

$\lim_{x \to y}\left|\frac{f(x)-f(y)}{x-y}\right| \le \lim_{x \to y}\left|x-y\right|$

$\Rightarrow f'(y)\ | \le 0$ [using the definition of $f'(y)$]

$\Rightarrow f'(y) = 0$ [since, modulus value can never be less than 0]

On integrating it, we get

$f(y) = c$ (constant)

Given, $f(0) = 1$ gives $c = 1$

$\therefore\ f(y) = 1\ \forall y \in R$

From given options, $f(x) > 0\ \forall x \in R$ is satisfied only.

Question35

Let $f(x)$ be a differentiable function at $x = a$ with $f'a = 2$ and $f(a) = 4$.Then, $\lim_{x \to a}\frac{xf(a)-af(x)}{x-a}$ equals

[2021, 26 Feb. Shift-II]

Options:

A. $2a+4$

B. $4-2a$

C. $2a-4$

D. $a+4$

Answer: B

Solution:

Solution:

$\lim_{x \to a}\frac{xf(a)-af(x)}{x-a}$

$= \lim_{x \to a}\frac{xf(a)-af(x)+af(a)-af(a)}{x-a}$

$= \lim_{x \to a}\frac{(x-a)f(a)-a[f(x)-f(a)]}{x-a}$

$= \lim_{x \to a}\frac{(x-a)f(a)}{x-a} - a\lim_{x \to a}\frac{f(x)-f(a)}{x-'a}$

$= f(a) - af'(a)$

$= 4 - a(2)$ [Given, $f(a) = 4, f'(a) = 2$]

$= 4 - 2a$

Question36

A function f is defined on $[-3, 3]$ as $f(x) = \begin{cases} \min\{|x|, 2-x^2\} & -2 \le x \le 2 \\ [|x|] & , 2 < |x| \le 3. \end{cases}$

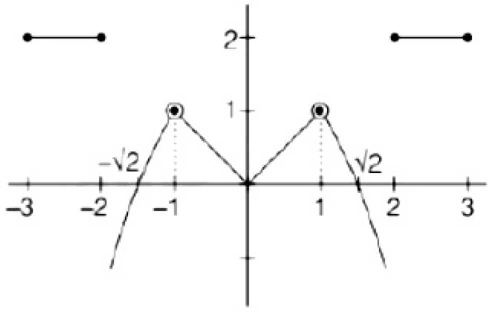
where, $[x]$ denotes the greatest integer $\le x$. The number of points, where f is not differentiable in $(-3, 3)$ is ....

[2021, 25 Feb. Shift-II]

Answer: 5

Solution:

Solution:
For this particular problem, try to draw graph in the region $(-3, 3)$, it will be as follows,



Thus, points of discontinuity will be at $-2, 2$ because the curve breaks at these points and at $-1, 0, 1$ because curve has sharp points.
∴ Point of discontinuity are $-2, -1, 0, 1, 2$ i.e. 5 points.

Question37

The number of points at which the function $f(x) = |2x+1| - 3x + 2| + x^2 + x - 2|\ x \in R$ is not differentiable, is ....
[2021, 25 Feb. Shift-1]

Answer: 2

Solution:

Solution:
Given,
$f(x) = |2x+1| - 3x + 2| + x^2 + x - 2|$
$= |2x+1| - 3x + 2| + |x+2|xx - 1|$
Here, critical points are $x = \frac{-1}{2}, -2, 1$

$$\therefore f(x) = \begin{cases} x^2 + 2x + 3 & x < -2 \\ -x^2 - 6x - 5 & -2 < x < \frac{-1}{2} \\ -x^2 - 2x - 3 & \frac{-1}{2} < x < 1. \end{cases}$$

Now, $f'(x) = \begin{cases} 2x + 2 & x < -2 \\ -2x - 6 & -2 < x < \frac{-1}{2} \\ -2x - 2 & \frac{-1}{2} < x < 1 \\ 2x & x > 1. \end{cases}$

Now, $f'(x)$ at 1, -2 and $-1/2$.
For $x = 1$,
$f'(x) = 2x = 2 \times 1 = 2$
and $-2x - 2 = -(2 \times 1) - 2 = -4$ both are not equal.
$\therefore$ Non-differentiable at $x = 1$
Similarly, for $x = -2$,
$f'(x) = 2x + 2 = 2 \times (-2) + 2 = -2$
and $-2x - 6 = -2 \times (-2) - 6 = -2$ both are equal.
$\therefore$ Differentiable at $x = -2$
and for $x = -1/2$, $f'(x) = -2x - 6$
$= -2 \times \left(\frac{-1}{2}\right) - 6 = -5$ and
$-2x - 2 = -2 \times \left(\frac{-1}{2}\right) - 2 = -1$ both are not equal.
$\therefore$ Non-differentiable at $x = -1/2$
$\therefore$ The number of points at which $f(x)$ is non-differentiable is 2 .

Question38

If $f : R \to R$ is a function defined by $f(x) = [x-1]\cos\left(\frac{2x-1}{2}\right)\pi$, where [.] denotes the greatest integer function, then f is
[2021, 24 Feb. Shift-1]

Options:

A. discontinuous only at $x = 1$

B. discontinuous at all integral values of x except at $x = 1$

C. continuous only at $x = 1$

D. continuous for every real x

Answer: D

Solution:

Solution:
Given, $f(x) = [x-1]\cos\left(\frac{2x-1}{2}\right)\pi$ where $[\cdot]$ is greatest integer function and $f : R \to R$
$\because$ It is a greatest integer function then we need to check its continuity at $x \in I$ except these it is continuous.
Let $x = n$ where $n \in 1$
Then, LHL $= \lim_{x \to n^-} [x-1]\cos\left(\frac{2x-1}{2}\right)\pi$
$= (n-2)\cos\left(\frac{2n-1}{2}\right)\pi = 0$
RHL $= \lim_{x \to n^+} [x-1]\cos\left(\frac{2x-1}{2}\right)\pi$

$= (n-2)\cos\left(\frac{2n-1}{2}\right)\pi = 0$

and $f(n) = 0$.

Here, $\lim_{x \to n^-} f(x) = \lim_{x \to n^+} f(x) = f(n)$

∴ It is continuous at every integers.

Therefore, the given function is continuous for all real x.

Question39

If $f : R \to R$ is a function defined by $f(x) = [x-1]\cos\left(\frac{2x-1}{2}\right)\pi$, where [.] denotes the greatest integer function, then f is:
24 Feb 2021 Shift 1

Options:

A. discontinuousatall integral values of x exceptat $x = 1$

B. continuous only at $x = 1$

C. continuous for every real x

D. discontinuous only at $x = 1$

Answer: C

Solution:

Solution:

For $x = n, n \in Z$

$LHL = \lim_{x \to n^-} f(x) = \lim_{x \to n^-} [x-1]\cos\left(\frac{2x-1}{2}\right)\pi = 0$

$RHL = \lim_{x \to n^+} f(x) = \lim_{x \to n^+} [x-1]\cos\left(\frac{2x-1}{2}\right)\pi = 0$

$f(n) = 0$

$\Rightarrow LHL = RHL = f(n)$

$\Rightarrow f(x)$ is continuous for every real x.

Question40

If $f(x) = \begin{cases} \frac{1}{|x|} & |x| \geq 1;\ ax^2 + b, & |x| < 1 \end{cases}$. is differentiable at every point of the domain, then the values of a and b are respectively
[2021, 18 March shift-1]

Options:

A. $\frac{1}{2}, \frac{1}{2}$

B. $\frac{1}{2}, -\frac{3}{2}$

C. $\frac{5}{2}, -\frac{3}{2}$

D. $-\frac{1}{2}, \frac{3}{2}$

Answer: D

Solution:

Solution:

Given, $f(x) = \left\{ \frac{1}{|x|} \quad \left|x\right| \geq 1; ax^2 + b,, \left|x\right| < 1. \right.$

$$\Rightarrow f(x) = \begin{cases} \frac{1}{|x|} & x \leq -1 \text{ or } x \geq 1 \\ ax^2 + b & -1 < x < 1. \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} \frac{-1}{x} & x \leq -1 \\ ax^2 + b & -1 < x < 1 \\ \frac{1}{x} & x \geq 1. \end{cases}$$

Given, $f(x)$ is differentiable at every point of domain. $\therefore f'(x) = \begin{cases} \frac{1}{x^2} & x < -1 \\ 2ax & -1 < x < 1 \\ \frac{-1}{x^2} & x > 1. \end{cases}$

$\because f(x)$ is differentiatble at $x = 1$
$\therefore$(LHD at $x = 1$) = (RHD at $x = 1$)
$\Rightarrow f'(1^-) = f'(1^+)$
$\Rightarrow 2a = -1 \Rightarrow a = -\frac{1}{2}$
As, we know that, a function is differentiable at $x = a$, if it is continuous at $x = a$.
Hence, $f(x)$ is also continuous at $x = 1$.
i.e., (LH L at $x = 1$) = (RHL at $x = 1$) = $f(1)$
$\Rightarrow a + b = 1$
$\Rightarrow \left(-\frac{1}{2}\right) + b = 1$
$\Rightarrow b = \frac{3}{2}$
Hence, $a = -\frac{1}{2}, b = \frac{3}{2}$
Note You can also (or apply) continuity and differentiability at $x = -1$.

Question41

Let f : R → R be a function defined as

$$\mathbf{f(x) =} \begin{cases} \frac{\sin(a+1)x + \sin 2x}{2x} & \text{if } x < 0 \\ \frac{b}{\sqrt{x + bx^3} - \sqrt{x}} & \text{if } x < 0 \\ bx^{5/2} & \text{if } x > 0. \end{cases}$$

If f is continuous at x = 0, then the value of a + b is equal to [2021, 18 March Shift-II]

Options:

A. $-\frac{5}{2}$

B. -2

C. -3

D. $-\frac{3}{2}$

Answer: D

Solution:

Solution:

Given, $f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin 2x}{2x} & x < 0 \\ b & x = 0 \\ \dfrac{\sqrt{x+bx^3} - \sqrt{x}}{bx^{5/2}} & x > 0. \end{cases}$

$\because$ $f(x)$ is continuous at $x = 0$.

$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$ ...(i)

$\because \quad f(0) = b$ ......(ii)

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\frac{\sin(a+1)x + \sin 2x}{2x} \right)$

$\Rightarrow \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\frac{\sin(a+1)x}{2x} + \frac{\sin 2x}{2x} \right)$

$= \lim_{x \to 0^-} \left(\frac{\sin(a+1)x}{(a+1)x} \times \left(\frac{a+1}{2} \right) + \frac{\sin 2x}{2x} \right)$

$= \frac{a+1}{2} + 1$. . . (iii)

Again, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\frac{\sqrt{x+bx^3} - \sqrt{x}}{bx^{5/2}} \right)$

$= \lim_{x \to 0^+} \frac{\left(\sqrt{x+bx^3} - \sqrt{x}\right)\left(\sqrt{x+bx^3} + \sqrt{x}\right)}{bx^{5/2}\left(\sqrt{x+bx^3} + \sqrt{x}\right)}$

$= \lim_{x \to 0^+} \frac{(x+bx^3 - x)}{bx^{5/2}\left(\sqrt{x+bx^3} + \sqrt{x}\right)}$

$= \lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{x}\left(\sqrt{1+bx^2} + 1\right)}$

$\Rightarrow \lim_{x \to 0^+} f(x) = \frac{1}{2}$ . . . (iv)

From Eq. (i), (ii), (iii) and (iv)

$\frac{1}{2} = b = \frac{a+1}{2} + 1$

$\Rightarrow b = \frac{1}{2}, a = -2$

$\therefore a + b = \frac{-3}{2}$

Question42

Let $f : R \to R$ satisfy the equation $f(x+y) = f(x) \cdot f(y)$ for all $x, y \in R$ and $f(x) \neq 0$ for any $x \in R$. If the function f is differentiable at $x = 0$ and $f'(0) = 3$, then $\lim_{h \to 0} \frac{1}{h}(f(h) - 1)$ is equal to............

[2021, 18 March Shift-II]

Answer: 3

Solution:

Solution:
Method 1
Given, $f(x+y) = f(x) \cdot f(y)\ \forall x, y \in R$
$\therefore f(x) = a^x$
$\Rightarrow f'(x) = a^x \cdot \log(a)$
Now, $f'(0) = \log(a)$
$\Rightarrow 3 = \log(a)$
$\Rightarrow a = e^3$
$\therefore f(x) = (e^3)^x = e^{3x}$
$f(h) = e^{3h}$
Now, $\lim_{h \to 0}\left(\frac{f(h)-1}{h}\right) = \lim_{h \to 0}\left(\frac{e^{3h}-1}{h}\right)$
$= \lim_{h \to 0}\left(\frac{e^{3h}-1}{3h} \times 3\right)$
$= 3 \times 1 = 3$
Method(2)
Let $L = \lim_{h \to 0} \frac{1}{h}(f(h)-1)\left(\frac{0}{0} \text{ form }\right)$
$f(x+y) = f(x) + f(y)$
Put $x = y = 0$
$\therefore f(0) = f(0) \cdot f(0)$
$\Rightarrow [f(0)]^2 = f(0)$
$\Rightarrow [f(0)]^2 - f(0) = 0$
$\Rightarrow f(0)[f(0) - 1] = 0$
$\Rightarrow f(0) = 0, f(0) = 1$
Rejected because $f(x) \neq 0, \forall x \in R$
$\therefore f(0) = 1$
Using L-Hospital Rule,
$L = \lim_{h \to 0} \frac{f'(h) - 0}{1}$
$= f'(0) = 3$

Question43

If the function $f(x) = \frac{\cos(\sin x) - \cos x}{x^4}$ is continuous at each point in its domain and $f(0) = \frac{1}{k}$, then k is ....
[2021, 17 March Shift-1]

Answer: 6

Solution:

$f(x) = \frac{\cos(\sin x) - \cos x}{x^4}$
As, $f(x)$ is continuous everywhere, so
$f(0) = \lim_{x \to 0} f(x)$
$= \lim_{x \to 0} \frac{\cos(\sin x) - \cos x}{x^4}$
On expanding the numerator and only identifying the coefficient of x^4 will give us the required limit.
$\cos(\sin x) = \left(1 - \frac{\sin^2 x}{2} + \frac{\sin^4 x}{24}\right)$
$= 1 - \frac{1}{2}\left(x - \frac{x^3}{6}\right)^2 + \frac{1}{24}(x)^4$

$$= 1 - \frac{1}{2}\left(x^2 - \frac{x^4}{3}\right) + \frac{x^4}{24}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^4}{24}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$$\therefore \frac{\cos(\sin x) - \cos x}{x^4}$$

$$= \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^4}{24}\right) - \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)}{x^4}$$

$$= \frac{1}{6}$$

$$\therefore\ f(0) = \frac{1}{6} = \frac{1}{k}$$

Hence, $k = 6$.

Question44

Consider the function f : R → R defined by Then, f is

$$f(x) = \begin{cases} \left[2 - \sin\left(\frac{1}{x}\right)\right]|x|, & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

[2021, 17 March Shift-II]

Options:

A. monotonic on $(-\infty, 0) \cup (0, \infty)$

B. not monotonic on $(-\infty, 0)$ and $(0, \infty)$

C. monotonic on $(0, \infty)$ only

D. monotonic on $(-\infty, 0)$ only

Answer: B

Solution:

Solution:
Method (1)

Given, $f(x) = \begin{cases} \left[2 - \sin\left(\frac{1}{x}\right)\right]|x|, & x \neq 0 \\ 0 & x = 0 \end{cases}.$

Here, $f'(x)$ is an oscillating function which is non-monotonic in $(-\infty, 0) \cup (0, \infty)$.
Method (II)

$$\because\ f(x) = \begin{cases} -\left(2 - \sin\frac{1}{x}\right)x & x < 0 \\ 0 & x = 0 \\ \left(2 - \sin\frac{1}{x}\right)x & x > 0. \end{cases}$$

From above we observe that, $f(x)$ is continuous and $f\left(\frac{1}{\pi}\right) = f\left(\frac{2}{\pi}\right) = \frac{2}{\pi}$ So, $f(x)$ is non-monotonic in $(0, \infty)$.
Further, $\lim_{x \to -\infty}(f) \to \infty$ and $\lim_{x \to \infty} f(x) \to \infty$
and $f(0) = 0$
Hence, $f(x)$ is non-monotonic on $(-\infty, 0) \cup (0, \infty)$.

Question45

Let the functions $f : R \rightarrow R$ and $g : R \rightarrow R$ be defined as

$$f(x) = \begin{cases} x + 2 & x < 0 \\ x^2 & x \geq 0. \end{cases}$$

$$\text{and } g(x) = \begin{cases} x^3 & x < 1 \\ 3x - 2 & x \geq 1. \end{cases}$$

Then, the number of points in R, where $(f og)(x)$ is not differentiable is equal to
[2021, 16 March Shift-1]

Options:

A. 3

B. 1

C. 0

D. 2

Answer: B

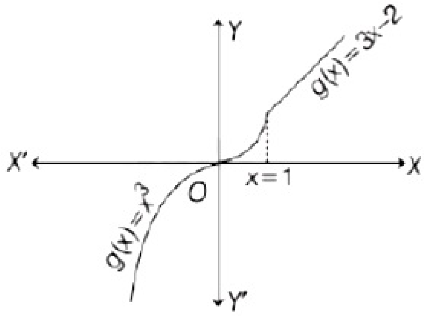
Solution:

Solution:

$$f(x) = \begin{cases} x + 2 & x < 0 \\ x^2 & x \geq 0. \end{cases}$$

$$g(x) = \begin{cases} x^3 & x < 1 \\ 3x - 2 & x \geq 1. \end{cases}$$

$$f[g(x)] = \begin{cases} g(x) + 2 & g(x) < 0 \\ g^2(x) & g(x) \geq 0. \end{cases}$$



When $g(x) < 0 \Rightarrow g(x) = x^3, x < 0$

When $g(x) \geq 0 \Rightarrow g(x) = \begin{cases} x^3 & 0 \leq x < 1 \\ 3x - 2 & x \geq 1. \end{cases}$

$$f[g(x)] = \begin{cases} x^3 + 2 & x < 0 \\ x^6 & 0 \leq x < 1 \\ (3x - 2)^2 & x \geq 1. \end{cases}$$

As, polynomial function is continuous everywhere in its domain. So, $f[g(x)]$ will be continuous everywhere at $x < 0$, $0 < x < 1$ and $x > 1$. We will check the behaviour of $f og(x)$ only at boundary points which is $x = 0$ and $x = 1$.

$\lim_{x = 0,} x^6 = 0^- \rightarrow 0^-(x^3 + 2) = 2$

Clearly, $L^+HL \neq RHL$ at $x = 0$

So, $f \circ g(x)$ is discontinuous at $x = 0$.

$\lim_{x \to 1^+} (3x-2)^{2x \to 1^-} (\lim^6 = 1$
Also $f(1) = 1$ fog (x) is continuous at $x = 1$
Derivative test at $x = 1$,

$$LHD = \lim_{h \to 0} \frac{f(1) - f(1-h)}{h}$$
$$= \lim_{h \to 0} \frac{1 - (1-h)^6}{h}$$
$$= \lim_{h \to 0} 6(1-h)^5 = 6$$
$$RHD = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h \to 0} \frac{[3(1+h) - 2]^2 - 1}{h}$$
$$= \lim_{h \to 0} 2[3(1+h) - 2] \cdot 3 = 6$$

∴ fog (x) is continuous and differentiable at $x = 1$.

Question46

Let f : R → R and g : R → R be defined as

f (x) = $\begin{cases} x + a & x < 0 \\ |x - 1 & x \geq 0. \end{cases}$ **and**

g(x) = $\begin{cases} x + 1 & x < 0 \\ (x-1)^2 + b & x \geq 0. \end{cases}$

where a, b are non-negative real numbers. If (gof) (x) is continuous for all x ∈ R, then a + b is equal to..........
[2021, 16 March Shift-II]

Answer: 1

Solution:

$$g(x) = \begin{cases} x + 1 & x < 0 \\ (x-1)^2 + b & x \geq 0. \end{cases}$$

$$g[f(x)] = \begin{cases} f(x) + 1 & f(x) < 0 \\ [f(x) - 1]^2 + b, & f(x) \geq 0. \end{cases}$$

$f(x) < 0$
Case I $x + a < 0$ and $x < 0 \Rightarrow x < -a$
Case II $|x - 1| < 0$ and $x \geq 0 \Rightarrow$ Not possible $f(x) \geq 0$
Case I $x + a \geq 0$ and $x < 0 \Rightarrow x \in [-a, 0)$
Case II $|x - 1| \geq 0$ and $x \geq 0 \Rightarrow x \geq 0$

$$g[f(x)] = \begin{cases} x + a + 1 & x < -a \\ (x + a - 1)^2 + b & -a \leq x < 0 \\ (|x - 1| - 1)^2 + b & x \geq 0. \end{cases}$$

This is continuous function.
Since, $g[f(x)]$ is continuous for all $x \in R$
So, $g(f(x))$ will be continuous at $x = -a$ and $x = 0$
Now, at $x = -a$
$LHL = RHL =$ value of function
$\Rightarrow 1 = 1 + b = 1 + b \Rightarrow b = 0$

At x = 0
LH L = RH L = value of function
$\Rightarrow (a-1)^2 + b = b$
$\Rightarrow (a-1)^2 = 0$
$\Rightarrow a = 1$
Hence, $a + b = 1$

Question47

Let $\alpha \in R$ be such that the function $f(x) = \begin{cases} \frac{\cos^{-1}(1-\{x\}^2)\sin^{-1}(1-\{x\})}{\{x\}-\{x\}^3} & x \neq 0 \\ \alpha & x = 0. \end{cases}$

is continuous at x = 0, where {x} = x − [x], [x] is the greatest integer less than or equal to x.
Then,
[2021, 16 March Shift-II]

Options:

A. $\alpha = \frac{\pi}{\sqrt{2}}$

B. $\alpha = 0$

C. no such α exists

D. $\alpha = \frac{\pi}{4}$

Answer: C

Solution:

Solution:

Given, $f(x) = \begin{cases} \frac{\cos^{-1}(1-\{x\}^2)\sin^{-1}(1-\{x\})}{\{x\}-\{x\}^3} & x \neq 0 \\ \alpha & x = 0. \end{cases}$

$\{x\} = x - [x]$
So, when $x \to 0^+$
$\Rightarrow \{x\} = x - 0 = x$
And, when $x \to 0^-$
$\Rightarrow \{x\} = x + 1$
LH L = $\lim_{x \to 0^-} f(x)$

$= \lim_{x \to 0} \frac{\cos^{-1}[1-(1+x)^2]\sin^{-1}[1-(1+x)]}{(1+x)-(1+x)^3}$

$= \lim_{x \to 0} \frac{\cos^{-1}(-x^2-2x)\sin^{-1}(-x)}{(1+x)(1+1+x)(1-1-x)}$

$= \lim_{x \to 0} \frac{\cos^{-1}(-x^2-2x)}{(1+x)(x+2)}$

$= \frac{\cos^{-1}(0)}{1.2} = \frac{\pi}{4}$

RHL $= \lim_{x \to 0^+} f(x)$

$= \lim_{x \to 0} \frac{\cos^{-1}(1-x^2)\sin^{-1}(1-x)}{x(1-x)(1+x)}$

$= \frac{\pi}{2} \lim_{x \to 0} \frac{\cos^{-1}(1-x^2)}{x}$

Applying L-Hospital Rule,

$= \frac{\pi}{2} \lim_{x \to 0} \frac{(-1)(-2x)}{\sqrt{1-(1-x^2)^2}}$

$= \frac{\pi}{2} \cdot 2 \lim_{x \to 0} \frac{x}{\sqrt{2x^2 - x^4}}$

$= \pi \lim_{x \to 0} \frac{1}{\sqrt{2 - x^2}}$

$= \frac{\pi}{\sqrt{2}}$

LH L = $\frac{\pi}{4}$ and RH L = $\frac{\pi}{\sqrt{2}}$

Hence, LHL ≠ RHL

So, the function will be discontinuous for every value of $\alpha \in R$.

∴ No such α exist.

Question48

Let $f : S \to S$, where $S = (0, \infty)$ be a twice differentiable function, such that $f(x+1) = xf(x)$. If $g : S \to R$ be defined as $g(x) = \log_e f(x)$, then the value of $g''(5) - g''(1)|$ is equal to [2021, 16 March Shift-II]

Options:

A. $\frac{205}{144}$

B. $\frac{197}{144}$

C. $\frac{187}{144}$

D. 1

Answer: A

Solution:

Solution:

We have, $f : S \to S$, $S = (0, \infty)$

$f(x+1) = x \cdot f(x)$

$g : S \to R$

$g(x) = \log_e f(x)$

To find $g''(5) - g''(1)|$

$\Rightarrow g(x+1) = \log_e f(x+1)$

$\Rightarrow g(x+1) = \log[x \cdot f(x)]$

$\Rightarrow g(x+1) = \log x + \log f(x)$

$\Rightarrow g(x+1) = \log x + g(x)$

$\Rightarrow g(x+1) - g(x) = \log x$

$\Rightarrow g'(x+1) - g'(x) = 1 / x$

$\Rightarrow g''(x+1) - g''(x) = \frac{-1}{x^2}$

$x = 1, g''(2) - g''(1) = -1$ . . . (i)

$x = 2, g''(3) - g''(2) = -1 / 4$ . . . (ii)

$x = 3, g''(4) - g''(3) = -1 / 9$ . . . (iii)

$x = 4, g''(5) - g''(4) = -1 / 16$ . . . (iv)

Adding Eqs. (i), (ii), (iii) and (iv),

$g''(5) - g''(1) = -1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16}$

$= -\left(\frac{144 + 36 + 16 + 9}{144}\right)$

$= \frac{-205}{144}$

So, $|g''(5) - g''(1)| = \frac{205}{144}$

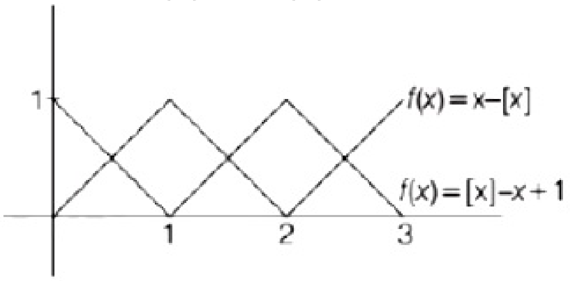
Question49

Let f : [0, 3] rightarrow R be defined by
f (x) = min{x − [x], 1 + [x] − x]where [x] is the greatest integer less than or equal to x. Let P denote the set containing all x in(0, 3), where f is discontinuous and Q denote the set containing all x in(0, 3), where f is not differentiable.
Then the sum of number of elements in P and Q is equal to ....
[2021, 27 July Shift-1]

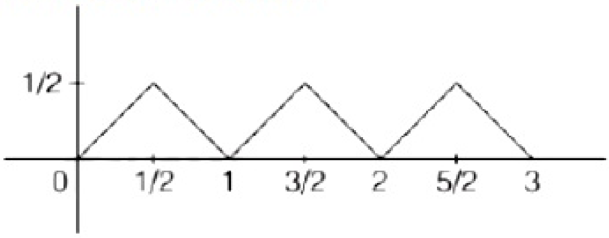
Answer: 5

Solution:

$f(x) = \min\{x - [x], 1 + [x] - x\}$
$f(x) = \min(\{x\}, 1 - \{x\})$



So, the graph of $f(x)$ will be



f is continuous everywhere for $0 \le x \le 3$. But f is non-differentiable at $x = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ and $x = 1, 2$

So, if set A denotes the points of discontinuity, then $n(A) = 0$.
And if set B denotes the points of non-differentiable, then
$n(B) = 5$
$\therefore\ n(A) + n(B) = 0 + 5 = 5$

Question50

Let f : $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \rightarrow R$ be defined as

$$f(x) = \begin{cases} (1+|\sin x|)^{\frac{3a}{|\sin x|}} & \frac{-\pi}{4} < x < 0 \\ b_1 & x = 0 \\ e^{\frac{\cot 4x}{\cot 2x}}, & 0 < x < \frac{\pi}{4}. \end{cases}$$

If f is continuous at x = 0, then the value of $6a + b^2$ is equal to [2021, 27 July Shift I]

Options:

A. $1 - e$

B. e -1

C. $1 + e$

D. e

Answer: C

Solution:

Solution:

$f : \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \to R$

$$f(x) = \begin{cases} (1+|\sin x|)^{\frac{3a}{|\sin x|}} & -\frac{\pi}{4} < x < 0 \\ b & x = 0 \\ e^{\frac{\cot 4x}{\cot 2x}} & 0 < x < \frac{\pi}{4}. \end{cases}$$

Given f (x) is continuous at x = 0

LHL at x = 0

Put x = 0 − h

we get $\lim\limits_{h \to 0} (1 - \sin h)^{\frac{3a}{\sin h}}$

$e^{\lim\limits_{h \to 0} (1 - \sinh - 1) \cdot \frac{3a}{-\sinh}} = e^{3a}$

$\lim\limits_{x \to 0^-} (1+|\sin x|)^{\frac{3a}{|\sin x|}}$

$= e^{\lim\limits_{x \to 0} |\sin x| \frac{3a}{|\sin x|}} = e^{3a}$

RH L at x = 0

$\lim\limits_{x \to 0^+} e^{\frac{\cot 4x}{\cot 2x}}$

Put x = 0 + h

we get $\lim\limits_{h \to 0} \frac{\cot 4h}{e^{\cot 2h}}$

$\lim\limits_{h \to 0} \frac{\cos 4h}{e \cos 2h} \times \frac{\sin 2h}{\sin 4h}$

$e^{\frac{\cos 4h}{\cos 2h} \times \frac{\frac{\sin 2h}{2h} \times 2h}{\frac{\sin 4h}{4h} \times 4h}} = e^{1/2}$

As, f (x) is continuous at x = 0.

So, LH L = f (0) = RH L

$e^{3a} = b = e^{\frac{1}{2}}$

$\therefore\ a = \frac{1}{6}, b = \sqrt{e}$

$\therefore\ 6a + b^2 = 6\left(\frac{1}{6}\right) + (\sqrt{e})^2$

$= 1 + e$

Question51

Let f : [0, ∞) → [0, 3] be a function defined by

$$f(x) = \begin{cases} \max\{\sin t : 0 \le t \le x\} & 0 \le x < \pi \\ 2 + \cos x & x > \pi. \end{cases}$$

Then which of the following is true?
[2021, 27 July Shift-11]

Options:

A. f is continuous everywhere but not differentiable exactly at one point in (0, ∞)

B. f is differentiable everywhere in (0, ∞)

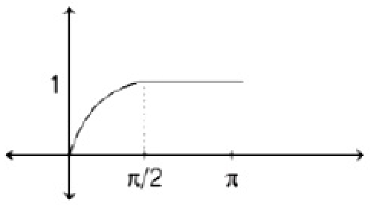
C. f is not continuous exactly at two points in (0, ∞)

D. f is continuous everywhere but not differentiable exactly at two points in (0, ∞)

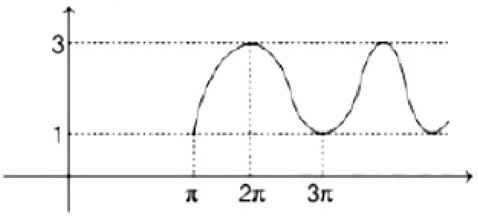
Answer: B

Solution:

Solution:
Graph of max(sint : 0 ≤ t ≤ x) in x ∈ [0, π]

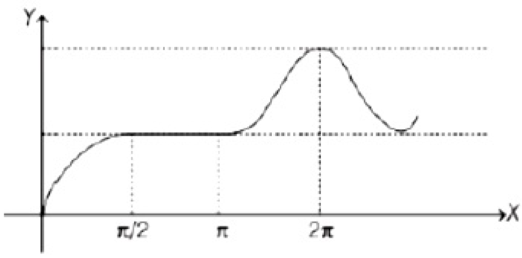


and graph of $2 + \cos x$ for $x \in [\pi, \infty]$



So, graph of

$$f(x) = \begin{cases} \max[\sin t : 0 \le t \le x], & 0 \le x \le \pi \\ 2 + \cos x, & x > \pi \end{cases}$$



So, $f(x)$ is differentiable everywhere in $(0, \infty)$.

Question52

Let $f : (a, b) \to R$ be twice differentiable function such that $f(x) = \int_a^x g(t) dt$ for a differentiable function $g(x)$. If $f(x) = 0$ has exactly five distinct roots in (a, b), then $g(x)g'(x) = 0$ has at least
[2021, 27 July Shift-II]

Options:

A. twelve roots in (a, b)

B. five roots in (a, b)

C. seven roots in (a, b)

D. three roots in (a, b)

Answer: C

Solution:

Solution:

We have, $f(x) = \int_a^x g(t) dt$

So, $f'(x) = g(x)$ and $f''(x) = g'(x)$

$$\left\{ \begin{array}{c} \because f(x) = \int_{g(x)}^{h(x)} F(t) \cdot dt \Rightarrow f'(x) \\ = F[h(x)] \cdot h'(x) - F[g(x)] \cdot g'(x) \end{array} \right\}$$

Now, $g'(x)g(x) = 0$

$\Rightarrow f''(x)f'(x) = 0$

If $f(x)$ has five roots, then $f'(x)$ has atleast 4 roots and $f''(x)$ has atleast 3 roots.

So, $f''(x) \cdot f'(x) = 0$ has atleast 7 roots. Hence, the minimum number of roots of the equation $g'(x)g(x) = 0$ is 7 .

Question53

Let $f : R \to R$ be defined as

$$f(x) = \left\{ \begin{array}{ll} \dfrac{\lambda x^2 - 5x + 6|}{\mu(5x - x^2 - 6)} & x < 2 \\ e^{\frac{\tan(x-2)}{x - [x]}} & x > 2 \\ \mu & x = 2. \end{array} \right.$$

where, [x] is the greatest integer less than or equal to x. If f is continuous at x = 2, then $\lambda + \mu$ is equal to
[2021, 25 July Shift-1]

Options:

A. $e\, e(-e + 1)$

B. $e(e - 2)$

C. 1

D. $2e - 1$

Answer: A

Solution:

Solution:

We have $f(x) = \begin{cases} \dfrac{\lambda|x^2 - 5x + 6|}{\mu(5x - x^2 - 6)} & x < 2 \\ e^{\frac{\tan(x-2)}{x-[x]}} & x > 2 \\ \mu & x = 2. \end{cases}$

$f(x)$ is continuous at $x = 2$.

$\therefore \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$

$\therefore \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{\lambda}{\mu} \frac{|(x-3)(x-2)|}{-(x-3)(x-2)}$

$= \lim_{x \to 2^-} \frac{\lambda}{\mu} \frac{(x-3)(x-2)}{-(x-3)(x-2)} = -\frac{\lambda}{\mu}$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} e^{\frac{\tan(x-2)}{x-2}} = e$

As, $f(x)$ is continuous.

So, $LHL = f(2) = RHL$

$-\frac{\lambda}{\mu} = \mu = e$

$\lambda = -e^2$

$\mu = e$

$\lambda + \mu = e(-e + 1)$

Question54

Let $f : [0, \infty) \to [0, \infty)$ be defined as $f(x) = \int_0^x [y] dy$ where, $[x]$ is the greatest integer less than or equal to x. Which of the following is true? [2021, 25 July Shift-1]

Options:

A. f is continuous at every point in $[0, \infty)$ and differentiable except at the integer points

B. f is both continuous and differentiable except at the integer points in $[0, \infty)$

C. f is continuous everywhere except at the integer points in $[0, \infty)$

D. f is differentiable at every point in $[0, \infty)$

Answer: A

Solution:

Solution:

$f : [0, \infty) \to [0, \infty)$

$f(x) = \int_0^x [y] dy$

Let $x = 1 + f$, $0 < f < 1$

$f(x) = \int_0^1 [y] dy + \int_1^2 [y] dy + \int_2^3 [y] dy + \dots \int_{1-1}^{1} [y] dy$

$+ \int_{1+f}^{1-y} [y d y$

$f(x) = 0 + 1 + 2 + \dots (1 - 1) + 1 \cdot f$

$= \frac{(l-1)(l-1+1)}{2} + l \cdot f$

$= \frac{|(l-1)}{2} + 1 \cdot f$

$f(x) = \frac{[x]([x]-1)}{2} + [x][x\}$

$f(x) = \frac{[x]([x]-1)}{2} + [x](x-[x])$

$f(I) = \frac{|(|-1)}{2}$

$\lim_{x \to I^-} f(x) = \lim_{h \to 0} \frac{I(|-1)}{2} + I(I + h - l)$

$= \frac{|(|-1)}{2}$

$\lim_{x \to I^-} f(x) = \lim_{h \to 0} \frac{|(|-1)(|-2)}{2} + (|-1)(|+h-I+1)$

$= \frac{(1-1)(1-2)}{2} + (1-1)$

$= \frac{(1-1)|}{2}$

$\therefore f(x)$ is continuous and differentiable except at integer points.

Question55

If $f(x) = \begin{cases} \int_0^x (5+|1-t|)dt & x > 2 \\ 5x+1 & x \le 2. \end{cases}$ **, then**

[2021, 25 July Shift-11]

Options:

A. $f(x)$ is not continuous at $x = 2$

B. $f(x)$ is everywhere differentiable

C. $f(x)$ is continuous but not differentiable at $x = 2$

D. $f(x)$ is not differentiable at $x = 1$

Answer: C

Solution:

Solution:

$f(x) = \{ \int_0^x (5+|1-t|dt,, x > 2; 5x+1,, x \le 2.$

$\int_0^x 5+\left|1-t\right|dt$

$= \int_0^1 5+(1-t)dt + \int_1^x 5+(t-1)dt$

$= \int_0^1 (6-t)dt + \int_1^x (4+t)dt$

$= \left[6t - \frac{t^2}{2}\right]_0^1 + \left[4t + \frac{t^2}{2}\right]_1^x = 1 + 4x + \frac{x^2}{2}$

$\Rightarrow f(x) = \begin{cases} 1 + 4x + \frac{x^2}{2} & x > 2 \\ 5x+1 & x \le 2. \end{cases}$

At $x = 2$

$LHL = \lim_{x \to 2^-} (5x+1) = 11$

$\text{RHL} = \lim_{x \to 2^+} \left(1 + 4x + \frac{x}{2}\right)^2 = 1 + 8 + 2 = 11$

$\therefore f(2) = 11$

So, f (x) is continuous at $x = 2$.

$$f'(x) = \begin{cases} 4 + x & x > 2 \\ 5 & x \le 2 \end{cases}$$

Now, LHD at $x = 2$ is $\frac{d}{dx}(5x + 1)_{x=2} = 5$

RHD at $x = 2$ is $4 + 2 = 6$

Here, LHD ≠ RHD

So, function is not differentiable at $x = 2$.

Question56

Consider the function $f(x) = \begin{cases} \dfrac{P(x)}{\sin(x-2)} & x \ne 2 \\ 7 & x = 2. \end{cases}$

where, P(x) is a polynomial such that $P'''(x)$ is always a constant and P(3) = 9. If f (x) is continuous at x = 2, then P(5) is equal to ....
[2021, 25 July Shift-11]

Answer: 39

Solution:

Solution:

$$f(x) = \begin{cases} P(x) / \sin(x-2) & x \ne 2 \\ 7 & x = 2. \end{cases}$$

Given, that $P''(x)$ is always a constant.

⇒P(x) is a 2 degree polynomial.

f (x) is continuous at $x = 2$

$\lim_{x \to 2^+} P(x) / \sin(x-2) = 7$

$\Rightarrow \lim_{x \to 2^+} (x-2)(ax+b) / \sin(x-2) = 7$

⇒ $2a + b = 7$ . . . (i)

Now, $P(x) = (x-2)(ax+b)$

$P(3) = 9$(given)

⇒ $3a + b = 9$

Subtracting Eq. (ii) from Eq. (i),

$a = 2$

From Eq. (i), $b = 3$

Hence, $P(x) = (x-2)(2x+3)$

So, $P(5) = (5-2)(2 \times 5 + 3) = 3 \times 13 = 39$

Question57

Let f : R → R be defined as $f(x) = \begin{cases} \dfrac{x^3}{(1-\cos 2x)^2} \log_e\left(\dfrac{1 + 2xe^{-2x}}{(1 - xe^{-x})^2}\right) & x \ne 0 \\ \alpha & x = 0. \end{cases}$

If f is continuous at x = 0, then α is equal to

[2021, 22 July Shift-II]

Options:

A. 1

B. 3

C. 0

D. 2

Answer: A

Solution:

Solution:

$$f(x) = \begin{cases} \dfrac{x^3}{(1-\cos 2x)^2}\log_e\left[\dfrac{1+2xe^{-2x}}{(1-xe^{-x})^2}\right] & x \neq 0 \\ \alpha & x = 0. \end{cases}$$

For continuity, $\lim\limits_{x \to 0} \dfrac{x^3}{4\sin^4 x}[\log_e(1+2xe^{-2x})$

$-\log_e(1-xe^{-x})^2] = \alpha(\text{ by expansion })\ldots(\text{ i })$

$\because \log(1+2xe^{-2x}) = 2xe^{-2x} - \dfrac{(2xe^{-2x})^2}{2} + \ldots$ and $\log(1-xe^{-x}) = -xe^{-x} - \dfrac{(xe^{-x})^2}{2} - \ldots$

On putting the values in Eq. (i), we get

$\lim\limits_{x \to 0}\left(\dfrac{1}{4}\cdot\dfrac{x}{x}\right)\left(\dfrac{x^3}{\sin^4 x}\right)[2xe^{-2x} - 2(-xe^{-x})]$

$= \lim\limits_{x \to 0}\left(\dfrac{1}{4x}\right)\left(\dfrac{x}{\sin x}\right)^4(2xe^{-2x} + 2xe^{-x})$

$= \lim\limits_{x \to 0}\left(\dfrac{1}{4x}\right)\cdot\left(\dfrac{x}{\sin x}\right)^4\cdot 2x\cdot(e^{-2x} + e^{-x})$

$= \left(\dfrac{1}{4}\right)\cdot(1)\cdot(2)\cdot(2) \Rightarrow \alpha = 1$

Question58

Let $f: R \to R$ be a function defined

$f(x) = \begin{cases} 3\left(1 - \frac{|x|}{2}\right) \quad \text{if} \end{cases} \left|x\right| \leq 2;\ 0,, \quad \text{if} \ , \left|x\right| > 2.$

Let $g: R \to R$ be given by $g(x) = f(x+2) - f(x-2)$
If n and m denote the number of points in R,
where g is not continuous and not differentiable respectively, then n + m is equal to ....
[2021, 22 July Shift-II]

Answer: 4

Solution:

Solution:

$f(x) = \begin{cases} 3\left(\frac{1-|x|}{2}\right) \quad \text{if} \end{cases} \left|x\right| \leq 2;\ 0, \quad \text{if} \ , \left|x\right| > 2.$

$g(x) = f(x+2) - f(x-2)$

$$f(x) = \begin{cases} 0 & x < -2 \\ \frac{3}{2}(1+x) & -2 \le x < 0 \\ \frac{3}{2}(1-x) & 0 \le x < 2 \\ 0 & x > 2. \end{cases}$$

$$f(x+2) = \begin{cases} 0 & x < -4 \\ \frac{3}{2}(3+x) & -4 \le x < -2 \\ \frac{3}{2}(-1-x) & -2 \le x < 0 \\ 0 & x > 4. \end{cases}$$

$$f(x-2) = \begin{cases} 0 & x < 0 \\ \frac{3}{2}(x-1) & 0 \le x < 2 \\ \frac{3}{2}(-1-x) & 2 \le x < 4 \\ 0 & x > 4. \end{cases}$$

$g(x) = f(x+2) + f(x-2)$

$$= \begin{cases} \frac{3x}{2} + 6 & -4 \le x \le 2 \\ -\frac{3x}{2} & -2 < x < 2 \\ \frac{3x}{2} - 6 & 2 \le x \le 4 \\ 0 & |x| > 4. \end{cases}$$

So, $n = 0$ and $m = 4$

$\therefore\ m + n = 4$

Question59

Let a function $f : R \to R$ be defined as $f(x) = \begin{cases} \sin x - e^x & \text{if } x \le 0 \\ a + [-x] & \text{if } 0 < x < 1 \\ 2x - b & \text{if } x \ge 1. \end{cases}$

where, [x] is the greatest integer less than or equal to x. If f is continuous on R, then (a + b) is
[2021, 20 July Shift-1]

Options:

A. 4

B. 3

C. 2

D. 5

Answer: B

Solution:

Solution:

$$f(x) = \begin{cases} \sin x - e^x & x < 0 \\ a + [-x] & 0 < x < 1 \\ 2x - b & x \geq 1. \end{cases}$$

f (x) is continuous.

So, $\lim f(0^-) = 0 - e^0 = -1$

$\lim f(0^+) = a - 1$

$\Rightarrow a - 1 = -1 \Rightarrow a = 0$

$\lim f(1^-) = \lim_{h \to 0} a + [-1 - h] = a - 1$

$\lim f^{-1}(1^+) = 2(1 + h) - b = 2 - b$

$\therefore 2 - b = a - 1 \Rightarrow b = 2 + 1 = 3$

$\therefore a + b = 3$

Question60

Let a function g : [0, 4] → R be defined as

$$g(x) = \begin{cases} \max_{0 \leq t \leq x}(t^3 - 6t^2 + 9t - 3) & 0 \leq x \leq 3 \\ 4 - x & 3 < x \leq 4. \end{cases}$$

then the number of points in the interval (0, 4) where g(x) is not differentiable, is ....

[2021, 20 July Shift-II]

Answer: 1

Solution:

Solution:

Let $f(x) = x^3 - 6x^2 + 9x - 3$

$f'(x) = 3x^2 - 12x + 9$

$f'(x) = 0$ gives

$3x^2 - 12x + 9 = 0$

$\Rightarrow 3(x - 1)(x - 3) = 0$

$\therefore x = 1$ or $x = 3$

Now, $f(1) = 1$ and $f(3) = -3$

$$g(x) = \begin{cases} f(x) & 0 \leq x \leq 1 \\ 1 & 1 \leq x \leq 3 \\ 4 - x & 3 < x \leq 4. \end{cases}$$

g(x) is continuous.

$$g'(x) = \begin{cases} 3(x - 1)(x - 3) & 0 \leq x \leq 1 \\ 0 & 1 \leq x < 3 \\ -1 & 3 < x \leq 4. \end{cases}$$

g(x) is non-differentiable at $x = 3$.

So, the number of points in (0, 4) where g(x) is not differentiable is 1.

Question61

The function $f(x) = x^2 - 2x - 3 \left| \cdot e^{9x^2 - 12x + 4|} \right.$ is not differentiable at exactly

[2021, 31 Aug. Shift-1]

Options:

A. four points

B. three points

C. two points

D. one point

Answer: C

Solution:

Solution:

$$f(x) = \begin{cases} (x-3)(x+1)e^{(3x-2)^2} & x > 3 \\ -(x-3)(x+1)e^{(3x-2)^2} & -1 \le x \le 3 \\ (x-3)(x+1)e^{(3x-2)^2} & x < -1. \end{cases}$$

At $x = -1$, let LHD be α , then its clear that RHD be $-\alpha$.
Similarly, at $x = 3$, if LHD is β , then RHD at $x = 3$ will be $-\beta$.
So, $f(x)$ is not differentiable at $x = -1, x = 3$
At, all other points $f(x)$ will be differentiable.

Question62

If the function $f(x) = \begin{cases} \frac{1}{x}\log_e\left(\frac{1+\frac{x}{a}}{1-\frac{x}{b}}\right) & x < 0 \\ k & x = 0 \\ \frac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1}-1} & x > 0. \end{cases}$ **is continuous at** $x = 0$**, then** $\frac{1}{a} + \frac{1}{b} + \frac{4}{k}$ **is equal to**
[2021,31 Aug. Shift-1]

Options:

A. -5-2

B. 5

C. -4

D. 4

Answer: A

Solution:

Solution:
$f(x)$ is continuous at $x = 0$
LHL at $x = 0 = f(0) =$ RHL at $x = 0$

$$\lim_{x \to 0^-} \frac{\ln\left(\frac{1+\frac{x}{a}}{1-\frac{x}{b}}\right)}{x} = \frac{\lim_{x \to 0^-} \left(\frac{1}{a}\right) \ln\left(1+\frac{x}{a}\right)}{\left(\frac{1}{a}\right)x}$$

$$-\frac{\lim_{x \to 0^-} \left(-\frac{1}{b}\right) \ln\left(1-\frac{x}{b}\right)}{\left(-\frac{1}{b}\right)x} = \left(\frac{1}{a}+\frac{1}{b}\right)$$

$$\lim_{x \to 0^+} \frac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1}-1}$$

$$\lim_{x \to 0^+} \frac{-2\sin^2 x}{\sqrt{x^2+1}-1} = \lim_{x \to 0^+} -\left(\frac{2\sin^2 x}{x^2}\right)$$

$$\left(\sqrt{x^2+1}+1\right) = -4$$

$$\Rightarrow \frac{1}{a}+\frac{1}{b} = -4 = k$$

$$\left(\frac{1}{a}+\frac{1}{b}\right)+\left(\frac{4}{k}\right) = -4-1 = -5$$

Question63

Let a, b ∈ R, b ≠ 0. Define a function
$f(x) = begin cases a \sin \frac{\pi}{2}(x-1), \quad for \ x \leq 0 \backslash \frac{\tan 2x - \sin 2x}{bx^3}, \quad for \ x > 0 end cases$
If f is continuous at x = 0, then 10 - ab is equal to
[2021, 26 Aug. Shift-1]

Answer: 14

Solution:

For continuity
LHL at 0 = f (0) = RHL at 0

$$\text{LH L} = \lim_{x \to 0^-} a \sin \frac{\pi}{2}(x-1)$$

$$= -a \sin \frac{\pi}{2} = -a \ . \ . \ . \ (i)$$

$$\text{RHL} = \lim_{x \to 0^+} \frac{\tan 2x - \sin 2x}{bx^3}$$

$$= \lim_{x \to 0^+} \frac{\sin 2x(1-\cos 2x)}{bx^3 \cdot \cos 2x}$$

$$= \lim_{x \to 0^+} 2\left(\frac{\sin 2x}{2x}\right) \frac{(2\sin^2 x)}{x^2} \cdot \frac{1}{b \cos 2x} = \frac{4}{b} \ . \ . \ . \ (i)$$

From Eqs. (i) and (ii), we get

$$-a = \frac{4}{b}$$

$\Rightarrow ab = -4$
$\Rightarrow 10 - ab = 14$

Question64

Let [t] denote the greatest integer less than or equal to t. Let
$f(x) = x - [x]$, $g(x) = 1 - x + [x]$, and $h(x) = \min\{f(x), g(x)\}$, $x \in [-2, 2]$.

Then h is
[2021, 26 Aug. Shift-II]

Options:

A. continuous in [−2, 2] but not differentiable at more than four points in (−2, 2)

B. not continuous at exactly three points in [−2, 2]

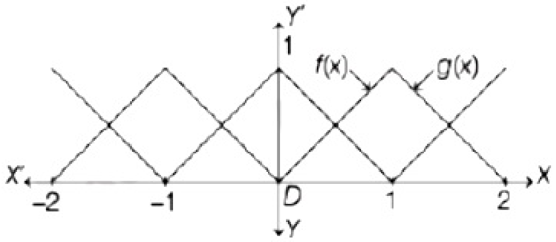
C. continuous in [−2, 2] but not differentiable at exactly three points in (−2, 2)

D. not continuous at exactly four points in [−2, 2]

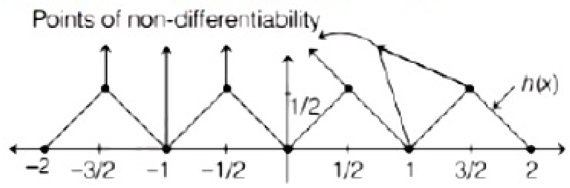
Answer: A

Solution:

Solution:
We have, $f(x) = x - [x] = \{x\}$
and $g(x) = 1 - x + [x] = 1 - \{x\}$



Again, $h(x) = \min[f(x), g(x)]$, so graph of $h(x)$ will be



From graph, it is clear that h(x) is continuous in [−2, 2] but not differentiable at $x = \frac{-3}{2}, -1, \frac{-1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}$ in(−2, 2)

Question65

Let $x^k + y^k = a^k$, (a, k > 0) and $\frac{dy}{dx} + \left(\frac{y}{x}\right)^{\frac{1}{3}} = 0$, then k is:
[Jan. 7, 2020 (I)]

Options:

A. $\frac{3}{2}$

B. $\frac{4}{3}$

C. $\frac{2}{3}$

D. $\frac{1}{3}$

Answer: C

Solution:

$k \cdot x^{k-1} + k \cdot y^{k-1}\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\left(\frac{x}{y}\right)^{k-1}$

$\Rightarrow \frac{dy}{dx} + \left(\frac{x}{y}\right)^{k-1} = 0$

$\Rightarrow k - 1 = -\frac{1}{3}$

$\Rightarrow k = 1 - \frac{1}{3} = \frac{2}{3}$

Question66

The value of c in the Lagrange's mean value theorem forthe function $f(x) = x^3 - 4x^2 + 8x + 11$, when $x \in [0, 1]$ is:
[Jan. 7, 2020 (II)]

Options:

A. $\frac{4-\sqrt{5}}{3}$

B. $\frac{4-\sqrt{7}}{3}$

C. $\frac{2}{3}$

D. $\frac{\sqrt{7}-2}{3}$

Answer: B

Solution:

Solution:
Since, $f(x)$ is a polynomial function.
∴ It is continuous and differentiable in [0,1]
Here, $f(0) = 11$, $f(1) = 1 - 4 + 8 + 11 = 16$
$f'(x) = 3x^2 - 8x + 8$
$\therefore f'(c) = \frac{f(1) - f(0)}{1 - 0} = \frac{16 - 11}{1}$
$= 3c^2 - 8c + 8$
$\Rightarrow 3c^2 - 8c + 3 = 0$
$\Rightarrow c = \frac{8 \pm 2\sqrt{7}}{6} = \frac{4 \pm \sqrt{7}}{3}$
$\therefore c = \frac{4 - \sqrt{7}}{3} \in (0, 1)$

Question67

Let [t] denote the greatest integer ≤t and $\lim_{x \to 0} x\left[\frac{4}{x}\right] = A$. Then the function, $f(x) = [x^2]\sin(\pi x)$ is discontinuous, when x is equal to :
[Jan. 9, 2020 (II)]

Options:

A. $\sqrt{A+1}$

B. $\sqrt{A+5}$

C. $\sqrt{A+21}$

D. $\sqrt{A}$

Answer: A

Solution:

Solution:

$\lim_{x \to 0} x\left[\frac{4}{x}\right] = A \Rightarrow \lim_{x \to 0} x\left[\frac{4}{x} - \left\{\frac{4}{x}\right\}\right] = A$

$\Rightarrow \lim_{x \to 0} 4 - x\left\{\frac{4}{x}\right\} = A \Rightarrow 4 - 0 = A$

As, $f(x) = [x^2]\sin(\pi x)$ will be discontinuous at non- integers And, when $x = \sqrt{A+1} \Rightarrow x = \sqrt{5}$, which is not an integer.

Hence, $f(x)$ is discontinuous when x is equal to $\sqrt{A+1}$

Question68

If the function f defined on $\left(-\frac{1}{3}, \frac{1}{3}\right)$ by

$$f(x) = \begin{cases} \frac{1}{x}\log_e\left(\frac{1+3x}{1-2x}\right), & \text{when } x \neq 0 \\ k, & \text{when } x = 0 \end{cases}$$

is continuous, then k is equal to _______.
[NA Jan. 7, 2020 (II)]

Answer: 5

Solution:

Solution:

$\lim_{x \to 0} f(x) = \lim_{x \to 0}\left(\frac{1}{x}\ln\left(\frac{1+3x}{1-2x}\right)\right)$

$= \lim_{x \to 0}\left(\frac{\ln(1+3x)}{x} - \frac{\ln(1-2x)}{x}\right)$

$= \lim_{x \to 0}\left(\frac{3\ln(1+3x)}{3x} - \frac{2\ln(1-2x)}{-2x}\right)$

$= 3 + 2 = 5$

$\because f(x)$ will be continuous

$\therefore k = f(0) = \lim_{x \to 0} f(x) = 5$

Question69

If c is a point at which Rolle's theorem holds for the function, $f(x) = \log_e\left(\frac{x^2+a}{7x}\right)$ in the interval [3, 4], where $\alpha \in R$, then $f''(c)$ is equal to:

[Jan. 8, 2020 (I)]

Options:

A. $-\frac{1}{12}$

B. $\frac{1}{12}$

C. $-\frac{1}{24}$

D. $\frac{\sqrt{3}}{7}$

Answer: B

Solution:

Solution:
Since, Rolle's theorem is applicable
$\therefore f(a) = f(b)$
$f(3) = f(4) \Rightarrow \alpha = 12$
$f'(x) = \frac{x^2 - 12}{x(x^2 + 12)}$
As $f'(c) = 0$ (by Rolle's theorem)
$x = \pm\sqrt{12}, \therefore c = \sqrt{12}, \therefore f''(c) = \frac{1}{12}$

Question70

If $f(x) = \begin{cases} \frac{\sin(a+2)x + \sin x}{x} & : x < 0 \\ b & : x = 0 \\ \frac{(x + 3x^2)^{1/3} - x^{1/3}}{x^{4/3}} & : x > 0 \end{cases}$

is continuous at x = 0, then a + 2b is equal to:
[Jan. 9, 2020 (I)]

Options:

A. 1

B. –1

C. 0

D. –2

Answer: C

Solution:

Solution:
$LHL = \lim_{x \to 0} \frac{\sin(a+2)x + \sin x}{x}$
$= \lim_{x \to 0} \left(\frac{\sin(a+2)x}{(a+2)x}\right)(a+2) + \lim_{x \to 0} \frac{\sin x}{x} = a + 3$

$f(0) = b$

RH L $= \lim_{h \to 0} \left(\frac{(1 + 3h)^{\frac{1}{3}} - 1}{h} \right) = 1$

∵ Function $f(x)$ is continuous

$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

$\therefore a + 3 = 1 \Rightarrow a = -2$

and $b = 1$

Hence, $a + 2b = 0$

Question71

Let f and g be differentiable functions on R such that fog is the identity function. If for some a, b ∈ R, $g'(a) = 5$ and $g(a) = b$, then $f'(b)$ is equal to:
[Jan. 9,2020 (II)]

Options:

A. $\frac{1}{5}$

B. 1

C. 5

D. $\frac{2}{5}$

Answer: A

Solution:

Solution:

It is given that functions f and g are differentiable and $f og$ is identity function.

$\therefore (f og)(x) = x \Rightarrow f(g(x)) = x$

Differentiating both sides, we get

$f'(g(x)) . g'(x) = 1$

Now, put $x = a$, then

$f'(g(a)) . g'(a) = 1$

$f'(b) . 5 = 1$

$f'(b) = \frac{1}{5}$

Question72

Let S be the set of all functions $f : [0, 1] \to R$, which are continuous on [0,1] and differentiable on (0,1) . Then for every f in S, there exists a $c \in (0, 1)$, depending on f, such that:
[Jan. 8, 2020 (II)]

Options:

A. $|f(c) - f(1)| < (1 - c) \, |f'(c)|$

B. $\frac{f(1) - f(c)}{1 - c} = f'(c)$

C. $|f(c) + f(1)| < (1 + c) \, |f'(c)|$

D. Bonus

Answer: D

Solution:

Solution:
For a constant function f(x), option (1), (3) and (4) doesn’t hold and by LMVT theorem, option (2) is incorrect.

Question73

Let the function, $f : [-7, 0] \to R$ be continuous on [-7,0]and differentiable on (-7,0) . If $f(-7) = -3$ and $f'(x)$d "2, for all $x \in (-7, 0)$, then for all such functions f , $f'(-1) + f(0)$ lies in the interval: [Jan. 7, 2020 (I)]

Options:

A. $(-\infty, 20]$

B. [-3,11]

C. $(-\infty, 11]$

D. [-6,20]

Answer: A

Solution:

Solution:
From, LMVT for $x \in [-7, -1]$
$\frac{f(-1) - f(-7)}{(-1 + 7)} \le 2 \Rightarrow \frac{f(-1) + 3}{6} \le 2 \Rightarrow f(-1) \le 9$
From, LMVT for $x \in [-7, 0]$
$\frac{f(0) - f(-7)}{(0 + 7)} \le 2$
$\frac{f(0) + 3}{7} \le 2 \Rightarrow f(0) \le 11$
$\therefore f(0) + f(-1) \le 20$

Question74

Let S be the set of points where the function, $f(x) = |2 - |x - 3||, x \in R$, is not differentiable. Then $\sum_{x \in S} f(f(x))$ is equal to ________. [NA Jan. 7,2020 (I)]

Answer: 3

Solution:

$\because f(x)$ is non differentiable at $x = 1, 3, 5$
[$\because x - 3|$ is not differentiable at $x = 3$]
$\sum f(f(x)) = f(f(1)) + f(f(3)) + f(f(5))$
$= 1 + 1 + 1 = 3$

Question75

If $x = 2\sin\theta - \sin 2\theta$ and $y = 2\cos\theta - \cos 2\theta$, $\theta \in [0, 2\pi]$, then $\frac{d^2y}{dx^2}$ at $\theta = \pi$ is :
[Jan. 9, 2020 (II)]

Options:

A. $\frac{3}{4}$

B. $\frac{3}{8}$

C. $\frac{3}{2}$

D. $-\frac{3}{4}$

Answer: B

Solution:

Solution:
It is given that
$x = 2\sin\theta - \sin 2\theta$ .....(i)
$y = 2\cos\theta - \cos 2\theta$ ....(ii)
Differentiating (i) w.r.t. theta, we get
$\frac{dx}{d\theta} = 2\cos\theta - 2\cos 2\theta$
Differentiating (ii) w.r.t. θ; we get
$\frac{dy}{d\theta} = -2\sin\theta + 2\sin 2\theta$
From (ii)\÷(i), we get
$\therefore \frac{dy}{dx} = \frac{\sin 2\theta - \sin\theta}{\cos\theta - \cos 2\theta}$
$= \frac{2\sin\frac{\theta}{2} . \cos\frac{3\theta}{2}}{2\sin\frac{\theta}{2} . \sin\frac{3\theta}{2}} = \cot\frac{3\theta}{2}$ ......(iii)
Again, differentiating eqn. (iii), we get
$\frac{d^2y}{dx^2} = \frac{-3}{2}\text{cosec}^2\frac{3\theta}{2} . \frac{d\theta}{dx}$
$\frac{d^2y}{dx^2} = \frac{\frac{-3}{2}\text{cosec}^2\frac{3\theta}{2}}{2(\cos\theta - \cos 2\theta)}$
$\frac{d^2y}{dx^2(\theta = \pi)} = -\frac{3}{4(-1-1)} = \frac{3}{8}$

Question76

If $y(\alpha) = \sqrt{2\left(\frac{\tan\alpha + \cot\alpha}{1 + \tan^2\alpha}\right) + \frac{1}{\sin^2\alpha}}$, $\alpha \in \left(\frac{3\pi}{4}, \pi\right)$, then $\frac{dy}{d\alpha}$ at $\alpha = \frac{5\pi}{6}$ is:
[Jan. 7, 2020 (I)]

Options:

A. 4

B. $\frac{4}{3}$

C. -4

D. $-\frac{1}{4}$

Answer: A

Solution:

Solution:

$y(\alpha) = \sqrt{\dfrac{\frac{2\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha}}{\sec^2\alpha}} = \sqrt{\dfrac{2\cos^2\alpha}{\sin\alpha\cos\alpha} + \dfrac{1}{\sin^2\alpha}}$

$= \sqrt{2\cot\alpha + \operatorname{cosec}^2\alpha} = \sqrt{2\cot\alpha + 1 + \cot^2\alpha}$

$= |1 + \cot\alpha| = -1 - \cot\alpha \left[\because \alpha \in \left(\frac{3\pi}{4}, \pi\right)\right]$

$\frac{dy}{d\alpha} = \operatorname{cosec}^2\alpha \Rightarrow \left(\frac{dy}{d\alpha}\right)_{\alpha = \frac{5\pi}{6}} = 4$

Question77

Let y = y(x) be a function of x satisfying $y\sqrt{1-x^2} = k - x\sqrt{1-y^2}$ where k is a constant and $y\left(\frac{1}{2}\right) = -\frac{1}{4}$. Then $\frac{dy}{dx}$ at $x = \frac{1}{2}$, is equal to:
[Jan. 7, 2020 (II)]

Options:

A. $-\frac{\sqrt{5}}{4}$

B. $-\frac{\sqrt{5}}{2}$

C. $\frac{2}{\sqrt{5}}$

D. $\frac{\sqrt{5}}{2}$

Answer: B

Solution:

Solution:

Given, $x = \frac{1}{2}, y = \frac{-1}{4} \Rightarrow xy = \frac{-1}{8}$

$y \cdot \frac{1 \cdot (-2x)}{2\sqrt{1-x^2}} + y'\sqrt{1-x^2}$

$= -\left\{1 \cdot \sqrt{1-y^2} + \frac{x \cdot (-2y)}{2\sqrt{1-y^2}} y'\right\}$

$\Rightarrow -\frac{xy}{\sqrt{1-x^2}} + y'\sqrt{1-x^2} = -\sqrt{1-y^2} + \frac{xy \cdot y'}{\sqrt{1-y^2}}$

$$\Rightarrow y'\left(\sqrt{1-x^2}-\frac{xy}{\sqrt{1-y^2}}\right)=\frac{xy}{\sqrt{1-x^2}}-\sqrt{1-y^2}$$

$$\Rightarrow y'\left(\frac{\sqrt{3}}{2}+\frac{1}{8\cdot\frac{\sqrt{15}}{4}}\right)=\frac{-1}{8\cdot\sqrt{\frac{3}{2}}}-\frac{\sqrt{15}}{4}$$

$$\Rightarrow y'\left(\frac{\sqrt{45}+1}{2\sqrt{15}}\right)=-\frac{(1+\sqrt{45})}{4\sqrt{3}}$$

$$\therefore y'=-\frac{\sqrt{5}}{2}$$

Question78

For all twice differentiable functions $f: R \to R$, with $f(0)=f(1)=f'(0)=0$
[Sep. 06, 2020 (II)]

Options:

A. $f''(x) \neq 0$ at every point $x \in (0,1)$

B. $f''(x) = 0$, for some $x \in (0,1)$

C. $f''(0) = 0$

D. $f''(x) = 0$, at every point $x \in (0,1)$

Answer: B

Solution:

Solution:
Let $f: R \to R$, with $f(0)=f(1)=0$ and $f'(0)=0$
$\because f(x)$ is differentiable and continuous and
$f(0)=f(1)=0$
Then by Rolle's theorem, $f'(c)=0,\ c\in(0,1)$
Now again
$\because f'(c)=0,\ f'(0)=0$
Then, again by Rolle's theorem,
$f''(x)=0$ for some $x\in(0,1)$

Question79

If $y^2+\log_e(\cos^2 x)=y,\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, then :
[Sep. 03, 2020 (I)]

Options:

A. $y''(0)=0$

B. $|y'(0)|+|y''(0)|=1$

C. $|y''(0)|=2$

D. $|y'(0)|+|y''(0)|=3$

Answer: C

Solution:

Solution:
$y^2 + 2\log_e(\cos x) = y$ ....(i)
$\Rightarrow 2yy' - 2\tan x = y'$ ....(ii)
From (i), $y(0) = 0$ or 1
$\therefore y'(0) = 0$
Again differentiating (ii) we get,
$2(y')^2 + 2yy' - 2\sec^2 x = y'$
Put $x = 0$, $y(0) = 0, 1$ and $y'(0) = 0$
we get, $|y''(0)| = 2$.

Question80

Let $f(x) = x \cdot \left[\frac{x}{2}\right]$, for $-10 < x < 10$, where $[t]$ denotes the greatest integer function. Then the number of points of discontinuity of f is equal to ______.
[NA Sep. 05, 2020 (I)]

Options:

Answer: 0

Solution:

Solution:
We know $[x]$ discontinuous for $x \in Z$
$f(x) = x\left[\frac{x}{2}\right]$ may be discontinuous where $\frac{x}{2}$ is an integer.
So, points of discontinuity are,
$x = \pm 2, \pm 4, \pm 6, \pm 8$ and 0
but at $x = 0$
$\lim_{x \to 0^+} f(x) = 0 = f(0) = \lim_{x \to 0^-} f(x)$
So, $f(x)$ will be discontinuous at $x = \pm 2, \pm 4, \pm 6$ and ± 8 .

Question81

If a function f (x) defined by

$$f(x) = \begin{cases} ae^x + be^{-x}, & -1 \le x < 1 \\ cx^2, & 1 \le x \le 3 \\ ax^2 + 2cx, & 3 < x \le 4 \end{cases}$$

be continuous for some $a, b, c \in R$ and $f'(0) + f'(2) = e$, then the value of a is :
[Sep. 02, 2020 (I)]

Options:

A. $\frac{1}{e^2 - 3e + 13}$

B. $\frac{e}{e^2 - 3e - 13}$

C. $\frac{e}{e^2 + 3e + 13}$

D. $\frac{e}{e^2 - 3e + 13}$

Answer: D

Solution:

Solution:
Since, function f (x) is continuous at x = 1, 3
$\therefore f(1) = f(1^+)$
$\Rightarrow ae + be^{-1} = c$ ...(i)
$f(3) = f(3^+)$
$\Rightarrow 9c = 9a + 6c \Rightarrow c = 3a$ ...(ii)From (i) and (ii),
$b = ae(3 - e)$...(iii)

$$f'(x) = \begin{cases} ae^x - be^{-x} & -1 < x < 1 \\ 2cx & 1 < x < 3 \\ 2ax + 2c & 3 < x < 4 \end{cases}$$

$f'(0) = a - b,\ f'(2) = 4c$
Given, $f'(0) + f'(2) = e$
$a - b + 4c = e$
From eqs. (i), (ii),(iii) and (iv),
$a - 3ae + ae^2 + 12a = e$
$\Rightarrow 13a - 3ae + ae^2 = e$
$\Rightarrow a = \frac{e}{e^2 - 3e + 13}$

Question82

Let f : R → R be a function defined by $f(x) = \max\{x, x^2\}$. Let S denote the set of all points in R, where f is not differentiable. Then: [Sep. 06, 2020 (II)]

Options:

A. {0, 1}

B. {0}

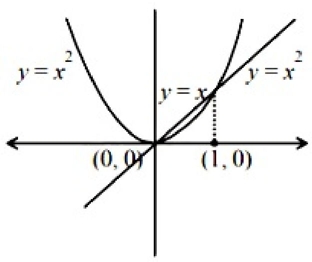
C. π(an empty set)

D. {1}

Answer: A

Solution:

Solution:



$f(x) = \max\{x, x^2\}$

$$\Rightarrow f(x) = \begin{cases} x^2, & x < 0 \\ x, & 0 \le x < 1 \\ x^2, & x \ge 1 \end{cases}$$

$\therefore$ f(x) is not differentiable at x = 0, 1

Question83

If the function $f(x) \begin{cases} k_1(x-\pi)^2 - 1, & x \le \pi \\ k_2 \cos x, & x > \pi \end{cases}$ is twice dif-ferentiable, then the ordered pair (k_1, k_2) is equal to:
[Sep. 05, 2020 (I)]

Options:

A. $\left(\frac{1}{2}, 1\right)$

B. (1,0)

C. $\left(\frac{1}{2}, -1\right)$

D. (1,1)

Answer: A

Solution:

Solution:
f (x) is differentiable then, f (x) is also continuous.
$\therefore \lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^-} f(x) = f(\pi)$

$\Rightarrow -1 = -K_2 \Rightarrow K_2 = 1$

$$\therefore f'(x) = \begin{cases} 2K_1(x-\pi) & : \ x \le \pi \\ -K_2 \sin x & \ x > \pi \end{cases}$$

Then, $\lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^-} f(x) = 0$

$$f''(x) = \begin{cases} 2K_1 & ; \ x \le \pi \\ -K_2 \cos x & ; \ x > \pi \end{cases}$$

Then, $\lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^-} f(x)$

$\Rightarrow 2K_1 = K_2 \Rightarrow K_1 = \frac{1}{2}$

So, $(K_1, K_2) = \left(\frac{1}{2}, 1\right)$

Question84

Let f be a twice differentiable function on (1, 6). If $f(2) = 8, f'(2) = 5, f'(x) \ge 1$ and $f''(x) \ge 4$, for all $x \in (1, 6)$, then
[Sep. 04, 2020 (I)]

Options:

A. $f(5)+f'(5) \le 26$

B. $f(5)+f'(5) \ge 28$

C. $f'(5)+f''(5) \le 20$

D. $f(5) \le 10$

Answer: B

Solution:

Solution:
Let f be twice differentiable function
$\because f'(x) \ge 1$
$\Rightarrow \frac{f(5)-f(2)}{3} \ge 1$
$\Rightarrow f(5) \ge 3+f(2)$
$\Rightarrow f(5) \ge 3+8 \Rightarrow f(5) \ge 11$
and also $f''(x) \ge 4$
$\Rightarrow \frac{f'(5)-f'(2)}{5-2} \ge 4 \Rightarrow f'(5) \ge 12+f'(2)$
$\Rightarrow f'(5) \ge 17$
Hence, $f(5)+f'(5) \ge 28$

Question85

Suppose a differentiable function $f(x)$ satisfies the identity $f(x+y)=f(x)+f(y)+xy^2+x^2y$, for all real x and y. If $\lim_{x \to 0} \frac{f(x)}{x}=1$, then $f'(3)$ is equal to _______. [NA Sep. 04, 2020 (I)]

Answer: 10

Solution:

Solution:
$f(x+y)=f(x)+f(y)+xy^2+x^2y$
Differentiate w.r.t. x :
$f'(x+y)=f'(x)+0+y^2+2xy$
Put $y=-x$
$f'(0)=f'(x)+x^2-2x^2$ ....(i)
$\because \lim_{x \to 0} \frac{f(x)}{x}=1 \Rightarrow f(0)=0$
$\therefore f'(0)=1$ ....(ii)
From equations (i) and (ii),
$f'(x)=(x^2+1) \Rightarrow f'(3)=10$

Question86

The function

$$f(x) = \begin{cases} \frac{\pi}{4} + \tan^{-1}x, & |x| \le 1 \\ \frac{1}{2}(|x| - 1), & |x| > 1 \end{cases}$$

is :

[Sep. 04, 2020 (II)]

Options:

A. continuous on $R - \{1\}$ and differentiable on $R - \{-1, 1\}$

B. both continuous and differentiable on $R - \{1\}$.

C. continuous on $R - \{-1\}$ and differentiable on $R - \{-1, 1\}$

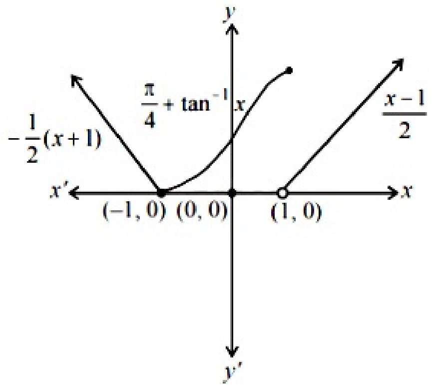
D. both continuous and differentiable on $R - \{-1\}$.

Answer: A

Solution:

Solution:

$$f(x) = \begin{cases} \frac{-x-1}{2}, & x < -1 \\ \frac{\pi}{4} + \tan^{-1}x, & -1 \le x \le 1 \\ \frac{1}{2}(x - 1), & x > 1 \end{cases}$$



It is clear from above graph that,
f (x) is discontinuous at x = 1.
i.e. continuous on R – {1}
f (x) is non-differentiable at x = – 1, 1
i.e. differentiable on R – {–1, 1}.

Question87

The derivative of $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x = \frac{1}{2}$ is :

[Sep. 05, 2020 (II)]

Options:

A. $\frac{2\sqrt{3}}{5}$

B. $\frac{\sqrt{3}}{12}$

C. $\frac{2\sqrt{3}}{3}$

D. $\frac{\sqrt{3}}{10}$

Answer: D

Solution:

Solution:

Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$\therefore u = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$

$= \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$

$\therefore \frac{du}{dx} = \frac{1}{2} \times \frac{1}{(1+x^2)}$

Let $v = \tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$

Put $x = \sin\phi \Rightarrow \phi = \sin^{-1}x$

$v = \tan^{-1}\left(\frac{2\sin\phi\cos\phi}{\cos 2\phi}\right) = \tan^{-1}(\tan 2\phi)$

$= 2\phi = 2\sin^{-1}x$

$\frac{dv}{dx} = 2\frac{1}{\sqrt{1-x^2}}$

$\frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{1-x^2}{4(1+x^2)}$

$\therefore \left(\frac{du}{dv}\right)_{\left(x=\frac{1}{2}\right)} = \frac{\sqrt{3}}{10}$

Question88

If $(a+\sqrt{2}b\cos x)(a-\sqrt{2}b\cos y) = a^2 - b^2$, where $a > b > 0$ then $\frac{dx}{dy}$ at $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ is

[Sep. 04, 2020 (I)]

Options:

A. $\frac{a-2b}{a+2b}$

B. $\frac{a-b}{a+b}$

C. $\frac{a+b}{a-b}$

D. $\frac{2a+b}{2a-b}$

Answer: C

Solution:

Solution:

$(a+\sqrt{2}b\cos x)(a-\sqrt{2}b\cos y) = a^2 - b^2$

Differentiating both sides,

$(-\sqrt{2}b\sin x)(a-\sqrt{2}b\cos y) + (a+\sqrt{2}b\cos x)$

$(\sqrt{2}b\sin y)\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{(\sqrt{2}b\sin x)(a-\sqrt{2}b\cos y)}{(a+\sqrt{2}b\cos x)(\sqrt{2}b\sin y)}$

$\therefore \left[\frac{dy}{dx}\right]\left(\frac{\pi}{4},\frac{\pi}{4}\right) = \frac{a-b}{a+b} \Rightarrow \frac{dx}{dy} = \frac{a+b}{a-b}$

Question89

If $y = \sum_{k=1}^{6} k\cos^{-1}\left\{\frac{3}{5}\cos kx - \frac{4}{5}\sin kx\right\}$, then $\frac{dy}{dx}$ at $x = 0$ is ______.

[NA Sep. 02, 2020 (II)]

Answer: 91

Solution:

Solution:

$y = \sum_{k=1}^{6} k\cos^{-1}\left\{\frac{3}{5}\cos kx - \frac{4}{5}\sin kx\right\}$

Let $\cos a = \frac{3}{5}$ and $\sin a = \frac{4}{5}$

$\therefore y = \sum_{k=1}^{6} k\cos^{-1}\{\cos a\cos kx - \sin a\sin kx\}$

$= \sum_{k=1}^{6} k\cos^{-1}(\cos(kx+a))$

$= \sum_{k=1}^{6} k(kx+a) = \sum_{k=1}^{6} (k^2x + ak)$

$\therefore \frac{dy}{dx} = \sum_{k=1}^{6} k^2 = \frac{6(7)(13)}{6} = 91$

Question90

Let f : R → R be a function defined as

$$f(x) = \begin{cases} 5, & \text{if } x \le 1 \\ a+bx, & \text{if } 1 < x < 3 \\ b+5x, & \text{if } 3 \le x < 5 \\ 30, & \text{if } x \ge 5 \end{cases}$$

Then, f is :

[Jan 09, 2019 (I)]

Options:

A. continuous if a = 5 and b = 5

B. continuous if a = – 5 and b = 10

C. continous if a = 0 and b = 5

D. not continuous for any values of a and b

Answer: D

Solution:

Solution:
Let $f(x)$ is continuous at $x = 1$, then
$f(1^-) = f(1) = f(1^+)$
$\Rightarrow 5 = a + b$ ....(1)
Let $f(x)$ is continuous at $x = 3$, then
$f(3^-) = f(3) = f(3^+)$
$\Rightarrow a + 3b = b + 15$ ......(2)
Let $f(x)$ is continuous at $x = 5$, then
$f(5^-) = f(5) = f(5^+)$
$\Rightarrow b + 25 = 30$
$\Rightarrow b = 30 - 25 = 5$
From (1), $a = 0$
But $a = 0$, $b = 5$ do not satisfy equation (2)
Hence, f(x) is not continuous for any values of a and b

Question91

Let f be a differentiable function such that $f(1) = 2$ and $f'(x) = f(x)$ for all $x \in R$. If $h(x) = f(f(x))$, then $h'(1)$ is equal to : [Jan. 12, 2019 (II)]

Options:

A. $2e^2$

B. $4e$

C. $2e$

D. $4e^2$

Answer: B

Solution:

Solution:
Since, $f'(x) = f(x)$
Then, $\frac{f'(x)}{f(x)} = 1$
$\Rightarrow \frac{f'(x)}{f(x)} = dx \Rightarrow \frac{f'(x)}{f(x)} dx = \int dx$
$\Rightarrow \ln|f(x)| = x + c$
$f(x) = \pm e^{x+c}$ .....(1)
Since, the given condition
$f(1) = 2$
From eq $^n(1)$ $f(x) = e^{x+c} = e^c e^x$
Then, $f(1) = e^c . e^1$
$\Rightarrow 2 = e^c . e$
$\Rightarrow \frac{2}{e} = e^c$
Then, from eq $^n(1)$
$f(x) = \frac{2}{e} e^x$
$\Rightarrow f'(x) = \frac{2}{e} e^x$
Now $h(x) = f(f(x))$
$\Rightarrow h'(x) = f'(f(x)) . f'(x)$
$h'(1) = f'(2) . f'(1) = \frac{2}{e} e^2 . \frac{2}{e} . e = 4e$

Question92

Let $f(x) = \begin{cases} -1, & -2 \le x < 0 \\ x^2 - 1, & 0 \le x \le 2 \end{cases}$ and $g(x) = |f(x)| + f(|x|)$. Then, in the interval $(-2, 2)$, g is :
[Jan. 11, 2019 (I)]

Options:

A. differentiable at all points

B. not continuous

C. not differentiable at two points

D. not differentiable at one point

Answer: D

Solution:

Solution:

$f(x) = \begin{cases} -1, & -2 \le x < 0 \\ x^2 - 1, & 0 \le x \le 2 \end{cases}$

Then, $f(|x|) = \begin{cases} -1, & -2 \le |x| < 0 \\ |x|^2 - 1, & 0 \le |x| \le 2 \end{cases}$

$\Rightarrow f(|x|) = x^2 - 1, -2 \le x \le 2$

$g(x) = \begin{cases} -1 + x^2 - 1, & -2 \le x < 0 \\ (x^2 - 1) + |x^2 - 1|, & 0 \le x \le 2 \end{cases}$

$= \begin{cases} x^2, & -2 \le x < 0 \\ 0, & 0 \le x < 1 \\ 2(x^2 - 1), & 1 \le x \le 2 \end{cases}$

$g'(0^-) = 0, g'(0^+) = 0, g'(1^-) = 0, g'(1^+) = 4$
$\Rightarrow$ g(x) is non-differentiable at $x = 1$
$\Rightarrow$ g(x) is not differentiable at one point.

Question93

If $x\log_e(\log_e x) - x^2 + y^2 = 4 (y > 0)$, then $\frac{dy}{dx}$ at $x = e$ is equal to :
[Jan. 11, 2019 (I)]

Options:

A. $\frac{(1 + 2e)}{2\sqrt{4 + e^2}}$

B. $\frac{(2e - 1)}{2\sqrt{4 + e^2}}$

C. $\frac{(1 + 2e)}{\sqrt{4 + e^2}}$

D. $\frac{e}{\sqrt{4+e^2}}$

Answer: B

Solution:

Solution:
Consider the equation,
$x\log_e(\log_e x) - x^2 + y^2 = 4$
Differentiate both sides w.r.t. x,
$\log_e(\log_e x) + x \cdot \frac{1}{x \cdot \log_e x} - 2x + 2y\frac{dy}{dx} = 0$
$\log_e(\log_e x) + \frac{1}{\log_e x} - 2x + 2y\frac{dy}{dx} = 0$
When $x = e$, $y = \sqrt{4+e^2}$. Put these values in (1),
$0 + 1 - 2e + 2\sqrt{4+e^2}\frac{dy}{dx} = 0$
$\frac{dy}{dx} = \frac{2e-1}{2\sqrt{4+e^2}}$

Question94

Let K be the set of all real values of x where the function $f(x) = \sin|x| - |x| + 2(x-\pi)\cos|x|$ is not differentiable. Then the set K is equal to [Jan. 11, 2019 (II)]

Options:

A. ϕ (an empty set)

B. {π}

C. {0}

D. {0, π}

Answer: A

Solution:

Solution:
$f(x) = \sin|x| - |x| + 2(x-\pi)\cos|x|$
There are two cases,
Case (1), $x > 0$
$f(x) = \sin x - x + 2(x-\pi)\cos x$
$f'(x) = \cos x - 1 + 2(1-0)\cos x - 2\sin(x-\pi)$
$f'(x) = 3\cos x - 2(x-\pi)\sin x - 1$
Then, function $f(x)$ is differentiable for all $x > 0$
Case (2) $x < 0$
$f(x) = -\sin x + x + 2(x-\pi)\cos x$
$f'(x) = -\cos x + 1 - 2(x-\pi)\sin x + 2\cos x$
$f'(x) = \cos x + 1 - 2(x-\pi)\sin x$
Then, function $f(x)$ is differentiable for all $x < 0$
Now check for $x = 0$
$f'(0^+) R.H.D. = 3 - 1 = 2$
$f'(0^-) L.H.D. = 1 + 1 = 2$
L.H.D. = R.H.D.
Then, function $f(x)$ is differentiable for $x = 0$. So it is differentiable everywhere
Hence, $k = \phi$

Question95

Let f (x) = $\begin{cases} \max\{|x|, x^2\}, & |x| \le 2 \\ 8 - 2|x|, & 2 < |x| \le 4 \end{cases}$

Let S be the set of points in the interval (-4,4) at which f is not differentiable. Then S:
[Jan 10, 2019 (I)]

Options:

A. is an empty set

B. equals $\{-2, -1, 0, 1, 2\}$

C. equals $\{-2, -1, 1, 2\}$

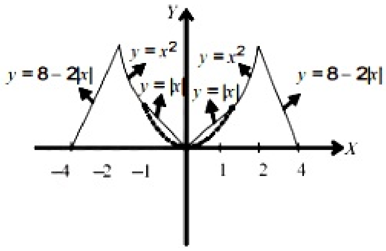
D. equals $\{-2, 2\}$

Answer: B

Solution:

Solution:

Given f (x) = $\begin{cases} \max\{|x|, x^2\}, & |x| \le 2 \\ 8 - 2|x|, & 2 < |x| \le 4 \end{cases}$



∵f (x) is not differentiable at -2,-1,0,1 and 2.
∴S = $\{-2, -1, 0, 1, 2\}$

Question96

Let f : $(-1, 1) \to R$ be a function defined by $f(x) = \max\left\{-|x|, -\sqrt{1-x^2}\right\}$. If K be the set of all points at which f is not differentiable, then K has exactly:
[Jan. 10, 2019 (II)]

Options:

A. five elements

B. one element

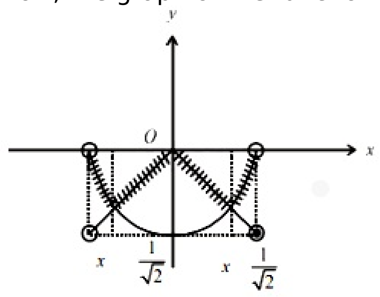
C. three elements

D. two elements

Answer: C

Solution:

Solution:
Consider the function
$f(x) = \max\{-|x|, -\sqrt{1-x^2}\}$
Now, the graph of the function



From the graph, it is clear that f(x) is not differentiable at x $= 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

Then, $K = \left\{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$

Hence, K has exactly three elements.

Question97

Let S be the set of all points in $(-\pi, \pi)$ at which the function $f(x) = \min\{\sin x, \cos x\}$ is not differentiable. Then S is a subset of which of the following? [Jan. 12, 2019 (I)]

Options:

A. $\left\{-\frac{\pi}{4}, 0, \frac{\pi}{4}\right\}$

B. $\left\{-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{4}\right\}$

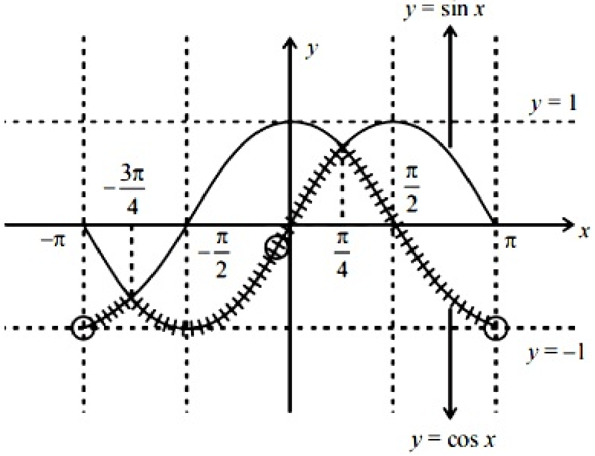
C. $\left\{-\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}\right\}$

D. $\left\{-\frac{3\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$

Answer: B

Solution:

Solution:
$f(x) = \min\{\sin x, \cos x\}$



$\because f(x)$ is not differentiable at $x = -\frac{3\pi}{4}, \frac{\pi}{4}$

$\therefore S = \left\{-\frac{3\pi}{4}, \frac{\pi}{4}\right\}$

$\Rightarrow S \subseteq \left\{-\frac{3\pi}{4}, \frac{-\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{4}\right\}$

Question98

For $x > 1$, if $(2x)^{2y} = 4e^{2x-2y}$, then $(1+\log_e 2x)^2 \frac{dy}{dx}$ is equal to:
[Jan. 12, 2019 (I)]

Options:

A. $\frac{x\log_e 2x - \log_e 2}{x}$

B. $\log_e 2x$

C. $\frac{x\log_e 2x + \log_e 2}{x}$

D. $x\log_e 2x$

Answer: A

Solution:

Solution:
Consider the equation,
$(2x)^{2y} = 4e^{2x-2y}$
Taking log on both sides
$2y\ln(2x) = \ln 4 + (2x - 2y)$ ......(1)
Differentiating both sides w.r.t. x,
$2y\frac{1}{2x}2 + 2\ln(2x)\frac{dy}{dx} = 0 + 2 - 2\frac{dy}{dx}$
$2\frac{dy}{dx}\left(1 + \ln(2x) = 2 - \frac{2y}{x} = \frac{2x-2y}{x}\right.$ ......(2)
From (1) and (2),
$\frac{dy}{dx}(1+\ln 2x) = 1 - \frac{1}{x}\left(\frac{\ln 2 + x}{1+\ln 2x}\right)$
$(1+\ln 2x)^2\frac{dy}{dx} = 1 + \ln(2x) - \left(\frac{x+\ln 2}{x}\right)$
$= \frac{x\ln(2x) - \ln 2}{x}$

Question99

Let $f: R \to R$ be a function such that
$f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3), x \in R$. Then $f(2)$ equals:
[Jan 10, 2019 (I)]

Options:

A. – 4

B. 30

C. – 2

D. 8

Answer: C

Solution:

Solution:
Let $f(x) = x^3 + ax^2 + bx + c$
$f'(x) = 3x^2 + 2ax + b \Rightarrow f'(1) = 3 + 2a + b$
$f''(x) = 6x + 2a \Rightarrow f''(2) = 12 + 2a$
$f'''(x) = 6 \Rightarrow f''(3) = 6$
$\because f(x) = x^3 + f'(1)x^2 + f''(2)x + f'''(3)$
$\therefore f'(1) = a \Rightarrow 3 + 2a + b = a \Rightarrow a + b = -3$ ......(1)
also $f''(2) = b \Rightarrow 12 + 2a = b \Rightarrow 2a - b = -12$ ......(2)
and $f''(3) = c \Rightarrow c = 6$
Add (1) and (2)
$3a = -15 \Rightarrow a = -5 \Rightarrow h = 2$
$\Rightarrow f(x) = x^3 - 5x^2 + 2x + 6$
$\Rightarrow f(2) = 8 - 20 + 4 + 6 = -2$

Question100

If $x = 3\tan t$ and $y = 3\sec t$, then the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$, is: [Jan. 09, 2019 (II)]

Options:

A. $\frac{1}{3\sqrt{2}}$

B. $\frac{1}{6\sqrt{2}}$

C. $\frac{3}{2\sqrt{2}}$

D. $\frac{1}{6}$

Answer: B

Solution:

Solution:
$\because x = 3\tan t \Rightarrow \frac{dx}{dt} = 3\sec^2 t$

and $y = 3\sec t \Rightarrow \frac{dy}{dt} = 3\sec t \cdot \tan t$

$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \therefore \frac{dy}{dx} = \frac{\tan t}{\sec t} = \sin t$

$\therefore \frac{d^2y}{dx^2} = \frac{d}{dt}(\sin t) \cdot \frac{dt}{dx}$

$= \cos t \cdot \frac{1}{3\sec^2 t}$

$\therefore \frac{d^2y}{dx^2}\left(\text{at } t = \frac{\pi}{4}\right) = \frac{1}{3} \cdot \left(\frac{1}{\sqrt{2}}\right)^3$

$= \frac{1}{6\sqrt{2}}$

Question101

If the function f defined on $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ by

$$f(x) = \begin{cases} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}, & x \neq \frac{\pi}{4} \\ k, & x = \frac{\pi}{4} \end{cases}$$

is continuous, then k is equal to:
[April 09, 2019 (I)]

Options:

A. 2

B. $\frac{1}{2}$

C. 1

D. $\frac{1}{\sqrt{2}}$

Answer: B

Solution:

Solution:
Since, $f(x)$ is continuous, then

$$\lim_{x \to \frac{\pi}{4}} f(x) = f\left(\frac{\pi}{4}\right)$$

$$\lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\cos x - 1}{\cot x - 1} = k$$

Now by L- hospital's rule

$$\lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\sin x}{\operatorname{cosec}^2 x} = k \Rightarrow \frac{\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)}{(\sqrt{2})^2} = k \Rightarrow k = \frac{1}{2}$$

Question102

If $f(x) = [x] - \left[\frac{x}{4}\right]$, $x \in R$, where [x] denotes the greatest integer function, then:
[April 09, 2019 (II)]

Options:

A. f is continuous at $x = 4$.

B. $\lim_{x \to 4+} f(x)$ exists but $\lim_{x \to 4} f(x)$ does not exist.

C. Both $\lim_{x \to 4-} f(x)$ and $\lim_{x \to 4+} f(x)$ exist but are not equal.

D. $\lim_{x \to 4-} f(x)$ exists but $\lim_{x \to 4+} f(x)$ does not exist.

Answer: A

Solution:

Solution:

L.H.L. $\lim_{x \to 4^-} \left([x] - \left[\frac{x}{4} \right] \right) = 3 - 0 = 3$

R.H.L. $\lim_{x \to 4^+} [x] - \left[\frac{x}{4} \right] = 4 - 1 = 3$

$f(4) = [4] - \left[\frac{4}{4} \right] = 4 - 1 = 3$

$\because LHL = f(4) = RHL$

$\therefore f(x)$ is continuous at $x = 4$

Question103

If the function

$$f(x) = \begin{cases} a|\pi - x| + 1, & x \le 5 \\ b|x - \pi| + 3, & x > 5 \end{cases}$$

is continuous at $x = 5$, then the value of $a - b$ is: [April 09, 2019 (II)]

Options:

A. $\frac{2}{\pi + 5}$

B. $\frac{-2}{\pi + 5}$

C. $\frac{2}{\pi - 5}$

D. $\frac{2}{5 - \pi}$

Answer: D

Solution:

Solution:

R.H.L. $\lim_{x \to 5^+} b|(x - \pi)| + 3 = (5 - \pi)b + 3$

$f(5) =$ L.H.L. $\lim_{x \to 5^-} a|(\pi - x)| + 1 = a(5 - \pi) + 1$

$\because$ function is continuous at $x = 5$

$\therefore LHL = RHL$

$(5 - \pi)b + 3 = (5 - \pi)a + 1$

$\Rightarrow 2 = (a - b)(5 - \pi) \Rightarrow a - b = \frac{2}{5 - \pi}$

Question104

Let $f : [-1, 3] \to R$ be defined as

$$f(x) = \begin{cases} |x| + [x], & -1 \le x < 1 \\ x + |x|, & 1 \le x < 2 \\ x + [x], & 2 \le x \le 3 \end{cases}$$

where [t] denotes the greatest integer less than or equal to t. Then, f is

discontinuous at :
[April 08, 2019 (II)]

Options:

A. only one point

B. only two points

C. only three points

D. four or more points

Answer: C

Solution:

Solution:
Given function is,

$$f(x) = \begin{cases} |x| + [x], & -1 \le x < 1 \\ x + |x|, & 1 \le x < 2 \\ x + [x], & 2 \le x \le 3 \end{cases}$$

$$= \begin{cases} -x - 1, & -1 \le x < 0 \\ x, & 0 \le x < 1 \\ 2x, & 1 \le x < 2 \\ x + 2, & 2 \le x < 3 \\ 6, & x = 3 \end{cases}$$

$\Rightarrow f(-1) = 0, f(-1^{+}) = 0$
$f(0^{-}) = -1, f(0) = 0, f(0^{+}) = 0$
$f(1^{-}) = 1, f(1) = 2, f(1^{+}) = 2$
$f(2^{-}) = 4, f(2) = 4, f(2^{+}) = 4;$
$f(3^{-}) = 5, f(3) = 6$
$f(x)$ is discontinuous at $x = \{0, 1, 3\}$
Hence, $f(x)$ is discontinuous at only three points.

Question105

If

$$f(x) = \begin{cases} \frac{\sin(p+1)x + \sin x}{x} & ,x < 0 \\ q & ,x = 0 \\ \frac{\sqrt{x + x^2} - \sqrt{x}}{x^{3/2}} & ,x > 0 \end{cases}$$ **is continuous at $x = 0$,**

then the ordered pair (p, q) is equal to:
[April 10,2019 (I)]

Options:

A. $\left(-\frac{3}{2}, -\frac{1}{2}\right)$

B. $\left(-\frac{1}{2}, \frac{3}{2}\right)$

C. $\left(-\frac{3}{2}, \frac{1}{2}\right)$

D. $\left(\frac{5}{2}, \frac{1}{2}\right)$

Answer: C

Solution:

Solution:

$$f(x) = \begin{cases} \frac{\sin(p+1)x + \sin x}{x} & , x < 0 \\ q & , x = 0 \\ \frac{\sqrt{x + x^2} - \sqrt{x}}{x^{3/2}} & , x > 0 \end{cases}$$ is continuous at $x = 0$,

Therefore ,$f(0) = f(0) = f(0^+)$ .....(1)

$f(0-) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{\sin(p+1)(-h) + \sin(-h)}{-h}$

$= \lim_{h \to 0} \left[\frac{-\sin(p+1)h}{-h} + \frac{\sin h}{h} \right]$

$= \lim_{h \to 0} \frac{\sin(p+1)h}{h(p+1)} \times (p+1) + \lim_{h \to 0} \frac{\sin h}{h}$

$= (p+1) + 1 = p + 2$ ....(2)

And $f(0^+) = \lim_{h \to 0} f(0+h) = \frac{\sqrt{h^2 + h} - \sqrt{h}}{h^{3/2}}$

$= \lim_{h \to 0} \frac{(h)^{\frac{1}{2}}[\sqrt{h+1} - 1]}{h\left(h^{\frac{1}{2}}\right)}$

$= \lim_{h \to 0} \frac{\sqrt{h+1} - 1}{h} \times \frac{\sqrt{h+1} + 1}{\sqrt{h+1} + 1} = \lim_{h \to 0} \frac{h + 1 - 1}{h(\sqrt{h+1} + 1)}$

$= \lim_{h \to 0} \frac{1}{\sqrt{h+1} + 1} = \frac{1}{1+1} = \frac{1}{2}$ ....(3)

Now, from equation (1),

$f(0^-) = f(0) = f(0^+) \Rightarrow p + 2 = q = \frac{1}{2}$

$\Rightarrow q = \frac{1}{2}$ and $p = \frac{-3}{2}$

$\therefore (p, q) = \left(-\frac{3}{2}, \frac{1}{2}\right)$

Question106

Let $f(x) = \log_e(\sin x)$, $(0 < x < \pi)$ and $g(x) = \sin^{-1}(e^{-x})$, $(x \geq 0)$.
If α is a positive real number such that $a = (fog)'(\alpha)$ and $b = (fog)(\alpha)$, then:
[April 10, 2019 (II)]

Options:

A. $a\alpha^2 + b\alpha + a = 0$

B. $a\alpha^2 - b\alpha - a = 1$

C. $a\alpha^2 - b\alpha - a = 0$

D. $a\alpha^2 + b\alpha - a = -2a^2$

Answer: B

Solution:

Solution:
$f(x) = \ln(\sin x)$, $g(x) = \sin^{-1}(e^{-x})$
$\Rightarrow f(g(x)) = \ln(\sin(\sin^{-1}e^{-x})) = -x$
$\Rightarrow f(g(\alpha)) = -\alpha$
But given that $(f o g)(\alpha) = b$
$\therefore -\alpha = b$ and $f'(g(\alpha)) = a$, i.e., $a = -1$
$\therefore a\alpha^2 - b\alpha - a = -\alpha^2 + \alpha^2 - (-1)$
$\Rightarrow a\alpha^2 - b\alpha - a = 1.$

Question107

Let $f : R \to R$ be differentiable at $c \in R$ and $f(c) = 0$. If $g(x) = |f(x)|$, then at $x = c$, g is: [April 10, 2019 (I)]

Options:

A. not differentiable if f '(c) = 0

B. differentiable if f "(c) ≠ 0

C. differentiable if f ' (c) = 0

D. not differentiable

Answer: C

Solution:

Solution:
$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$
$\Rightarrow g'(c) = \lim_{x \to c} \frac{|f(x)| - |f(c)|}{x - c}$
Since, $f(c) = 0$ Then, $g'(c) = \lim_{x \to c} \frac{|f(x)|}{x - c}$
$\Rightarrow g'(c) = \lim_{x \to c} \frac{f(x)}{x - c}$; if $f(x) > 0$
and $g'(c) = \lim_{x \to c} \frac{-f(x)}{x - c}$; if $f(x) < 0$
$\Rightarrow g'(c) = f'(c) = -f'(c)$
$\Rightarrow 2f'(c) = 0 \Rightarrow f'(c) = 0$
Hence, $g(x)$ is differentiable if $f'(c) = 0$

Question108

Let $f(x) = 15 - |x - 10|$; $x \in R$. Then the set of all values of xat which the function, $g(x) = f(f(x))$ is not differentiable, is: [April 09, 2019 (I)]

Options:

A. {5, 10, 15}

B. {10, 15}

C. {5, 10, 15, 20}

D. {10}

Answer: A

Solution:

Solution:
Since, $f(x) = 15 - |(10 - x)|$
$\therefore g(x) = f(f(x)) = 15 - |10 - [15 - |10 - x|]|$
$= 15 - ||10 - x| - 5|$
$\therefore$ Then, the points where function g(x) is Non-differentiable are
$10 - x = 0$ and $|10 - x| = 5$
$\Rightarrow x = 10$ and $x - 10 = \pm 5$
$\Rightarrow x = 10$ and $x = 15, 5$

Question109

If $f(1) = 1$, $f'(1) = 3$, then the derivative of $f(f(f(x))) + (f(x))^2$ at $x = 1$ is :
[April 08, 2019 (II)]

Options:

A. 33

B. 12

C. 15

D. 9

Answer: A

Solution:

Solution:
Let $g(x) = f(f(f(x))) + (f(x))^2$
Differentiating both sides w.r.t. x, we get
$g'(x) = f'(f(f(x)))f'(f(x))f'(x) + 2f(x)f'(x)$
$g'(1) = f'(f(f(1)))f'(f(1))f'(1) + 2f(1)f'(1)$
$= f'(f(1))f'(1)f'(1) + 2f(1)f'(1)$
$= 3 \times 3 \times 3 + 2 \times 1 \times 3 = 27 + 6 = 33$

Question110

If $e^y + xy = e$, the ordered pair $\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}\right)$ at $x = 0$ isequal to :

[April 12, 2019 (I)]

Options:

A. $\left(\frac{1}{e}, -\frac{1}{e^2}\right)$

B. $\left(-\frac{1}{e}, \frac{1}{e^2}\right)$

C. $\left(\frac{1}{e}, \frac{1}{e^2}\right)$

D. $\left(-\frac{1}{e}, -\frac{1}{e^2}\right)$

Answer: B

Solution:

Solution:
Given, $e^y + xy = e$...(i)
Putting $x = 0$ in (i), $\Rightarrow e^y = e \Rightarrow y = 1$
On differentiating (i) w. r. to x
$e^y\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$...(ii)
Putting $y = 1$ and $x = 0$ in (ii),
$e\frac{dy}{dx} + 0 + 1 = 0 \Rightarrow \frac{dy}{dx} = -\frac{1}{e}$
On differentiating (ii) w. r. to x,
$e^y\frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot e^y \cdot \frac{dy}{dx} + x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = 0$
Putting $y = 1$, $x = 0$ and $\frac{dy}{dx} = -\frac{1}{e}$ in (iii),
$e\frac{d^2y}{dx^2} + \frac{1}{e} - \frac{2}{e} = 0 \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{e^2}$
Hence, $\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}\right) \equiv \left(-\frac{1}{e}, \frac{1}{e^2}\right)$

Question111

The derivative of $\tan^{-1}\left(\frac{\sin x - \cos x}{\sin x + \cos x}\right)$, with respect to $\frac{x}{2}$ where left$\left(x \in \left(0, \frac{\pi}{2}\right)\right)$ is:

[April 12, 2019 (II)]

Options:

A. 1

B. $\frac{2}{3}$

C. $\frac{1}{2}$

D. 2

Answer: D

Solution:

Solution:
$f(x) = \tan^{-1}\left(\frac{\tan x - 1}{\tan x + 1}\right)$
$= -\tan^{-1}\left(\tan\left(\frac{\pi}{4} - x\right)\right) \quad \left[\because \frac{\pi}{4} - x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)\right]$
So, $f(x) = -\left(\frac{\pi}{4} - x\right) = x - \frac{\pi}{4}$
Let $y = \Rightarrow f(y) = 2y - \frac{\pi}{4}$

Now, differentiate w.r.t. y, $\frac{d f(y)}{d y} = 2$.

Question112

If $2y = \left(\cot^{-1}\left(\frac{\sqrt{3}\cos x + \sin x}{\cos x - \sqrt{3}\sin x}\right)\right)^2$,$x \in \left(0, \frac{\pi}{2}\right)$ then $\frac{dy}{dx}$ is equal to :

[April 08, 2019 (I)]

Options:

A. $\frac{\pi}{6} - x$

B. $x - \frac{\pi}{6}$

C. $\frac{\pi}{3} - x$

D. None

Answer: D

Solution:

Solution:

$$2y = \left[\cot^{-1}\left(\frac{\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x}{\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x}\right)\right]^2$$

$$\Rightarrow 2y = \left[\cot^{-1}\left(\frac{\cos\left(\frac{\pi}{6} - x\right)}{\sin\left(\frac{\pi}{6} - x\right)}\right)\right]^2$$

$$\Rightarrow 2y = \left[\cot^{-1}\left(\cot\left(\frac{\pi}{6} - x\right)\right)\right]^2 \ \because \frac{\pi}{6} - x \in \left(-\frac{\pi}{3}, \frac{\pi}{6}\right)$$

$$\Rightarrow 2y = \begin{cases} \left(\frac{7\pi}{6} - x\right)^2 & , \text{ if } \frac{\pi}{6} - x \in \left(\frac{-\pi}{3}, 0\right) \\ \left(\frac{\pi}{6} - x\right)^2 & , \text{ if } \frac{\pi}{6} - x \in \left(0, \frac{\pi}{0}\right) \end{cases}$$

$$\Rightarrow \frac{dy}{dx} = \begin{cases} x - \frac{7\pi}{6} \text{ if } x \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right) \\ x - \frac{\pi}{6} \text{ if } x \in \left(0, \frac{\pi}{6}\right) \end{cases}$$

Question113

If the function f defined as $f(x) = \frac{1}{x} - \frac{k-1}{e^{2x} - 1}$ $x \neq 0$, is continuous at $x = 0$, then the ordered pair $(k, f(0))$ is equal to?

[Online April 16, 2018]

Options:

A. (3,1)

B. (3,2)

C. $\left(\frac{1}{3}, 2\right)$

D. (2,1)

Answer: A

Solution:

Solution:
If the function is continuous at x = 0, then
$\lim_{x \to 0} f(x)$ will exist and $f(0) = \lim_{x \to 0} f(x)$

Now, $\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\frac{1}{x} - \frac{k-1}{e^{2x}-1} \right)$

$= \lim_{x \to 0} \left(\frac{e^{2x} - 1 - kx + x}{(x)(e^{2x}-1)} \right)$

$= \lim_{x \to 0} \left[\frac{\left(1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots..\right) - 1 - kx + x}{(x)\left(\left(1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots...\right) - 1\right)} \right]$

$= \lim_{x \to 0} \left[\frac{(3-k)x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \ldots..}{\left(2x^2 + \frac{4x^3}{2!} + \frac{8x^3}{3!} + \ldots.\right)} \right]$

For the limit to exist, power of x in the numerator should be greater than or equal to the power of x in the denominator.
Therefore, coefficient of x in numerator is equal to zero
$\Rightarrow 3 - k = 0$
$\Rightarrow k = 3$
So the limit reduces to

$\lim_{x \to 0} \frac{(x^2)\left(\frac{4}{2!} + \frac{8x}{3!} + \ldots.\right)}{(x^2)\left(2 + 4x2! + \frac{8x^2}{3!} + \ldots.\right)}$

$= \lim_{x \to 0} \frac{\frac{4}{2!} + \frac{8x}{3!} + \ldots.}{2 + \frac{4x}{2!} + \frac{8x^2}{3!} + \ldots..} = 1$

Hence $f(0) = 1$

Question114

Let $f(x) = \begin{cases} (x-1)^{\frac{1}{2-x}}, & x > 1, x \neq 2 \\ k, & x = 2 \end{cases}$ The value of k for which f is continuous at x = 2 is
[Online April 15, 2018]

Options:

A. e^{-2}

B. e

C. e^{-1}

D. 1

Answer: C

Solution:

Solution:
Since f(x) is continuous at x = 2.
$\therefore \lim_{x \to 2} f(x) = f(2)$

$\Rightarrow \lim_{x \to 2} (x-1)^{\frac{1}{2-x}} = k$
$\therefore e^{l} = k$
where $l = \lim_{x \to 2} (x-1-1) \times \frac{1}{2-x} = \lim_{x \to 2} \frac{x-2}{2-x}$
$= \lim_{x \to 2} \left(\frac{x-2}{x-2}\right)$
$\Rightarrow k = e^{-1}$

Question115

Let $S = \{ t \in R : f(x) = |x-\pi|(e^{|x|}-1)\sin|x|$. is not differentiable at t }. Then the set S is equal to : [2018]

Options:

A. {0}

B. {π}

C. {0, π}

D. ϕ(an empty set)

Answer: D

Solution:

Solution:
$f(x) = |x-\pi|(e^{|x|}-1)\sin|x|$
Check differentiability of f(x) at x = π and x = 0
at x = π:

$R.H.D. = \lim_{h \to 0} \frac{|\pi+h-\pi|(e^{|x+h|}-1)\sin|\pi+h|-0}{h}$

$L.H.D = \lim_{h \to 0} \frac{|\pi-h-\pi|(e^{|\pi-h|}-1)\sin|\pi-h|-0}{-h} = 0$

$\because RHD = LHD$
Therefore, function is differentiable at x = π
at x = 0 :

$R.H.D = \lim_{h \to 0} \frac{|h-\pi|(e^{|h|}-1)\sin|h|-0}{h} = 0$

$L.H.D. = \lim_{h \to 0} \frac{|-h-\pi|(e^{-h|}-1)\sin|-h|-0}{-h} = 0$

∴ RHD = LHD
Therefore, function is differentiable.
at x = 0.
Since, the function f(x) is differentiable at all the points including p and 0.
i.e., f(x) is every where differentiable .
Therefore, there is no element in the set S.
⇒S = ϕ(an empty set)

Question116

Let S = { $(\lambda, \mu) \in R \times R$: $f(t) = (|\lambda|e^{|t|} - \mu)$ .sin(2 | t|), $t \in R$, . is a differentiable function }. Then S is a subest of?
[Online April 15, 2018]

Options:

A. $R \times [0, \infty)$

B. $(-\infty, 0) \times R$

C. $[0, \infty) \times R$

D. $R \times (-\infty, 0)$

Answer: A

Solution:

Solution:
S = { $(\lambda, \mu) \in R \times R$: $f(t) = (|\lambda|e^{|t|} - \mu)\sin(2|t|)$, $t \in R$
$f(t) = (|\lambda|e\_^{|t|}\mu)\sin(2|t|)$

$$= \begin{cases} (|\lambda|e^{t} - \mu)\sin 2t, & t > 0 \\ (|\lambda|e^{-t} - \mu)(-\sin 2t), & t < 0 \end{cases}$$

$$f'(t) = \begin{cases} (|\lambda|e^{t})\sin 2t + (|\lambda|e^{t} - \mu)(2\cos 2t), & t > 0 \\ |\lambda|e^{-t}\sin 2t + (|\lambda|e^{-t} - \mu)(-2\cos 2t), & t < 0 \end{cases}$$

As, f (t) is differentiable
∴ LHD = RHD at t = 0
$\Rightarrow |\lambda| .\sin 2(0) + (|\lambda|e^{0} - \mu)2\cos(0)$
$= |\lambda| e^{-0} . \sin 2(0) - 2\cos(0)\ (|\lambda|e^{-0} - \mu)$
$\Rightarrow 0 + (|\lambda| - \mu)2 = 0 - 2(|\lambda| - \mu)$
$\Rightarrow 4(|\lambda| - \mu) = 0$
$\Rightarrow |\lambda| = \mu$
So, $S \equiv (\lambda, \mu) = \{\lambda \in R \ \&\ \mu \in [0, \infty)\}$
Therefore set S is subset of $R \times [0, \infty)$

Question117

If $x = \sqrt{2^{\mathrm{cosec}^{-1}t}}$ and $y = \sqrt{2^{\sec^{-1}t}}$ $(|t| \geq 1)$, then $\frac{dy}{dx}$ is equal to.
[Online April 16, 2018]

Options:

A. $\frac{y}{x}$

B. $-\frac{y}{x}$

C. $-\frac{x}{y}$

D. $\frac{x}{y}$

Answer: B

Solution:

Solution:

Here, $\frac{dx}{dt} = \frac{1}{2\sqrt{2^{\operatorname{cosec}^{-1}t}}} 2^{\operatorname{cosec}-1_t} \log 2 \cdot \frac{-1}{x\sqrt{x^2-1}}$

$\frac{dy}{dt} = \frac{1}{2\sqrt{2^{\sec^{-1}t}}} 2^{\sec-1_t} \log 2 \cdot \frac{1}{x\sqrt{x^2-1}}$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sqrt{2^{\operatorname{cosec}^{-1}t}}}{\sqrt{2^{\sec^{-1}t}}} \frac{2^{\sec^{-1}t}}{2^{\operatorname{cosec}^{-1}t}}$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\sqrt{\frac{2^{\sec-1_t}}{2^{\operatorname{cosec}-1_t}}} = \frac{-y}{x}$

Question118

If $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$, then $\lim_{x \to 0} \frac{f'(x)}{x}$

[Online April 15, 2018]

Options:

A. Exists and is equal to – 2

B. Does not exist

C. Exist and is equal to 0

D. Exists and is equal to 2

Answer: A

Solution:

Solution:

$f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$

$= \cos x(x^2 - 2x^2) - x(2\sin x - 2x\tan x) + 1(2x\sin x - x^2\tan x)$

$= -x^2\cos x - 2x\sin x + 2x^2\tan x + 2x\sin x - x^2\tan x$

$= x^2\tan x - x^2\cos x = x^2(\tan x - \cos x)$

$\Rightarrow f'(x) = 2x(\tan x - \cos x) + x^2(\sec^2 x + \sin x)$

$\therefore \lim_{x \to 0} \frac{f'(x)}{x} = \lim_{x \to 0} \frac{2x(\tan x - \cos x) + x^2(\sec^2 x + \sin x)}{x}$

$= \lim_{x \to 0} (\tan x - \cos x) + x(\sec^2 x + \sin x)$

$= 2(0 - 1) + 0 = -2$

So, $\lim_{x \to 0} \frac{f'(x)}{x} = -2$

Question119

If $f(x) = \sin^{-1}\left(\frac{2 \times 3^x}{1 + 9^x}\right)$, then $f'\left(-\frac{1}{2}\right)$ equals.

[Online April 15, 2018]

Options:

A. $\sqrt{3}\log_e\sqrt{3}$

B. $-\sqrt{3}\log_e\sqrt{3}$

C. $-\sqrt{3}\log_e 3$

D. $\sqrt{3}\log_e 3$

Answer: A

Solution:

Solution:

Since $f(x) = \sin^{-1}\left(\frac{2 \times 3^x}{1 + 9^x}\right)$

Suppose $3^x = \tan t$

$\Rightarrow f(x) = \sin^{-1}\left(\frac{2\tan t}{1 + \tan^2 t}\right) = \sin^{-1}(\sin 2t) = 2t$

$= 2\tan^{-1}(3x)$

So, $f'(x) = \frac{2}{1 + (3^x)^2} \times 3^x . \log_e 3$

$\therefore f'\left(-\frac{1}{2}\right) = \frac{2}{1 + \left(3^{-\frac{1}{2}}\right)^2} \times 3^{-\frac{1}{2}} . \log_e 3$

$= \frac{1}{2} \times \sqrt{3} \times \log_e 3 = \sqrt{3} \times \log_e\sqrt{3}$

Question120

If $x^2 + y^2 + \sin y = 4$, then the value of $\frac{d^2y}{dx^2}$ at the point (-2,0) is
[Online April 15, 2018]

Options:

A. – 34

B. – 32

C. – 2

D. 4

Answer: A

Solution:

Solution:

Given, $x^2 + y^2 + \sin y = 4$

After differentiating the above equation w. r. t. x we get

$2x + 2y\frac{dy}{dx} + \cos y\frac{dy}{dx} = 0$ ....(1)

$\Rightarrow 2x + (2y + \cos y)\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-2x}{2y + \cos y}$

At$(-2, 0)$, $\left(\frac{dy}{dx}\right)_{(-2,0)} = \frac{-2\times-2}{2\times0+\cos0}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(-2,0)} = \frac{4}{0+1}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(-2,0)} = 4$ .... (2)

Again differentiating equation (1) w. r. t to x, we get

$2+2\left(\frac{dy}{dx}\right)^2+2y\frac{d^2y}{dx^2}-\sin y\left(\frac{dy}{dx}\right)^2+\cos y\frac{d^2y}{dx^2}=0$

$\Rightarrow 2+(2-\sin y)\left(\frac{dy}{dx}\right)^2+(2y+\cos y)\frac{d^2y}{dx^2}=0$

$\Rightarrow (2y+\cos y)\frac{d^2y}{dx^2} = -2-(2-\sin y)\left(\frac{dy}{dx}\right)^2$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2-(2-\sin y)\left(\frac{dy}{dx}\right)^2}{2y+\cos y}$

So, at (– 2, 0),

$\frac{d^2y}{dx^2} = \frac{-2-(2-0)\times4^2}{2\times0+1}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2-2\times16}{1}$

$\Rightarrow \frac{d^2y}{dx^2} = -34$

Question121

The value of k for which the function

$$\mathbf{f(x)} = \begin{cases} \left(\frac{4}{5}\right)^{\frac{\tan 4x}{\tan 5x}}, & 0 < x < \frac{\pi}{2} \\ k+\frac{2}{5}, & x = \frac{\pi}{2} \end{cases}$$

is continuous at $x = \frac{\pi}{2}$, is:

[Online April 9, 2017]

Options:

A. $\frac{17}{20}$

B. $\frac{2}{5}$

C. $\frac{3}{5}$

D. $-\frac{2}{5}$

Answer: C

Solution:

Solution:

$\lim_{x\to\pi/2} f(x) = f(\pi/2)$

$\Rightarrow k+2/5 = 1 \Rightarrow k = 1-\frac{2}{5} \Rightarrow k = \frac{3}{5}$

Question122

If for $x \in \left(0, \frac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is$\sqrt{x}$. g(x), then g(x) equals:
[2017]

Options:

A. $\frac{3}{1+9x^3}$

B. $\frac{9}{1+9x^3}$

C. $\frac{3x\sqrt{x}}{1-9x^3}$

D. $\frac{3x}{1-9x^3}$

Answer: B

Solution:

Solution:

Let $F(x) = \tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ where $x \in \left(0, \frac{1}{4}\right)$

$= \tan^{-1}\left(\frac{2.(3x^{3/2})}{1-(3x^{3/2})^2}\right) = 2\tan^{-1}(3x^{3/2})$

As $3x^{3/2} \in \left(0, \frac{3}{8}\right)$

$\left[\because 0 < x < \frac{1}{4} \Rightarrow 0 < x^{3/2} < \frac{1}{8} \Rightarrow 0 < 3x^{3/2} < \frac{3}{8}\right]$

So $\frac{dF(x)}{dx} = 2 \times \frac{1}{1+9x^3} \times 3 \times \frac{3}{2} \times x^{1/2} = \frac{9}{1+9x^3}\sqrt{x}$

On comparing

$\therefore g(x) = \frac{9}{1+9x^3}$

Question123

If $2x = y^{\frac{1}{5}} + y^{-\frac{1}{5}} =$ and $(x^2-1)\frac{d^2y}{dx^2} + \lambda x\frac{dy}{dx} + ky = 0$then $\lambda + k$ is equal to [Online April 9, 2017]

Options:

A. – 23

B. – 24

C. 26

D. – 26

Answer: B

Solution:

Solution:

$y^{1/5} + y^{-1/5} = 2x$

$\Rightarrow \left(\frac{1}{5}y^{-4/5} - \frac{1}{5}y^{-6/5}\right)\frac{dy}{dx} = 2$

$\Rightarrow y'(y^{1/5} - y^{-1/5}) = 10y$

$\Rightarrow y^{1/5} + y^{-1/5} = 2x$

$\Rightarrow y^{1/5} - y^{-1/5} = \sqrt{4x^2 - 4}$

$\Rightarrow y'\left(2\sqrt{x^2 - 1}\right) = 10y$

$\Rightarrow y''\left(2\sqrt{x^2 - 1}\right) + y'2\frac{2x}{2\sqrt{x^2 - 1}} = 10y'$

$\Rightarrow y''(x^2 - 1) + xy' = 5\sqrt{x^2 - 1}(y')$

$\Rightarrow y''(x^2 - 1) + xy' - 25y = 0$

$\lambda = 1, k = -25$

Question124

Let f be a polynomial function such that $f(3x) = f'(x), f''(x)$, for all $x \in R$. Then :
[Online April 9, 2017]

Options:

A. $f(b) + f'(b) = 28$

B. $f''(b) - f'(b) = 0$

C. $f''(b) - f'(b) = 4$

D. $f(b) - f'(b) + f''(b) = 10$

Answer: B

Solution:

Solution:

Let $f(x) = ax^3 + bx^2 + cx + d$

$\Rightarrow f(3x) = 27ax^3 + 9bx^2 + 3cx + d$

$\Rightarrow f'(x) = 3ax^2 + 2bx + c$

$\Rightarrow f''(x) = 6ax + 2b$

$\Rightarrow f(3x) = f'(x)f''(x)$

$\Rightarrow 27a = 18a^2$

$\Rightarrow a = \frac{3}{2}, b = 0, c = 0, d = 0$

$\Rightarrow f(x) = \frac{3}{2}x^3,$

$f'(x) = \frac{9}{2}x^2, f'(x) = 9x$

$\Rightarrow f'(2) = 18$

and $f''(2) = 18$

$\Rightarrow f''(b) - f'(b) = 0$

Question125

If $y = \left[x + \sqrt{x^2 - 1}\right]^{15} + \left[x - \sqrt{x^2 - 1}\right]^{15}$, then $(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx}$ is equal to
[Online April 8, 2017]

Options:

A. $12y$

B. $224y^2$

C. $225y^2$

D. $225y$

Answer: D

Solution:

Solution:

$y = \{x + \sqrt{x^2 - 1}\}^{15} + \{x - \sqrt{x^2 - 1}\}^{15}$

Differentiate w.r.t. 'x'

$\frac{dy}{dx} = 15(x + \sqrt{x^2 - 1})^{14}\left[1 + \frac{x}{\sqrt{x^2 - 1}}\right] + 15(x - \sqrt{x^2 - 1})^{14}\left(1 - \frac{x}{\sqrt{x^2 - 1}}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{15}{\sqrt{x^2 - 1}} . y$

$\Rightarrow \sqrt{x^2 - 1} . \frac{dy}{dx} = 15y$

Again differentiating both sides w.r.t. x

$\frac{x}{\sqrt{x^2 - 1}} . \frac{dy}{dx} + \sqrt{x^2 - 1}\frac{d^2y}{dx^2} = 15\frac{dy}{dx}$

$\Rightarrow x\frac{dy}{dx} + (x^2 - 1)\frac{d^2y}{dx^2}$

$= 15\sqrt{x^2 - 1} . \frac{15}{\sqrt{x^2 - 1}} . y = 225y$

Question126

Let a, b ∈ R, (a ≠ 0). if the function f defined as

$$f(x) = \begin{cases} \frac{2x^2}{a} & , \quad 0 \le x < 1 \\ a & , \quad 1 \le x < \sqrt{2} \\ \frac{2b^2 - 4b}{x^3} & , \quad \sqrt{2} \le x \le \infty \end{cases}$$

is continuous in the interval [0, ∞), then an ordered pair (a, b) is: [Online April 10, 2016]

Options:

A. $(-\sqrt{2}, 1 - \sqrt{3})$

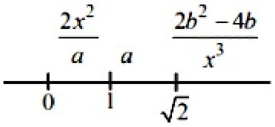
B. $(\sqrt{2}, -1 + \sqrt{3})$

C. $(\sqrt{2}, 1 - \sqrt{3})$

D. $(-\sqrt{2}, 1 + \sqrt{3})$

Answer: C

Solution:



Continuity at $x = 1$

$\frac{2}{a} = a \Rightarrow a = \pm\sqrt{2}$

Continuity at $x = \sqrt{2} a = \sqrt{2}$

$a = \frac{2b^2 - 4b}{2\sqrt{2}}$

Put $a = \sqrt{2}$

$2 = b^2 - 2b \Rightarrow b^2 - 2b - 2 = 0$

$b = \frac{2 \pm \sqrt{4 + 4.2}}{2} = 1 \pm \sqrt{3}$

So, $(a, b) = (\sqrt{2, 1} - \sqrt{3})$

Question127

If the function

$f(x) = \begin{cases} -x, & x < 1 \\ a + \cos^{-1}(x + b), & 1 \le x \le 2 \end{cases}$

is differentiable atx = 1, then $\frac{a}{b}$ is equal to: [Online April 9,2016]

Options:

A. $\frac{\pi + 2}{2}$

B. $\frac{\pi - 2}{2}$

C. $\frac{-\pi - 2}{2}$

D. $-1 - \cos^{-1}(2)$

Answer: A

Solution:

Solution:

$f(x) = \begin{cases} -x, & x < 1 \\ a + \cos^{-1}(x + b), & 1 \le x \le 2 \end{cases}$

f (x) is continuous

$\Rightarrow \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} a + \cos^{-1}(x + b) = f(x)$

$\Rightarrow -1 = a + \cos^{-1}(1 + b)$

$\cos^{-1}(1 + b) = -1 - a$ ......(a)

f (x) is differentiate

$\Rightarrow$ LHD = RHD

$\Rightarrow -1 = \frac{-1}{\sqrt{1 - (1 + b)^2}}$

$\Rightarrow 1 - (1 + b)^2 = 1 \Rightarrow b = -1$ ....(b)

From (a) $\Rightarrow \cos^{-1}(0) = -1 - a$

$\therefore -1 - a = \frac{\pi}{2}$

$a = -1 - \frac{\pi}{2} \Rightarrow a = \frac{-\pi - 2}{2}$ ....(c)

$\therefore \frac{a}{b} = \frac{\pi + 2}{2}$

Question128

For $x \in R$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then [2016]

Options:

A. g'(0) = – cos(log2)

B. g is differentiable at x = 0 and g'(0) = – sin(log2)

C. g is not differentiable at x = 0

D. g'(0) = cos(log2)

Answer: D

Solution:

Solution:
(d) $g(x) = f(f(x))$
In the neighbourhood of $x = 0$
$f(x) = |\log 2 - \sin x| = (\log 2 - \sin x)$
$\therefore g(x) = |\log 2 - \sin|\log 2 - \sin x\|$
$= (\log 2 - \sin(\log 2 - \sin x))$
∴g(x) is differentiable
and $g'(x) = -\cos(\log 2 - \sin x)(-\cos x)$
$\Rightarrow g'(0) = \cos(\log 2)$

Question129

Let k be a non–zero real number

$$\text{If } f(x) = \begin{cases} \dfrac{(e^x - 1)}{\sin\left(\frac{x}{k}\right)\log\left(1 + \frac{x}{4}\right)_{12}} , & x \neq 0 \\ 12 , & x \neq 0 \end{cases}$$

is a continuous function then the value of k is: [Online April 11, 2015]

Options:

A. 4

B. 1

C. 3

D. 2

Answer: C

Solution:

Solution:
Since $f(x)$ is a continuous function therefore limit of $f(x)$ at $x \to 0 =$ value of $f(x)$ at 0

$$\therefore \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{(e^x - 1)^2}{\sin\left(\frac{x}{k}\right) \log\left(1 + \frac{x}{4}\right)}$$

$$= \lim_{x \to 0} \frac{x^2 \left(\frac{e^x - 1}{x}\right)^2}{\frac{x}{R}\left[\frac{\sin\left(\frac{x}{R}\right)}{\frac{x}{R}}\right] \cdot \frac{\log\left(1 + \frac{x}{4}\right)}{\left(\frac{x}{4}\right)}} \times \left(\frac{x}{4}\right)$$

$$= \lim_{x \to 0} \frac{x^2 \left(\frac{e^x - 1}{x}\right)^2 4k}{\frac{\sin\frac{x}{k}}{\frac{x}{k}} \cdot \frac{\log\left(1 + \frac{x}{4}\right)}{\frac{x}{4}}}$$

on applying limit we get
$4k = 12 \Rightarrow k = 3$

Question130

If the function.

g(x) = $\begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx + 2, & 3 < x \le 5 \end{cases}$

is differentiable, then the value of k + m is: [2015]

Options:

A. $\frac{10}{3}$

B. 4

C. 2

D. $\frac{16}{5}$

Answer: C

Solution:

Solution:
Since g (x) is differentiable, it will be continuous at x = 3
$\therefore \lim_{x \to 3^-} g(x) = \lim_{x \to 3^+} g(x)$
$2k = 3m + 2$ ....(1)
Also g(x) is differentiable at x = 0
$\therefore \lim_{x \to 3^-} g'(x) = \lim_{x \to 3^+} g'(x)$
$\frac{k}{2\sqrt{3+1}} = m$
$k = 4m$ ....(2)
(1) and (2), we get Solving
$m = \frac{2}{5}, k = \frac{8}{5}$
$k + m = 2$

Question131

If Rolle's theorem holds for the function $f(x) 2x^3 + bx^2 + cx, x \in [-1, 1]$, at the point $x = \frac{1}{2}$, then $2b + c$ equals:
[Online April 10, 2015]

Options:

A. –3

B. –1

C. 2

D. 1

Answer: B

Solution:

Solution:
Conduction for Rolls theorem
$f(1) = f(-1)$
and $f'\left(\frac{1}{2}\right) = 0$
$c = -2$ and $b = \frac{1}{2}$
$2b + c = -1$

Question132

If the function

$$f(x) = \begin{cases} \frac{\sqrt{2+\cos x} - 1}{(\pi - x)^2}, & x \neq \pi \\ k, & x = \pi \end{cases}$$

is continuous at $x = \pi$, then k equals:
[Online April 19, 2014]

Options:

A. 0

B. $\frac{1}{2}$

C. 2

D. $\frac{1}{4}$

Answer: D

Solution:

Solution:
Since $f(x) = \frac{\sqrt{2+\cos x} - 1}{(\pi - x)^2}$ is
Continuous at $x = \pi$
$\therefore L.H.L = R.H.L = f(\pi)$
Let $(\pi - x) = \theta$, $\theta \to 0$ when $x \to \pi$

$\therefore \lim_{\theta \to 0} \frac{\sqrt{2 - \cos\theta} - 1}{\theta^2}$

$= \lim_{\theta \to 0} \frac{(2 - \cos\theta) - 1}{\theta^2} \times \frac{1}{\sqrt{2 - \cos\theta} + 1}$

$= \lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta^2} \cdot \frac{1}{2} (\because \cos 0 = 1)$

$= \frac{1}{2} \lim_{\theta \to 0} \frac{2\sin^2\theta / 2}{\theta^2} = \frac{2}{2} \lim_{\theta \to 0} \frac{\sin^2\theta / 2}{\frac{\theta^2}{4} \cdot 4}$

$= \frac{1}{4} \left(\because \lim_{x \to 0} \frac{\sin x}{x} = 1 \right)$

Question133

If f (x) is continuous and $f\left(\frac{9}{2}\right) = \frac{2}{9}$, then $\lim_{x \to 0} f\left(\frac{1 - \cos 3x}{x^2}\right)$ is equal to:

[Online April 9, 2014]

Options:

A. $\frac{9}{2}$

B. $\frac{2}{9}$

C. 0

D. $\frac{8}{9}$

Answer: B

Solution:

Solution:

Given that $f\left(\frac{9}{2}\right) = \frac{2}{9}$

$\lim_{x \to 0} f\left(\frac{1 - \cos 3x}{x^2}\right) = \lim_{x \to 0} \left(\frac{x^2}{1 - \cos 3x}\right)$

$= \lim_{x \to 0} \left(\frac{x^2}{2\sin^2\frac{3x}{2}}\right) = \frac{1}{2} \lim_{x \to 0} \left(\frac{\frac{9}{4} \cdot x^2 \cdot \frac{4}{9}}{\sin^2\frac{3x}{2}}\right)$

$= \frac{4}{9 \times 2} \lim_{x \to 0} \left(\frac{1}{\frac{\sin^2\frac{3x}{2}}{\left(\frac{3x}{2}\right)^2}}\right)$

$= \frac{2}{9} \left[\frac{\lim_{x \to 0} 1}{\lim_{x \to 0} \frac{\sin^2\frac{3x}{2}}{\left(\frac{3x}{2}\right)^2}}\right] \left\{ \lim_{x \to 0} \frac{\sin x}{x} = 1 \right\}$

$= \frac{2}{9} \cdot \left[\frac{1}{1}\right] = \frac{2}{9}$

Question134

Let f : R → R be a function such that $|f(x)| \le x^2$, for all x ∈ R. Then, at x = 0, f is :
[Online April 19, 2014]

Options:

A. continuous but not differentiable.

B. continuous as well as differentiable.

C. neither continuous nor differentiable.

D. differentiable but not continuous.

Answer: B

Solution:

Solution:
Let $|f(x)| \le x^2, \forall x \in R$
Now, at $x = 0$, $|f(0)| \le 0$
$\Rightarrow f(0) = 0$
$\therefore f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h - 0} = \lim_{h \to 0} \frac{f(h)}{h}$ .....(1)
Now, $\left|\frac{f(h)}{h}\right| \le |h| (\because |f(x)| \le x^2)$
$\Rightarrow -|h| \le \frac{f(h)}{h} \le |h|$
$\Rightarrow \lim_{h \to 0} \frac{f(h)}{h} \to 0$ .....(2)
(using sandwich Theorem)
∴ from (1) and (2), we get f'(0) = 0,
i.e. – f (x) is differentiable, at x = 0
Since, differentiability ⇒ Continutity
$\therefore |f(x)| \le x^2$, for all $x \in R$ is continuous as well as differentiable at $x = 0$

Question135

Let f , g : R → R be two functions defined by $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$,and

g(x) = xf (x)
Statement I: f is a continuous function at x = 0.
Statement II: g is a differentiable function at x = 0.
[Online April 12, 2014]

Options:

A. Both statement I and II are false.

B. Both statement I and II are true.

C. Statement I is true, statement II is false.

D. Statement I is false, statement II is true.

Answer: B

Solution:

Solution:

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and $g(x) = xf(x)$

For $f(x)$

$LHL = \lim_{h \to 0^-} \left\{ -h\sin\left(-\frac{1}{h}\right) \right\}$

$= 0 \times$ a finite quantity between -1 and 1 = 0

$RHL = \lim_{h \to 0^+} h\sin\frac{1}{h} = 0$

Also, $f(0) = 0$

Thus $LHL = RHL = f(0)$

$\therefore f(x)$ is continuous at $x = 0$

$$g(x) = \begin{cases} x^2\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

For $g(x)$

$LHL = \lim_{h \to 0^-} \left\{ -h^2\sin\left(\frac{1}{h}\right) \right\}$

$= 0^2 \times$ a finite quantity between -1 and 1 = 0

$RHL = \lim_{h \to 0^+} h^2\sin\left(\frac{1}{h}\right) = 0$

Also $g(0) = 0$

$\therefore g(x)$ is continuous at $x = 0$

Question136

If $f(x) = x^2 - x + 5$, $x > \frac{1}{2}$, and g(x) is its inverse function,then $g'(7)$ equals:
[Online April 12, 2014]

Options:

A. $-\frac{1}{3}$

B. $\frac{1}{13}$

C. $\frac{1}{3}$

D. $-\frac{1}{13}$

Answer: C

Solution:

Solution:

$f(x) = y = x^2 - x + 5$

$x^2 - x + \frac{1}{4} - \frac{1}{4} + 5 = y$

$\left(x - \frac{1}{2}\right)^2 + \frac{19}{4} = y$

$\left(x - \frac{1}{2}\right)^2 = y - \frac{19}{4}$

$x - \frac{1}{2} = \pm\sqrt{y - \frac{19}{4}}$

$x = \frac{1}{2} \pm \sqrt{y - \frac{19}{4}}$

As $x > \frac{1}{2}$

$x = \frac{1}{2} + \sqrt{y - \frac{19}{4}}$

$g(x) = \frac{1}{2} + \sqrt{x - \frac{19}{4}}$

$g'(x) = \frac{1}{2\sqrt{x - \frac{19}{4}}}$

$g'(7) = \frac{1}{2\sqrt{7 - \frac{19}{4}}} = \frac{1}{2\frac{\sqrt{28 - 19}}{2}} = \frac{1}{3}$

Question137

If f and g are differentiable functions in [0, 1] satisfying f (0) = 2 = g(1), g(0) = 0 and f (1) = 6, then for some c ∈ [0, 1]
[2014]

Options:

A. f'(c) = g'(c)

B. f'(c) = 2g'(c)

C. 2f'(c) = g'(c)

D. 2f'(c) = 3g'(c)

Answer: B

Solution:

Solution:
Since, f and g both are continuous function on [0,1]and differentiable on (0,1) then ∃c ∈ (0, 1) such that

$f'(c) = \frac{f(1) - f(0)}{1} = \frac{6 - 2}{1} = 4$

and $g'(c) = \frac{g(1) - g(0)}{1} = \frac{2 - 0}{1} = 2$

Thus, we get $f'(c) = 2g'(c)$

Question138

Let $f(x) = x|x|$, $g(x) = \sin x$ and $h(x) = (gof)(x)$. Then
[Online April 11, 2014]

Options:

A. h(x) is not differentiable at x = 0.

B. h(x) is differentiable at x = 0, but h'(x) is not continuous at x = 0

C. h'(x) is continuous at x = 0 but it is not differentiable at x = 0

D. h'(x) is differentiable at x = 0

Answer: C

Solution:

Solution:
Let $f(x) = x|x| = x|x|, g(x) = \sin x$
and $h(x) = gof(x) = g[f(x)]$

$$\therefore h(x) = \begin{cases} \sin x^2, & x \geq 0 \\ -\sin x^2, & x < 0 \end{cases}$$

$$\text{Now, } h'(x) = \begin{cases} 2x\cos x^2, & x \geq 0 \\ -2x\cos x^2, & x < 0 \end{cases}$$

Since, L.H.L and R.H.L at x = 0 of h'(x) is equal to 0 therefore h'(x) is continuous at x = 0
Now, suppose h'(x) is differentiable

$$\therefore h''(x) = \begin{cases} 2(\cos x^2 + 2x^2(-\sin x^2)), & x \geq 0 \\ 2(-\cos x^2 + 2x^2 \sin x^2), & x < 0 \end{cases}$$

Since, L.H.L and R.H.L at x = 0 of h"(x) are different therefore h"(x) is not continuous.
⇒ h"(x) is not differentiable
⇒ our assumption is wrong
Hence h'(x) is not differentiable at x = 0.

Question139

Let for i = 1, 2, 3, $p_i(x)$ be a polynomial of degree 2 in x, $p_i'(x)$and $p''_i(x)$ be the first and second order derivatives of $p_i(x)$respectively. Let,

$$\mathbf{A(x)} = \begin{bmatrix} p_1(x) & p_1'(x) & p_1''(x) \\ p_2(x) & p_2(x) & p_2''(x) \\ p_3(x) & p_3'(x) & p_3''(x) \end{bmatrix}$$

and B(x) = $[A(x)]^T A(x)$. Then determinant of B(x)
[Online April 11, 2014]

Options:

A. is a polynomial of degree 6 in x.

B. is a polynomial of degree 3 in x.

C. is a polynomial of degree 2 in x.

D. does not depend on x.

Answer: A

Solution:

Solution:
Let $p_1(x) = a_1x^2 + b_1x + c_1$
$p_2(x) = a_2x^2 + b_2x + c_2$
and $p_3(x) = a_3x^2 + b_3x + c_3$
where a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 are real numbers.

$$\therefore A(x) = \begin{bmatrix} a_1x^2 + b_1x + c_1 & 2a_1x + b_1 & 2a_1 \\ a_2x^2 + b_2x + c_2 & 2a_2x + b_2 & 2a_2 \\ a_3x^2 + b_3x + c_3 & 2a_3x + b_3 & 2a_3 \end{bmatrix}$$

$$B(x) = \begin{bmatrix} a_1x^2 + b_1x + c_1 & a_2x^2 + b_2x + c_2 & a_3x^2 + b_3x + c_3 \\ 2a_1x + b_1 & 2a_2x + b_2 & 2a_3x + b_2 \\ 2a_1 & 2a_2 & 2a_3 \end{bmatrix}$$

$$\times \begin{bmatrix} a_1x^2 + b_1x + c_1 & 2a_1x + b_1 & 2a_1 \\ a_2x^2 + b_2x + c_2 & 2a_2x + b_2 & 2a_2 \\ a_3x^2 + b_3x + c_3 & 2a_3x + b_3 & 2a_3 \end{bmatrix}$$

It is clear from the above multiplication, the degree of determinant of B(x) can not be less than 4.

Question140

If the Rolle's theorem holds for the function $f(x) = 2x^3 + ax^2 + bx$ in the interval [-1,1] for the pointc $= \frac{1}{2}$, then the value of $2a + b$ is:
[Online April 9, 2014]

Options:

A. 1

B. – 1

C. 2

D. – 2

Answer: B

Solution:

Solution:
$f(x) = 2x^3 + ax^2 + bx$
let, $a = -1, b = 1$
Given that $f(x)$ satisfy Rolle's theorem in interval [-1,1]
$f(x)$ must satisfy two conditions.
(1) $f(a) = f(b)$
(2) $f'(c) = 0$ (c should be between a and b)
$f(a) = f(1) = 2(1)^3 + a(1)^2 + b(1) = 2 + a + b$
$f(b) = f(-1) = 2(-1)^3 + a(-1)^2 + b(-1)$
$= -2 + a - b$
$f(a) = f(b)$
$2 + a + b = -2 + a - b$
$2b = -4$
$b = -2$
(given that $c = \frac{1}{2}$)
$f'(x) = 6x^2 + 2ax + b$
at $x = \frac{1}{2}, f'(x) = 0$
$\frac{3}{2} + a + b = 0$
$\frac{3}{2} + a - 2 = 0$
$a = 2 - \frac{3}{2} = \frac{1}{2}$
$2a + b = 2 \times \frac{1}{2} - 2 = 1 - 2 = -1$

Question141

Consider the function :
$f(x) = [x] + |1 - x|, -1 \le x \le 3$ where [x] is the greatest integer function.
Statement 1: f is not continuous at $x = 0, 1, 2$ and 3.

Statement 2 :

$$f(x) = \begin{cases} -x, & -1 \le x < 0 \\ 1 - x, & 0 \le x < 1 \\ 1 + x, & 1 \le x < 2 \\ 2 + x, & 2 \le x \le 3 \end{cases}$$

[Online April 25, 2013]

Options:

A. Statement 1 is true ; Statement 2 is false,

B. Statement 1 is true; Statement 2 is true; Statement 2 is not correct explanation for Statement 1.

C. Statement 1 is true; Statement 2 is true; Statement It is a correct explanation for Statement 1.

D. Statement 1 is false; Statement 2 is true.

Answer: A

Solution:

Solution:
Let $f(x) = [x] + |1 - x|, -1 \le x \le 3$
where $[x] =$ greatest integer function.
f is not continuous at $x = 0, 1, 2, 3$
But in statement- $2 f(x)$ is continuous at $x = 3$.
Hence, statement- 1 is true and 2 is false.

Question142

Let f be a composite function of x defined by $f(u) = \frac{1}{u^2 + u - 2}$, $u(x) = \frac{1}{x - 1}$. Then the number of points x where f is discontinuous is : [Online April 23, 2013]

Options:

A. 4

B. 3

C. 2

D. 1

Answer: B

Solution:

Solution:
$\mu(x) = \frac{1}{x - 1}$, which is discontinous at $x = 1$
$f(u) = \frac{1}{u^2 + u - 2} = \frac{1}{(u + 2)(u - 1)}$
which is discontinous at $u = -2, 1$

when $u = -2$, then $\frac{1}{x-1} = -2 \Rightarrow x = \frac{1}{2}$

when $u = 1$, then $\frac{1}{x-1} = 1 \Rightarrow x = 2$

Hence given composite function is discontinous at three points, $x = 1, \frac{1}{2}$ and 2 .

Question143

Let $f(x) = -1 + |x-2|$, and $g(x) = 1 - |x|$; then the set of all points where fog is discontinuous is :
[Online April 22, 2013]

Options:

A. {0, 2}

B. {0, 1, 2}

C. {0}

D. an empty set

Answer: D

Solution:

Solution:

$fog = f(g(x)) = f(1 - |x|)$

$= -1 + |1 - |x| - 2|$

$= -1 + |-|x| - 1| = -1 + ||x| + 1|$

Let $fog = y$

$\therefore y = -1 + ||x| + 1|$

$\Rightarrow y = \begin{cases} -1 + x + 1, & x \geq 0 \\ -1 - x + 1, & x < 0 \end{cases}$

$\Rightarrow y = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

LHL at $(x = 0) = \lim_{x \to 0}(-x) = 0$

RHL at $(x = 0) = \lim_{x \to 0}(x) = 0$

When $x = 0$, then $y = 0$

Hence, LHL at $(x = 0) =$ RHL at $(x = 0)$

= value of y at $(x = 0)$

Hence y is continuous at $x = 0$.

Clearly at all other point y continuous. Therefore, the set of all points where fog is discontinuous is an empty set.

Question144

If $y = \sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to :
[2013]

Options:

A. $\frac{1}{\sqrt{2}}$

B. $\frac{1}{2}$

C. 1

D. $\sqrt{2}$

Answer: A

Solution:

Solution:
Let $y = \sec(\tan^{-1}x) = \sec\left(\sec^{-1}\sqrt{1+x^2}\right)$
$\Rightarrow y = \sqrt{1+x^2}$
$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{1+x^2}} . 2x$
At $x = 1$
$\frac{dy}{dx} = \frac{1}{\sqrt{2}}$

Question145

If the curves $\frac{x^2}{\alpha} + \frac{y^2}{4} = 1$ and $y^3 = 16x$ intersect at right angles, then a value of α is :
[Online April 23, 2013]

Options:

A. 2

B. $\frac{4}{3}$

C. $\frac{1}{2}$

D. $\frac{3}{4}$

Answer: B

Solution:

Solution:
$\frac{x^2}{\alpha} + \frac{y^2}{4} = 1 \Rightarrow \frac{2x}{\alpha} + \frac{2y}{4} . \frac{dy}{dx} = 0$
$\Rightarrow \frac{dy}{dx} = \frac{-4x}{\alpha y}$ ....(i)
$y^3 = 16x \Rightarrow 3y^2 . \frac{dy}{dx} = 16 \Rightarrow \frac{dy}{dx} = \frac{16}{3y^2}$ ......(ii)
Since curves intersects at right angles
$\therefore \frac{-4x}{\alpha y} \times \frac{16}{3y^2} = -1 \Rightarrow 3\alpha y^3 = 64x$
$\Rightarrow \alpha = \frac{64x}{3 \times 16x} = \frac{4}{3}$

Question146

For $a > 0$, $t \in \left(0, \frac{\pi}{2}\right)$, let $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$ Then, $1 + \left(\frac{dy}{dx}\right)^2$ equals:

[Online April 22, 2013]

Options:

A. $\frac{x^2}{y^2}$

B. $\frac{y^2}{x^2}$

C. $\frac{x^2+y^2}{y^2}$

D. $\frac{x^2+y^2}{x^2}$

Answer: D

Solution:

Solution:
Let $x = \sqrt{a^{\sin^{-1}t}}$
$\Rightarrow x^2 = a^{\sin^{-1}t}$
$\Rightarrow 2\log x = \sin^{-1}t \,.\, \log a$
$\Rightarrow \frac{2}{x} = \frac{\log a}{\sqrt{1-t^2}} \cdot \frac{dt}{dx}$
$\Rightarrow \frac{2\sqrt{1-t^2}}{x\log a} = \frac{dt}{dx}$ ....(1)
Now, let $y = \sqrt{a^{\cos^{-1}t}}$
$\Rightarrow 2\log y = \cos^{-1}t \,.\, \log a$
$\Rightarrow \frac{2}{y} \cdot \frac{dy}{dx} = \frac{-\log a}{\sqrt{1-t^2}} \cdot \frac{dt}{dx}$
$\Rightarrow \frac{2}{y} \cdot \frac{dy}{dx} = \frac{-\log a}{\sqrt{1-t^2}} \times \frac{2\sqrt{1-t^2}}{x\log a}$ (from (1)
$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$
Hence, $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{-y}{x}\right)^2 = \frac{x^2+y^2}{x^2}$

Question147

Let $f(x) = \frac{x^2-x}{x^2+2x} x \neq 0, -2$. Then $\frac{d}{dx}[f^{-1}(x)]$ (wherever it is defined) is equal to :
[Online April 9, 2013]

Options:

A. $\frac{-1}{(1-x)^2}$

B. $\frac{3}{(1-x)^2}$

C. $\frac{1}{(1-x)^2}$

D. $\frac{-3}{(1-x)^2}$

Answer: B

Solution:

Solution:
Let $y = \frac{x^2 - x}{x^2 + 2x}$
$\Rightarrow (x^2 + 2x)y = x^2 - x$
$\Rightarrow x(x + 2)y = x(x - 1)$
$\Rightarrow x[(x + 2)y - (x - 1)] = 0$
$\because x \neq 0, \therefore (x + 2)y - (x - 1) = 0$
$\Rightarrow xy + 2y - x + 1 = 0$
$\Rightarrow x(y - 1) = -(2y + 1)$
$\therefore x = \frac{2y + 1}{1 - y} \Rightarrow f^{-1}(x) = \frac{2x + 1}{1 - x}$
$\frac{d}{dx}(f^{-1}(x)) = \frac{2(1 - x) - (2x + 1)(-1)}{(1 - x)^2}$
$= \frac{2 - 2x + 2x + 1}{(1 - x)^2} = \frac{3}{(1 - x)^2}$

Question148

If $f(x) = \sin(\sin x)$ and $f''(x) + \tan x\, f'(x) + g(x) = 0$, theng(x) is : [Online April 23, 2013]

Options:

A. $\cos^2 x \cos(\sin x)$

B. $\sin^2 x \cos(\cos x)$

C. $\sin^2 x \sin(\cos x)$

D. $\cos^2 x \sin(\sin x)$

Answer: D

Solution:

Solution:
$f(x) = \sin(\sin x)$
$\Rightarrow f'(x) = \cos(\sin x) . \cos x$
$\Rightarrow f''(x) = -\sin(\sin x) . \cos^2 x + \cos(\sin x) . (-\sin x) = -\cos^2 x . \sin(\sin x) - \sin x . \cos(\sin x)$
Now $f''(x) + \tan x . f'(x) + g(x) = 0$
$\Rightarrow g(x) = \cos^2 x . \sin(\sin x) + \sin x . \cos(\sin x) - \tan x . \cos x . \cos(\sin x)$
$\Rightarrow g(x) = \cos^2 x . \sin(\sin x)$

Question149

If $f : R \to R$ is a function defined by $f(x) = [x] \cos\left(\frac{2x - 1}{2}\right)\pi$, where [x] denotes the greatest integer function, then f is. [2012]

Options:

A. continuous for every real x.

B. discontinuous only at x = 0

C. discontinuous only at non-ero integral values of x.

D. continuous only at x = 0.

Answer: A

Solution:

Solution:

Let $f(x) = [x]\cos\left(\frac{2x-1}{2}\right)$

We know that [x] is discontinuous at all integral points and cos x is continuous at x ∈ R

So, check at $x = n, n \in I$

$L.H.L = \lim_{x \to n^-}[x]\cos\left(\frac{2x-1}{2}\right)\pi$

$= (n-1)\cos\left(\frac{2n-1}{2}\right)\pi = 0$

(∵[x] is the greatest integer function)

$R.H.L = \lim_{x \to n^+}[x]\cos\left(\frac{2x-1}{2}\right)\pi$

$= n\cos\left(\frac{2n-1}{2}\right)\pi = 0$

Now, value of the function at x = n is

$f(n) = 0$

Since, L.H.L = R.H.L. = f (n)

therefore $f(x) = [x]\cos\left(\frac{2x-1}{2}\right)$ is continuous for every real x.

Question150

Let $f : [1, 3] \to R$ be a function satisfying $\frac{x}{[x]} \le f(x) \le \sqrt{6-x}$, for all $x \ne 2$ and $f(2) = 1$,where R is the set of all real numbers and [x] denotes the largest integer less than or equal to x.
Statement 1: $\lim_{x \to 2^-} f(x)$ exists.
Statement 2: f is continuous at x = 2.
[Online May 19, 2012]

Options:

A. Statement 1 is true, Statement 2 is true, Statement 2 is a correct explanation for Statement 1.

B. Statement 1 is false, Statement 2 is true.

C. Statement 1 is true, Statement 2 is true, Statement 2 is not a correct explanation for Statement 1.

D. Statement 1 is true, Statement 2 is false.

Answer: D

Solution:

Solution:

Consider $\frac{x}{[x]} \le f(x) \le \sqrt{6-x}$

$\Rightarrow \lim_{x \to 2^-}\frac{x}{[x]} = \frac{2}{1} = 2$

$\Rightarrow \lim_{x \to 2^-}\sqrt{6-x} = 2$

therefore $\lim_{x \to 2^-} f(x) = 2$ [By Sandwich theorem]

Now $\lim_{x \to 2^+}\frac{x}{[x]} = 1, \lim_{x \to 2^+}\sqrt{6-x} = 2$

Hence by Sandwich theorem $\lim_{x \to 2^+} f(x)$ does not exists.

Therefore f is not continuous at x = 2. Thus statement-1 is true but statement- 2 is not true

----

Question151

Statement 1: A function $f : R \to R$ is continuous at x_0 if and only if $\lim_{x \to x_0} f(x)$ exists and $\lim_{x \to x_0} f(x) = f(x_0)$

Statement 2: A function $f : R \to R$ is discontinuous at x_0 if and only if, $\lim_{x \to x_0} f(x)$ exists and $\lim_{x \to x_0} f(x) \neq f(x_0)$.

[Online May 12, 2012]

Options:

A. Statement 1 is true, Statement 2 is true, Statement 2 is not a correct explanation of Statement 1.

B. Statement 1 is false, Statement 2 is true.

C. Statement 1 is true, Statement 2 is true, Statement 2 is a correct explanation of Statement 1.

D. Statement 1 is true, Statement 2 is false.

Answer: D

Solution:

Solution:
Statement - 1 is true.
It is the definition of continuity.
Statement - 2 is false

Question152

Consider the function, $f(x) = |x - 2|' + |x - 5|, x \in R$.
Statement- 1: $f'(4) = 0$
Statement - 2: f is continuous in [2, 5], differentiable in (2,5) and $f(2) = f(5)$.
[2012]

Options:

A. Statement-1 is false, Statement-2 is true.

B. Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.

C. Statement-1 is true, statement-2 is true; statement-2 is not a correct explanation for Statement-1.

D. Statement-1 is true, statement-2 is false.

Answer: C

Solution:

Solution:

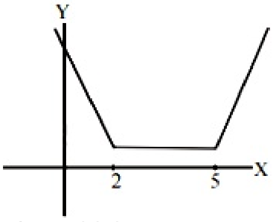
$f(x) = |x-2| = \begin{cases} x-2, & x-2 \geq 0 \\ 2-x, & x-2 \leq 0 \end{cases}$

$= \begin{cases} x-2, & x \geq 2 \\ 2-x, & x \leq 2 \end{cases}$

Similarly,

$f(x) = |x-5| = \begin{cases} x-5, & x \geq 5 \\ 5-x, & x \leq 5 \end{cases}$

$\therefore f(x) = |x-2| + |x-5|$
$= \{x-2+5-x = 3, 2 \leq x \leq 5\}$
Thus $f(x) = 3, 2 \leq x \leq 5$
$f'(x) = 0, 2 < x < 5$
$f'(4) = 0$
$\because$ Statement-1 is true



Since $f(x) = 3, 2 \leq x \leq 5$ is constant function.
So, it continuous in 2,5 and differentiable in (2,5)
$\because f(2) = 0 + |2-5| = 3$
and $f(5) = |5-2| + 0 = 3$
statement- 2 is also true.

Question153

If $f(x) = a|\sin x| + be^{|x|} + c|x|^3$, where a, b, c ∈ R, is differentiable at x = 0, then
[Online May 26, 2012]

Options:

A. a = 0, b and c are any real numbers

B. c = 0, a = 0, b is any real number

C. b = 0, c = 0, a is any real number

D. a = 0, b = 0, c is any real number

Answer: D

Solution:

Solution:
$|\sin x|$ and $e^{|x|}$ are not differentiable at $x = 0$ and $|x|^3$ is differentiable at $x = 0$
$\therefore$ for $f(x)$ to be differentiable at $x = 0$, we must have $a = 0$, $b = 0$ and c is any real number.

Question154

If $x + |y| = 2y$, then y as a function of x, at x = 0 is
[Online May 7, 2012]

Options:

A. differentiable but not continuous

B. continuous but not differentiable

C. continuous as well as differentiable

D. neither continuous nor differentiable

Answer: B

Solution:

Solution:
Given $x + |y| = 2y$
$\Rightarrow x + y = 2y$ or $x - y = 2y$
$\Rightarrow x = y$ or $x = 3y$
This represent a straight line which passes through origin.
Hence, $x + |y| = 2y$ is continuous at $x = 0$.
Now, we check differentiability at $x = 0$
$x + |y| = 2y \Rightarrow x + y = 2y, y \geq 0$
$x - y = 2y, y < 0$

Thus, $f(x) = \begin{Bmatrix} x, & y < 0 \\ x/3, & y \geq 0 \end{Bmatrix}$

Now, L.H.D. $= \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{-h}$

$= \lim_{h \to 0^-} \frac{x + h - x}{-h} = -1$

R.H.D $= \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0^+} \frac{\frac{x+h}{3} - \frac{x}{3}}{h} = \lim_{h \to 0^+} \frac{1}{3} = \frac{1}{3}$

Since, L.H.D $\neq$ R.H.D. at $x = 0$
$\therefore$ given function is not differentiable at $x = 0$

Question155

If $f'(x) = \sin(\log x)$ and $y = f\left(\frac{2x+3}{3-2x}\right)$, then $\frac{dy}{dx}$ equals [Online May 12, 2012]

Options:

A. $\sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right]$

B. $\frac{12}{(3-2x)^2}$

C. $\frac{12}{(3-2x)^2}\sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right]$

D. $\frac{12}{(3-2x)^2}\cos\left[\log\left(\frac{2x+3}{3-2x}\right)\right]$

Answer: C

Solution:

Solution:
Let $f'(x) = \sin[\log x]$ and $y = f\left(\frac{2x+3}{3-2x}\right)$

Now, $\frac{dy}{dx} = f'\left(\frac{2x+3}{3-2x}\right) \cdot \frac{d}{dx}\left(\frac{2x+3}{3-2x}\right)$

$= \sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right] \frac{[(6-4x)-(-4x-6)]}{(3-2x)^2}$

$= \frac{12}{(3-2x)^2} . \sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right]$

Question156

Consider a quadratic equation $ax^2 + bx + c = 0$, where$2a + 3b + 6c = 0$ and let $g(x) = a\frac{x^3}{3} + b\frac{x^2}{2} + cx$.
Statement 1: The quadratic equation has at least one root in the interval (0, 1).
Statement 2: The Rolle's theorem is applicable to function g(x) on the interval [0, 1].
[Online May 19, 2012]

Options:

A. Statement 1 is false, Statement 2 is true.

B. Statement 1 is true, Statement 2 is false.

C. Statement 1 is true, Statement 2 is true, Statement 2 is not a correct explanation for Statement 1.

D. Statement 1 is true, Statement 2 is true, , Statement 2 is a correct explanation for Statement 1.

Answer: D

Solution:

Solution:

Let $g(x) = \frac{ax^3}{3} + b . \frac{x^2}{2} + cx$

$g'(x) = ax^2 + bx + c$

Given: $ax^2 + bx + c = 0$ and $2a + 3b + 6c = 0$

Statement-2:

(i) $g(0) = 0$ and $g(1) = \frac{a}{3} + \frac{b}{2} + c = \frac{2a + 3b + 6c}{6}$

$= \frac{0}{6} = 0$

$\Rightarrow g(0) = g(1)$

(ii) g is continuous on [0,1] and differentiable on (0,1)

$\therefore$ By Rolle's theorem $\exists k \in (0, 1)$ such that $g'(k) = 0$

This holds the statement 2 . Also, from statement-2,we can say $ax^2 + bx + c = 0$ has at least one root in (0,1).

Thus statement- 1 and 2 both are true and statement- 2 is a correct explanation for statement-1.

Question157

Define f (x) as the product of two real functions

$f_1(x) = x, x \in R$, and $f_2(x) = \begin{cases} \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

as follows:

$$f(x) = \begin{cases} f_1(x) \cdot f_2(x), & \text{if } x = 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Statement -1: f (x) is continuous on R
Statement -2: $f_1(x)$ and $f_2(x)$ are continuous on R
[2011RS]

Options:

A. Statement -1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

B. Statement-1 is true, Statement-2 is true; Statement-2 is NOT a correct explanation for Statement-1

C. Statement-1 is true, Statement-2 is false

D. Statement-1 is false, Statement-2 is true

Answer: C

Solution:

Solution:

Given that $f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

At $x = 0$

LH L = $\lim_{h \to 0^-} \left\{ -h\sin\left(-\frac{1}{h}\right) \right\}$

= 0 × a finite quantity between -1 and 1 = 0

RH L = $\lim_{h \to 0^+} h\sin\frac{1}{h} = 0$

Also, $f(0) = 0$ Thus LH L = RH L = $f(0)$

∴ $f(x)$ is continuous on R.

but $f_2(x)$ is not continuous at $x = 0$

Question158

The values of p and q for which the function

$$f(x) = \begin{cases} \dfrac{\sin(p+1)x + \sin x}{x} & ,x < 0 \\ q & ,x = 0 \\ \dfrac{\sqrt{x + x^2} - \sqrt{x}}{x^{3/2}} & ,x > 0 \end{cases}$$

is continuous for all x in R,are
[2011]

Options:

A. $p = \frac{5}{2}, q = \frac{1}{2}$

B. $p = -\frac{3}{2}, q = \frac{1}{2}$

C. $p = \frac{1}{2}, q = \frac{3}{2}$

D. $p = \frac{1}{2}, q = -\frac{3}{2}$

Answer: B

Solution:

Solution:

$\underset{(at x = 0)}{L.H.L} = \lim_{x \to 0^-} f(x)$

$= \lim_{h \to 0^-} \frac{\sin\{(p+1)(-h)\} - \sin h}{-h} = p + 1 + 1 = p + 2$

$R.H.L = \lim_{x \to 0^+} f(x)$

$= \lim_{h \to 0} \frac{\sqrt{x + x^2} - \sqrt{x}}{x^{3/2}} \times \frac{\sqrt{x + x^2} + \sqrt{x}}{\sqrt{x + x^2} + \sqrt{x}} = \frac{1}{1+1} = \frac{1}{2}$

$f(0) = 2$

Given that $f(x)$ is continuous at $x = 0$

$\therefore p + 2 = q = \frac{1}{2}$

$\Rightarrow p = -\frac{3}{2}, q = \frac{1}{2}$

Question159

If function $f(x)$ is differentiable at $x = a$, then $\lim_{x \to a} \frac{x^2 f(a) - a^2 f(x)}{x - a}$ is : [2011RS]

Options:

A. $-a^2 f'(a)$

B. $af(a) - a^2 f'(a)$

C. $2af(a) - a^2 f'(a)$

D. $2af(a) + a^2 f'(a)$

Answer: C

Solution:

Solution:

$\lim_{x \to a} \frac{x^2 f(a) - a^2 f(x)}{x - a}$

Applying L-Hospital rule

$= \lim_{x \to a} \frac{2x f(a) - a^2 f'(x)}{1} = 2af(a) - a^2 f'(a)$

Question160

$\frac{d^2x}{dy^2}$ equals: [2011]

Options:

A. $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

B. $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$

C. $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

D. $\left(\frac{d^2y}{dx^2}\right)^{-1}$

Answer: C

Solution:

Solution:

$$\frac{d^2x}{dy^2}=\frac{d}{dy}\left(\frac{dx}{dy}\right)=\frac{d}{dx}\left(\frac{dx}{dy}\right)\frac{dx}{dy}=\frac{d}{dx}\left(\frac{1}{dy/dx}\right)\frac{dx}{dy}$$

$$=-\frac{1}{\left(\frac{dy}{dx}\right)^2}\cdot\frac{d^2y}{dx^2}\cdot\frac{1}{\frac{dy}{dx}}\left[\because\frac{d\left(\frac{1}{x}\right)}{dx}=-\frac{1}{x^2}\right]$$

$$=-\frac{1}{\left(\frac{dy}{dx}\right)^3}\frac{d^2y}{dx^2}$$

Question161

Let $f:(-1,1)\to R$ be a differentiable function with $f(0)=-1$ and $f'(0)=1$. Let $g(x)=[f(2f(x)+2)]^2$. Then $g'(0)=$ [2010]

Options:

A. –4

B. 0

C. –2

D. 4

Answer: A

Solution:

Solution:

Given that $g(x)=[f(2f(x))+2]^2$

$\therefore g'(x)=2(f(2f(x)+2))\left(\frac{d}{dx}(f(2f(x)+2))\right)$

$=2f(2f(x)+2)f'(2f(x))+2).(2f'(x))$

$\Rightarrow g'(0)=2f(2f(0)+2.f'(2f(0)+2).$

$2f'(0)=4f(0)(f'(0))^2=4(-1)(1)^2=-4$

Question162

Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then $y'(1)$ equals
[2009]

Options:

A. 1

B. log 2

C. –log 2

D. –1

Answer: D

Solution:

Solution:
$x^{2x} - 2x^x \cot y - 1 = 0$
$\Rightarrow 2\cot y = x^x - x^{-x}$
Let $u = x^x$
$\Rightarrow 2\cot y = u - \frac{1}{u}$
Differentiating both sides with respect to x, we get
$-2\operatorname{cosec}^2 y \frac{dy}{dx} = \left(1 + \frac{1}{u^2}\right)\frac{du}{dx}$
Now $u = x^x$ Taking log both sides
$\Rightarrow \log u = x \log x$
$\Rightarrow \frac{1}{u}\frac{du}{dx} = 1 + \log x$
$\Rightarrow \frac{du}{dx} = x^x(1 + \log x)$
∴ We get
$-2\operatorname{cosec}^2 y \frac{dy}{dx} = (1 + x^{-2x}) . x^x(1 + \log x)$
$\Rightarrow \frac{dy}{dx} = \frac{(x^x + x^{-x})(1 + \log x)}{-2(1 + \cot^2 y)}$ ......(i)
Put $n = 1$ in eqn. $x^{2x} - 2x^x \cot y - 1 = 0$, gives
$1 - 2\cot y - 1 = 0$
$\Rightarrow \cot y = 0$
∴ Putting $x = 1$ and $\cot y = 0$ in eqn. (i), we get
$y'(1) = \frac{(1+1)(1+0)}{-2(1+0)} = -1$

Question163

Let $f(x) = x\,|x|$ and $g(x) = \sin x$.
Statement-1 : gof is differentiable at $x = 0$ and its derivative is continuous at that point.
Statement-2 : gof is twice differentiable at $x = 0$.
[2009]

Options:

A. Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

B. Statement-1 is true, Statement-2 is false.

C. Statement-1 is false, Statement-2 is true.

D. Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

Answer: B

Solution:

Solution:
Given that $f(x) = xx|$ and $g(x) = \sin x$
So that
$gof(x) = g(f(x)) = g(x\,|\,x|) = \sin x\,|\,x|$

$$= \begin{cases} \sin(-x^2), & \text{if } x < 0 \\ \sin(x^2) & \text{if } x \geq 0 \end{cases} = \begin{cases} -\sin x^2, & \text{if } x < 0 \\ \sin(x^2) & \text{if } x \geq 0 \end{cases}$$

$$\therefore (gof)'(x) = \begin{cases} -2x\cos x^2, & \text{if } x < 0 \\ 2x\cos x^2, & \text{if } x \geq 0 \end{cases}$$

Here we observe $L(gof)'(0) = 0 = R(gof)'(0)$
$\Rightarrow$ go f is differentiable at $x = 0$
and $(gof)'$ is continuous at $x = 0$

$$\text{Now } (gof)''(x) = \begin{cases} -2\cos x^2 + 4x^2 \sin x^2, & x < 0 \\ 2\cos x^2 - 4x^2 \sin x^2, & x \geq 0 \end{cases}$$

Here
$L(gof)''(0) = -2$ and $R(gof)''(0) = 2$
$\because L(gof)''(0) \neq R(gof)''$
$\Rightarrow$ go f (x) is not twice differentiable at x = 0.
$\therefore$ Statement - 1 is true but statement -2 is false.

Question164

Let $f(x) = \begin{cases} (x-1)\sin\dfrac{1}{x-1} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$

Then which one of the following is true?
[2008]

Options:

A. f is neither differentiable at x = 0 nor at x =1

B. f is differentiable at x = 0 and at x =1

C. f is differentiable at x = 0 but not at x = 1

D. f is differentiable at x = 1 but not at x = 0

Answer: C

Solution:

Solution:
Given that,

$$f(x) = \begin{cases} (x-1)\sin\dfrac{1}{x-1}, & \text{if } x \neq 1 \\ 0, & \text{if } x = 1 \end{cases}$$

At $x = 1$

$$\text{R.H.D.} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{h \sin \frac{1}{h} - 0}{h} = \lim_{h \to 0} \sin \frac{1}{h} = \text{a finite number}$$

Let this finite number be l

$$\text{L.H.D.} = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{-h \sin\left(\frac{1}{-h}\right)}{-h} = \lim_{h \to 0} \sin\left(\frac{1}{-h}\right)$$

$$= -\lim_{h \to 0} \sin\left(\frac{1}{h}\right) = -(\text{ a finite number }) = -l$$

Thus $R.H.D \neq L.H.D$

∴ f is not differentiable at x = 1

At x = 0 $f'(0) = \sin \frac{1}{(x-1)} - \frac{x-1}{(x-1)^2} \cos\left(\frac{1}{x-1}\right) \Big]_{x=0}$

$= -\sin 1 + \cos 1$

∴ f is differentiable at $x = 0$

Question165

The function f : R / {0} → R given by $f(x) = \frac{1}{x} - \frac{2}{e^{2x} - 1}$ can be made continuous at x = 0 by defining f (0) as [2007]

Options:

A. 0

B. 1

C. 2

D. – 1

Answer: B

Solution:

Solution:

Given, $f(x) = \frac{1}{x} - \frac{2}{e^{2x} - 1}$ is continuous at $x = 0$

$$\Rightarrow f(0) = \lim_{x \to 0} \frac{1}{x} - \frac{2}{e^{2x} - 1}$$

$$= \lim_{x \to 0} \frac{(e^{2x} - 1) - 2x}{x(e^{2x} - 1)};\ \left[\frac{0}{0}\text{form}\right]$$

∴ Applying, L'Hospital rule

Differentiate two times, we get

$$f(0) = \lim_{x \to 0} \frac{4e^{2x}}{2(xe^{2x}2 + e^{2x}.1) + e^{2x}.2}$$

$$= \lim_{x \to 0} \frac{4e^{2x}}{4xe^{2x} + 2e^{2x} + 2e^{2x}} \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{x \to 0} \frac{4e^{2x}}{4(xe^{2x} + e^{2x})} = \frac{4.e^0}{4(0 + e^0)} = 1$$

Question166

Let f : R → R be a function defined by $f(x) = \min\{x + 1, |x| + 1\}$, Then

which of the following is true?
[2007]

Options:

A. f (x) is differentiable everywhere

B. f (x) is not differentiable at x = 0

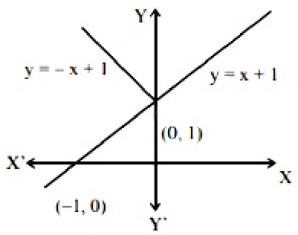
C. f (x) ≥ 1 for all x ∈ R

D. f (x) is not differentiable at x = 1

Answer: A

Solution:

Solution:
$f(x) = \min\{x + 1, |x| + 1\}$
$\Rightarrow f(x) = x + 1\ \forall x \in R$



Since f (x) = x + 1 is polynomial function
Hence, f (x) is differentiable everywhere for all x ∈ R.

Question167

A value of c for which conclusion of Mean Value Theorem holds for the function $f(x) = \log_e x$ on the interval [1,3] is
[2007]

Options:

A. $\log_3 e$

B. $\log_e 3$

C. $2\log_3 e$

D. $\frac{1}{2}\log_3 e$

Answer: C

Solution:

Solution:
Using Lagrange's Mean Value Theorem
Let f (x) be a function defined on [a, b]
then, $f'(c) = \frac{f(b) - f(a)}{b - a}$ ......(i)
c ∈ [a, b]
∴ Given $f(x) = \log_e x \therefore f'(x) = \frac{1}{x}$
∴ equation (i) become

$\frac{1}{c} = \frac{f(3) - f(1)}{3 - 1}$

$\Rightarrow \frac{1}{c} = \frac{\log_e 3 - \log_e 1}{2} = \frac{\log_e 3}{2}$

$\Rightarrow c = \frac{2}{\log_e 3} \Rightarrow c = 2\log_3 e$

Question168

The set of points where $f(x) = \frac{x}{1+|x|}$ is differentiable is [2006]

Options:

A. $(-\infty, 0) \cup (0, \infty)$

B. $(-\infty, -1) \cup (-1, \infty)$

C. $(-\infty, \infty)$

D. $(0, \infty)$

Answer: C

Solution:

Solution:

$$f(x) = \begin{cases} \frac{x}{1-x}, & x < 0 \\ \frac{x}{1+x}, & x \geq 0 \end{cases}$$

$f(x) = x1 - x$ is not define at $x \neq 1$ but here $x < 0$ and $f(x) = \frac{x}{1+x}$ is not define at $x = -1$ but here $x > 0$. So, $f(x)$ is continuous for $x \in R$.

and $$f(x) = \begin{cases} \frac{x}{(1-x)^2}, & x < 0 \\ \frac{x}{(1+x)^2}, & x \geq 0 \end{cases}$$

∴ f'(x) exist at everywhere.

Question169

If $x^m \cdot y^n = (x+y)^{m+n}$, then $\frac{dy}{dx}$ is [2006]

Options:

A. $\frac{y}{x}$

B. $\frac{x+y}{xy}$

C. xy

D. $\frac{x}{y}$

Answer: A

Solution:

Solution:
$x^m . y^n = (x + y)^{m+n}$
taking log both sides
$\Rightarrow m \ln x + n \ln y = (m + n) \ln(x + y)$
Differentiating both sides, we get
$\therefore \frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\left(1 + \frac{dy}{dx}\right)$
$\Rightarrow \left(\frac{m}{x} - \frac{m+n}{x+y}\right) = \left(\frac{m+n}{x+y} - \frac{n}{y}\right)\frac{dy}{dx}$
$\Rightarrow \frac{my - nx}{x(x+y)} = \left(\frac{my - nx}{y(x+y)}\right)\frac{dy}{dx}$
$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$

Question170

If f is a real valued differentiable function satisfying $|f(x) - f(y)| \le (x - y)^2$, x, y ∈ R and f (0) = 0, then f (1)equals [2005]

Options:

A. – 1

B. 0

C. 2

D. 1

Answer: B

Solution:

Solution:
Given that $|f(x) - f(y)| \le (x - y)^2, x, y \in R$....... (i) and $f(0) = 0$
$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
$f'(x)\Big| = \lim_{h \to 0}\left|\frac{f(x+h) - f(x)}{h}\right| \le \lim_{h \to 0}\left|\frac{(h)^2}{h}\right|$
$\Rightarrow f'(x) \mid \le 0 \Rightarrow f''(x) = 0$
$\Rightarrow f(x) =$ constant
As $f(0) = 0$
$\Rightarrow f(1) = 0$

Question171

Suppose f (x) is differentiable at x = 1 and $\lim_{h \to 0} \frac{1}{h} f(1 + h) = 5$, then f ′(1) equals [2005]

Options:

A. 3

B. 4

C. 5

D. 6

Answer: C

Solution:

Solution:
(c) $f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$;
Given that function is differentiable so it is continuous also
and $\lim_{h \to 0} \frac{f(1+h)}{h} = 5$ and hence $f(1) = 0$
Hence, $f'(1) = \lim_{h \to 0} \frac{f(1+h)}{h} = 5$

Question172

Let f be differentiable for all x. If f (1) = – 2 and f '(x) ≥ 2 for x ∈ [1, 6], then [2005]

Options:

A. $f(6) \geq 8$

B. $f(6) < 8$

C. $f(6) < 5$

D. $f(6) = 5$

Answer: A

Solution:

Solution:
As $f(1) = -2$ & $f'(x) \geq 2\ \forall x \in [1, 6]$
Applying Lagrange's mean value theorem
$\frac{f(6) - f(1)}{5} = f'(c) \geq 2$
$\Rightarrow f(6) \geq 10 + f(1)$
$\Rightarrow f(6) \geq 10 - 2 \Rightarrow f(6) \geq 8.$

Question173

If the equation $a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x = 0 a_1 \neq 0, n \geq 2$, has a positive root $x = \alpha$, then the equation $n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + ... + a_1 = 0$ has a positive root, which is [2005]

Options:

A. greater than α

B. smaller than α

C. greater than or equal to α

D. equal to α

Answer: B

Solution:

Solution:
Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_1 x = 0$
The other given equation,
$n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \ldots\ldots\ldots + a_1 = 0 = f'(x)$
Given $a_1 \neq 0 \Rightarrow f(0) = 0$
Again $f(x)$ has root α, $\Rightarrow f(\alpha) = 0$
$\therefore f(0) = f(\alpha)$
$\therefore$ By Rolle's theorem $f'(x) = 0$ has root between $(0, \alpha)$
Hence $f'(x)$ has a positive root smaller than α.

Question174

Let $f(x) = \frac{1 - \tan x}{4x - \pi}$, $x \neq \frac{\pi}{4}$, $x \in \left[0, \frac{\pi}{2}\right]$. If $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$, then $f\left(\frac{\pi}{4}\right)$ is [2004]

Options:

A. –1

B. $\frac{1}{2}$

C. $-\frac{1}{2}$

D. 1

Answer: C

Solution:

Solution:
Given that $f(x) = \frac{1 - \tan x}{4x - \pi}$ is continuous in $\left[0, \frac{\pi}{2}\right]$

$\therefore f\left(\frac{\pi}{4}\right) = \lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{x \to \frac{\pi}{4}^+} f(x)$

$\lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{h \to 0} f\left(\frac{\pi}{4} + h\right)$

$= \lim_{h \to 0} \frac{1 - \tan\left(\frac{\pi}{4} + h\right)}{4\left(\frac{\pi}{4} + h\right) - \pi}, h > 0 = \lim_{h \to 0} \frac{1 - \frac{1 + \tan h}{1 - \tan h}}{4h}$

$= \lim_{h \to 0} \frac{-2}{1 - \tan h} \cdot \frac{\tan h}{4h} = \frac{-2}{4} = -\frac{1}{2} \left[\because \lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1\right]$

Question175

If $x = e^{y + e^{y + \ldots . t0\infty}}$, x > 0, then $\frac{dy}{dx}$ is [2004]

Options:

A. $\frac{1+x}{x}$

B. $\frac{1}{x}$

C. $\frac{1-x}{x}$

D. $\frac{x}{1+x}$

Answer: C

Solution:

Solution:
Given that $x = e^{y + e^{y + \ldots \ldots . \infty}} \Rightarrow x = e^{y+x}$.
Taking log both sides.
$\log x = y + x$ differentiating both side $\Rightarrow \frac{1}{x} = \frac{dy}{dx} + 1$
$\therefore \frac{dy}{dx} = \frac{1}{x} - 1 = \frac{1-x}{x}$

Question176

If 2a + 3b + 6c = 0, then at least one root of the equation $ax^2 + bx + c = 0$ lies in the interval [2004]

Options:

A. (1, 3)

B. (1, 2)

C. (2, 3)

D. (0, 1)

Answer: D

Solution:

Solution:
Let us define a function
$f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx$
Being polynomial, it is continuous and differentiable, also,
$f(0) = 0$ and $f(1) = \frac{a}{3} + \frac{b}{2} + c$
$\Rightarrow f(1) = \frac{2a + 3b + 6c}{6} = 0$(given)
$\therefore f(0) = f(1)$
$\therefore f(x)$ satisfies all conditions of Rolle's theorem therefore $f'(x) = 0$ has a root in (0,1)

i.e. $ax^2 + bx + c = 0$ has at lease one root in (0,1)

Question177

If $f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases}$

then $f(x)$ is
[2003]

Options:

A. discontinuous every where

B. continuous as well as differentiable for all x

C. continuous for all x but not differentiable at x = 0

D. neither differentiable nor continuous at x = 0

Answer: C

Solution:

Solution:

Given that $f(0) = 0$; $f(x) = xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}$
$R.H.L = \lim_{h \to 0}(0+h)e^{-2/h} = \lim_{h \to 0}\frac{h}{e^{2/h}} = 0$

$L.H.L = \lim_{h \to 0}(0-h)e^{-\left(\frac{1}{h}-\frac{1}{h}\right)} = 0$
therefore, $f(x)$ is continuous at $x = 0$.

Now, $R.H.D = \lim_{h \to 0}\frac{(0+h)e^{-\left(\frac{1}{h}+\frac{1}{h}\right)} - 0}{h} = 0$

L.H.D. $= \lim_{h \to 0}\frac{(0-h)e^{-\left(\frac{1}{h}-\frac{1}{h}\right)} - 0}{-h} = 1$
therefore, L.H.D. ≠ R.H.D.
$f(x)$ is not differentiable at $x = 0$

Question178

Let f(x) be a polynomial function of second degree.If f(1) = f(-1) and a,b,c are in A. P , then $f'(a), f'(b), f'(c)$ are in
[2003]

Options:

A. Arithmetic -Geometric Progression

B. A.P

C. G..P

D. H.P

Answer: B

Solution:

Solution:
$f(x) = ax^2 + bx + c$
$f(1) = f(-1)$
$\Rightarrow a + b + c = a - b + c$ or $b = 0$
$\therefore f(x) = ax^2 + c$ or $f'(x) = 2ax$
Now $f'(a)$; $f'(b)$ and $f'(c)$
are $2a(a)$; $2a(b)$; $2a(c)$
i.e. $2a^2$, $2ab$, $2ac$
$\Rightarrow$ If a, b, c are in A.P. then $f'(a)$; $f'(b)$ and $f'(c)$ are also in A.P.

Question179

If $f(x) = x^n$, then the value off $(1) - f'(1)1! + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \ldots\ldots \frac{(-1)^n f^n(1)}{n!}$ is [2003]

Options:

A. 1

B. 2^n

C. $2^n - 1$

D. 0.

Answer: D

Solution:

Solution:
Given that $f(x) = x^n \Rightarrow f(1) = 1$
$f'(x) = nx^{n-1} \Rightarrow f'(1) = n$
$f''(x) = n(n-1)x^{n-2} \Rightarrow f''(1) = n(n-1)$
....
$f^n(x) = n! \Rightarrow f^n(1) = n!$
$f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - f'''(1)3! + \ldots\ldots\ldots + \frac{(-1)^n f^n(1)}{n!}$
$= 1 - \frac{n}{1!} + \frac{n(n-1)}{2!} - \frac{n(n-1)(n-2)}{3!} + \ldots\ldots + (-1)^n \frac{n!}{n!}$
$= {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots\ldots\ldots + (-1)^n\, {}^nC_n$
$= (1-1)^n = 0$

Question180

Let $f(a) = g(a) = k$ and their nth derivativesf $^n(a)$, $g^n(a)$ exist and are not equal for some n. Further if $\lim_{x \to a} \frac{f(a)g(x) - f(a) - g(a)f(x) + f(a)}{g(x) - f(x)} = 4$ then the value of k is [2003]

Options:

A. 0

B. 4

C. 2

D. 1

Answer: B

Solution:

Solution:
$\lim_{x \to a} \frac{f(a)g'(x) - g(a)f'(x)}{g'(x) - f'(x)} = 4$ (By Applying L' Hospital rule)
$\lim_{x \to a} \frac{kg'(x) - kf'(x)}{g'(x) - f'(x)} = 4$
$\therefore k = 4$

Question181

f is defined in [-5, 5] as
f (x) = x if x is rational
= – x if x is irrational. Then
[2002]

Options:

A. f (x) is continuous at every x, except x = 0

B. f (x) is discontinuous at every x, except x = 0

C. f (x) is continuous everywhere

D. f (x) is discontinuous everywhere

Answer: B

Solution:

Solution:
Let a is a rational number other than 0 , in [-5,5] ,then $f(a) = a$ and $\lim_{x \to a} f(x) = -a$
$\therefore x \to a^-$ and $x \to a^+$ is tends to irrational number
$\therefore f(x)$ is discontinuous at any rational number
If a is irrational number, then
$f(a) = -a$ and $\lim_{x \to a} f(x) = a$
$\therefore f(x)$ is not continuous at any irrational number. For $x = 0$, $\lim_{x \to a} f(x) = f(0) = 0$
$\therefore f(x)$ is continuous at $x = 0$

Question182

If $f(x + y) = f(x) . f(y)\ \forall x . y$ and $f(5) = 2$, $f'(0) = 3$, then $f'(5)$ is [2002]

Options:

A. 0

B. 1

C. 6

D. 2

Answer: C

Solution:

Solution:
Given that $f(x+y) = f(x) \times f(y)$
Differentiate with respect to x, treating y as constant
$f'(x+y) = f'(x) f(y)$
Putting $x = 0$ and $y = x$, we get $f'(x) = f'(0) f(x)$;
$\Rightarrow f'(5) = 3 f(5) = 3 \times 2 = 6$

Question183

If $y = \left(x + \sqrt{1 + x^2}\right)^n$, then $(1 + x^2)\frac{d^2 y}{d x^2} + x\frac{d y}{d x}$ is [2002]

Options:

A. $n^2 y$

B. $-n^2 y$

C. $-y$

D. $2x^2 y$

Answer: A

Solution:

Solution:
Given that $y = \left(x + \sqrt{1 + x^2}\right)^n$ ....(i)
Differentiating both sides w.r. to x

$$\frac{dy}{dx} = n\left(x + \sqrt{1 + x^2}\right)^{n-1}\left(1 + \frac{1}{2}(1 + x^2)^{-1/2} . 2x\right)$$

$$\frac{dy}{dx} = n\left(x + \sqrt{1 + x^2}\right)^{n-1}\frac{\left(\sqrt{1 + x^2} + x\right)}{\sqrt{1 + x^2}}$$

$$= \frac{n\left(\sqrt{1 + x^2} + x\right)^n}{\sqrt{1 + x^2}}$$

or $\sqrt{1 + x^2}\frac{dy}{dx} = ny$ [from (i)]

$\Rightarrow \sqrt{1 + x^2} y_1 = ny \left(\because y_1 = \frac{dy}{dx}\right)$ Squaring both sides,
we get $(1 + x^2) {y_1}^2 = n^2 y^2$
Differentiating it w.r. to x,
$(1 + x^2) 2 y_1 y_2 + {y_1}^2 . 2x = n^2 . 2 y y_1$
$\Rightarrow (1 + x^2) y_2 + x y_1 = n^2 y$

Question184

If $2a + 3b + 6c = 0$, $(a, b, c \in R)$ then the quadratic equation $ax^2 + bx + c = 0$ has

[2002]

Options:

A. at least one root in [0, 1]

B. at least one root in [2, 3]

C. at least one root in [4, 5]

D. None of these

Answer: A

Solution:

Let $f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx$

$\Rightarrow f(0) = 0$ and $f(1) = \frac{a}{3} + \frac{b}{2} + c = \frac{2a + 3b + 6c}{6} = 0$

Also $f(x)$ is continuous and differentiable in [0,1] and $[0,1]$ So by Rolle's theorem, $f'(x) = 0$.

i.e $ax^2 + bx + c = 0$ has at least one root in [0,1]
