CHAPTER 09

Gravitation

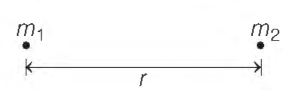
Gravitational Force between Two Point Masses

- It can be given as

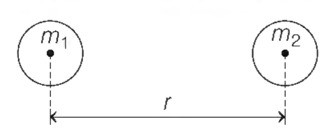
$$F = G\,\frac{m_1 m_2}{r^2}$$

- Direct formula $F = \dfrac{Gm_1 m_2}{r^2}$ can be applied under following three conditions:

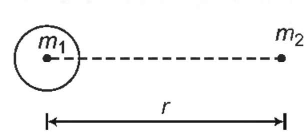
  (i) To find force between two point masses.



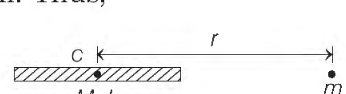
  (ii) To find force between two spherical bodies.



  (iii) To find force between a spherical body and a point mass.



- To find force between a point mass and a rod, integration is required. In this case, we cannot assume whole mass of the rod at its centre to find force between them. Thus,



$$F \neq \frac{GMm}{r^2}$$

Acceleration Due to Gravity

- On the surface of earth, $g = \dfrac{GM}{R^2} = 9.81\ \text{m/s}^2$
- At height h from the surface of earth,

$$g' = \frac{g}{\left(1 + \dfrac{h}{R}\right)^2} \approx g\left(1 - \frac{2h}{R}\right), \text{ if } h << R$$

- At depth d from the surface of earth,

$$g' = g\left(1 - \frac{d}{R}\right)$$

$g' = 0$, if $d = R$, i.e. at centre of earth

- Effect of rotation of earth at latitude ϕ,

$$g' = g - R\omega^2 \cos^2 \phi$$

At equator $\phi = 0$, $g' = g - R\omega^2$ = minimum value
At poles, $\phi = 90°$, $g' = g$ = maximum value.
At equator, effect of rotation of earth is maximum and value of g is minimum.
At pole, effect of rotation of earth is zero and value of g is maximum.

Field Strength

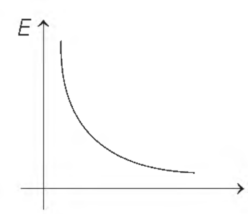
- Gravitational field strength at a point in gravitational field is defined as

$$\mathbf{E} = \frac{\mathbf{F}}{m} = \text{gravitational force per unit mass}$$

- **Due to a point mass**

$$E = \frac{GM}{r^2} \qquad \text{(Towards the mass)}$$

or $$E \propto \frac{1}{r^2}$$



- **Due to a solid sphere**

Inside points,

$$E_i = \frac{GM}{R^3} r$$

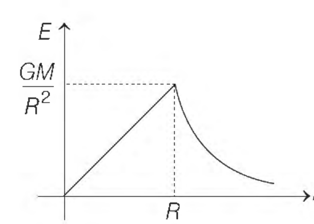
At $r = 0$, i.e. at centre $E = 0$

At $r = R$, i.e. on surface $E = \dfrac{GM}{R^2}$

Outside points, $E_o = \dfrac{GM}{r^2}$ or $E_o \propto \dfrac{1}{r^2}$

At $r = R$, i.e. on surface $E = \dfrac{GM}{R^2}$

As $r \to \infty, E \to 0$



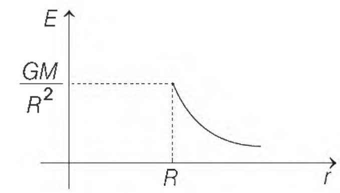
- **Due to a spherical shell**

  Inside points, $E_i = 0$

  Outside points, $E_o = \dfrac{GM}{r^2}$

  Just outside the surface, $E = \dfrac{GM}{R^2}$

  On the surface, E - r graph is discontinuous.



- **On the axis of a ring**

  $$E_x = \frac{GMx}{(R^2 + x^2)^{3/2}}$$

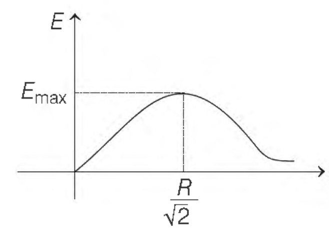
  At $\quad x = 0, E = 0$ i.e. at centre $E = 0$

  If $\quad x >> R, E \approx \dfrac{GM}{x^2}$

  i.e. ring behaves as a point mass.

  As $x \to \infty, E \to 0$

  $$E_{\max} = \frac{2\,GM}{3\sqrt{3}R^2} \quad \text{at } x = \frac{R}{\sqrt{2}}$$



Gravitational Potential

- Gravitational potential at a point in a gravitational field is defined as the negative of work done by gravitational force in moving a unit mass from infinity to that point. Thus,

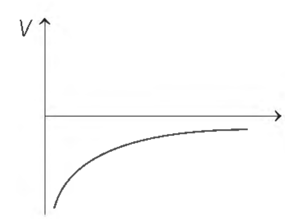
  $$V_P = -\frac{W_{\infty \to P}}{m}$$

- **Due to a point mass**

  $$V = -\frac{Gm}{r}$$

  $$V \to -\infty \text{ as } r \to 0$$

  and $\quad V \to 0$ as $r \to \infty$



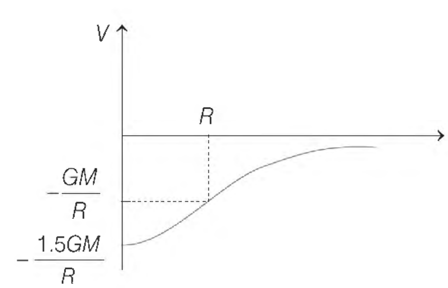
- **Due to a solid sphere**

  Inside points $V_i = -\dfrac{GM}{R^3}(1.5R^2 - 0.5r^2)$

  At $r = R$, i.e. on surface $V = -\dfrac{GM}{R}$

  At $r = 0$, i.e. at centre $V = -1.5\dfrac{GM}{R}$

  V-r graph is parabolic for inside points and potential at centre is 1.5 times the potential at surface.

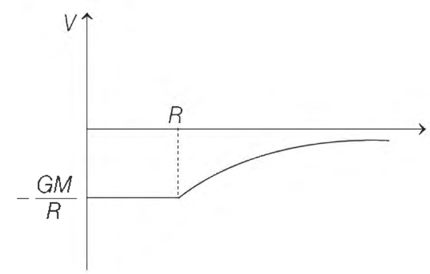


Outside points $V_o = -\frac{GM}{r}$

At $r = R$, i.e. on surface $V = -\frac{GM}{R}$

As $r \to \infty, V \to 0$

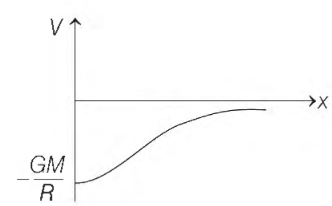
- **Due to a spherical shell**



Inside points, $V_i = -\frac{GM}{R}$ = constant

Outside points, $V_o = -\frac{GM}{r}$

- **On the axis of a ring** $V_x = -\frac{GM}{\sqrt{R^2 + x^2}}$



At $x = 0$, i.e. at centre, $V = -\frac{GM}{R}$

This is the minimum value.

As $x \to \infty, V \to 0$

Gravitational Potential Energy

- This is negative of work done by gravitational forces in making the system from infinite separation to the present position.
- Gravitational potential energy of two point masses is

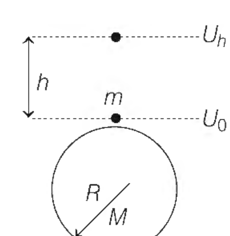
$$U = -\frac{Gm_1 m_2}{r}$$

- To find gravitational potential energy of more than two point masses, we have to make pairs of masses but neither of the pair should be repeated. For example, in case of four point masses,

$$U = -G\left[\frac{m_4 m_3}{r_{43}} + \frac{m_4 m_2}{r_{42}} + \frac{m_4 m_1}{r_{41}} + \frac{m_3 m_2}{r_{32}} + \frac{m_3 m_1}{r_{31}} + \frac{m_2 m_1}{r_{21}}\right]$$

For n point masses, total number of pairs will be $\frac{n(n-1)}{2}$.

- If a point mass m is placed on the surface of earth, the potential energy here can be given as



$$U_0 = -\frac{GMm}{R}$$

and potential energy at height h is

$$U_h = -\frac{GMm}{(R+h)}$$

The difference in potential energy will be

$$\Delta U = U_h - U_0$$

or $$\Delta U = \frac{mgh}{1+\frac{h}{R}}$$

If $h << R, \Delta U \approx mgh$

- **Maximum height attained by a particle** Suppose a particle of mass m is projected vertically upwards with a speed v and we want to find the maximum height h attained by the particle. Then, we can use conservation of mechanical energy, i.e.

  Decrease in kinetic energy = increase in gravitational potential energy of particle

  $\therefore$ $$\frac{1}{2}mv^2 = \Delta U$$

  or $$\frac{1}{2}mv^2 = \frac{mgh}{1+\frac{h}{R}}$$

  Solving this, we get $$h = \frac{v^2}{2g - \frac{v^2}{R}}$$

  From this, we can see that $h \approx \frac{v^2}{2g}$, if v is small.

Relation between Field Strength E and Potential V

***Case* 1** Conversion of V into $\mathbf{E}$

(i) If V is a function of only one variable (say r), then

$$E = -\frac{dV}{dr} = -\text{ slope of } V\text{-}r \text{ graph}$$

$$= -dV/dx = -\text{ slope of } V\text{-}x \text{ graph}$$

(ii) If V is a function of more than one variables, then

$$\mathbf{E} = -\left[\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right]$$

***Case* 2** Conversion of $\mathbf{E}$ into V

(i) $dV = -\mathbf{E}\cdot d\mathbf{r}$ (More than one variables)

(ii) $dV = -Edr$ or $-Edx$ (One variable)

Escape Velocity

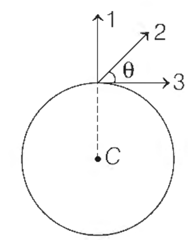
- From the surface of earth,

$$v_e = \sqrt{2gR} = \sqrt{\frac{2GM}{R}} \qquad \left(\text{as } g = \frac{GM}{R^2}\right)$$

$$\approx 11.2 \text{ km/s}$$

- The value of escape velocity is 11.2 km/s from the surface of earth. From some height above the surface of earth, this value will be less than 11.2 km/s.

- Escape velocity is independent of the direction in which it is projected. In the figure shown, body is given 11.2 km/s along three different paths. In each case, it will escape to infinity, but following different paths.



  For example, along path-1 it will follow a straight line.
- If velocity of a particle is v_e, then its total mechanical energy is zero. As the particle moves towards infinity, its kinetic energy decreases and potential energy increases, but total mechanical energy remains constant. At any point
$$E = K + U = 0 \quad \Rightarrow \quad K = -U$$
  For example, if $K = 100$ J on the surface of earth, then $U = -100$J. At some height suppose K becomes 60 J, then U will become -60J. At infinity $K = 0$, so U is also zero. Hence, speed at infinity will be zero.
- If velocity of the particle is less than v_e, then total mechanical energy is negative and it does not escape to infinity.
- If velocity of the particle is more than v_e, then total mechanical energy is positive. In this case, at infinity some kinetic energy and speed are left in the particle. Although its potential energy becomes zero.

Motion of Satellites

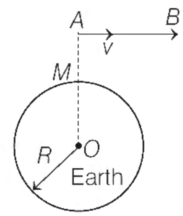
- Orbital speed $v_o = \sqrt{\dfrac{GM}{r}}$
- Time period $T = \dfrac{2\pi}{\sqrt{GM}} r^{3/2}$
- Kinetic energy $K = \dfrac{GMm}{2r}$
- Potential energy $U = -\dfrac{GMm}{r}$
- Total mechanical energy $E = -\dfrac{GMm}{2r}$
- Near the surface of earth, $r \approx R$ and $v_o = \sqrt{\dfrac{GM}{R}} = \sqrt{gR} = \dfrac{v_e}{\sqrt{2}} = 7.9 \text{ kms}^{-1}$.

  This is the maximum speed of any earth's satellite.
- Time period of such a satellite would be
$$T = \frac{2\pi}{\sqrt{GM}} R^{3/2} = 2\pi\sqrt{\frac{R}{g}}$$
$$= 84.6 \text{ min}$$
  This is the minimum time period of any earth's satellite.

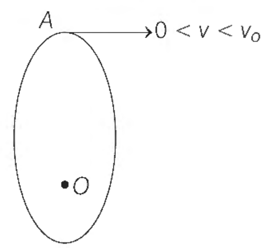
Trajectory of a Body Projected from Point *A* in the Direction *AB* with Different Initial Velocities

Let a body be projected from point A with velocity v in the direction AB. For different values of v, the paths are different. Here, the possible cases are:

(i) If $v = 0$, path is a straight line from A to M.

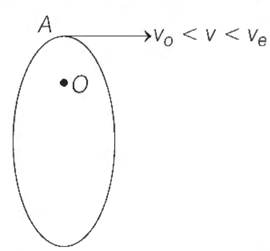


(ii) If $0 < v < v_o$, path is an ellipse with centre O of the earth as a focus.



(iii) If $v = v_o$, path is a circle with O as the centre.

(iv) If $v_o < v < v_e$, path is again an ellipse with O as a focus.



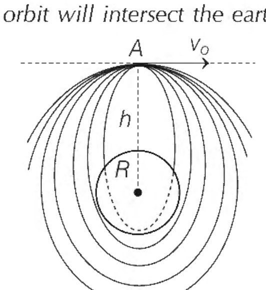
(v) If $v = v_e$, body escapes from the gravitational pull of the earth and path is a parabola.

(vi) If $v > v_e$, body again escapes but now the path is a hyperbola.

Here, v_o = orbital speed $\left(\sqrt{\frac{GM}{r}}\right)$ at A for a circular orbit and v_e = escape velocity from A.

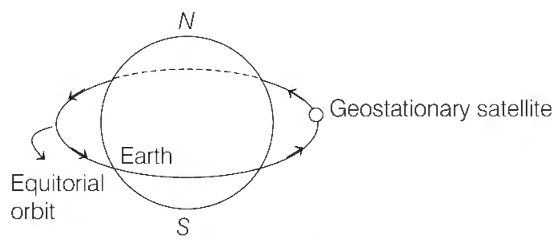
Note *(i) From case (i) to (iv), total mechanical energy is negative. Hence, these are the closed orbits. For case (v), total energy is zero and for case (vi) ,total energy is positive. In these two cases orbits are open.*

(ii) If v is not very large, the elliptical orbit will intersect the earth and the body will fall back to earth.



Geostationary or Parking Satellites

A satellite which appears to be at a fixed position at a definite height to an observer on earth is called geostationary or parking satellite. They rotate from west to east.



Height from earth's surface = 36000 km

Time period = 24 h

Orbital velocity = 3.1 km/s

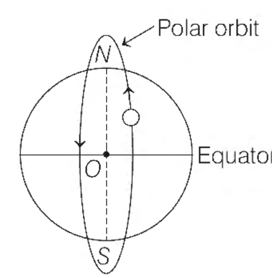
$$\text{Angular velocity} = \frac{2\pi}{24} = \frac{\pi}{12} \text{ rad/h}$$

These satellites are used in communication purpose.

INSAT 2B and INSAT 2C are geostationary satellites of India.

Polar Satellites

These are those satellites which revolve in polar orbits around earth.



Height from earth's surface ≈ 880 km

Time period ≈ 90 min

Orbital velocity ≈ 8 km/s

$$\text{Angular velocity} \approx \frac{2\pi}{90} = \frac{\pi}{45} \text{ rad/min}$$

These satellites revolve around the earth in polar orbits.
These satellites are used in forecasting weather, studying the upper region of the atmosphere, in mapping, etc.
PSLV series satellites are polar satellites of India.

Kepler's Laws

- Kepler's three empirical laws describe the motion of planets.

  First law Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.

  Second law The radius vector drawn from the sun to a planet, sweeps out equal areas in equal time interval, i.e. areal velocity is constant. This law is derived from law of conservation of angular momentum.

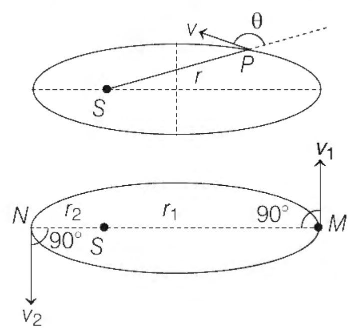
$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant.}$$

Here, L is angular momentum and m is mass of planet.

Third law $T^2 \propto r^3$, where r is semi-major axis of elliptical path.

Note *Circle is a special case of an ellipse. Therefore, second and third laws can also be applied for circular path. In third law,* r *is radius of circular path.*

- Most of the problems of planetary motion are solved by two conservation laws:
  (i) conservation of angular momentum about centre of the sun and
  (ii) conservation of mechanical (potential + kinetic) energy



Hence, the following two equations are used in most of the cases,

$$mvr \sin\theta = \text{constant} \quad \ldots\text{(i)}$$

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = \text{constant} \quad \ldots\text{(ii)}$$

At aphelion (or M) and perihelion (or N) positions, $\theta = 90°$.
Hence, Eq. (i) can be written as

$$mvr \sin 90° = \text{constant}$$

or $$mvr = \text{constant} \quad \ldots\text{(iii)}$$

Further, since mass of the planet (m) also remains constant, so, Eq. (i) can also be written as

$$vr \sin\theta = \text{constant} \quad \ldots\text{(iv)}$$

or $$v_1 r_1 = v_2 r_2 \quad (\because \theta = 90°)$$

$$r_1 > r_2$$

$\Rightarrow$ $$v_1 < v_2$$

- If the law of force obeys the inverse square law, then

$$\left(F \propto \frac{1}{r^2},\ F = \frac{-dU}{dr}\right) \Rightarrow K = \frac{|U|}{2} = |E|$$

The same is true for electron-nucleus system because there also, the electrostatic force $F_e \propto \frac{1}{r^2}$.