Linear Equation (IMP)

- A diff. eq^n of the form $\frac{dy}{dx} + Py = Q$ where P & Q are function of x is said to be linear eq^n

i.e. [Degree = 1]

$$\boxed{I.F. = e^{\int P\,dx}}$$

General sol^n: $$\boxed{y e^{\int P\,dx} = \int (Q e^{\int P\,dx})\,dx + c}$$

Similarly, $\frac{dx}{dy} + Px = Q$, P & Q are functions of y

$$I.F. = e^{\int P\,dy}$$

$$G.S. = x e^{\int P\,dy} = \int (Q e^{\int P\,dy})\,dy + c$$

Q:- $\frac{dy}{dx} + \frac{y}{x} = \frac{\log x}{x}$

$$I.F. = e^{\int \frac{1}{x}\,dx} = e^{\log x} = x$$

$$G.S. = yx = \int \frac{\log x}{\cancel{x}} \cdot \cancel{x} + c$$

$$yx = x\log x - x + c$$

$$yx = x\log x + c - x$$

Q:- $\frac{dy}{dx} + 2xy = e^{-x^2}$; $y(0) = 1$

$$I.F. = e^{\int 2x\,dx} = e^{x^2}$$

$$G.S. = y \cdot e^{x^2} = \int (e^{-x^2} \cdot e^{x^2})\,dx + c$$

$$ye^{x^2} = x + c$$

$$1 \cdot e^0 = 0 + c$$

$$\boxed{ye^{x^2} = x + 1}$$

Q:- $(x + 2y^3)\frac{dy}{dx} = y$.

y^3 is not linear, so we find $\frac{dx}{dy}$ ✓

$$dy\,x + 2y^3 dy = y\,dx$$

$$\frac{dx}{dy} = \frac{x + 2y^3}{y} = \frac{x}{y} + 2y^2$$

$$\frac{dx}{dy} - \frac{x}{y} = 2y^2$$

$$\text{I.F.} = e^{\int -\frac{1}{y} dy}$$

$$\text{I.F.} = e^{-\log y} = +\frac{1}{y}$$

$$\text{G.S.} = x \cdot \left(+\frac{1}{y}\right) = \int 2y^2\left(+\frac{1}{y}\right) dy + c$$

$$= +\frac{x}{y} = \int 2y + c$$

$$= \frac{x}{y} = y^2 + c$$

$$= \cancel{2xy + y^4 = 2c}$$

$$= \cancel{y^4 - 2xy = c'}$$

$$\boxed{y^2 - \frac{x}{y} + c = 0}$$

Higher order differential eq^n.

D.E. with constant co-efficient

- A D.E. of the form $a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = Q$

where $a_0, a_1, a_2, \dots a_n$ are constant and Q be the function of x is said to be D.E. with co-efficient constant.

Let $\frac{d}{dx} = D$

$$(a_0 D^n + a_1 D^{n-1} + \dots + a_n) y = Q$$

$$f(D) y = Q$$

- This eq^n may contain two type of solution
(1) is called complementary function (c.f.) and
(2) is called particular integral (P.I.).

NOTE:-

If $Q = 0$ then C.F. is called G.S.

If $Q \neq 0$ then P.I. must be existed then

G.S. = C.F. + P.I.

*Methods to find C.F.

① Replace D by m and $f(m) = 0$

② Find roots

Case: 1 Roots are real and distinct (unequal)

$$m = \alpha, \beta, \gamma$$

$$y = c_1 e^{\alpha x} + c_2 e^{\beta x} + c_3 e^{\gamma x} + \dots$$

Q:- $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

$D^2 - 5D + 6 = 0$

$m^2 - 5m + 6 = 0$

$m = \frac{+5 \pm \sqrt{25-24}}{2} = \frac{6}{2}, \frac{4}{2} = 3, 2$

$= C_1 e^{2x} + c_2 e^{3x}$

Case: 2 Roots are real and equal

Eg:- α, α, β -: roots

$y = (c_1 + c_2 x) e^{\alpha x} + c_3 e^{\beta x}$

Eg:- roots:- $\alpha, \alpha, \beta, \beta$

$y = (c_1 + c_2 x) e^{\alpha x} + (c_3 + c_4 x) e^{\beta x}$

Q:- $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

$D^2 - 6D + 9 = 0$

$m^2 - 6m + 9 = 0$

$m = \frac{6 \pm \sqrt{36-36}}{2} = 3, 3$

$y = (c_1 + c_2 x) e^{3x}$

Case: 3 Roots are imaginary

roots: $m = \alpha \pm i\beta$

$$y = (c_1 \cos\beta x + c_2 \sin\beta x)e^{\alpha x}$$

Q:- $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

$D^2 - 2D + 2 = 0$

$m^2 - 2m + 2 = 0$

$m = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

$\alpha = 1, \beta = 1$

$$y = (c_1 \cos x + c_2 \sin x)e^{x}$$

Newton's Law of Cooling

$\theta \rightarrow$ temp.

$t \rightarrow$ time

$\frac{d\theta}{dt} \propto -(\theta - \theta_s)$; θ_s = air temperature

= surrounding temperature

$$\frac{d\theta}{dt} = -k(\theta - \theta_s)$$

$$\int \frac{d\theta}{\theta - \theta_s} = \int -k\,dt$$

$$\ln(\theta - \theta_s) = -kt + \ln c$$

$$\therefore \boxed{\theta - \theta_s = c \cdot e^{-kt}}$$

growth

$$\frac{dx}{dt} \propto x$$

$$\frac{dx}{dt} = kx$$

$$\int \frac{dx}{x} = \int k\,dt$$

$$\ln x = kt + \ln c$$

$$\boxed{x = ce^{kt}}$$

decay

$$\boxed{x = ce^{-kt}}$$

Q:- A body originally at 60°C. cools down to 40°C in 15min. When kept in air at a temp. of 25°C what will be temp. of body at end of 30min is ___ °C

$$\frac{d\theta}{dt} = -k(\theta_1 - \theta_s)$$

$$\frac{60° - 40°}{15} = -k[\theta_1 - 25]$$

$$\frac{\cancel{20}\,4}{\cancel{15}\,3} = -k[\theta_1 - 25]$$

$$\frac{4}{3} = -k[40 - 25]$$

$$\frac{4}{3} = -k[15]$$

$$k = -0.0888$$

$$\frac{40° - ?}{15} = -k[\theta_2 - 25]$$

$$\frac{40 - x}{15} = +0.0888[x - 25]$$

$$40 - x = \overset{332}{1.\,x} - \overset{33.33}{\cancel{66.66}}$$

$$\overset{73.33}{\cancel{126.6}} = \overset{2.332}{x}$$

~~x = 24.6 °C~~

$$\boxed{x = 31.44°C}$$

$\theta\,°C$	t (min.)
$60°C$	0
$40°C$	15min
?	30min

$$\frac{d\theta}{dt} = -K(\theta - \theta_s)$$

$$\int \frac{d\theta}{\theta - \theta_s} = -\int k\,dt$$

$$\ln(\theta - \theta_s) = -kt + \ln c$$

$$\theta - \theta_s = c \cdot e^{-kt} \quad \text{---(1)}$$

$$\theta - 25 = ce^{-kt}$$

$$60 - 25 = c$$

$$\boxed{c = 35}$$

$$40 - 25 = 35e^{-k \cdot 15}$$

$$k = 0.0564$$

$$\theta - 25 = 35e^{-0.0564t}$$

$$\theta - 25 = 35e^{-0.0564 \cdot 30}$$

$$\boxed{\theta = 31.44°C}$$

Find complementary func.

Q:- $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

$$D^3 - 3D + 2 = 0$$

$$m^3 - 3m + 2 = 0$$

$$
\begin{array}{r|l}
 & m^2 - m - 4 \\
m-1 & m^3 - 3m + 2 \\
 & -\,m^3 - m^2 \\
 & \quad + \\
\hline
 & m^2 - 3m
\end{array}
$$

$$
\begin{array}{r}
m^2 - 3m \\
\underline{\overset{-}{m^2} \overset{-}{+ m}} \\
-4m + 2 \\
\underline{-4m + 4} \\
\underline{6}
\end{array}
$$

$$D^3 - 3D + 2 = 0$$

$$(D-1)D^2 + D(D-1) - 2(D-1) = 0$$

$$D = 1, 1, -2$$

$$\text{c.f.} = c_1 e^{x} + c_2 e^{x} x + c_3 e^{-2x}$$

Q:- $\dfrac{d^3y}{dx^3} - 4\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} - 2y = 0$

$$D^3 - 4D^2 + 5D - 2 = 0$$

$$
\begin{array}{r|l}
 & D^2 - 3D + 2 \\
\hline
D - 1 & D^3 - 4D^2 + 5D - 2 \\
 & \underline{-\,D^3 \overset{+}{-} D^2} \\
 & -3D^2 + 5D \\
 & \underline{\mp 3D^2 \pm 3D} \\
 & 2D - 2 \\
 & \underline{2D - 2} \\
 & \underline{0}
\end{array}
$$

$$D = 1, 1, 2$$

$$\text{c.f.} = (c_1 + c_2 x)e^{x} + c_3 e^{2x}$$

Q:- $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{dy}{dx} - y = 0$

$m^4 - 2m^3 + 2m - 1 = 0$

$m^2(\cancel{m^2 - 2m}) +$

$\cancel{m^3}(m - \cancel{2}) + 2\,\cancel{(m - 1/2)}$

$$
\begin{array}{r|l}
 & m^3 - m^2 - m + 1 \\
\hline
m - 1 & m^4 - 2m^3 + 2m - 1 \\
 & m^4 - m^3 \\
 & -m^3 + 2m \\
 & -m^3 + m^2 \\
 & -m^2 + 2m \\
 & -m^2 + m \\
 & m - 1 \\
 & m - 1 \\
 & 0
\end{array}
$$

$$
\begin{array}{r|l}
 & m^2 - 1 \\
\hline
m - 1 & m^3 - m^2 - m + 1 \\
 & m^3 - m^2 \\
 & 0 - m + 1 \\
 & -m + 1 \\
 & 0
\end{array}
$$

$(m-1)^2(m^2-1)$

$m = 1, 1, 1, -1$

$c.f. = (c_1 + c_2 x + c_3 x^2)e^{x} + c_4 e^{-x}$

Q:- $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$

$m^3 + 6m^2 + 11m + 6 = 0$

$-1 + 6 - 11 + 6$

$$
\begin{array}{r|l}
 & m^2 + 5m + 6 \\
\hline
m + 1 & m^3 + 6m^2 + 11m + 6 \\
 & m^3 + m^2 \\
 & 5m^2 + 11m \\
 & 5m^2 + 5m \\
 & 6m + 6
\end{array}
$$

$(m+1)(m^2+5m+6)$

$-1\ ,\ m = \dfrac{-5 \pm \sqrt{25-24}}{2}$

$-1\ ,\ m = \dfrac{-5 \pm 1}{2} = -3, -2$

$-1, -2, -3$

$c.f. = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$

Q:- $\dfrac{d^4y}{dx^4} - \dfrac{d^3y}{dx^3} - 9\dfrac{d^2y}{dx^2} - 11\dfrac{dy}{dx} - 4y = 0$

$m^4 - m^3 - 9m^2 - 11m - 4 = 0$

sum of even terms = -12

sum of odd terms = -12

$m=-1$ is root

$\cancel{16+8}-36+22-4$

$\cancel{1}+1-9+11-4$

$$\begin{array}{r|rrrrr}
m=-1 & 1 & -1 & -9 & -11 & -4 \\
 & 0 & -1 & 2 & 7 & 4 \\
\hline
 & 1 & -2 & -7 & -4 & \boxed{0} \\
m=-1 & 0 & -1 & 3 & 4 & \\
\hline
 & 1 & -3 & -4 & \boxed{0} &
\end{array}$$

$$\begin{array}{r|l}
 & m^3-2m^2-7m-4 \\
\hline
m+1 & m^4-m^3-9m^2-11m-4 \\
 & m^4+m^3 \\
 & -\quad - \\
\hline
 & -2m^3-9m^2 \\
 & -2m^3-2m^2 \\
 & +\quad + \\
\hline
 & -7m^2-11m \\
 & -7m^2-7m \\
 & +\quad + \\
\hline
 & -4m-4 \\
 & -4m-4 \\
 & +\quad + \\
\hline
 & 0
\end{array}$$

$$\begin{array}{r|l}
 & m^2-3m-4 \\
\hline
m+1 & m^3-2m^2-7m-4 \\
 & m^3+m^2 \\
 & -\quad - \\
\hline
 & -3m^2-7m \\
 & -3m^2-3m \\
 & +\quad + \\
\hline
 & -4m-4 \\
 & -4m-4 \\
 & +\quad + \\
\hline
 & 0
\end{array}$$

$m = -1, -1, \cancel{\dfrac{+3 \pm \sqrt{9-16}}{2}}$

$-1, 4$

$$\begin{array}{r|rrr}
 & 1 & -3 & -4 \\
m=-1 & 0 & -1 & 4 \\
\hline
 & 1 & -4 & \boxed{0}
\end{array}$$

$m-4=0, m=4$

$C.f. = (c_1 + c_2 x + c_3 x^2)e^{-x} + c_4 e^{4x}$

Q:- $(D^2-3D+4)y=0$

$m^2-3m+4=0$

$m = \frac{+3 \pm \sqrt{9-\cancel{8}16}}{2}$

$m = \frac{+3 \pm \sqrt{7}j}{2}$

c.f. $= \left(c_1 \cos\frac{\sqrt{7}}{2}x + c_2 \sin\frac{\sqrt{7}x}{2}\right) e^{3/2 x}$

Q:- $(D^4+2D^2+1)y=0$

$m^4+2m^2+1=0$

$m^2(m^2+1)+(m^2+1)=0$

$(m^2+1)(m^2+1)=0$

$\cancel{m^2 = \pm 1}$ $m = \pm i$, $m = \pm i$

$m = i, i, -i, -i$

c.f. $= (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x \cancel{e}$

c.f. $= (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$

Bernaulli's equation

- A differential eq^n of form $\boxed{\frac{dy}{dx} + Py = Qy^n}$

where P and Q are functions of x is said to be Bernaolli's eq^n.

As it is non-linear

$$\frac{dy}{dx} + Py = Qy^n$$

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

Let $y^{1-n} = z$

$$\frac{dz}{dx} = (1-n)\, y^{-n}\frac{dy}{dx}$$

$$y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dz}{dx}$$

$$\frac{1}{(1-n)}\frac{dz}{dx} + Pz = Q$$

$$\boxed{\frac{dz}{dx} + P(1-n)z = Q(1-n)}$$ linear

$$\text{I.F.} = e^{\int(1-n)P\,dx}$$

$$\text{G.S.} = y^{1-n}\cdot \text{I.F.} = \int Q(1-n)e^{\int(1-n)P\,dx} + c$$

$$\boxed{\text{G.S.} = y^{1-n}\,\text{I.F.} = \int Q(1-n)\,\text{I.F.} + c}$$

Q:- $\frac{dy}{dx} + \frac{y}{x} = x^2 y^3$

$\frac{1}{y^3}\frac{dy}{dx} + \frac{y}{xy^3} = x^2$

$y^{-3}\frac{dy}{dx} + \frac{y^{-2}}{x} = x^2$

$y^{-2} = z$

$\frac{dz}{dx} = -2y^{-1}\frac{dy}{dx}$

$I.F. = e^{\int(1-n)P\,dx}$

$= e^{\int(1-3)\frac{1}{x}\,dx}$

$= e^{\int -2/x\,dx}$

$= e^{-2\log x}$

$= e^{\log 1/x^2}$

$I.F. = 1/x^2$

Q:- $\frac{dy}{dx} = \frac{1}{xy + x^2y^3}$

$\frac{dy}{dx} + Py = Qy^n$

$dy(xy) + x^2y^3 = dx$

$\frac{dx}{dy} = xy + x^2y^3$

$\frac{dx}{dy} + Px = Qx^n$

$\frac{dx}{dy} - xy = x^2y^3$

$I.F. = e^{\int(1-n)P\,dy}$

$= e^{\int(1-2)(-y)\,dy}$

$I.F. = e^{\int y\,dy} = e^{\frac{y^2}{2}}$

Q:- $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

$$\frac{dy}{dx} + Py = Qy^n$$

$$\frac{dy}{dx} + x \cdot 2\sin y\cos y = x^3 \cdot \cos^2 y$$

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{x 2\sin \cancel{2} y \cos y}{\cos^{\cancel{2}} y} = x^3$$

$$\sec^2 y \frac{dy}{dx} + 2x\tan y = x^3$$

$$z = \tan y$$

$$\frac{dz}{dx} = \sec^2 y \frac{dy}{dx}$$

$$\frac{dz}{dx} + 2x \cdot z = x^3$$

$$P = 2x, \quad n = 0$$

$$\text{I.F.} = e^{\int (1-n) P\,dx} = e^{\int 2x\,dx} = \underline{\underline{e^{x^2}}}$$

*** Particular Integral**

If $\frac{1}{f(D)} \cdot X$ is that function of x, not containing arbritary constant which when operated upon $f(D)$ gives the value x.

$$f(D)y = X$$

$$f(D)\left[\frac{1}{f(D)} \cdot X\right] = X$$

→ Thus $\frac{1}{f(D)} \cdot X$ satisfies the equation $f(D)y = X$ is therefore it's particular integral.

→

$$\frac{1}{D} \cdot X = \int X \, dx$$

D = differential operator

$$\frac{1}{D^2} \cdot X = \int\int x \, dx \, dx$$

$$\frac{1}{D^3} \cdot X = \int\int\int x \, dx \, dx \, dx$$

→ $$\boxed{\frac{1}{D-\alpha} \cdot X = e^{\alpha x} \int x e^{-\alpha x} \cdot dx}$$

$$f(D)y = X$$

X can be algebraic, logarithmic, exponential, trignometric etc. but <u>not inverse trigonometry</u>

Q:- Find P.I. of $\frac{1}{D^2-3D+2} e^{3x}$

$$\frac{e^{3x}}{(D-2)(D-1)}$$

$$\frac{1}{(D-1)} \cdot \left[\frac{e^{3x}}{(D-2)}\right]$$

$$\frac{1}{(D-1)}\left(e^{2x}\int e^{3x}\cdot e^{-2x}\,dx\right)$$

$$\frac{1}{D-1}\left(e^{2x}\right)\cdot e^{x}$$

$$\frac{e^{3x}}{D-1}$$

$$e^{x}\int e^{3x}\cdot e^{-x}\,dx$$

$$e^{x}\cdot\int e^{2x}\,dx$$

$$\frac{e^{x}\cdot e^{2x}}{2} = \frac{e^{3x}}{2}$$

Case:1 To find P.I. of the form $\frac{1}{f(D)}\cdot e^{ax} = \frac{1}{f(a)}e^{ax}$; $f(a)\neq$

Q:- $\frac{e^{3x}}{D^2-3D+2}$

$$= \frac{e^{3x}}{9-9+2} = \frac{e^{3x}}{2}$$

Case:-2 If $f(a)=0$ then to find P.I. of the form

$$\frac{1}{f(D)}e^{ax} = \frac{x e^{ax}}{f'(a)} \quad \text{if } f(a)=0$$

$$= \frac{x^2 e^{ax}}{f''(a)} \quad \text{if } f'(a)=0$$

Q:- $\frac{1}{D^3-3D+2} e^x$

$\frac{x\cdot e^x}{3D^2-3} = \frac{x e^x}{6D} = \frac{x^2 e^x}{6}$

$\frac{x^2 e^x}{6D}$

$\int \frac{x^2 e^x}{6}$

$\frac{1}{6}\left[x^2 \int e^x - \int 2x \int e^x \, dx\right]$

$\frac{1}{6}\left[x^2 e^x - \int 2x e^x \, dx\right]$

$\frac{1}{6}\left[x^2 e^x - 2\left[x \int e^x - \int\int e^x \, dx \, dx\right]\right]$

$\frac{1}{6}\left[x^2 e^x - 2x e^x + 2e^x\right]$

$\frac{1}{6} e^x \left[x^2 - 2x + 2\right]$

Q:- $(D^2-7D+6)y = e^{6x}$

$y = \frac{e^{6x}}{(D^2-7D+6)}$

$= \frac{e^{6x}}{(D-1)(D-6)} \qquad = \frac{x e^{6x}\cdot 6}{2D-7}$

$= \cancel{\frac{e^{6x}}{36-42+6}} \qquad = \frac{x e^{6x}\cdot 6}{5}$

Q:- $(D^2+4D+4)y = e^{2x}+e^{-2x}$

$$y = \frac{e^{2x}}{D^2+4D+4} + \frac{e^{-2x}}{D^2+4D+4}$$

$$= \frac{e^{2x}}{(D+2)(D+2)}$$

$$= \frac{e^{2x}}{4+8+4} + \cancel{\frac{e^{-2x}}{4-8+4}} \qquad \frac{x\, e^{-2x}\cdot(-2)}{2D+4}$$

$$= \frac{e^{2x}}{16} + \cancel{e^{-2x}} \qquad \frac{x\left(-2e^{-2x}-2x\,e^{-2x}(-2)\right)}{2}$$

$$= \frac{e^{2x}}{16} + e^{-2x}\left[-2x-2x^2\right]$$

Q:- $(D^2-3D+2)y = \cosh x$

Q:-The number N of the bacteria in a culture of grew at rate proportional to N. The value of N was initially 100 and increase to 332 in one hour. What will be the value of N after ~~0~~ 1½ hours?

$$\frac{dN}{\partial t} \propto N \quad \text{---(1)}$$

$$\cancel{\int \frac{dN}{dt} = \int KN}$$

$$\cancel{N = KN + C}$$

$$\frac{dN}{dt} = NK$$

$$\cancel{N = \frac{KN^2}{2} t + c}$$

$$\cancel{c = 100}$$

$$\cancel{332 = \frac{K \cdot (1)}{2} + 100}$$

$$\cancel{332 = \frac{K1}{2} + 100}$$

$$\cancel{K = 464}$$

$$\cancel{N = \frac{464(1.5)}{2} + 100}$$

$$\int \frac{dN}{N} = \int dt \cdot k$$

$$\ln N = kt + \ln c$$

$$N = ce^{+kt}$$

At $t=0$, $N=100$

$$100 = c$$

At $t=1$, $N=332$

$$\therefore 332 = 100e^{k}$$

$$k = 1.19 \cong 1.2$$

At $t=1.5$, $N=?$

$$N = 100e^{(1.2)(1.5)}$$

$$N = 604.96 \cong 605$$

Q:- Given that $\ddot{x} + 3x = 0$ and initial conditions is $x(0) = 1,\ \dot{x}(0) = 0$ then what is $x(1) = ?$

$$\frac{d^2x}{dy^2} + 3x = 0$$

$$\cancel{D^2 + 3 = 0}$$

$$(D^2 + 3)x = 0$$

$$\cancel{\int \frac{d^2x}{dy^2} = \int -3x}$$

$$\frac{dx}{dy} = -3xy + c$$

$$\cancel{x = -\frac{3xy^2}{2} + cy + c'}$$

$$\boxed{0 = c}$$

$$1 = c'$$

$$x = -\frac{3xy^2}{2} + 1$$

$$x = -\frac{3x + 1}{2} = -1/2$$

$$x + \frac{3}{2}x = 1$$

$$\cancel{\frac{5x}{2} = 1}$$

$$x = \frac{2}{5} = 0.4$$

$$\Rightarrow (D^2 + 3)x = 0$$

$$m^2 + 3 = 0$$

$$D = \pm\sqrt{3}\,i$$

$$x = (c_1 \cos\sqrt{3}x' + c_2 \sin\sqrt{3}x')e^0$$

$$x' = c_1 - \sin\sqrt{3}x' \cdot \sqrt{3} + c_2 \cos\sqrt{3}x' \cdot \sqrt{3}$$

$$\boxed{1 = c_1}$$

$$0 = \sqrt{3}\,c_2 \qquad \boxed{c_2 = 0}$$

$$x = \cos\sqrt{3}x'$$

$$x(1) = \cos\sqrt{3} = 0.999$$

Q:- For $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3e^{2x}$. Find P.I.

$$(D^2 + 4D + 3)y = 3e^{2x}$$

$$y = \frac{3e^{2x}}{D^2+4D+3}$$

$$P.I. = \frac{3e^{2x}}{(D+1)(D+3)}$$

$$P.I. = \frac{\cancel{3}e^{2x}}{(\cancel{3})(5)}$$

$$\boxed{P.I. = \frac{e^{2x}}{5}}$$

Q:- If the characteristics roots of differential eqⁿ

$\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + y = 0$ has two equal roots

what is the value of α?

✓(a) ± 1 (c) $0, 0$

(b) $\pm i$ (d) $\pm 1/2$

Solⁿ:- $(D^2 + 2\alpha D + 1)y = 0$

$$m^2 + m \cdot 2\alpha + 1 = 0$$

$$\text{roots} = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4}}{2} = -1\cdot\alpha \pm \sqrt{\alpha^2 - 1}$$

$$\sqrt{\alpha^2 - 1} = 0 \qquad \alpha^2 = 1$$

$$\boxed{\alpha = \pm 1}$$

Case: 3 To find P.I. of the form $\frac{1}{f(D)}\sin ax$ and $\frac{1}{f(D)}\cos ax$ is replace $D^2 = -a^2$, $\phi(-a^2) \neq 0$

Q:- Find P.I. of form $\frac{1}{D^2-2D+5}\sin 2x$

$$P.I. = \frac{\sin 2x}{-4-2D+5}$$

$$= \frac{\sin 2x}{-2D+1} \times \frac{1+2D}{1+2D}$$

$$= \frac{(1+2D)\sin 2x}{4D^2-1}$$

$$\frac{\sin 2x}{4D^2-1} + \frac{2D\sin 2x}{4D^2-1}$$

$$\frac{\sin 2x}{-16-1} + 2D\frac{\sin 2x}{4(-4)-1}$$

$$\frac{\sin 2x}{-17} + 2D\frac{\sin 2x}{-17}$$

$$\frac{\sin 2x}{-17} - \frac{2}{17}\cos 2x \cdot 2$$

$$-\frac{4\cos 2x}{17} - \frac{\cos 2x}{17}$$

$$= \frac{(1+2D)\sin 2x}{1-4D^2}$$

$$= \frac{(1+2D)\sin 2x}{1+16}$$

$$= \frac{1}{17}\sin 2x + \frac{2D\sin 2x}{17}$$

$$= \frac{1}{17}\sin 2x + \frac{2\cdot 2\cos 2x}{17}$$

$$P.I. = \frac{1}{17}\sin 2x + \frac{4\cos 2x}{17}$$

Case:-4 To find P.I. of form $\frac{1}{f(D)} x^m$

P.I. $= [f(D)]^{-1} x^m$. Expand $[f(D)]^{-1}$ in ascending power of D as far as the term in D^n and operate on x^m terms by terms since $(m+1)^{th}$ and higher derivative of x^m are zero. (we do not consider the term beyound D^m).

Q:- $(D^2+4)y = x^2$

$$y = \frac{x^2}{(D^2+4)}$$

$$\text{P.I.} = x^2 (D^2+4)^{-1}$$

$$y = \frac{x^2}{4\left(\frac{D^2}{4}+1\right)}$$

$$y = \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1} x^2$$

$$\text{P.I.} = \frac{1}{4}\left[1-\frac{D^2}{4}\right] \cdot x^2$$

$$\text{P.I.} = \frac{1}{4}\left[x^2 - \frac{2}{4}\right]$$

$$\text{P.I.} = \frac{1}{4}\left[x^2 - \frac{1}{2}\right] = \frac{x^2}{4} - \frac{1}{8}$$

NOTE:

$(1+D)^{-1} = 1 - D + D^2 - D^3 + \dots$

$(1-D)^{-1} = 1 + D + D^2 + D^3 + \dots$

$(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \dots$

$(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \dots$

Q:- $(D^3+8)y = x^4+x+1$

$$y = \frac{x^4+x+1}{D^3+8}$$

$$= \frac{1 \quad x^4+x+1}{8\left(\frac{D^3}{8}+1\right)}$$

$$= \frac{x^4}{8\left(1+\frac{D^3}{8}\right)} + \frac{x}{8\left(\frac{D^3}{8}+1\right)} + \frac{1}{8\left(\frac{D^3}{8}+1\right)}$$

$$y = \frac{x^4}{8}\left(1+\frac{D^3}{8}\right)^{-1} + \frac{x}{8}\left[1+\frac{D^3}{8}\right]^{-1} + \frac{1}{8}\left[1+\frac{D^3}{8}\right]^{-1}$$

P.I. $= \frac{x^4}{8}\left[1-\frac{D^3}{8}\right] + \frac{x}{8}[1] + \frac{1}{8}$ $= \cancel{x^4+x/8}$

$$= \frac{x^4}{8} - \frac{3x}{64} + \frac{x}{8} + \frac{1}{8}$$

P.I. $= \frac{x^4}{8} + \frac{5x}{64} + \frac{1}{8}$

~~Q//~~ $y = \frac{1}{8}\left(1+\frac{D^3}{8}\right)^{-1}(x^4+x+1)$

P.I. $= \frac{1}{8}\left[1-\frac{D^3}{8}\right][x^4+x+1]$

P.I. $= \frac{1}{8}\left[x^4+x+1-\frac{1}{\cancel{8}}\overset{3}{\cancel{24}}x\right]$

P.I. $= \frac{1}{8}[x^4-2x+1]$

Case: 5 To find P.I. of the form $\frac{1}{f(D)} \cdot e^{ax} \cdot V$

where V is the function of x [V can't be exponential]

then $$\frac{1}{f(D)} \cdot e^{ax} \cdot V = e^{ax} \cdot \frac{1}{f(D+a)} \cdot V$$

Q:- $(D^2 - 2D + 1) y = x^2 e^x$

$$y = \frac{1}{(D^2 - 2D + 1)} x^2 \cdot e^x$$

$$P.I. = e^x \cdot \frac{1}{((D+1)^2 - 2(D+1) + 1)} \cdot x^2$$

$$= e^x \frac{x^2}{D^2 + \cancel{2D} + 1 - \cancel{2D} - 2 + 1}$$

$$= \frac{e^x \cdot x^2}{D^2}$$

$$= e^x \int \frac{x^3}{3}$$

$$P.I. = \frac{e^x \cdot x^4}{12}$$