Definite Integration & its Application

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JEE (Advanced) Syllabus

a : .

JEE (Main) Syllabus

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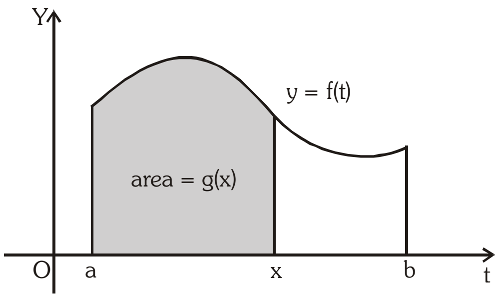
DEFINITE INTEGRATION & ITS APPLICATION

A definite integral is denoted by $\int_a^b f(x)dx$ which represent the algebraic area bounded by the curve y = f(x), the ordinates x = a, x = b and the x axis.

1. THE FUNDAMENTAL THEOREM OF CALCULUS :

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus : differential calculus and integral calculus. Differential calculus arose from the tangent problem, whereas integral calculus arose from a seemingly unrelated problem, the area problem. Newton's teacher at Cambridge, Isaac Barrow (1630-1677), discovered that these two problems are actually closely related. In fact, he realized that differentiation and integration are inverse processes. The Fundamental Theorem of Calculus gives the precise inverse relationship between the derivative and the integral.

It was Newton and Leibnitz who exploited this relationship and used it to develop calculus into a systematic mathematical method. In particular, they saw that the Fundamental Theorem enabled them to compute areas and integrals very easily without having to compute them as limits of sums.



Part 1 : The First Fundamental Theorem of Calculus : If f is continuous on [a, b], then the function g defined by $g(x)=\int_a^x f(t)dt \quad , \qquad a \le x \le b$

is continuous on [a, b] and differentiable on (a, b), and g'(x) = f(x).

Corollary : If f is continuous on [a, b], then $\int_a^b f(x)dx = F(b)-F(a)$ where F is any antiderivative off, that is a function such that F '= f.

Part 2 : The Second Fundamental Theorem of Calculus
This part is sometimes referred as **Newton-Leibnitz** axiom.
Let f be a real-valued function on a closed interval [a, b] and F an antiderivative of f in [a, b] :
F' (x) = f (x).

If f is Riemann integrable on [a, b] then $\int_a^b f(x)dx = f(b)-f(a)$

The second part is somewhat stronger then the corollary because it does not assume that f is continuous.

Note : If $\int_a^b f(x)dx = 0 \Rightarrow$ then the equation f(x) = 0 has atleast one root lying in (a, b) provided f is a continuous function in (a,b).

SOLVED EXAMPLE

Example 1 : Evaluate $\int_1^2 \frac{dx}{(x+1)(x+2)}$

Solution : $\because$ $\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$ (by partial fractions)

$$\int_1^2 \frac{dx}{(x+1)(x+2)} = \left[\ell n(x+1) - \ell n(x+2)\right]_1^2 = \ell n3 - \ell n4 - \ell n2 + \ell n3 = \ell n\left(\frac{9}{8}\right)$$

Problems for Self Practice -01:

Evaluate the following

(1) $\int_1^2 \frac{5x^2}{x^2+4x+3}\,dx$ (2) $\int_0^{\frac{\pi}{4}} \left(2\sec^2 x + x^3 + 2\right) dx$ (3) $\int_0^{\frac{\pi}{3}} \frac{x}{1+\sec x}\,dx$

Answers : (1) $5 - \frac{5}{2}\left(9\ell n\frac{5}{4} - \ell n\frac{3}{2}\right)$ (2) $\frac{\pi^4}{1024} + \frac{\pi}{2} + 2$

(3) $\frac{\pi^2}{18} - \frac{\pi}{3\sqrt{3}} + 2\,\ell n\left(\frac{2}{\sqrt{3}}\right)$

2. PROPERTIES OF DEFINITE INTEGRAL :

Property -1 : $\int_a^b f(x)dx = \int_a^b f(t)\,dt$ provided f is same

Property -2 : $\int_a^b f(x)dx = -\int_b^a f(x)\,dx$

Property -3 : $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where c may lie inside or outside the interval [a,b].

This property is to be used when f is piecewise continuous in (a, b).

SOLVED EXAMPLE

Example 2 : Evaluate $\int_2^8 |x-5|\, dx$.

Solution : $\int_2^8 |x-5|\, dx = \int_2^5 (-x+5)\, dx + \int_5^8 (x-5)\, dx = 9$

Example 3 : Show that $\int_0^2 (2x+1)\, dx = \int_0^5 (2x+1) + \int_5^2 (2x+1)$

Solution : L.H.S. = $x^2 + x]_0^2$ = 4 + 2 = 6; R.H.S. = 25 + 5 – 0 + (4 + 2) – (25 + 5) = 6

$\therefore$ L.H.S. = R.H.S

Example 4 : If $f(x) = \begin{cases} x^2, & 0 < x < 2 \\ 3x-4, & 2 \le x < 3 \end{cases}$ then evaluate $\int_0^3 f(x)dx$

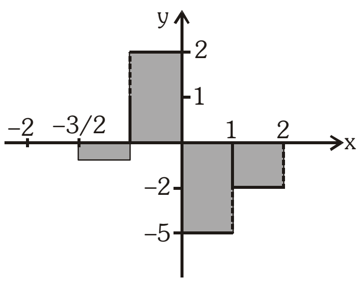
Solution : $\int_0^3 f(x)dx = \int_0^2 f(x)dx + \int_2^3 f(x)dx = \int_0^2 x^2 dx + \int_2^3 (3x-4)dx$

$$= \left(\frac{x^3}{3}\right)_0^2 + \left(\frac{3x^2}{2} - 4x\right)_2^3 = \frac{8}{3} + \frac{27}{2} - 12 - 6 + 8 = 37/6$$

Example 5 : If $f(x) = \begin{cases} 3[x] - 5\dfrac{|x|}{x}, & x \ne 0 \\ 2, & x = 0 \end{cases}$ then find $\int_{-3/2}^{2} f(x)dx$ (where [.] denotes the greatest integer function)

Solution : $3[x] - 5\dfrac{|x|}{x} = 3[x] - 5$, if $x > 0$

= 3[x] + 5, if x < 0



$$\Rightarrow \int_{-3/2}^{2} f(x)dx = \int_{-3/2}^{-1} (-1)dx + \int_{-1}^{0} (2)dx + \int_{0}^{1} (-5)dx + \int_{1}^{2} (-2)dx$$

$$= -1\left(-1 + \frac{3}{2}\right) + 2(1) + 1(-5) + (-2) = -\frac{1}{2} + 2 - 5 - 2 = -\frac{11}{2}$$

Example 6 : Find the value of $\int_1^2 (x^{[x^2]} + [x^2]^x)dx$, where [.] denotes the greatest integer function.

Solution : We have, I = $\int_1^2 (x^{[x^2]} + [x^2]^x)dx = \int_1^{\sqrt{2}} (x+1)dx + \int_{\sqrt{2}}^{\sqrt{3}} (x^2 + 2^x)dx + \int_{\sqrt{3}}^{2} (x^3 + 3^x)dx$

$= \left(\frac{x^2}{2} + x\right)_1^{\sqrt{2}} + \left(\frac{x^3}{3} + \frac{2^x}{\log 2}\right)_{\sqrt{2}}^{\sqrt{3}} + \left(\frac{x^4}{4} + \frac{3^x}{\log 3}\right)_{\sqrt{3}}^{2}$

$= \frac{5}{4} + \sqrt{3} + \frac{\sqrt{2}}{3} + \frac{1}{\log 2}(2^{\sqrt{3}} - 2^{\sqrt{2}}) + \frac{1}{\log 3}(3^2 - 3^{\sqrt{3}})$

Example 7 : Evaluate : $\int_{-10}^{20} [\cot^{-1} x]dx$. Here [.] is the greatest integer function.

Solution : I = $\int_{-10}^{20} [\cot^{-1} x]dx$, we know $\cot^{-1} x \in (0, \pi)\ \forall\ x \in R$

Thus $[\cot^{-1} x] = \begin{cases} 3, & x \in (-\infty, \cot 3) \\ 2, & x \in (\cot 3, \cot 2) \\ 1, & x \in (\cot 2, \cot 1) \\ 0 & x \in (\cot 1, \infty) \end{cases}$

Hence I = $\int_{-10}^{\cot 3} 3dx + \int_{\cot 3}^{\cot 2} 2dx + \int_{\cot 2}^{\cot 1} 1dx + \int_{\cot 1}^{20} 0dx$ = 30 + cot1 + cot2 + cot3 **Ans.**

<u>Problems for Self Practice</u> -02:

Evaluate the following

(1) $\int_0^2 |x^2 + 2x - 3|\ dx$. (2) $\int_0^3 [x]\ dx$, where [x] is integral part of x. (3) $\int_0^9 \left[\sqrt{t}\right] dt$.

(4) $\int_0^4 \{x\}dx$, where {.} denotes fractional part of x. **(5)** $\int_0^{\pi/2} |\sin x - \cos x|dx$

(6) If $f(x) = \begin{cases} 2 & 0 \le x \le 1 \\ x + [x] & 1 \le x < 3 \end{cases}$, where [.] denotes the greatest integer function. Evaluate $\int_0^2 f(x)dx$

Answers : (1) 4 (2) 3 (3) 13 (4) 2 (5) $2(\sqrt{2} - 1)$ (6) $\frac{9}{2}$

Property -4 : $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a}[f(x)+f(-x)]dx = \begin{cases} 0 & ;\text{if } f(x) \text{ is an odd function} \\ 2\int_0^a f(x)dx & ;\text{if } f(x) \text{ is an even function} \end{cases}$

Example 8 : Evaluate $\int_{-1/2}^{1/2} \cos x\, \ell n\left(\frac{1+x}{1-x}\right)dx$

Solution : $f(-x) = \cos(-x)\, \ell n\left(\frac{1-x}{1+x}\right) = -\cos \ell n\left(\frac{1+x}{1-x}\right) = -f(x) \Rightarrow f(x)$ is odd

Hence, the value of the given integral = 0.

Example 9 : If $f(x) = \begin{vmatrix} \cos x & e^{x^2} & 2x\cos^2 x/2 \\ x^2 & \sec x & \sin x + x^3 \\ 1 & 2 & x+\tan x \end{vmatrix}$, then find the value of $\int_{-\pi/2}^{\pi/2} (x^2+1)(f(x)+f''(x))dx$

Solution : As, $f(x) = \begin{vmatrix} \cos x & e^{x^2} & 2x\cos^2 x/2 \\ x^2 & \sec x & \sin x + x^3 \\ 1 & 2 & x+\tan x \end{vmatrix}$

$\Rightarrow$ $f(-x) = -f(x)$ $\Rightarrow$ $f(x)$ is odd

$\Rightarrow$ $f'(x)$ is even $\Rightarrow$ $f''(x)$ is odd

Thus, $f(x) + f''(x)$ is odd function let,

$\phi(x) = (x^2 + 1).\{f(x) + f''(x)\}$

$\Rightarrow$ $\phi(-x) = -\phi(x)$ i.e. $\phi(x)$ is odd $\therefore$ $\int_{-\pi/2}^{\pi/2} \phi(x)dx = 0$

Example 10 Evaluate $\int_{-1}^{1} \frac{e^x + e^{-x}}{1+e^x}\,dx$

Solution $\int_{-1}^{1} \frac{e^x+e^{-x}}{1+e^x}\,dx = \int_0^1 \left(\frac{e^x+e^{-x}}{1+e^x} + \frac{e^{-x}+e^x}{1+e^{-x}}\right)dx$

$= \int_0^1 \left(\frac{e^x+e^{-x}}{1+e^x} + \frac{e^x(e^{-x}+e^x)}{e^x+1}\right)dx = \int_0^1 (e^x+e^{-x})\,dx = e - 1 + \frac{(e^{-1}-1)}{-1} = \frac{e^2-1}{e}$

Problems for Self Practice -03:

Evaluate the following

(1) $\int_{-1}^{1} |x|\, dx.$ (2) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\, dx.$ (3) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^x}\, dx.$

(4) $\int_{-\pi/2}^{\pi/2} (x^2 \sin^3 x + \cos x)dx$ (5) $\int_{-\pi/2}^{\pi/2} \ell n\left[2\left(\frac{4-\sin\theta}{4+\sin\theta}\right)\right]d\theta$

Answers : (1) 1 (2) 0 (3) 1 (4) 2 (5) $\pi\ell n2$

Property -5 : $\int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx$, In particular $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$

Example 11 : Prove that $\int_0^{\frac{\pi}{2}} \frac{g(\sin x)}{g(\sin x)+g(\cos x)}\,dx = \int_0^{\frac{\pi}{2}} \frac{g(\cos x)}{g(\sin x)+g(\cos x)}\,dx = \frac{\pi}{4}$.

Solution : Let $I = \int_0^{\frac{\pi}{2}} \frac{g(\sin x)}{g(\sin x)+g(\cos x)}\,dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{g\left(\sin\left(\frac{\pi}{2}-x\right)\right)}{g\left(\sin\left(\frac{\pi}{2}-x\right)\right)+g\left(\cos\left(\frac{\pi}{2}-x\right)\right)} = \int_0^{\frac{\pi}{2}} \frac{g(\cos x)}{g(\cos x)+g(\sin x)}\,dx$$

on adding, we obtain

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{g(\sin x)}{g(\sin x)+g(\cos x)} + \frac{g(\cos x)}{g(\cos x)+g(\sin x)}\right)dx = \int_0^{\frac{\pi}{2}} dx \quad \Rightarrow I = \frac{\pi}{4}$$

Example 12 : If f, g, h be continuous functions on [0, a] such that f(a – x) = –f(x), g(a – x) = g(x) and 3h(x) – 4h(a – x) = 5, then prove that $\int_0^a f(x)g(x)h(x)dx = 0$

Solution : I = $\int_0^a f(x)g(x)h(x)dx = \int_0^a f(a-x)g(a-x)h(a-x)dx = -\int_0^a f(x)g(x)h(a-x)dx$

7I = 3I + 4I = $\int_0^a f(x)g(x)\{3h(x)-4h(a-x)\}dx = 5\int_0^a f(x)g(x)dx = 0$

(since f(a – x) g(a – x) = –f(x)g(x)) $\Rightarrow$ I = 0

Example 13 : Evaluate $\int_{-\pi}^{\pi} \frac{x\sin x}{e^x+1}dx$

Solution : I = $\int_{-\pi}^{0} \frac{x\sin x}{e^x+1}dx + \int_0^{\pi} \frac{x\sin x}{e^x+1}dx = I_1 + I_2$ where $I_1 = \int_{-\pi}^{0} \frac{x\sin x}{e^x+1}dx$

Put x = –t $\Rightarrow$ dx = – dt

$\Rightarrow$ $I_1 = \int_{\pi}^{0} \frac{(-t)\sin(-t)(-dt)}{e^{-t}+1} = \int_0^{\pi} \frac{t\sin t\,dt}{e^{-t}+1} = \int_0^{\pi} \frac{e^t t\sin t\,dt}{e^t+1} = \int_0^{\pi} \frac{e^x x\sin x\,dx}{e^x+1}$

Hence I = $I_1 + I_2$ = $\int_0^{\pi} \frac{e^x x\sin x}{e^x+1}dx + \int_0^{\pi} \frac{x\sin x}{e^x+1}dx$

I = $\int_0^{\pi} x\sin x dx = \int_0^{\pi} (\pi-x)\sin(\pi-x)dx = \pi\int_0^{\pi} \sin x dx - I$

$\Rightarrow$ 2I = $\pi\int_0^{\pi} \sin x dx = \pi\left|-\cos x\right|_0^{\pi} = 2\pi \Rightarrow I = \pi$

Example 14 : Evaluate

(i) $\int_0^2 \frac{dx}{(17+8x-4x^2)[e^{6(1-x)}+1]}$

(ii) $\int_0^1 \cot^{-1}(1-x+x^2)dx$

(iii) $\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x+\cos x}dx$

(iv) $\int_0^{\pi/2} \frac{2^{\sin x}}{2^{\sin x}+2^{\cos x}}dx$

Solution : (i) Let I = $\int_0^2 \frac{dx}{(17+8x-4x^2)[e^{6(1-x)}+1]}$

Also I = $\int_0^2 \frac{dx}{(17+8x-4x^2)[e^{-6(1-x)}+1]}$ $\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

Adding, we get

$$2I = \int_0^2 \frac{1}{17+8x-4x^2}\left(\frac{1}{e^{6(1-x)}+1}+\frac{1}{e^{-6(1-x)}+1}\right)dx = \int_0^2 \frac{1}{17+8x-4x^2}dx = -\frac{1}{4}\int_0^2 \frac{dx}{x^2-2x-17/4}$$

=

$$-\frac{1}{4}\int_0^2 \frac{dx}{(x-1)^2-21/4} = -\frac{1}{4}\times\frac{1}{2\times\frac{\sqrt{21}}{2}}\left[\log\left|\frac{x-1-\frac{\sqrt{21}}{2}}{x-1+\frac{\sqrt{21}}{2}}\right|\right]_0^2 = -\frac{1}{4\sqrt{21}}\left[\log\left|\frac{2x-2-\sqrt{21}}{2x-2+\sqrt{21}}\right|\right]_0^2$$

$$\Rightarrow \quad I = -\frac{1}{8\sqrt{21}}\left[\log\left|\frac{2-\sqrt{21}}{2+\sqrt{21}}\right| - \log\left|\frac{2+\sqrt{21}}{\sqrt{21}-2}\right|\right] = -\frac{1}{4\sqrt{21}}\left[\log\left|\frac{\sqrt{21}-2}{2+\sqrt{21}}\right|\right]$$

(ii) $$I = \int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx = \int_0^1 \tan^{-1}\left(\frac{x+(1-x)}{1-x(1-x)}\right)dx$$

$$= \int_0^1 [\tan^{-1}x + \tan^{-1}(1-x)]\ dx = \int_0^1 \tan^{-1}x\ dx + \int_0^1 \tan^{-1}(1-x)\ dx$$

$$= 2\int_0^1 \tan^{-1}x\ dx = 2\left[x\ \tan^{-1}x - \frac{1}{2}\log(1+x^2)\right]_0^1 = 2\frac{\pi}{4} - \log 2 = \frac{\pi}{2} - \log 2$$

(iii) $$I = \int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x}dx \qquad .....(i)$$

$$I = \int_0^{\pi/2} \frac{a\sin(\pi/2-x) + b\cos(\pi/2-x)}{\sin(\pi/2-x)+\cos(\pi/2-x)}dx = \int_0^{\pi/2} \frac{a\cos x + b\sin x}{\sin x + \cos x}dx \qquad .....(ii)$$

$$\therefore \quad 2I = \int_0^{\pi/2} \frac{(a+b)(\sin x + \cos x)}{\sin x + \cos x}dx = \int_0^{\pi/2}(a+b)dx = (a+b)\pi/2 \Rightarrow I = (a+b)\pi/4$$

(iv) $$I = \int_0^{\pi/2} \frac{2^{\sin x}}{2^{\sin x}+2^{\cos x}}dx = \int_0^{\pi/2} \frac{2^{\sin(\pi/2-x)}}{2^{\sin x(\pi/2-x)}+2^{\cos(\pi/2-x)}}dx = \int_0^{\pi/2} \frac{\cos x}{2^{\cos x}\quad 2^{\sin x}}dx$$

$$2I = \int_0^{\pi/2} dx = \frac{\pi}{2} \quad \Rightarrow \quad I = \frac{\pi}{4}$$

Problems for Self Practice -04:

Evaluate the following

(1) $\int_0^{\pi} \frac{x}{1+\sin x}\,dx.$ (2) $\int_0^{\frac{\pi}{2}} \frac{x}{\sin x+\cos x}\,dx.$ (3) $\int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx.$

(4) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\tan x}}$ (5) $\int_1^5 \frac{\sqrt{x}}{\sqrt{6-x}+\sqrt{x}}\,dx$

Answers : (1) π (2) $\frac{\pi}{2\sqrt{2}}\log_e\left(1+\sqrt{2}\right)$ (3) $\frac{\pi^2}{16}$ (4) $\frac{\pi}{12}$ **(5)** 2

Property -6 :

$$\int_0^{2a} f(x)dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx = \begin{cases} 2\int_0^{a} f(x)\,dx \ ; \ \text{if } f(2a-x)=f(x) \\ 0 \ ; \ \text{if } f(2a-x)=-f(x) \end{cases}$$

Example 15 : Evaluate (i) $\int_0^{\pi} \sin^3 x\cos^3 x\,dx$ (ii) $\int_0^{\pi} \frac{dx}{1+2\sin^2 x}\,dx$ (iii) $\int_0^{\pi} \frac{xdx}{1+\cos^2 x}$

Solution : (i) Let $f(x) = \sin^3x\cos^3x \Rightarrow$ $f(\pi-x) = -f(x)$ $\therefore$ $\int_0^{\pi} \sin^3 x\cos^3 x\,dx = 0$

(ii) Let $f(x) = \frac{1}{1+2\sin^2 x}$ $\Rightarrow$ $f(\pi-x)=f(x)$ $\Rightarrow$ $\int_0^{\pi} \frac{dx}{1+2\sin^2 x}$

$$= 2\int_0^{\frac{\pi}{2}} \frac{dx}{1+2\sin^2 x} = 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\,dx}{1+\tan^2 x+2\tan^2 x} = 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\,dx}{1+3\tan^2 x} = \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^{\frac{\pi}{2}}$$

$\because$ $\tan\frac{\pi}{2}$ is undefined, we take limit

$$= \frac{2}{\sqrt{3}}\left[\operatorname*{Lt}_{x\to\frac{\pi}{2}^-} \tan^{-1}\left(\sqrt{3}\tan x\right) - \tan^{-1}\left(\sqrt{3}\tan 0\right)\right] = \frac{2}{\sqrt{3}}\,\frac{\pi}{2} = \frac{\pi}{\sqrt{3}}$$

Alternatively : $\int_0^{\pi} \frac{dx}{1+2\sin^2 x} = \int_0^{\pi} \frac{\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x+2}\,dx = \int_0^{\pi} \frac{\operatorname{cosec}^2 x\,dx}{\cot^2 x+3}$

Observe that we are not converting in terms of tan x as it is not continuous in $(0, \pi)$

$$= -\frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)\right]_0^{\pi} = -\frac{1}{\sqrt{3}}\left[\underset{x\to\pi^-}{Lt}\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right) - \underset{x\to 0^+}{Lt}\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)\right]$$

$$= -\frac{1}{\sqrt{3}}\left[-\frac{\pi}{2}-\frac{\pi}{2}\right] = \frac{\pi}{\sqrt{3}}$$

(iii) Let I $= \int_0^{\pi}\frac{xdx}{1+\cos^2 x} = \int_0^{\pi}\frac{(\pi-x)dx}{1+\cos^2(\pi-x)} = \int_0^{\pi}\frac{\pi dx}{1+\cos^2 x} - I$

$$\Rightarrow 2I = \int_0^{\pi}\frac{\pi dx}{1+\cos^2 x} = 2\pi\int_0^{\pi/2}\frac{dx}{1+\cos^2 x} = 2\pi\int_0^{\pi/2}\frac{\sec^2 x dx}{2+\tan^2 x}$$

Let tan x = t so that for $x \to 0$, $t \to 0$ and for $x \to \pi/2$, $t \to \infty$. Hence we can write,

$$I = \pi\int_0^{\infty}\frac{dt}{2+t^2} = \pi\frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{t}{\sqrt{2}}\right]_0^{\infty} = \frac{\pi^2}{2\sqrt{2}}$$

Example 16 : Prove that $\int_0^{\pi/2}\log(\sin x)dx = \int_0^{\pi/2}\log(\cos x)dx = -\frac{\pi}{2}\log 2$

Solution : Let $I = \int_0^{\pi/2}\log(\sin x)dx$ .... (i)

then $I = \int_0^{\pi/2}\log\sin\left(\frac{\pi}{2}-x\right)dx = \int_0^{\pi/2}\log(\cos x)dx$ .... (ii)

adding (i) and (ii), we get

$$2I = \int_0^{\pi/2}\log\sin x\,dx + \int_0^{\pi/2}\log\cos x\,dx = \int_0^{\pi/2}(\log\sin x + \log\cos x)dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2}\log(\sin x\cos x)dx = \int_0^{\pi/2}\log\left(\frac{2\sin x\cos x}{2}\right)dx$$

$$= \int_0^{\pi/2}\log\left(\frac{\sin 2x}{2}\right)dx = \int_0^{\pi/2}\log(\sin 2x)dx - \int_0^{\pi/2}(\log 2)dx = \int_0^{\pi/2}\log\sin 2x\,.dx - (\log 2)(x)_0^{\pi/2}$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2}\log(\sin 2x)dx - \frac{\pi}{2}\log 2 \quad \text{......... (iii)}$$

Let $I_1 = \int_0^{\pi/2} \log(\sin 2x)dx$, putting 2x = t, we get

$$I_1 = \int_0^{\pi} \log(\sin t)\frac{dt}{2} = \frac{1}{2}\int_0^{\pi} \log(\sin t)dt = \frac{1}{2}.2\int_0^{\pi/2} \log(\sin t)dt$$

$I_1 = \int_0^{\pi/2} \log(\sin x)dx$ $\therefore$ (iii) becomes ; $2I = I - \frac{\pi}{2}\log 2$

Hence $\int_0^{\pi/2} \log\sin x\, dx = -\frac{\pi}{2}\log 2$

Example 17 : Find the value of $\int_0^{\pi/2} (2\log\sin x - \log\sin 2x)dx$

Solution : $I = \int_0^{\pi/2} (2\log\sin x - \log 2\sin x\cos x)dx = \int_0^{\pi/2} (2\log\sin x - \log 2 - \log\sin x - \log\cos x)dx$

$= \int_0^{\pi/2} \log\sin x dx - \int_0^{\pi/2} \log 2 dx - \int_0^{\pi/2} \log\cos x dx = -(\pi/2)\log 2$

<u>Problems for Self Practice</u> -05:

Evaluate the following

(1) $\int_0^{\infty} \left(\frac{\ell n\left(x+\frac{1}{x}\right)}{1+x^2}\right)dx.$ (2) $\int_0^{1} \frac{\sin^{-1}x}{x}\,dx.$ (3) $\int_0^{\pi} x\ell n \sin x\, dx.$

(4) $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{(1+e^x)(1+x^2)}$ (5) $\int_0^{\pi/2} \ell n\left(\sin^2 x\cos x\right)dx$ **(6)** $\int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x\cos x}dx$

(7) $\int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x}\, dx$

Answers : (1) $\pi\ \ell n 2$ (2) $\frac{\pi}{2}\ \ell n 2$ (3) $-\frac{\pi^2}{2}\ \ell n 2$ (4) $\frac{\pi}{3}$

(5) $-\left(\frac{3\pi}{2}\right)\ell n 2$ (6) 0 (7) $\frac{4}{3}$

Property -7 : If f(x) is a periodic fuction with period T (i.e. f(T + x) = f(x))

(i) $\int_0^{nT} f(x)dx = n\int_0^{T} f(x)dx$, $(n \in I)$

Note that : $\int_x^{T+x} f(t)dt$ will be independent of x and equal to $\int_0^{T} f(t)dt$

(ii) $\int_a^{a+nT} f(x)\,dx = n\int_0^{T} f(x)dx,\ n \in I$ (iii) $\int_{a+nT}^{b+nT} f(x)dx = \int_a^{b} f(x)dx$, $n \in I$.

(iv) $\int_{mT}^{nT} f(x)dx = (n-m)\int_0^{T} f(x)dx$, $(n, m \in I)$ (v) $\int_{nT}^{a+nT} f(x)\,dx = \int_0^{a} f(x)dx,\ n \in I$

(vi) $\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^{b} f(x)dx,\ n \in I$

Example 18 Evaluate (i) $\int_{-1}^{2} e^{\{x\}}\,dx$. (ii) $\int_0^{n\pi+v} |\cos x|\,dx$, $\frac{\pi}{2} < v < \pi$ and $n \in z$.

(iii) $\int_0^{4\pi} |\cos x|dx$ (iv) Evaluate $\int_0^{16\pi/3} |\sin x|dx$

(v) $\int_0^{2n\pi} [\sin x + \cos x]dx$. Here [.] is the greatest integer function.

Solution (i) $\int_{-1}^{2} e^{\{x\}}\,dx = \int_{-1}^{-1+3} e^{\{x\}}\,dx = 3\int_0^{1} e^{\{x\}}\,dx = 3\int_0^{1} e^{\{x\}}\,dx = 3(e-1)$

(ii) $\int_0^{n\pi+v} |\cos x|dx = \int_0^{v} |\cos x|\,dx + \int_v^{n\pi+v} |\cos x|\,dx = \int_0^{\frac{\pi}{2}} \cos x - \int_{\pi/2}^{v} \cos x\,dx + n\int_0^{\pi} |\cos x|\,dx$

$= (1-0) - (\sin v - 1) + 2n\int_0^{\frac{\pi}{2}} \cos x\,dx = 2 - \sin v + 2n(1-0) = 2n + 2 - \sin v$

(iii) Note that |cos x| is a periodic function with period π. Hence the given integral.

$$I = 4\int_0^{\pi}|\cos x|\,dx = 4\left[\int_0^{\frac{\pi}{2}}\cos x dx - \int_{\frac{\pi}{2}}^{\pi}\cos x dx\right] = 4\left[[\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}\right] = 4[1+1] = 8$$

(iv) $$\int_0^{16\pi/3}|\sin x|\,dx = \int_0^{5\pi}|\sin x|\,dx + \int_{5\pi}^{5\pi+\pi/3}|\sin x|\,dx = 5\int_0^{\pi}|\sin x|\,dx + \int_0^{\pi/3}|\sin x|\,dx$$

$$= 5[-\cos x]_0^{\pi} + [-\cos x]_0^{\pi/3} = 10 + \left(-\frac{1}{2}+1\right) = \frac{21}{2}$$

(v) Let $I = \int_0^{2n\pi}[\sin x + \cos x]dx = n\int_0^{2\pi}[\sin x + \cos x]dx$

($\because$ [sinx + cosx] is periodic function with period 2π]

$$[\sin x + \cos x] = \begin{cases} 1, & 0 \le x \le \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{4} \\ -1, & \dfrac{3\pi}{4} < x \le \pi \\ -2, & \pi < x \le \dfrac{3\pi}{2} \\ -1, & \dfrac{3\pi}{2} < x \le \dfrac{7\pi}{4} \\ 0, & \dfrac{7\pi}{4} < x \le 2\pi \end{cases}$$

Hence $$I = n\left[\int_0^{\pi/2}1dx + \int_{\pi/2}^{3\pi/4}0dx + \int_{3\pi/4}^{\pi}-1dx + \int_{\pi}^{3\pi/2}-2dx + \int_{3\pi/2}^{7\pi/4}-1dx + \int_{7\pi/4}^{2\pi}0dx\right]$$

$$I = n\left[\frac{\pi}{2} + 0 - \pi + \frac{3\pi}{4} - 3\pi + 2\pi - \frac{7\pi}{4} + \frac{3\pi}{2} + 0\right] = -n\pi$$

Problems for Self Practice -06:

Evaluate the following

(1) $\int_{-1}^{2} e^{\{3x\}}\,dx.$ (2) $\int_{0}^{2000\pi} \frac{dx}{1+e^{\sin x}}\,dx.$ (3) $\int_{\pi}^{\frac{5\pi}{4}} \frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx.$

(4) $\int_{-1.5}^{10} \{2x\}dx$, where {.} denotes fractional part of x. (5) $\int_{20\pi+\frac{\pi}{6}}^{20\pi+\frac{\pi}{3}} (\sin x+\cos x)dx$

Answers : (1) $3(e-1)$ (2) 1000π (3) $\frac{\pi}{4}$

(4) $\frac{23}{4}$ (5) $\left(\sqrt{3}-1\right)$

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3. REDUCTION FORMULAE AND WALLI'S FORMULA :

(a) If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$, then show that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

Proof : $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx \Rightarrow I_n = \left[-\sin^{n-1} x\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2} x\,.\cos^2 x\,dx = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,.(1-\sin^2 x)\,dx$

$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x\,dx \Rightarrow I_n + (n-1)I_n = (n-1)I_{n-2} \Rightarrow I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

Note : (i) $\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$

(ii) $I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)$ ..... I_0 or I_1 according as n is even or odd. $I_0 = \frac{\pi}{2}$, $I_1 = 1$

$$\text{Hence } I_n = \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\ldots\ldots\left(\frac{1}{2}\right).\frac{\pi}{2}, & \text{if n is even} \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\ldots\ldots\left(\frac{2}{3}\right).1, & \text{if n is odd} \end{cases}$$

(b) If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx$, then show that $I_n + I_{n-2} = \frac{1}{n-1}$

Solution $I_n = \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} . \tan^2 x\, dx = \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} (\sec^2 x - 1)\, dx$

$$= \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} \sec^2 x\, dx - \int_0^{\frac{\pi}{4}} (\tan x)^{n-2}\, dx = \left[\frac{(\tan x)^{n-1}}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2}$$

$I_n = \frac{1}{n-1} - I_{n-2}$ $\therefore$ $I_n + I_{n-2} = \frac{1}{n-1}$

(c) If $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x . \cos^n x\, dx$, then show that $I_{m,n} = \frac{m-1}{m+n} I_{m-2,\, n}$

Solution $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^{m-1} x\, (\sin x \cos^n x)\, dx = \left[-\frac{\sin^{m-1} x . \cos^{n+1} x}{n+1}\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\cos^{n+1} x}{n+1} (m-1) \sin^{m-2} x \cos x\, dx$

$$= \left(\frac{m-1}{n+1}\right) \int_0^{\frac{\pi}{2}} \sin^{m-2} x . \cos^n x . \cos^2 x\, dx = \left(\frac{m-1}{n+1}\right) \int_0^{\frac{\pi}{2}} \left(\sin^{m-2} x . \cos^n x - \sin^m x . \cos^n x\right) dx$$

$$= \left(\frac{m-1}{n+1}\right) I_{m-2,n} - \left(\frac{m-1}{n+1}\right) I_{m,n}$$

$$\Rightarrow \left(1 + \frac{m-1}{n+1}\right) I_{m,n} = \left(\frac{m-1}{n+1}\right) I_{m-2,n}$$

$$I_{m,n} = \left(\frac{m-1}{m+n}\right) I_{m-2,n}$$

Note : (i) $I_{m,n} = \left(\frac{m-1}{m+n}\right)\left(\frac{m-3}{m+n-2}\right)\left(\frac{m-5}{m+n-4}\right)$ .... $I_{0,n}$ or $I_{1,n}$ according as m is even or odd.

$I_{0,n} = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$ and $I_{1,n} = \int_0^{\frac{\pi}{2}} \sin x \cdot \cos^n x\, dx = \frac{1}{n+1}$

(ii) <u>Walli's Formula</u>

$$I_{m,n} = \begin{cases} \frac{(m-1)(m-3)(m-5)\,.........(n-1)(n-3)(n-5).......}{(m+n)(m+n-2)(m+n-4)........}\,\frac{\pi}{2} & \text{when both m, n are even} \\ \frac{(m-1)(m-3)(m-5)\,.........(n-1)(n-3)(n-5).......}{(m+n)(m+n-2)(m+n-4)........} & \text{otherwise} \end{cases}$$

SOLVED EXAMPLE

Example 19 : Evaluate (i) $\int_{-\pi/2}^{\pi/2} \sin^4 x \cos^6 x\, dx$ (ii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^2 x(\sin x + \cos x)\, dx$.

(iii) $\int_0^{\pi} x \sin^5 x \cos^6 x\, dx$ (iv) $\int_0^1 x^3(1-x)^5 dx$.

Solution : (i) $I = \int_{-\pi/2}^{\pi/2} \sin^4 x \cos^6 x\, dx = 2\int_0^{\pi/2} \sin^4 x \cos^6 x.dx = 2\frac{(3.1)(5.3.1)}{10.8.6.4.2}.\frac{\pi}{2} = \frac{3\pi}{256}$

(ii) Given integral = $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x \cos^2 x\, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^3 x\, dx$

$= 0 + 2\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x\, dx$ ($\because$ $\sin^3 x \cos^2 x$ is odd and $\sin^2 x \cos^3 x$ is even)

$= 2.\,\frac{1.2}{5.3.1} = \frac{4}{15}$

(iii) Let $I = \int_0^{\pi} x \sin^5 x \cos^6 x\, dx$

$I = \int_0^{\pi} (\pi - x)\sin^5(\pi - x)\cos^6(\pi - x)\, dx = \pi\int_0^{\pi} \sin^5 . \cos^6 x\, dx - \int_0^{\pi} x \sin^5 x . \cos^6 x\, dx$

$\Rightarrow \quad 2I = \pi . 2 \int_0^{\frac{\pi}{2}} \sin^5 x . \cos^6 x \, dx$

$I = \pi \dfrac{4.2.5.3.1}{11.9.7.5.3.1} \quad ; \quad I = \dfrac{8\pi}{693}$

(iv) Put $x = \sin^2\theta \Rightarrow \quad dx = 2 \sin\theta \cos\theta \, d\theta$

L . L $\quad : x = 0 \Rightarrow \quad \theta = 0$

U.L. $\quad : x = 1 \Rightarrow \quad \theta = \dfrac{\pi}{2}$

$\therefore \quad \int_0^1 x^3(1-x)^5 dx = \int_0^{\frac{\pi}{2}} \sin^6\theta(\cos^2\theta)^5 \, 2 . \sin\theta . \cos\theta \, d\theta = 2 . \int_0^{\frac{\pi}{2}} \sin^7\theta \cos^{11}\theta \, d\theta$

$= 2 . \dfrac{6.4.2.10.8.6.4.2}{18.16.14.12.10.8.6.4.2} \quad = \dfrac{1}{504}$

Problems for Self Practice -07:

Evaluate the following

(1) $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx$. (2) $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^4 x \, dx$. (3) $\int_0^1 x^6 \sin^{-1} x \, dx$.

(4) $\int_0^a x\left(a^2 - x^2\right)^{\frac{7}{2}} dx$. (5) $\int_0^2 x^{3/2}\sqrt{2-x} \, dx$.

Answers : (1) $\dfrac{8}{15}$ (2) $\dfrac{8}{315}$ (3) $\dfrac{\pi}{14} - \dfrac{16}{245}$

(4) $\dfrac{a^9}{9}$ (5) $\dfrac{\pi}{2}$

4. DERIVATIVE OF ANTIDERIVATIVE FUNCTION (Leibnitz Integral Formula) :

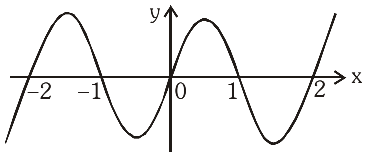
If h(x) & g(x) are differentiable functions of x then,

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f[h(x)].h'(x) - f[g(x)].g'(x)$$

SOLVED EXAMPLE

Example 20 Find the points of maxima/minima of $\int_0^{x^2} \frac{t^2-5t+4}{2+e^t}dt$

Solution : Let f(x) = $\int_0^{x^2} \frac{t^2-5t+4}{2+e^t}dt$



Graph of f'(x)

$f'(x) = \frac{x^4-5x^2+4}{2+e^{x^2}}2x - 0 = \frac{(x-1)(x+1)(x-2)(x+2)2x}{2+e^{x^2}}$

From the wavy curve, it is clear that f'(x) changes its sign at x = ± 2, ±1, 0 and hence the points of maxima are –1, 1 and of the minima are –2, 0, 2.

Example 21 : Evaluate $\frac{d}{dt}\int_{t^2}^{t^3} \frac{1}{\log x}dx$

Solution : $\frac{d}{dt}\int_{t^2}^{t^3} \frac{1}{\log x}dx = \frac{1}{\log t^3}.\frac{d}{dt}(t^3) - \frac{1}{\log t^2}.\frac{d}{dt}(t^2) = \frac{3t^2}{3\log t} - \frac{2t}{2\log t} = \frac{t(t-1)}{\log t}$

Example 22 : If $F(x) = \int_x^{x^2} \sqrt{\sin t}\, dt$, then find F′(x).

Solution : $F'(x) = 2x . \sqrt{\sin x^2} - 1 . \sqrt{\sin x}$

Example 23 : If $F(x) = \int_{e^{2x}}^{e^{3x}} \frac{t}{\log_e t}dt$, then find first and second derivative of F(x) with respect to $\ell n\, x$ at x = ℓn 2.

Solution : $\frac{dF(x)}{d(\ell n\, x)} = \frac{dF(x)}{dx}\frac{dx}{d(\ell n\, x)} = \left[3.e^{3x}.\frac{e^{3x}}{\ell n\, e^{3x}} - 2.e^{2x}\frac{e^{2x}}{\ell n\, e^{2x}}\right]x = e^{6x} - e^{4x}$.

$\frac{d^2F(x)}{d(\ell n\, x)^2} = \frac{d}{d(\ell n\, x)}(e^{6x}-e^{4x}) = \frac{d}{dx}(e^{6x}-e^{4x}) \times \frac{1}{d\ell nx/dx} = (6e^{6x} - 4e^{4x})\,x$

First derivative of F(x) at x = ℓn 2 (i.e. $e^x = 2$) is $2^6 - 2^4 = 48$

Second derivative of F(x) at x = ℓn 2 (i.e. $e^x = 2$) is $(6 . 2^6 - 4 . 2^4) . \ell n\, 2 = 5 . 2^6 . \ell n\, 2$.

Example 24 Evaluate $\underset{x\to\infty}{\text{Lt}} \dfrac{\left(\int_0^x e^{t^2}\,dt\right)^2}{\int_0^x e^{2t^2}\,dt}$.

Solution $\underset{x\to\infty}{\text{Lt}} \dfrac{\left(\int_0^x e^{t^2}\,dt\right)^2}{\int_0^x e^{2t^2}\,dt}$ $\left(\dfrac{\infty}{\infty}\text{ form}\right)$

Applying L′ Hospital rule

$$= \underset{x\to\infty}{\text{Lt}} \frac{2 . \int_0^x e^{t^2}dt . e^{x^2}}{1 . e^{2x^2}} = \underset{x\to\infty}{\text{Lt}} \frac{2 . \int_0^x e^{t^2}dt}{e^{x^2}} = \underset{x\to\infty}{\text{Lt}} \frac{2 . e^{x^2}}{2x . e^{x^2}} = 0$$

Example 25 If $f(x) = \int_{\log_e^x}^{x} \dfrac{dt}{x+t}$, then find f′(x).

Solution $f'(x) = \int_{\ell n\, x}^{x} \dfrac{-1}{(x+t)^2}\,dt + 1 . \dfrac{1}{2x} - \dfrac{1}{x}\,\dfrac{1}{(x+\ell n\, x)} = \left.\dfrac{1}{(x+t)}\right]_{\ell n\, x}^{x} + \dfrac{1}{2x} - \dfrac{1}{x\,(x+\ell n\, x)}$

$$= \frac{1}{2x} - \frac{1}{x+\ell n\, x} + \frac{1}{2x} - \frac{1}{x\,(x+\ell n\, x)} = \frac{1}{x} - \frac{x+1}{x\,(x+\ell n\, x)} = \frac{\ell n\, x - 1}{x\,(x+\ell n\, x)}$$

Alternatively : $f(x) = \int_{\ell n\, x}^{x} \dfrac{dt}{x+t} = \Big[\ \ell n\,(x+t)\ \Big]_{\ell n\, x}^{x}$ (treating ‘t’ as constant)

$f(x) = \ell n\, 2x - \ell n\,(x + \ell n\, x)$

$f'(x) = \dfrac{1}{x} - \dfrac{1}{(x+\ell n\, x)}\left(1+\dfrac{1}{x}\right) = \dfrac{\ell n\, x - 1}{x(x+\ell n\, x)}$

Example 26 Evaluate (i) $\int_0^1 \frac{x^b - 1}{\ell nx}$, 'b' being parameter.

(ii) $\int_0^1 \frac{\tan^{-1}(ax)}{x\sqrt{1-x^2}}\, dx$, 'a' being parameter.

Solution (i) Let $I(b) = \int_0^1 \frac{x^b - 1}{\ell nx}\, dx \Rightarrow \frac{dI(b)}{db} = \int_0^1 \frac{x^b \ell nx}{\ell nx}\, dx + 0 - 0$ (using modified Leibnitz Theorem)

$= \int_0^1 x^b\, dx = \left.\frac{x^{b+1}}{b+1}\right]_0^1 = \frac{1}{b+1} \Rightarrow I(b) = \ell n\,(b+1) + c$

$b = 0 \quad \Rightarrow \quad I(0) = 0$

$\therefore\ c = 0 \quad \therefore\ I(b) = \ell n\,(b+1)$

(ii) Let $I(a) = \int_0^1 \frac{\tan^{-1}(ax)}{x\sqrt{1-x^2}}\, dx$

$$\frac{dI(a)}{da} = \int_0^1 \frac{x}{(1+a^2x^2)} \frac{1}{x\sqrt{1-x^2}}\, dx = \int_0^1 \frac{dx}{(1+a^2x^2)\sqrt{1-x^2}}$$

Put $x = \sin t \quad \Rightarrow \quad dx = \cos t\, dt$

L.L. : $x = 0 \quad \Rightarrow \quad t = 0$

U.L. : $x = 1 \quad \Rightarrow \quad t = \frac{\pi}{2}$

$$\frac{dI(a)}{da} = \int_0^{\frac{\pi}{2}} \frac{1}{1+a^2\sin^2 t} \frac{1}{\cos t} \cos t\, dt = \int_0^{\frac{\pi}{2}} \frac{dt}{1+a^2\sin^2 t}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 t\, dt}{1+(1+a^2)\tan^2 t} = \frac{1}{\sqrt{1+a^2}} \tan^{-1}\left(\sqrt{1+a^2}\,\tan t\right)\Big]_0^{\frac{\pi}{2}} = \frac{1}{\sqrt{1+a^2}} \cdot \frac{\pi}{2}$$

$\Rightarrow \quad I(a) = \frac{\pi}{2}\ \ell n\left(a + \sqrt{1+a^2}\right) + c$

But $I(0) = 0 \quad \Rightarrow \quad c = 0$

$\Rightarrow \quad I(a) = \frac{\pi}{2}\ \ell n\left(a + \sqrt{1+a^2}\right)$

Problems for Self Practice -08:

(1) If $f(x) = \int_0^{x^3} \sqrt{\cos t}\, dt$, find $f'(x)$.

(2) If $f(x) = e^{g(x)}$ and $g(x) = \int_2^{x} \frac{t}{1+t^4}\, dt$, then find the value of $f'(2)$.

(3) If $x = \int_0^{y} \frac{dt}{\sqrt{1+4t^2}}$ and $\frac{d^2y}{dx^2} = Ry$, then find R

(4) If $f(x) = \int_x^{x^2} x^2 \sin t\, dt$, then find $f'(x)$.

(5) If $\phi(x) = \cos x - \int_0^{x} (x-t)\,\phi(t)\, dt$, then find the value of $\phi''(x) + \phi(x)$.

(6) Find the value of the function $f(x) = 1 + x + \int_1^{x} \left((\ell n t)^2 + 2\ell n t\right) dt$, where $f'(x)$ vanishes.

(7) Evaluate $\underset{x\to 0}{\text{Lt}} \frac{\int_0^{x^2} \cos t^2\, dt}{x \sin x}$.

(8) Evaluate $\int_0^{\pi} \ell n(1 + b\cos x)\, dx$, 'b' being parameter.

(9) If $f(x) = \int_{1/x}^{\sqrt{x}} \sin t\, dt$, then find $f'(1)$.

(10) $\int_{\pi/3}^{x} \sqrt{3-\sin^2 t}\, dt + \int_0^{y} \cos t\, dt = 0$, then evaluate $\frac{dy}{dx}$.

Answers :

(1) $3x^2\sqrt{\cos x^3}$ (2) $\frac{2}{17}$ (3) 4

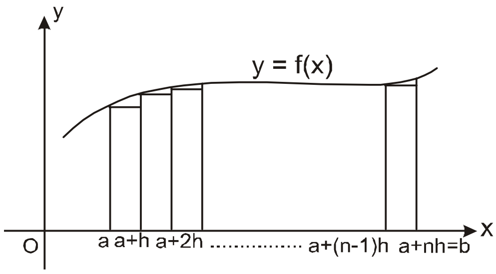
(4) $x^2(2x \sin x^2 - \sin x) + (\cos x - \cos x^2)\, 2x$ (5) $-\cos x$

(6) $1 + \frac{2}{e}$ (7) 1 (8) $\pi\, \ell n\left(\frac{1+\sqrt{1-b^2}}{2}\right)$

(9) $\frac{3}{2}\sin 1$ (10) $\frac{dy}{dx} = \frac{-\sqrt{3-\sin^2 x}}{\cos y}$

5. DEFINITE INTEGRAL AS LIMIT OF A SUM :

Let f(x) be a continuous real valued function defined on the closed interval [a, b] which is divided into n parts as shown in figure.



The point of division on x-axis are a, a + h, a + 2h ....a + (n – 1)h, a + nh, where $\frac{b-a}{n} = h$.

Let S_n denotes the area of these n rectangles.

Then, $S_n = hf(a) + hf(a + h) + hf(a + 2h) + \ldots\ldots + hf(a + (n-1)h)$

Clearly, S_n is area very close to the area of the region bounded by curve y = f(x), x–axis and the ordinates x = a, x = b.

Hence $\int_a^b f(x)\,dx = \underset{n\to\infty}{\text{Lt}}\ S_n$

$$\int_a^b f(x)\,dx = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} h\,f(a+rh) = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1}\left(\frac{b-a}{n}\right) f\left(a+\frac{(b-a)r}{n}\right)$$

Note :

1. We can also write

$S_n = hf(a + h) + hf(a + 2h) + \ldots\ldots + hf(a + nh)$ and $\int_a^b f(x)\,dx = \underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{n}\left(\frac{b-a}{n}\right) f\left(a+\left(\frac{b-a}{n}\right)r\right)$

2. If a = 0, b = 1, $\int_0^1 f(x)\,dx = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1}\frac{1}{n} f\left(\frac{r}{n}\right)$

An alternative way of describing $\int_a^b f(x)dx$ is that the definite integral $\int_a^b f(x)dx$ is a limiting case of the summation of an infinite series, provided $f(x)$ is continuous on [a,b]

i.e. $\int_a^b f(x)dx = \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh)$ where $h = \frac{b-a}{n}$. The converse is also true i.e., if we have an infinite series of the above form, it can be expressed as a definite integral.

Step I : Express the given series in the form $\sum \frac{1}{n} f\left(\frac{r}{n}\right)$

Step II : Then the limit is its sum when $n \to \infty$, i.e. $\lim\limits_{n\to\infty} \frac{1}{n} f\left(\frac{r}{n}\right)$

Step III : Replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx and $\lim\limits_{n\to\infty} \sum$ by the sign of $\int$

Step IV : The lower and the upper limit of integration are the limiting values of $\frac{r}{n}$ for the first and the last term of r respectively.

SOLVED EXAMPLE

Example 27 : Evaluate (i) $\lim\limits_{n\to\infty}\left(\frac{1}{2n+1}+\frac{1}{2n+2}+\ldots\ldots\ldots+\frac{1}{6n}\right)$

(ii) $\lim\limits_{n\to\infty}\left[\frac{\sqrt{n}}{\left(3+4\sqrt{n}\right)^2}+\frac{\sqrt{n}}{\sqrt{2}\left(3\sqrt{2}+4\sqrt{n}\right)^2}+\frac{\sqrt{n}}{\sqrt{3}\left(3\sqrt{3}+4\sqrt{n}\right)^2}+\ldots\ldots+\frac{1}{49n}\right]$

Solution : (i) Let $S_n = \frac{1}{2n+1}+\frac{1}{2n+2}+\ldots\ldots\ldots+\frac{1}{6n} = \sum_{r=1}^{4n}\frac{1}{2n+r} = \sum_{r=1}^{4n}\frac{1}{n}.\frac{1}{2+\left(\frac{r}{n}\right)}$

$$\Rightarrow \quad S = \lim_{n\to\infty} S_n = \int_0^4 \frac{dx}{2+x} = [\ell n\,|\,2+x\,|]_0^4 = \ell n6 - \ell n2 = \ell n3$$

(ii) Let $p = \lim\limits_{n\to\infty}\left[\frac{\sqrt{n}}{\left(3+4\sqrt{n}\right)^2}+\frac{\sqrt{n}}{\sqrt{2}\left(3\sqrt{2}+4\sqrt{n}\right)^2}+\ldots\ldots\ldots+\frac{\sqrt{n}}{\sqrt{n}\left(3\sqrt{n}+4\sqrt{n}\right)^2}\right]$

Analyzing the expression with the view of increasing integral value we get the expression in terms of r as

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n}}{\sqrt{r}\left(3\sqrt{r}+4\sqrt{n}\right)^2} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n\sqrt{\frac{r}{n}}\left(3\sqrt{\frac{r}{n}}+4\right)^2} = \int_0^1\frac{dx}{\sqrt{x}\left(3\sqrt{x}+4\right)^2}$$

Put $3\sqrt{x}+4=t, \qquad \therefore \quad \frac{3}{2\sqrt{x}}dx = dt$

Hence $p = \frac{2}{3}\int_4^7\frac{dt}{t^2} = \frac{2}{3}\left[-\frac{1}{t}\right]_4^7 = \frac{2}{3}\left(-\frac{1}{7}+\frac{1}{4}\right) = \frac{1}{14}$ **Ans.**

Example 28 : Evaluate $\underset{n\to\infty}{\text{Lt}} \left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$.

Solution : Let $y = \underset{n\to\infty}{\text{Lt}} \left(\frac{n!}{n^n}\right)^{\frac{1}{n}} \Rightarrow \ell n\, y = \underset{n\to\infty}{\text{Lt}} \frac{1}{n} \ell n \left(\frac{n!}{n^n}\right) = \underset{n\to\infty}{\text{Lt}} \frac{1}{n} \ell n \left(\frac{1.2.3........n}{n^n}\right)$

$= \underset{n\to\infty}{\text{Lt}} \frac{1}{n} \left[\ell n\left(\frac{1}{n}\right) + \ell n\left(\frac{2}{n}\right) + \ell n\left(\frac{3}{n}\right) + ... + \ell n\left(\frac{n}{n}\right)\right]$

$= \underset{n\to\infty}{\text{Lt}} \frac{1}{n} \sum_{r=1}^{n} \ell n\left(\frac{r}{n}\right) = \int_0^1 \ell nx\, dx = x\,\ell n x - x\Big]_0^1 = (0-1) - \underset{x\to0^+}{\text{Lt}} x\, \ell n\, x + 0$

$= -1 - 0 = -1 \qquad \Rightarrow \qquad y = \frac{1}{e}$

<u>Problems for Self Practice</u> -09:

Evaluate the following limits

(1) $\underset{n\to\infty}{\text{Lt}} \left[\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+2n}} + ... + \frac{1}{\sqrt{n^2+n^2}}\right]$

(2) $\underset{n\to\infty}{\text{Lt}} \left[\frac{1}{1+n} + \frac{1}{2+n} + \frac{1}{3+n} + ... + \frac{1}{5n}\right]$

(3) $\underset{n\to\infty}{\text{Lt}} \frac{1}{n^2} \left[\sin^3\frac{\pi}{4n} + 2\sin^3\frac{2\pi}{4n} + 3\sin^3\frac{3\pi}{4n} + ... + n\sin^3\frac{n\pi}{4n}\right]$

(4) $\underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} \frac{1}{\sqrt{n^2-r^2}}$

(5) $\underset{n\to\infty}{\text{Lt}} \frac{3}{n} \left[1 + \sqrt{\frac{n}{n+3}} + \sqrt{\frac{n}{n+6}} + \sqrt{\frac{n}{n+9}} + ... + \sqrt{\frac{n}{n+3(n-1)}}\right]$

(6) $\lim_{n\to\infty}\left[\frac{1}{n+2.1} + \frac{1}{n+2.2} + \frac{1}{n+2.3} .... \frac{1}{3n}\right]$

Answers : (1) $2\left(\sqrt{2}-1\right)$ (2) $\ell n\, 5$ (3) $\frac{\sqrt{2}}{9\pi^2}(52-15\pi)$

(4) $\frac{\pi}{2}$ (5) 2 (6) $\frac{1}{2}\ell n3$

6. ESTIMATION OF DEFINITE INTEGRAL :

(a) If f(x) is continuous in [a, b] and it's range in this interval is [m, M], then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

Further if f(x) is monotonically decreasing in (a, b), then $f(b)(b-a) < \int_a^b f(x)\,dx < f(a)(b-a)$ and if f(x) is monotonically increasing in (a, b), then $f(a)(b-a) < \int_a^b f(x)\,dx < f(b)(b-a)$

SOLVED EXAMPLE

Example 29 : Prove that (i) $4 \le \int_1^3 \sqrt{3+x^3}dx \le 2\sqrt{30}$ (ii) $1 < \int_0^{\pi/2} \frac{\sin x}{x}\,dx < \frac{\pi}{2}$

Solution : (i) Since the function f(x) = $\sqrt{3+x^3}$ increases monotonically on the interval [1, 3], m = 2, M = $\sqrt{30}$, b – a =2.

Hence, $2.2 \le \int_1^3 \sqrt{3+x^3}dx \le 2\sqrt{30} \Rightarrow 4 \le \int_1^3 \sqrt{3+x^3}dx \le 2\sqrt{30}$

(ii) Let $f(x) = \frac{\sin x}{x}$

$$f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{(\cos x)(x - \tan x)}{x^2} < 0$$

$\Rightarrow$ f(x) is monotonically decreasing function.

f(0) is not defined, so we evaluate

$\underset{x\to 0^+}{\text{Lt}} f(x) = \underset{x\to 0^+}{\text{Lt}} \frac{\sin x}{x} = 1$. Take $f(0) = \underset{x\to 0^+}{\text{Lt}} f(x) = 1$

$$f\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$$

$$\frac{2}{\pi}\cdot\left(\frac{\pi}{2}-0\right) < \int_0^{\pi/2} \frac{\sin x}{x}\,dx < 1\cdot\left(\frac{\pi}{2}-0\right) \quad \Rightarrow \quad 1 < \int_0^{\pi/2} \frac{\sin x}{x}\,dx < \frac{\pi}{2}$$

(b) If $f(x) \le \phi(x)$ for $a \le x \le b$ then $\int_a^b f(x)dx \le \int_a^b \phi(x)\,dx$

Example 30 For $x \in (0, 1)$ arrange $f_1(x) = \frac{1}{\sqrt{4-x^2}}$, $f_2(x) = \frac{1}{\sqrt{4-2x^2}}$ and $f_3(x) = \frac{1}{\sqrt{4-x^2-x^3}}$ in ascending order and hence prove that $\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi}{4\sqrt{2}}$.

Solution $\because$ $0 < x^3 < x^2$ $\Rightarrow$ $x^2 < x^2 + x^3 < 2x^2$

$\Rightarrow$ $-2x^2 < -x^2 - x^3 < -x^2$ $\Rightarrow$ $4 - 2x^2 < 4 - x^2 - x^3 < 4 - x^2$

$\Rightarrow$ $\sqrt{4-2x^2} < \sqrt{4-x^2-x^3} < \sqrt{4-x^2}$ $\Rightarrow$ $f_1(x) < f_3(x) < f_2(x)$ for $x \in (0, 1)$

$$\Rightarrow \int_0^1 f_1(x)\,dx < \int_0^1 f_3(x)\,dx < \int_0^1 f_2(x)\,dx$$

$$\sin^{-1}\left(\frac{x}{2}\right)\Big]_0^1 < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{1}{\sqrt{2}}\sin^{-1}\frac{x}{\sqrt{2}}\Big]_0^1 \Rightarrow \frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi}{4\sqrt{2}}$$

Example 31 : Prove that $\frac{\pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{\pi}{4\sqrt{2}}$

Solution : Since $4 - x^2 \ge 4 - x^2 - x^3 \ge 4 - 2x^2 > 0 \ \forall\ x \in [0, 1]$

$$\sqrt{4-x^2} \ge \sqrt{4-x^2-x^3} \ge \sqrt{4-2x^2} > 0 \ \forall\ x \in [0, 1]$$

$$\Rightarrow \quad 0 < \frac{1}{\sqrt{4-x^2}} \le \frac{1}{\sqrt{4-x^2-x^3}} \le \frac{1}{\sqrt{4-2x^2}} \ \forall\ x \in [0\ 1]$$

$$\Rightarrow \quad \int_0^1 \frac{dx}{\sqrt{4-x^2}} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \int_0^1 \frac{dx}{\sqrt{4-2x^2}} \ \forall x \in [0, 1]$$

$$\Rightarrow \quad \left[\sin^{-1}\frac{x}{2}\right]_0^1 \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^1$$

$$\Rightarrow \frac{\pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{\pi}{4\sqrt{2}}$$ **Ans.**

(c) $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$.

Example 32 : Prove that $\left|\int_{10}^{19} \frac{\sin x}{1+x^8}dx\right| < 10^{-7}$

Solution : To find $I = \left|\int_{10}^{19} \frac{\sin x}{1+x^8}dx\right| \le \int_{10}^{19}\left|\frac{\sin x}{1+x^8}\right|dx$ .... (i)

Since $|\sin x| \le 1 \text{ for } x \ge 10$

The inequality $\left|\frac{\sin x}{1+x^8}\right| \le \frac{1}{|1+x^8|}$ .... (ii)

also, $10 \le x \le 19$

$\Rightarrow$ $1 + x^8 > 10^8$

$\Rightarrow$ $\frac{1}{1+x^8} < \frac{1}{10^8}$ or $\frac{1}{|1+x^8|} < 10^{-8}$ .... (iii)

from (ii) and (iii) ;

$\left|\frac{\sin x}{1+x^8}\right| < 10^{-8}$ $\Rightarrow$ $\left|\int_{10}^{19} \frac{\sin x}{1+x^8}dx\right| < \int_{10}^{19} 10^{-8}dx$

$\therefore$ $\left|\int_{10}^{19} \frac{\sin x}{1+x^8}dx\right| < (19-10).10^{-8} < 10^{-7}$ **Ans.**

Example 33 : If $f(x)$ is integrable function such that $|f(x) - f(y)| \le |x^2 - y^2|$, $\forall$ x, y $\in$ [a,b] then prove that $\left|\int_a^b \frac{f(x)-f(a)}{x+a}dx\right| \le \frac{(a-b)^2}{2}$.

Solution : Given, $\left|\int_a^b \frac{f(x)-f(a)}{x+a}dx\right| \le \int_a^b \left|\frac{f(x)-f(a)}{x+a}\right|dx$

$$\le \int_a^b \left|\frac{x^2-a^2}{x+a}\right|dx = \int_a^b |x-a|\,dx = \int_a^b (x-a)dx = \frac{(a-b)^2}{2}$$

(d) If $f(x) \geq 0$ on the interval [a, b], then $\int_a^b f(x)dx \geq 0$.

Example 34 : If f (x) is a continous function such that $f(x) \geq 0 \; \forall \; x \in [2,10]$ and $\int_4^8 f(x)\,dx = 0$, then find f (6).

Solution : f (x) is above the x-axis or on the x-axis for all $x \in [2,10]$. If f(x) is greater than zero for any sub interval of [4,8], then $\int_4^8 f(x)dx$ must be greater than zero.

But $\int_4^8 f(x)dx = 0$

$\Rightarrow f(x) = 0 \; \forall \; x \in [4,8]$

$\Rightarrow f(6) = 0$.

Problems for Self Practice -10:

Prove the following :

(1) $\int_0^1 e^{-x}\cos^2 x \; dx < \int_0^1 e^{-x^2}\cos^2 x \; dx$

(2) $0 < \int_0^{\frac{\pi}{2}} \sin^{n+1} x \; dx < \int_0^{\frac{\pi}{2}} \sin^2 x \, dx$, n > 1

(3) $e^{-\frac{1}{4}} < \int_0^1 e^{x^2-x} \; dx < 1$

(4) $0 \leq \int_0^1 \frac{x^3 \cos x}{2+x^2} \; dx < \frac{1}{2}$

(5) $1 < \int_0^{\frac{\pi}{2}} \sqrt{\sin x} \; dx < \sqrt{\frac{\pi}{2}}$

(6) $0 < \int_0^2 \frac{x\,dx}{16+x^3} < \frac{1}{6}$

(7) $4 \leq \int_1^3 \sqrt{3+x^2}dx \leq 4\sqrt{3}$

(8) $\frac{\pi}{4} \leq \int_0^{2\pi} \frac{dx}{5+3\sin x} \leq \pi$

(9) $\frac{3}{5}\left(2^{1/3}-1\right) \leq \int_0^1 \frac{x^4}{(1+x^6)^{2/3}}dx \leq 1$

Miscelleneous Illustrations :

Example 35 : Evaluate : $\int\limits_0^{\pi} \frac{x^3 \cos^4 x \sin^2 x}{(\pi^2 - 3\pi x + 3x^2)} dx$

Solution : Let $I = \int\limits_0^{\pi} \frac{x^3 \cos^4 x \sin^2 x}{(\pi^2 - 3\pi x + 3x^2)} dx$ .... (i)

$= \int\limits_0^{\pi} \frac{(\pi - x)^3 \cos^4(\pi - x) \sin^2(\pi - x) dx}{\pi^2 - 3\pi(\pi - x) + 3(\pi - x)^2}$ (By. Prop.)

$= \int\limits_0^{\pi} \frac{(\pi^3 - x^3 - 3\pi^2 x + 3\pi x^2) \cos^4 x \sin^2 x}{(\pi^2 - 3\pi x + 3x^2)} dx$ .... (ii)

Adding (i) and (ii) we have

$2I = \int\limits_0^{\pi} \frac{(\pi^3 - 3\pi^2 x + 3\pi x^2) \cos^4 x \sin^2 x}{(\pi^2 - 3\pi x + 3x^2)} dx$

$\Rightarrow$ $2I = \pi \int\limits_0^{\pi} \cos^4 x \sin^2 x dx$ $\Rightarrow$ $2I = 2\pi \int\limits_0^{\pi/2} \cos^4 x \sin^2 x dx$

$\therefore$ $I = \pi \int\limits_0^{\pi/2} \cos^4 x \sin^2 x \, dx$

Using walli's formula, we get $I = \pi \frac{(3.1)(1)}{6.4.2} \frac{\pi}{2} = \frac{\pi^2}{32}$

Example 36 : Let f be an injective function such that f(x) f(y) + 2 = f(x) + f(y) + f(xy) for all non negative real x and y with f(0) = 1 and f'(1) = 2 find f(x) and show that $3\int f(x)dx - x(f(x)+2)$ is a constant.

Solution : We have f(x)f(y) + 2 = f(x) + f(y) + f(xy)

Putting x = 1 & y = 1

then f(1)f(1) + 2 = 3f(1)

we get f(1) = 1,2

f(1) $\neq$ 1 ($\because$ f(0) = 1 & function is injective)

then f(1) = 2

Replacing y by $\frac{1}{x}$ in (1) then

$$f(x)f\left(\frac{1}{x}\right)+2=f(x)+f\left(\frac{1}{x}\right)+f(1) \Rightarrow f(x)f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)$$

Hence f(x) is of the type

$f(x) = 1 \pm x^n$

$\because$ $f(1) = 2$

$\therefore$ $f(x) = 1 + x^n$

and $f'(x) = nx^{n-1} \Rightarrow f'(1) = n = 2$

$f(x) = 1 + x^2$

$\therefore$ $$3\int f(x)dx - x(f(x)+2) = 3\int (1+x^2)dx - x(1+x^2+2)$$

$$= 3\left(x+\frac{x^3}{3}\right) - x(3+x^2) + c = c = \text{constant}$$

Example 37 : Evaluate : $\int_{-1}^{1}[x[1+\sin\pi x]+1]dx$, [.] is the greatest integer function.

Solution : Let $I = \int_{-1}^{1}[x[1+\sin\pi x]+1]dx = \int_{-1}^{0}[x[1+\sin\pi x]+1]dx + \int_{0}^{1}[x[1+\sin\pi x]+1]dx$

Now $[1 + \sin\pi x] = 0$ if $-1 < x < 0$

$[1 + \sin\pi x] = 1$ if $0 < x < 1$

$\therefore$ $I = \int_{-1}^{0}1.dx + \int_{0}^{1}[x+1]dx = 1 + 1\int_{0}^{1}dx = 1 + 1 = 2.$

Example 38 : Find the limit, when $n \to \infty$ of

$$\frac{1}{\sqrt{(2n-1^2)}}+\frac{1}{\sqrt{(4n-2^2)}}+\frac{1}{\sqrt{(6n-3^2)}}+.....+\frac{1}{n}$$

Solution : Let $P = \lim\limits_{n\to\infty}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\frac{1}{\sqrt{6n-3^2}}+....+\frac{1}{n}\right]$

$$=\lim_{n\to\infty}\left[\frac{1}{\sqrt{1(2n)-1^2}}+\frac{1}{\sqrt{2(2n)-2^2}}+\frac{1}{\sqrt{3(2n)-3^2}}+....+\frac{1}{\sqrt{n(2n)-n^2}}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{r(2n)-r^2}}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n.\sqrt{2\frac{r}{n}-\left(\frac{r}{n}\right)^2}}=\int_0^1\frac{dx}{\sqrt{(2x-x^2)}}$$

Put x = t² ⇒ dx = 2t dt

$\therefore$ P = $\int_0^1 \frac{2t\,dt}{t\sqrt{2-t^2}} = \left[2\sin^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_0^1 = 2\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 2\left(\frac{\pi}{4}\right)$

Hence P = π/2.

Example 39 : If f(x) = $\begin{cases} 1-|x|, & |x| \le 1 \\ |x|-1, & |x| > 1 \end{cases}$, and g(x) = f(x – 1) + f(x + 1). Find the value of $\int_{-3}^{5} g(x)\,dx$.

Solution : Given,

f(x) = $\begin{cases} -x-1, & x < -1 \\ 1+x, & -1 \le x < 0 \\ 1-x, & 0 \le x \le 1 \\ x-1, & x > 1 \end{cases}$; f(x–1) = $\begin{cases} -x, & x-1 < -1 & \Rightarrow x < 0 \\ x, & -1 \le x-1 < 0 & \Rightarrow 0 \le x < 1 \\ 2-x, & 0 \le x-1 \le 1 & \Rightarrow 1 \le x \le 2 \\ x-2, & x-1 > 1 & \Rightarrow x > 2 \end{cases}$

Similarly

f(x+1) = $\begin{cases} -x-2, & x+1 < -1 & \Rightarrow x < -2 \\ x+2, & -1 \le x+1 < 0 & \Rightarrow -2 \le x < -1 \\ -x, & 0 \le x+1 \le 1 & \Rightarrow -1 \le x \le 0 \\ x, & x+1 > 1 & \Rightarrow x > 0 \end{cases}$

⇒ g(x) = f(x – 1) + f(x + 1) = $\begin{cases} -2x-2 & x < -2 \\ 2, & -2 \le x < -1 \\ -2x, & -1 \le x \le 0 \\ 2x, & 0 < x < 1 \\ 2, & 1 < x \le 2 \\ 2x-2, & 2 < x \end{cases}$

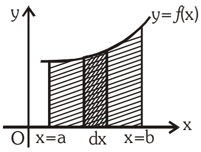
Clearly g(x) is even,

Now $\int_{-3}^{5} g(x)dx = 2\int_0^3 g(x)\,dx + \int_3^5 g(x)\,dx$

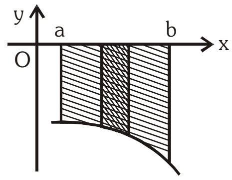
$= 2\left(\int_0^1 2x\,dx + \int_1^2 2dx + \int_2^3 (2x-2)\,dx\right) + \int_3^5 (2x-2)\,dx = 24$

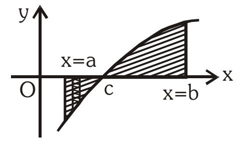
7. AREA UNDER THE CURVES :

(a) Area bounded by the curve y = f(x), the x-axis and the ordinates at x = a and x = b is given by A = $\int_a^b y\,dx$, where y = f (x) lies above the x-axis and b > a. Here vertical strip of thickness dx is considered at distance x.



(b) If y = f (x) lies completely below the x-axis then A is negative and we consider the magnitude only, i.e. A = $\left|\int_a^b y\,dx\right|$





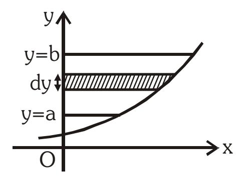
(c) If curve crosses the x-axis at x = c, then A = $\left|\int_a^c y\,dx\right| + \left|\int_c^b y dx\right|$

For right most digram, A = $-\int_a^c y\,dx + \int_c^b y dx$

(d) Sometimes integration w.r.t. y is very useful (horizontal strip) :

Area bounded by the curve, y-axis and the two abscissae at

y = a & y = b is written as $A = \int_a^b x dy$.



Note : If the curve is symmetric and suppose it has 'n' symmetric portions, then total area = n (Area of one symmetric portion).

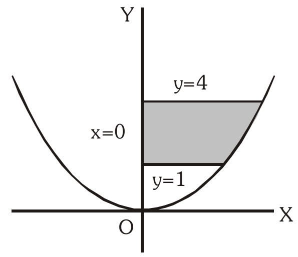
SOLVED EXAMPLE

Example 40 : Find the area bounded by y = $\sec^2$ x, x = $\frac{\pi}{6}$, x = $\frac{\pi}{3}$ & x-axis

Solution : Area bounded = $\int_{\pi/6}^{\pi/3} y dx = \int_{\pi/6}^{\pi/3} \sec^2 x dx$

$= [\tan x]_{\pi/6}^{\pi/3}$ = tan $\frac{\pi}{3}$ – tan $\frac{\pi}{6}$ = $\sqrt{3} - \frac{1}{\sqrt{3}}$ = $\frac{2}{\sqrt{3}}$ sq.units.

Example 41 : Find the area in the first quadrant bounded by y = $4x^2$, x = 0, y = 1 and y = 4.

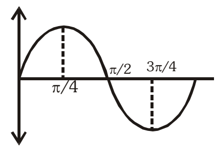


Solution : Required area $= \int_1^4 x\,dy = \int_1^4 \frac{\sqrt{y}}{2}\,dy = \frac{1}{2}\left[\frac{2}{3}y^{3/2}\right]_1^4$

$= \frac{1}{3}[4^{3/2} - 1] = \frac{1}{3}[8 - 1] = \frac{7}{3} = 2\frac{1}{3}$ sq.units.

Example 42 Find the area bounded by the curve y = sin2x, x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$

Solution : Required area $= \int_{\pi/4}^{\pi/2} \sin 2x dx + \left|\int_{\pi/2}^{3\pi/4} \sin 2x dx\right| = \left(-\frac{\cos 2x}{2}\right)\Bigg|_{\pi/4}^{\pi/2} + \left|\left(-\frac{\cos 2x}{2}\right)\Bigg|_{\pi/2}^{3\pi/4}\right|$



$= -\frac{1}{2}[-1-0] + \left|\frac{1}{2}(0+(-1))\right| = 1$ sq. unit

Problems for Self Practice -11:

(1) Find the area bounded by $y = x^2 + 2$ above x-axis between x = 2 & x = 3.

(2) Using integration, find the area of the curve $y = \sqrt{1-x^2}$ with co-ordinate axes bounded in first quadrant.

(3) Find the area bounded by the curve y = 2cosx and the x-axis from x = 0 to x = 2π.

(4) Find the area bounded by the curve y = x|x|, x-axis and the ordinates $x = -\frac{1}{2}$ and x=1.

(5) Find the area of the region bounded by the x-axis and the curves defined by y = tan x, $\left(\text{where} \frac{-\pi}{3} \le x \le \frac{\pi}{3}\right)$ and y = cot x $\left(\text{where} \frac{\pi}{6} \le x \le \frac{2\pi}{3}\right)$.

(6) Find the area of the region bounded by the parabola $(y-2)^2 = (x-1)$ and the tangent to it at ordinate y = 3 and x–axis.

(7) Find the area included between $y = \tan^{-1}x$, $y = \cot^{-1}x$ and y–axis.

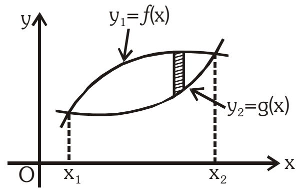
Answers : (1) $\frac{25}{3}$ sq. units (2) $\frac{\pi}{4}$ sq. units. (3) 8 sq. units.

(4) $\frac{3}{8}$ sq.units (5) $\ell n \frac{3}{2}$ (6) 9

(7) ℓn2

8. AREA ENCLOSED BETWEEN TWO CURVES :

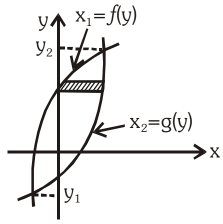
(a) Area bounded by two curves y = f (x) & y = g(x) such that f (x) > g(x) is



$$A = \int_{x_1}^{x_2} (y_1 - y_2)dy$$

$$A = \int_{x_1}^{x_2} [f(x) - g(x)]dx$$

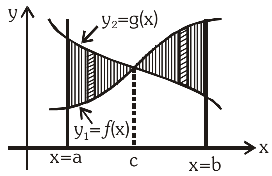
(b) In case horizontal strip is taken we have



$$A = \int_{y_1}^{y_2} (x_1 - x_2)dy$$

$$A = \int_{y_1}^{y_2} [f(y) - g(y)]dy$$

(c) If the curves $y_1 = f$ (x) and y_2 = g(x) intersect at x = c, then required area



A = $\int_a^c (g(x) - f(x))dx + \int_c^b (f(x) - g(x))dx = \int_a^b |f(x) - g(x)| dx$

Note : Required area must have all the boundaries indicated in the problem.

SOLVED EXAMPLE

Example 43 : Find the area bounded by the curve y = (x – 1) (x – 2) (x – 3) lying between the ordinates x = 0 and x = 3 and x-axis

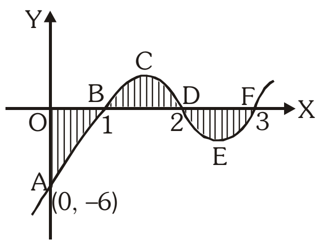
Solution : To determine the sign, we follow the usual rule of change of sign.

y = +ve for x > 3

y = –ve for 2 < x < 3

y = +ve for 1 < x < 2

y = –ve for x < 1.



$$\int_0^3 |y| dx = \int_0^1 |y| dx + \int_1^2 |y| dx + \int_2^3 |y| dx$$

= $\int_0^1 -y\,dx + \int_1^2 y\,dx + \int_2^3 -y\,dx$

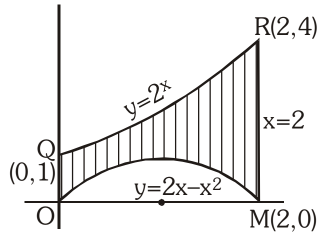
Now let F(x) $= \int (x-1)(x-2)(x-3)\,dx = \int (x^3 - 6x^2 + 11x - 6)\,dx = \frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 - 6x$.

$\therefore$ F(0) = 0, $F(1) = -\frac{9}{4}$, F(2) = –2, $F(3) = -\frac{9}{4}$.

Hence required Area = – [F(1) – F(0)] + [F(2) – F(1)] – [F(3) – F(2)] = $2\frac{3}{4}$ sq.units.

Example 44 : Compute the area of the figure bounded by the straight lines x = 0, x = 2 and the curves $y = 2^x$, $y = 2x - x^2$.

Solution : The required area $= \int_0^2 (y_1 - y_2)\,dx$



where $y_1 = 2^x$ and $y_2 = 2x - x^2$ = $\int_0^2 (2^x - 2x + x^2)dx$

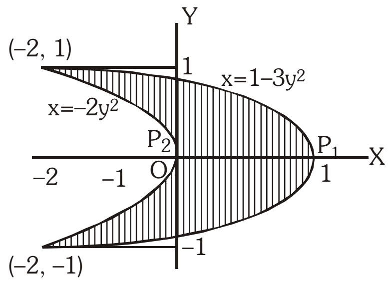
$= \left[\frac{2^x}{\ln 2} - x^2 + \frac{1}{3}x^3\right]_0^2 = \left(\frac{4}{\ln 2} - 4 + \frac{8}{3}\right) = \frac{1}{\ln 2} = \frac{3}{\ln 2} - \frac{4}{3}$ sq.units.

Example 45 : Compute the area of the figure bounded by the parabolas $x = -2y^2$, $x = 1 - 3y^2$.

Solution : Solving the equations $x = -2y^2$, $x = 1 - 3y^2$, we find that ordinates of the points of intersection of the two curves as $y_1 = -1$, $y_2 = 1$.

The points are (–2, –1) and (–2, 1).

The required area



$2\int_0^1 (x_1 - x_2)\,dy = 2\int_0^1 [(1-3y^2) - (-2y^2)]dy$

$= 2\int_0^1 (1-y^2)dy = 2\left[y - \frac{y^3}{3}\right]_0^1 = \frac{4}{3}$ sq.units.

Example 46 : Find the area bounded by the regions $y \ge \sqrt{x}$, $x > -\sqrt{y}$ & curve $x^2 + y^2 = 2$.

Solution : Common region is given by the diagram

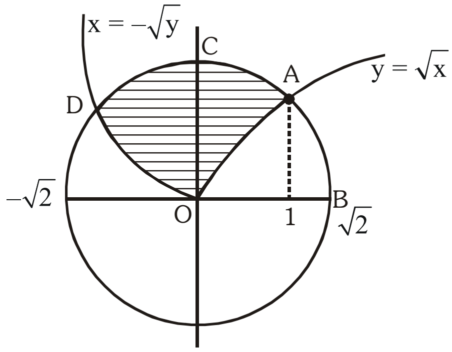
If area of region OAB = λ

then area of OCD = λ

Because $y = \sqrt{x}$ & $x = -\sqrt{y}$

will bound same area with x & y axes respectively.

$y = \sqrt{x} \Rightarrow y^2 = x$



$x = -\sqrt{y} \Rightarrow x^2 = y$ and hence both the curves are symmetric with respect to the line y = x

Area of first quadrant OBC = $\frac{\pi r^2}{4} = \frac{\pi}{2}$ ($\because$ $r = \sqrt{2}$)

Area of region OCA = $\frac{\pi}{2} - \lambda$

Area of shaded region = $(\frac{\pi}{2} - \lambda) + \lambda = \frac{\pi}{2}$ sq.units.

Example 47 : For any real t, $x = \frac{1}{2}(e^t + e^{-t})$, $y = \frac{1}{2}(e^t - e^{-t})$ is point on the hyperbola $x^2 - y^2 = 1$. Show that the area bounded by the hyperbola and the lines joining its centre to the points corresponding to t_1 and $-t_1$ is t_1.

Solution : It is a point on hyperbola $x^2 - y^2 = 1$.

$$\text{Area (PQRP)} = 2\int_1^{\frac{e^{t_1}+e^{-t_1}}{2}} y\,dx = 2\int_1^{\frac{e^{t_1}+e^{-t_1}}{2}} \sqrt{x^2-1}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ell n(x+\sqrt{x^2-1})\right]_1^{\frac{e^{t_1}+e^{-t_1}}{2}} = \frac{e^{2t_1} - e^{-2t_1}}{4} - t_1$$

$$\text{Area of } \Delta OPQ = 2\times \frac{1}{2}\left(\frac{e^{t_1}+e^{-t_1}}{2}\right)\left(\frac{e^{t_1}-e^{-t_1}}{2}\right) = \frac{e^{2t_1}+e^{-2t_1}}{4} - t_1.$$

$\therefore$ Required area = area ΔOPQ – area (PQRP)

$= t_1$

Example 48 : Find area contained by ellipse $2x^2 + 6xy + 5y^2 = 1$

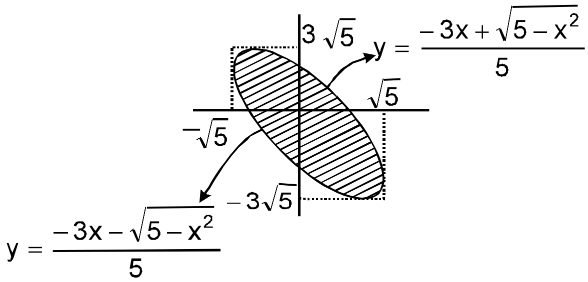
Solution : $5y^2 + 6xy + 2x^2 - 1 = 0$

$$y = \frac{-6x \pm \sqrt{36x^2 - 20(2x^2-1)}}{10}$$

$$y = \frac{-3x \pm \sqrt{5-x^2}}{5}$$

$\because$ y is real $\Rightarrow$ R.H.S. is also real.

$\Rightarrow$ $-\sqrt{5} \le x \le \sqrt{5}$



If $x = -\sqrt{5}$, $y = 3\sqrt{5}$

If $x = \sqrt{5}$, $y = -3\sqrt{5}$

If $x = 0$, $y = \pm \frac{1}{\sqrt{5}}$

If $y = 0$, $x = \pm \frac{1}{\sqrt{2}}$

Required area $= \int_{-\sqrt{5}}^{\sqrt{5}} \left(\frac{-3x+\sqrt{5-x^2}}{5} - \frac{-3x-\sqrt{5-x^2}}{5} \right) dx$

$= \frac{2}{5} \int_{-\sqrt{5}}^{\sqrt{5}} \sqrt{5-x^2}\, dx = \frac{4}{5} \int_{0}^{\sqrt{5}} \sqrt{5-x^2}\, dx$

Put $x = \sqrt{5} \sin\theta : dx = \sqrt{5} \cos\theta\, d\theta$

L.L : $x = 0 \Rightarrow \theta = 0$

U.L : $x = \sqrt{5} \Rightarrow \theta = \frac{\pi}{2}$ $= \frac{4}{5} \int_{\theta=0}^{\frac{\pi}{2}} \sqrt{5-5\sin^2\theta}\ \sqrt{5}\cos\theta d\theta = 4 \int_{0}^{\frac{\pi}{2}} \cos^2\theta d\theta = 4\ \frac{1}{2}\ \frac{\pi}{2} = \pi$

Example 49 : Find the area contained between the two arms of curves $(y - x)^2 = x^3$ between $x = 0$ and $x = 1$.

Solution : $(y - x)^2 = x^3 \Rightarrow y = x \pm x^{3/2}$

For arm $y = x + x^{3/2} \Rightarrow \frac{dy}{dx} = 1 + \frac{3}{2} x^{1/2} > 0 \quad x \geq 0.$

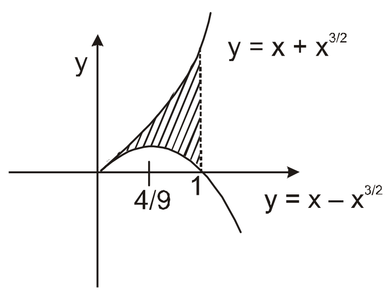
y is increasing function.

For arm

$y = x - x^{3/2} \Rightarrow \frac{dy}{dx} = 1 - \frac{3}{2} x^{1/2}$

$\frac{dy}{dx} = 0 \Rightarrow x = \frac{4}{9}$, $\frac{d^2y}{dx^2} = -\frac{3}{4} x^{-\frac{1}{2}} < 0$ at $x = \frac{4}{9}$

$\therefore$ at $x = \frac{4}{9}$, $y = x - x^{3/2}$ has maxima.



Figure

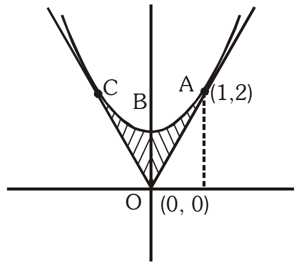
Required area $= \int_{0}^{1} (x + x^{3/2} - x + x^{3/2}) dx = 2 \int_{0}^{1} x^{3/2} dx = \left. \frac{2\, x^{5/2}}{5/2} \right]_0^1 = \frac{4}{5}$

Example 50 : Find the area bounded by $y = x^2 + 1$ and the tangents to it drawn from the origin

Solution : The parabola is even function & let the equation of tangent is $y = mx$

Now we calculate the point of intersection of parabola & tangent

$mx = x^2 + 1$



$x^2 - mx + 1 = 0 \Rightarrow D = 0$

$\Rightarrow m^2 - 4 = 0 \Rightarrow m = \pm 2$

Two tangents are possible $y = 2x$ & $y = -2x$

Intersection of $y = x^2 + 1$ & $y = 2x$ is $x = 1$ & $y = 2$

Area of shaded region OAB = $\int_0^1 (y_2 - y_1)\,dx = \int_0^1 ((x^2+1) - 2x)\,dx = \frac{1}{3}$ sq. units

Area of total shaded region = $2\left(\frac{1}{3}\right) = \frac{2}{3}$ sq. units

<u>Problems for Self Practice</u> -12:

(1) Find the area bounded by $y = \sqrt{x}$ and $y = x$.

(2) Find the area bounded by the curves $x = y^2$ and $x = 3 - 2y^2$.

(3) Find the area of the region bounded by the curves $x = \frac{1}{2}$, $x = 2$, $y = \log x$ and $y = 2^x$.

(4) Curves $y = \sin x$ and $y = \cos x$ intersect at infinite number of points forming regions of equal area between them calculate area of one such region.

(5) Find area common to circle $x^2 + y^2 = 2$ and the parabola $y^2 = x$.

(6) Find the area included between curves $y = \frac{4 - x^2}{4 + x^2}$ and $5y = 3|x| - 6$.

(7) Find the area bounded by the curve $|y| + \frac{1}{2} = e^{-|x|}$.

(8) Find the area enclosed by $|x| + |y| \le 3$ and $xy \ge 2$.

(9) Find are bounded by $x^2 + y^2 \le 2ax$ and $y^2 \ge ax$, $x \ge 0$.

Answers : (1) $\frac{1}{6}$ sq. units (2) 4 sq. units (3) $\frac{4-\sqrt{2}}{\log 2} - \frac{5}{2}\log 2 + \frac{3}{2}$ sq. units (4) $2\sqrt{2}$

(5) $\frac{\pi}{3} - \frac{\sqrt{3}}{2} - \frac{2}{3}$ (6) $2\pi - \frac{8}{5}$ (7) $2(1 - \ell n2)$ (8) $3 - 4\ell n2$ (9) $\left(\frac{3\pi - 8}{6}\right)a^2$

9. CURVE TRACING :

The following procedure is to be applied in sketching the graph of a function $y = f(x)$ which in turn will be extremely useful to quickly and correctly evaluate the area under the curves.

(a) Since area remains invariant even if the co-ordinate axes are shifted, hence shifting of origin in many cases proves to be very convenient in computing the area.

(b) Symmetry : The symmetry of the curve is judged as follows :

(i) If all the powers of y in the equation are even then the curve is symmetrical about the axis of x.

(ii) If all the powers of x are even, the curve is symmetrical about the axis of y.

(iii) If powers of x & y both are even, the curve is symmetrical about the axis of x as well as y.

(iv) If the equation of the curve remains unchanged on interchanging x and y, then the curve is symmetrical about $y = x$.

(v) If on interchanging the signs of x & y both, the equation of the curve is unaltered then there is symmetry in opposite quadrants.

(c) Find dy/dx & equate it to zero to find the points on the curve where you have horizontal tangents.

(d) Find the points where the curve crosses the x–axis & also the y–axis.

(e) Examine if possible the intervals when f(x) is increasing or decreasing.

(f) Examine what happens to 'y' when $x \to \infty$ or $-\infty$.

(g) **Asymptotes :**

Asymptote(s) is (are) line (s) whose distance from the curve tends to zero as point on curve moves towards infinity along branch of curve.

(i) If $\lim\limits_{x\to a} f(x) = \infty$ or $\mathop{\text{Lt}}\limits_{x\to a} f(x) = -\infty$, then $x = a$ is asymptote of $y = f(x)$

(ii) If $\lim\limits_{x\to\infty} f(x) = k$ or $\mathop{\text{Lt}}\limits_{x\to -\infty} f(x) = k$ then $y = k$ is asymptote of $y = f(x)$

(iii) If $\lim\limits_{x\to\infty} \frac{f(x)}{x} = m_1$, $\mathop{\text{Lt}}\limits_{x\to\infty} (f(x) - m_1x) = c$, then $y = m_1x + c_1$ is an asymptote (inclined to right).

(iv) If $\lim\limits_{x\to-\infty} \frac{f(x)}{x} = m_2$, $\lim\limits_{x\to-\infty} (f(x) - m_2x) = c_2$, then $y = m_2x + c_2$ is an asymptote (inclined to left).

9.1 Important Points

(i) Whole area of the ellipse, $x^2/a^2 + y^2/b^2 = 1$ is πab sq.units.

(ii) Area enclosed between the parabolas $y^2 = 4ax$ & $x^2 = 4by$ is 16ab/3 sq.units.

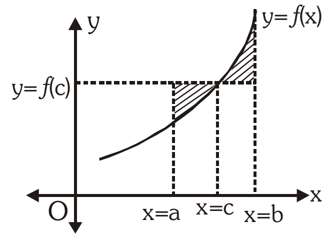
(iii) Area included between the parabola $y^2 = 4ax$ & the line $y = mx$ is $8a^2/3m^3$ sq.units.

(iv) The area of the region bounded by one arch of sin ax (or cos ax) and x-axis is 2/a sq.units.

(v) Average value of a function $y = f(x)$ over an interval $a \le x \le b$ is defined as : $y(av) = \frac{1}{b-a}\int_a^b f(x)dx$.

(vi) If y = f (x) is a monotonic function in (a, b), then the area bounded by the ordinates at x = a, x = b, y = f (x) and y = f (c) [where c $\in$ (a, b)] is minimum when $c = \frac{a+b}{2}$.

Proof : Let the function y = f (x) be monotonically increasing.



Required area A = $\int_a^c [f(c)-f(x)]dx + \int_c^b [f(x)-f(c)]dx$

For minimum area, $\frac{dA}{dc} = 0$

$\Rightarrow \quad [f'(c).c + f(c) - f'(c)a - f(c)] + [-f(c) - f'(c).b + f'(c).c + f(c)] = 0$

$\Rightarrow \quad f'(c)\left\{c - \frac{a+b}{2}\right\} = 0$

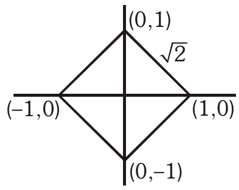
$\Rightarrow \quad c = \frac{a+b}{2} \qquad (\because f'(c) \neq 0)$

(vii) The area bounded by a curve & an axis is equal to the area bounded by the inverse of that curve & the other axis, i.e., the area bounded by y = f (x) and x-axis (say) is equal to the area bounded by y = f^{-1}(x) and y-axis.

SOLVED EXAMPLE

Example 51 : Find the area enclosed by |x – 1| + |y + 1| = 1.

Solution : Shift the origin to (1, –1).



X = x – 1 Y = y + 1

|X| + |Y| = 1

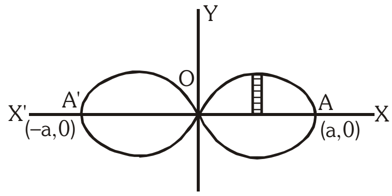
Area = $\sqrt{2} \times \sqrt{2}$ = 2 sq. units

Example 52 : Find the area of a loop as well as the whole area of the curve $a^2y^2 = x^2(a^2 - x^2)$.

Solution : The curve is symmetrical about both the axes. It cuts x-axis at (0, 0), (–a, 0), (a, 0)

Area of a loop = $2\int_0^a y\,dx = 2\int_0^a \frac{x}{a}\sqrt{a^2-x^2}\,dx$

$= -\frac{1}{a}\int_0^a \sqrt{a^2-x^2}(-2x)dx = \frac{1}{a}\left[\frac{2}{3}(a^2-x^2)^{3/2}\right]_0^a = \frac{2}{3}a^2$

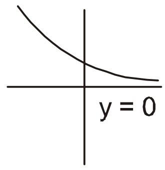


Total area = $2 \times \frac{2}{3}a^2 = \frac{4}{3}a^2$ sq.units.

Example 53 : Find asymptote of $y = e^{-x}$

Solution : $\lim_{x\to\infty} y = \lim_{x\to\infty} e^{-x} = 0$

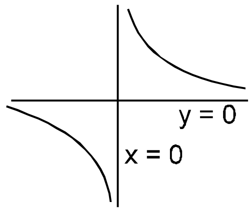
$\therefore$ y = 0 is asymptote.



Graph of $y = e^{-x}$

Example 54 : Find asymptotes of xy = 1 and draw graph.

Solution : $y = \frac{1}{x}$



$\lim_{x\to 0} y = \lim_{x\to 0} \frac{1}{x} = \infty \Rightarrow x = 0$ is asymptote.

$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{1}{x} = 0 \Rightarrow y = 0$ is asymptote.

Example 55 Find asymptotes of $y = x + \frac{1}{x}$ and sketch the curve (graph).

Solution : $\lim_{x\to 0} y = \lim_{x\to 0} \left(x + \frac{1}{x}\right) = +\infty$ or $-\infty$

$\Rightarrow$ x = 0 is asymptote.

$\lim_{x\to 0} y = \lim_{x\to 0} \left(x + \frac{1}{x}\right) = \infty$

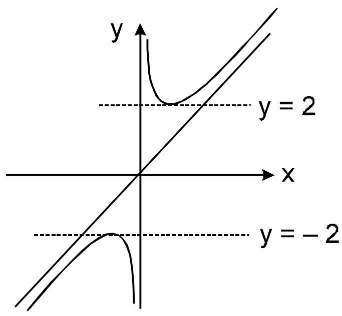
$\Rightarrow$ there is no asymptote of the type y = k

$\lim_{x\to\infty} \frac{y}{x} = \lim_{x\to\infty} \left(1 + \frac{1}{x^2}\right) = 1$

$\lim_{x\to\infty} (y - x) = \lim_{x\to\infty} \left(x + \frac{1}{x} - x\right) = \lim_{x\to\infty} \frac{1}{x} = 0$

$\therefore$ $y = x + 0 \Rightarrow y = x$ is asymptote.

A rough sketch is as follows



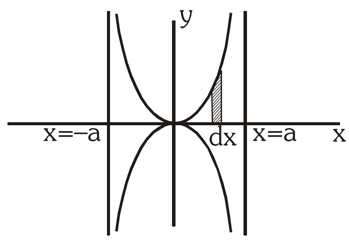
Example 56 Find the whole area included between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes.

Solution : (i) The curve is symmetric about both the axes (even powers of x & y)

(ii) Asymptotes are $x = \pm a$

$$A = 4\int_0^a y\,dx \quad = 4\int_0^a \frac{ax}{\sqrt{a^2 - x^2}}dx \quad = 4a\left|-\sqrt{a^2 - x^2}\right|_0^a$$

$= 4a^2$

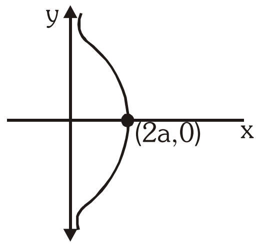


Example 57 : Find the area bounded by the curve $xy^2 = 4a^2(2a-x)$ and its asymptote.

Solution : (i) The curve is symmetrical about the x-axis as it contains even powers of y.

(ii) It passes through (2a,0).

(iii) Its asymptote is x = 0, i.e., y-axis.



$$A = 2\int_0^{2a} y\,dx = 2\int_0^{2a} 2a\sqrt{\frac{2a-x}{x}}\,dx$$

Put $x = 2a\sin^2\theta$

$$A = 16a^2\int_0^{\pi/2}\cos^2\theta\,d\theta = 4\pi a^2$$

Example 58 : Find the area of the region common to the circle $x^2 + y^2 + 4x + 6y - 3 = 0$ and the parabola $x^2 + 4x = 6y + 14$.

Solution : Circle is $x^2 + y^2 + 4x + 6y - 3 = 0$

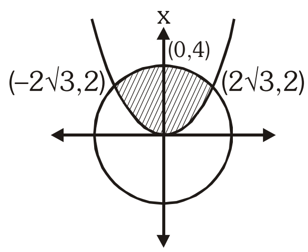
$\Rightarrow (x + 2)^2 + (y + 3)^2 = 16$

Shifting origin to (–2,–3).

$X^2 + Y^2 = 16$

equation of parabola $\rightarrow (x + 2)^2 = 6(y + 3)$

$\Rightarrow X^2 = 6Y$



Solving circle & parabola, we get $X = \pm 2\sqrt{3}$

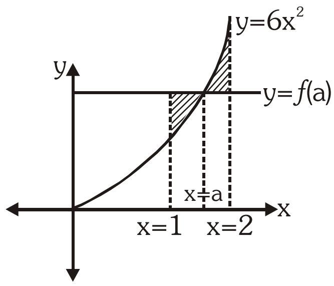
Hence they intersect at $\left(-2\sqrt{3},2\right)$ & $\left(2\sqrt{3},2\right)$

$$A = 2\left[\int_0^2 \sqrt{6Y}\,dY + \int_2^4 \sqrt{16-Y^2}\,dY\right]$$

$$= 2\left[\frac{2}{3}\sqrt{6}\left[Y^{3/2}\right]_0^2 + \left[\frac{1}{2}Y\sqrt{16-Y^2} + \frac{16}{2}\sin^{-1}\frac{Y}{4}\right]_2^4\right] = \left(\frac{4\sqrt{3}}{3} + \frac{16\pi}{3}\right) \text{ sq. units}$$

Example 59 : Find the value of 'a' for which area bounded by x = 1, x=2, $y=6x^2$ and $y=f(a)$ is minimum.

Solution : Let $b = f(a)$.



$$A = \int_1^a (b-6x^2)dx + \int_a^2 (6x^2-b)dx = \left|bx-2x^3\right|_1^a + \left|2x^3-bx\right|_a^2$$

$= 8a^3 - 18a^2 + 18$

For minimum area $\dfrac{dA}{da} = 0$

$\Rightarrow \quad 24a^2 - 36a = 0 \quad \Rightarrow \quad a = 1.5$

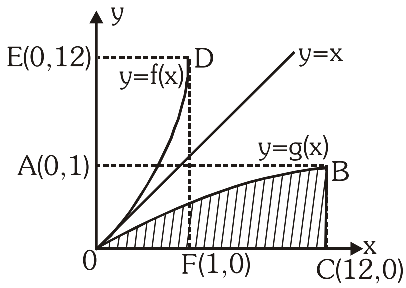
Alternatively, $y = 6x^2 \quad \Rightarrow \quad \frac{dy}{dx} = 12x$

Hence $y = f(x)$ is monotonically increasing. Hence bounded area is minimum when

$a = \left(\frac{1+2}{2}\right) = 1.5$

Example 60 : If $y = g(x)$ is the inverse of a bijective mapping $f : R \to R$, $f(x) = 6x^5 + 4x^3 + 2x$, find the area bounded by g(x), the x-axis and the ordinate at x = 12.

Solution : $f(x) = 12$



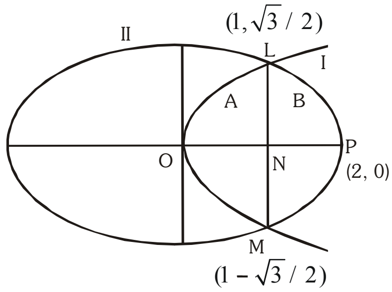
$\Rightarrow \quad 6x^5 + 4x^3 + 2x = 12 \quad \Rightarrow \quad x = 1$

$\int_0^{12} g(x)dx$ = area of rectangle OEDF – $\int_0^1 f(x)dx$

$= 1 \times 12 - \int_0^1 (6x^5 + 4x^3 + 2x)dx = 12 - 3 = 9$ sq. units.

Example 61 : Find the smaller of the areas bounded by the parabola $4y^2 - 3x - 8y + 7 = 0$ and the ellipse $x^2 + 4y^2 - 2x - 8y + 1 = 0$.

Solution : C_1 is $4(y^2 - 2y) = 3x - 7$



or $\quad 4(y-1)^2 = 3x - 3 = 3(x-1)$ .....(i)

Above is parabola with vertex at (1, 1)

C_2 is $(x^2 - 2x) + 4(y^2 - 2y) = -1$

or $\quad (x-1)^2 + 4(y-1)^2 = -1 + 1 + 4$

or $\quad \frac{(x-1)^2}{2^2} + \frac{(y-1)^2}{1^2} = 1$ ......(ii)

Above represents an ellipse with centre at (1, 1). Shift the origin to (1, 1) and this will not affect the magnitude of required area but will make the calculation simpler.

Thus the two curves are

$4Y^2 = 3X$ and $\frac{X^2}{2^2} + \frac{Y^2}{1} = 1$ They meet at $\left(1, \pm\frac{\sqrt{3}}{2}\right)$

Required area = 2(A + B) = $2\left[\int Y_1 dX + \int Y_2 dX\right]$

$= 2\left[\frac{\sqrt{3}}{2}\int_0^1 \sqrt{X}dX + \int_1^2 \frac{\sqrt{4-X^2}}{2}dX\right] = \left[\frac{\sqrt{3}}{6} + \frac{2\pi}{3}\right]$ sq.units.

Problems for Self Practice -13:

(1) Find the area inside the circle $x^2-2x + y^2 - 4y + 1 = 0$ and outside the ellipse $x^2-2x+4y^2-16y+13= 0$

(2) Find the value of 'a' ($0 < a < \frac{\pi}{2}$) for which the area bounded by the curve $f(x) = \sin^3 x + \sin x$, $y = f(a)$ between $x = 0$ & $x = \pi$ is minimum.

(3) Find the area bounded by the inverse of bijective function $f(x) = 4x^3 + 6x$, the x-axis and the ordinates $x = 0$ & $x = 44$.

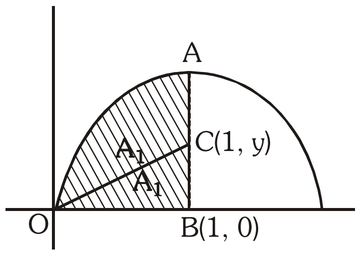
(4) Find the area of loop $y^2 = x(x-1)^2$.

Answers : (1) 2π sq. units **(2)** $\frac{\pi}{4}$ **(3)** 60 sq. units. **(4)** $\frac{8}{15}$

Miscellaneous Illustration :

Example 62 : Find the equation of line passing through the origin & dividing the curvilinear triangle with vertex at the origin, bounded by the curves $y = 2x - x^2$, $y = 0$ & $x = 1$ in two parts of equal areas.

Solution : Area of region OBA = $\int_0^1 (2x - x^2)dx$



$= \left[x^2 - \frac{x^3}{3}\right]_0^1 = \frac{2}{3}$

$\frac{2}{3} = A_1 + A_1 \Rightarrow A_1 = \frac{1}{3}$

Let pt. C has coordinates (1, y)

Area of ΔOCB = $\frac{1}{2} \times 1 \times y = \frac{1}{3}$

$y = \frac{2}{3}$

C has coordinates $\left(1, \frac{2}{3}\right)$

Line OC has slope $m = \frac{\frac{2}{3} - 0}{1 - 0} = \frac{2}{3}$

Equation of line OC is $y = mx \Rightarrow y = \frac{2}{3}x$.

Example 63 : Find the area bounded by x-axis and the curve given by x = asint, y = acost for $0 \le t \le \pi$.

Solution : Area = $\int_0^{\pi} y\frac{dx}{dt}.dt = \int_0^{\pi} a\cos t(a\cos t)dt = \frac{a^2}{2}\int_0^{\pi}(1+\cos 2t)dt = \frac{a^2}{2}\left|t+\frac{\sin 2t}{2}\right|_0^{\pi} = \frac{a^2}{2}|\pi| = \frac{\pi a^2}{2}$

Alternatively,

Area $= \int_0^{\pi} x\frac{dy}{dt}.dt = \left|\int_0^{\pi} a\sin t(-a\sin t)dt\right| = \frac{a^2}{2}\left|\int_0^{\pi}(\cos 2t-1)dt\right| = \frac{a^2}{2}\left|-\frac{\sin 2t}{2}-t\right|_0^{\pi} = \frac{\pi a^2}{2}$

Example 64 : Let A (m) be area bounded by parabola $y = x^2 + 2x - 3$ and the line y = mx + 1. Find the least area A(m).

Solution : Solving we obtain

$x^2 + (2 - m)x - 4 = 0$

Let α,β be roots $\Rightarrow \alpha + \beta = m - 2$, $\alpha\beta = -4$

$$A(m) = \left|\int_{\alpha}^{\beta}(mx+1-x^2-2x+3)dx\right| = \left|\int_{\alpha}^{\beta}(-x^2+(m-2)x+4)dx\right|$$

$$= \left|\left(-\frac{x^3}{3}+(m-2)\frac{x^2}{2}+4x\right)_{\alpha}^{\beta}\right| = \left|\frac{\alpha^3-\beta^3}{3}+\frac{m-2}{2}(\beta^2-\alpha^2)+4(\beta-\alpha)\right|$$

$$= |\beta-\alpha|.\left|-\frac{1}{3}(\beta^2+\beta\alpha+\alpha^2)+\frac{(m-2)}{2}(\beta+\alpha)+4\right|$$

$$= \sqrt{(m-2)^2+16}\left|-\frac{1}{3}\left((m-2)^2+4\right)+\frac{(m-2)}{2}(m-2)+4\right|$$

$$= \sqrt{(m-2)^2+16}\left|\frac{1}{6}(m-2)^2+\frac{8}{3}\right|$$

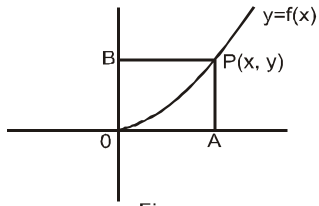
$$A(m) = \frac{1}{6}\left((m-2)^2+16\right)^{3/2}$$

Least $A(m) = \frac{1}{6}(16)^{3/2} = \frac{32}{3}$.

Example 65 : A curve y = f(x) passes through the origin and lies entirely in the first quadrant. Through any point P(x, y) on the curve, lines are drawn parallel to the coordinate axes. If the curve divides the area formed by these lines and coordinate axes in m : n, then show that $f(x) = cx^{m/n}$ or $f(x) = cx^{n/m}$ (c-being arbitrary).

Solution : Area (OAPB) = xy

Area (OAPO) = $\int_0^x f(t)dt$



Figure

Area (OPBO) = $xy - \int_0^x f(t)dt$

$$\frac{\text{Area (OAPO)}}{\text{Area (OPBO)}} = \frac{m}{n}$$

$$n\int_0^x f(t)dt = m\left(xy - \int_0^x f(t)dt\right)$$

$$n\int_0^x f(t)dt = mx\,f(x) - m\int_0^x f(t)dt$$

Differentiating w.r.t. x

nf(x) = m f(x) + mx f′(x) – m f(x)

$$\frac{f'(x)}{f(x)} = \frac{n}{m}\frac{1}{x}$$

f(x) = $cx^{n/m}$

similarly f(x) = $cx^{m/n}$

Exercise # 1

PART-I : SUBJECTIVE QUESTIONS

Section (A) : Definite Integration in terms of Indefinite Integration, using substitution and By parts

A-1. Evaluate :

(i) $\int_0^4 (x + x^{3/2})\, dx$ (ii) $\int_4^1 \frac{1}{x}\, dx$ (iii) $\int_0^1 \frac{\sqrt[3]{x^2} - \sqrt[4]{x}}{\sqrt{x}}\, dx$

(iv) $\int_0^1 x\cos(\tan^{-1} x)dx$ (v) $\int_a^b \sqrt{(x-a)(b-x)}\, dx$, a > b

(vi) $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2}$ (vii) $\int_{\sqrt{2}}^{\infty} \frac{dx}{x\sqrt{x^2 - 1}}$

(viii) $\int_0^{\pi/2} \sqrt{\cos\theta}\, \sin^3\theta\ d\theta$ (ix) $\int_0^4 \frac{x^2}{1+x}\, dx$

A-2. Evaluate :

(i) $\int_0^1 \sin^{-1} x\, dx$ (ii) $\int_1^2 \frac{\ell n\, x}{x^2}\, dx$ (iii) $\int_0^1 x e^x dx$

(iv) $\int_0^1 x^2 \sin^{-1} x\, dx$. (v) $\int_0^{\pi/2} \frac{x + \sin x}{1 + \cos x} dx$ (vi) $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

(vii) $\int_0^1 \frac{x\tan^{-1} x}{(1+x^2)^{3/2}}\, dx$

A-3. Evaluate :

(i) $\int_0^{\infty} \frac{dx}{e^x + e^{-x}}$ (ii) $\int_0^1 \frac{x}{1+\sqrt{x}}\, dx$ (iii) $\int_0^{\pi/2} \frac{\sin x\cos x}{\cos^2 x + 3\cos x + 2}\, dx$

(iv) $\int_0^{\pi/2} \frac{\sin 2\theta\ d\theta}{\sin^4\theta + \cos^4\theta}$ (v) $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x}\, dx$ (vi) $\int_0^{\pi/2} \frac{dx}{1 + 2\sin x + \cos x}$

(vii) $\int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}}$ (viii) $\int_0^{\infty} \frac{x\ \tan^{-1} x}{(1+x^2)^2} dx$

A-4. (i) Find the value of a such that $\int_0^a \frac{1}{e^x + 4e^{-x} + 5} dx = \ell n \sqrt[3]{2}$.

(ii) Find the value of $\int_0^{(\pi/2)^{1/3}} x^5 . \sin x^3 \; dx$

A-5. If $P = \int_0^\infty \frac{x^2}{1+x^4} dx ; Q = \int_0^\infty \frac{x\,dx}{1+x^4}$ and $R = \int_0^\infty \frac{dx}{1+x^4}$, then prove that :

(a) $Q = \frac{\pi}{4}$, (b) P = R, (c) $P - \sqrt{2}Q + R = \frac{\pi}{2\sqrt{2}}$

A-6. Suppose f is continuous, f (0) = 0, f (1) = 1, f ' (x) > 0 and $\int_0^1 f(x)\,dx = \frac{1}{3}$. Find the value of the definite integral $\int_0^1 f^{-1}(y)\,dy$.

A-7 Prove that $\int_0^\infty \frac{x}{(1+x)(1+x^2)} dx = \int_0^\infty \frac{dx}{(1+x)(1+x^2)} = \frac{\pi}{4}$

Section (B) : Definite Integration using Properties

B-1. Let f(x) = $\ell n \left(\frac{1-\sin x}{1+\sin x}\right)$, then show that $\int_a^b f(x)\,dx = \int_b^a \ell n \left(\frac{1+\sin x}{1-\sin x}\right) dx$.

B-2. Evaluate :

(i) $\int_0^2 [x^2]\,dx$ (where [.] denotes greatest integer function)

(ii) $\int_0^\pi \sqrt{1+\sin 2x}\,dx$ (iii) $\int_0^2 f(x)\,dx$ where $f(x) = \begin{cases} 2x+1 & 0 \le x < 1 \\ 3x^2 & 1 \le x \le 2 \end{cases}$

(iv) $\int_0^4 |x^2 - 4x + 3|\,dx$ (v) $\int_0^\infty [\cot^{-1} x]\,dx$ (where [.] denotes greatest integer function)

(vi) $\int_0^{\pi/2} |\sin x - \cos x|\,dx$

(vii) $\int_{-1}^1 [\cos^{-1} x]\; dx$ (where [.] denotes greatest integer function)

B-3. Evaluate

(i) $\int_0^{\pi/3} f(x)\,dx$ where f(x) = Minimum {tanx, cot x} $\forall\, x \in \left(0, \frac{\pi}{2}\right)$

(ii) $\int_{-1}^{1} f(x)\,dx$ where $f(x) = \min\{x+1, \sqrt{1-x}\}$

(iii) $\int_{-1}^{1} f(x)\,dx$ where f(x) = minimum (|x| , 1 – |x|, 1/4)

B-4 $\int_0^{\infty} \frac{x\log x}{(1+x^2)^2}dx$ equals-

B-5. Evaluate :

(i) $\int_{-1}^{1} e^{|x|}\,dx$ (ii) $\int_{-\pi/4}^{\pi/4} |\sin x|\,dx$ (iii) $\int_{-\pi/4}^{\pi/4} \frac{x+\pi/4}{2-\cos 2x}\,dx$

(iv) $\int_{-1}^{1} \sin^5 x\cos^4 x\,dx$ (v) $\int_{-\pi/2}^{\pi/2} \frac{g(x)-g(-x)}{f(-x)+f(x)}\,dx$

B-6. Evaluate

(i) $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ (ii) $\int_0^{\pi/2} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}\,dx$

(iii) $\int_0^{a} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$ (iv) $\int_0^{\pi/2} \frac{a\sin x+b\cos x}{\sin x+\cos x}\,dx$

(v) $\int_0^{\pi/2} \frac{\sin x-\cos x}{(\sin x+\cos x)^2}\,dx$

B-7. Evaluate :

(i) $\int_0^{2\pi} \{\sin(\sin x)+\sin(\cos x)\}\,dx$ (ii) $\int_0^{\pi} \frac{dx}{5+4\cos 2x}$

(iii) $\int_0^{\pi/2} (2\,\ell n\sin x-\ell n\sin 2x)\,dx$ (iv) $\int_0^{\infty} \ell n\left(x+\frac{1}{x}\right).\frac{dx}{1+x^2}$

B-8 $\int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$ $(n \in N)$ is equal to-

B-9. Evaluate :

(i) $\int_{-1}^{2} \{2x\}\, dx$ (where function {.} denotes fractional part function)

(ii) $\int_0^{10\pi} (|\sin x| + |\cos x|)\, dx$

(iii) $\frac{\int_0^n [x]\, dx}{\int_0^n \{x\}\, dx}$, where [x] and {x} are integral and fractional parts of x and $n \in N$

(iv) $\int_0^{2n\pi} \left(|\sin x| - \left[\left|\frac{\sin x}{2}\right|\right]\right) dx$ (where [ ] denotes the greatest integer function and $n \in I$)

(v) $\int_0^{100} 2^{\{x\}}\, dx$ (vi) $\int_0^{\pi} \sqrt{\frac{1+\cos\ 2x}{2}}\, dx$

B-10. Prove that $I = \int_0^{2\pi} \sin^2 x dx = \int_0^{2\pi} \cos^2 x dx = 2\int_0^{\pi} \sin^2 x dx = 4\int_0^{\pi/2} \sin^2 x dx$

B-11. If f(x) is a function defined $\forall\ x \in R$ and $f(x) + f(-x) = 0\ \forall\ x \in \left[-\frac{T}{2}, \frac{T}{2}\right]$ and has period T, then prove that

$\phi(x) = \int_a^x f(t)\, dt$ is also periodic with period T.

B-12. Show that $\int_0^{\infty} \frac{dx}{x^2+2x\cos\theta+1} = 2\int_0^1 \frac{dx}{x^2+2x\cos\theta+1}$

Section (C) : Leibnitz formula and Wallis' formula

C-1. (i) If $f(x) = 5^{g(x)}$ and $g(x) = \int_2^{x^2} \frac{t}{\ell n\,(1+t^2)}$ dt, then find the value of $f'(\sqrt{2})$.

(ii) The value of $\lim_{x\to 0} \frac{\frac{d}{dx}\int_0^{x^3}\sqrt{\cos t}\, dt}{1-\sqrt{\cos x}}$

(iii) Find the slope of the tangent to the curve $y = \int_x^{x^2} \cos^{-1} t^2$ at $x = \frac{1}{\sqrt[4]{2}}$

C-2. (i) If $f(x) = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\ dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\ dt$, then prove that $f'(x) = 0\ \forall x \in R$.

(ii) Find the value of x for which function $f(x) = \int_{-1}^{x} t\,(e^t - 1)\,(t - 1)\,(t - 2)^3\,(t - 3)^5\,dt$ has a local minimum

C-3. $y = \int_1^x x\sqrt{\ell n\, t}\,dt$. Find the value of $\frac{d^2y}{dx^2}$ at x = e

C-4. Evaluate : $\lim_{x\to+\infty} \frac{d}{dx}\int_{2\sin\frac{1}{x}}^{3\sqrt{x}} \frac{3t^4+1}{(t-3)(t^2+3)}dt$

C-5. If $y = x^{\int_1^x \ln t\,dt}$, find $\frac{dy}{dx}$ at x = e.

C-6. Evaluate :

(i) $\int_{-\pi/2}^{\pi/2} \sin^2 x\cos^2 x(\sin x + \cos x)\ dx$ (ii) $\int_0^{\pi} x\sin^5 x\ dx$

(iii) $\int_0^2 x^{3/2}\sqrt{2-x}\ dx$ (iv) $\int_0^{2\pi} x\ (\sin^2 x\cos^2 x)\ dx =$

Section (D) : Estimation & Mean value theorem

D-1. Prove the following inequalities : –

(i) $\frac{\sqrt{3}}{8} < \int_{\pi/4}^{\pi/3} \frac{\sin x}{x}\,dx < \frac{\sqrt{2}}{6}$ (ii) $4 \le \int_1^3 \sqrt{(3+x^3)}\ dx \le 2\sqrt{30}$

D-2. Show that

(i) $\frac{1}{10\sqrt{2}} < \int_0^1 \frac{x^9}{\sqrt{1+x}}dx < \frac{1}{10}$ (ii) $\frac{1}{2}\ell n2 < \int_0^1 \frac{\tan x}{1+x^2}dx < \frac{\pi}{2}$

D-3. (i) Show that $\int_0^2 \sin x.\cos\sqrt{x}\ dx = 2\sin c.\cos\sqrt{c}$ for some c∈(0, 2)

(ii) f(x) is a continuous function $\forall$ x ∈ R, then show that $\int_1^4 f(x)dx = 2\alpha f(\alpha^2)$ some $\alpha \in (1, 2)$

Section (E) : Integration as a limit of sum and reduction formula

E-1. Evaluate :

(i) $\lim\limits_{n\to\infty} \sum\limits_{r=0}^{n-1} \frac{1}{\sqrt{n^2 - r^2}}$

(ii) $\lim\limits_{n\to\infty} \frac{3}{n}\left[1 + \sqrt{\frac{n}{n+3}} + \sqrt{\frac{n}{n+6}} + \sqrt{\frac{n}{n+9}} + ... + \sqrt{\frac{n}{n+3(n-1)}}\right]$

(iii) $\lim\limits_{n\to\infty} \frac{1}{n^4}\left(\sum\limits_{r=1}^{2n}(3nr^2 + 2n^2r)\right)$

(iv) $\lim\limits_{n\to\infty}\left[\frac{1}{1-n^2} + \frac{2}{1-n^2} + ... + \frac{n}{1-n^2}\right]$

E-2. Consider a function $f(n) = \frac{1}{1+n^2}$. Let $\alpha_n = \frac{1}{n}\sum\limits_{r=1}^{n} f\left(\frac{r}{n}\right)$ and $\beta_n = \frac{1}{n}\sum\limits_{r=0}^{n-1} f\left(\frac{r}{n}\right)$ for n = 1,2,3,........

Also $\alpha = \lim\limits_{n\to\infty}\alpha_n$ & $\beta = \lim\limits_{n\to\infty}\beta_n$. Then prove **(a)** $\alpha_n < \beta_n$ **(b)** $\alpha = \beta$ **(c)** $\alpha_n < \frac{\pi}{4} < \beta_n$

E-3. (i) If $I_n = \int_0^{\pi/4} \tan^n x \, dx$, then show that $I_n + I_{n-2} = \frac{1}{n-1}$

(ii) $I_n = \int_0^{\pi/2} (\sin x)^n dx, \ n \in N$. Show that $I_n = \frac{n-1}{n} I_{n-2} \quad \forall n \geq 2$

E-4. Let $U_n = \int_0^{\frac{\pi}{2}} x \sin^n x dx$, then find the value of $\left(\frac{100U_{10} - 1}{U_8}\right)$.

Section (F) : Area Under Curve

F-1. Find the area enclosed between the curve $y = x^3 + 3$, $y = 0$, $x = -1$, $x = 2$.

F-2. (i) Find the area bounded by $x^2 + y^2 - 2x = 0$ and $y = \sin\frac{\pi x}{2}$ in the upper half of the circle.

(ii) Find the area bounded by the curve $y = 2x^4 - x^2$, x-axis and the two ordinates corresponding to the minima of the function.

(iii) Find area of the curve $y^2 = (7 - x)(5 + x)$ above x–axis and between the ordinates $x = -5$ and $x = 1$.

F-3. (i) Find the area of the region bounded by the curve $y^2 = 2y - x$ and the y-axis.

(ii) Find the area bounded by the y-axis and the curve $x = e^y \sin \pi y$, $y = 0$, $y = 1$.

F-4. (i) The line $3x + 2y = 13$ divides the area enclosed by the curve, $9x^2 + 4y^2 - 18x - 16y - 11 = 0$ into two parts. Find the ratio of the larger area to the smaller area.

(ii) Find the area bounded by the curves $y = \sqrt{1-x^2}$ and $y = x^3 - x$. Also find the ratio in which the y-axis divided this area.

(iii) Find the area of the region bounded by $y = \{x\}$ and $2x - 1 = 0$, $y = 0$, ({ } stands for fraction part)

F-5. (i) Find the area included between the parabolas $y^2 = x$ and $x = 3 - 2y^2$.

(ii) Find the area of the region $\{(x , y) : 0 \le y \le x^2 + 1 , 0 \le y \le x + 1 , 0 \le x \le 2\}$.

F-6. Prove that area bounded by $y = x^2 - 2|x|$ and $y = -1$ is equal to $\frac{1}{3}$ (Area of rectangle ABCD) where points A, B, C, D are $(-1, -1)$, $(-1, 0)$, $(1, 0)$ & $(1, -1)$

F-7. Let $f(x) = \sqrt{\tan x}$. Show that area bounded by $y = f(x)$, $y = f(c)$, $x = 0$ and $x = a$, $0 < c < a < \frac{\pi}{2}$ is minimum when $c = \frac{a}{2}$

F-8. (i) Draw graph of $y = (\tan x)^n$, $n \in N$, $x \in \left[0, \frac{\pi}{4}\right]$. Hence show $0 < (\tan x)^{n+1} < (\tan x)^n$, $x \in \left(0, \frac{\pi}{4}\right)$

(ii) Let A_n be the area bounded by the curve $y = (\tan x)^n$ and the lines $x = 0$, $y = 0$ and $x = \pi/4$. Prove that for $n > 2$, $A_n + A_{n-2} = 1/(n - 1)$ and deduce that $1/(2n + 2) < A_n < 1/(2n - 2)$.

F-9. Consider two curves $C_1 : y = \frac{1}{x}$ and $C_2 : y = \ln x$ on the xy plane. Let D_1 denotes the region surrounded by C_1, C_2 and the line $x = 1$ and D_2 denotes the region surrounded by C_1, C_2 and the line $x = a$. If $D_1 = D_2$. Find the value of 'a'.

F-10. For what value of 'a' is the area bounded by the curve $y = a^2x^2 + ax + 1$ and the straight line $y = 0$, $x = 0$ & $x = 1$ the least ?

F-11. For what value of 'a' is the area of the figure bounded by the lines, $y = \frac{1}{x}$, $y = \frac{1}{2x-1}$, $x = 2$ & $x = a$ equal to $\ln\frac{4}{\sqrt{5}}$?

PART-II : OBJECTIVE QUESTIONS

Section (A) : D.I. in terms of Indefinite Intigration, using substitution and By parts

A-1. If $\int_1^x \frac{dt}{|t|\sqrt{t^2-1}} = \frac{\pi}{6}$, then x can be equal to :

(A) $\frac{2}{\sqrt{3}}$ (B) $\sqrt{3}$ (C) 2 (D) $\frac{4}{\sqrt{3}}$

A-2. The value of the integral $\int_0^1 \frac{dx}{x^2+2x\cos\alpha+1}$, where $0 < \alpha < \frac{\pi}{2}$, is equal to:

(A) $\sin\alpha$ (B) $\alpha\sin\alpha$ (C) $\frac{\alpha}{2\sin\alpha}$ (D) $\frac{\alpha}{2}\sin\alpha$

A-3. If f(x) = $\begin{cases} x & x<1 \\ x-1 & x\geq 1 \end{cases}$, then $\int_0^2 x^2 f(x)\,dx$ is equal to :

(A) 1 (B) $\frac{4}{3}$ (C) $\frac{5}{3}$ (D) $\frac{5}{2}$

A-4. If f(0) = 1, f(2) = 3, f′(2) = 5 and f′(0) is finite, then $\int_0^1 x\,.\,f''(2x)\,dx$ is equal to

(A) zero (B) 1 (C) 2 (D) 3

A-5. $\int_0^{\pi} |1+2\cos x|\,dx$ is equal to :

(A) $\frac{2\pi}{3}$ (B) π (C) 2 (D) $\frac{\pi}{3}+2\sqrt{3}$

A-6. The value of $\int_{-1}^{3} (|x-2|+[x])\,dx$ is ([x] stands for greatest integer less than or equal to x)

(A) 7 (B) 5 (C) 4 (D) 3

A-7. $\int_{\ell n\pi-\ell n2}^{\ell n\pi} \frac{e^x}{1-\cos\left(\frac{2}{3}e^x\right)}\,dx$ is equal to

(A) $\sqrt{3}$ (B) $-\sqrt{3}$ (C) $\frac{1}{\sqrt{3}}$ (D) $-\frac{1}{\sqrt{3}}$

A-8. If $I_1 = \int_e^{e^2} \frac{dx}{\ell n x}$ and $I_2 = \int_1^2 \frac{e^x}{x}\,dx$, then

(A) $I_1 = I_2$ (B) $2I_1 = I_2$ (C) $I_1 = 2I_2$ (D) $I_1 + I_2 = 0$

A-9. $\int_0^{\pi/4} \frac{x.\sin x}{\cos^3 x}\,dx$ equals to :

(A) $\frac{\pi}{4}+\frac{1}{2}$ (B) $\frac{\pi}{4}-\frac{1}{2}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{4}+1$

A-10. The value of the definite integral $\int_{\frac{3}{2}}^{\frac{9}{4}} \left[\sqrt{2x-\sqrt{5(4x-5)}} + \sqrt{2x+\sqrt{5(4x-5)}}\right] dx$ is equal to

(A) $4\sqrt{5}-\frac{2\sqrt{2}}{5}$ (B) $4\sqrt{5}$ (C) $4\sqrt{3}-\frac{4}{3}$ (D) $\frac{3\sqrt{5}}{\sqrt{8}}$

A-11. If $\int_{\ell n2}^{x} \frac{dx}{\sqrt{e^x-1}} = \frac{\pi}{6}$, then x is equal to

(A) 4 (B) ℓn 8 (C) ℓn 4 (D) ℓn 2

A-12. $\int_0^{\infty} \frac{x^2+1}{x^4+7x^2+1} dx =$

(A) π (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

A-13. Consider the following statements :

S_1 : The value of $\int_0^{2\pi} \cos^{-1}(\cos x)\, dx$ is π^2

S_2 : If $\frac{d}{dx} f(x) = g(x)$ for $a \le x \le b$, then $\int_a^b f(x)g(x)dx$ equals to $\frac{[f(b)]^2-[f(a)]^2}{2}$.

State, in order, whether S_1, S_2 are true or false

(A) TT (B) TF (C) FT (D) FF

A-14. If $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$, then $\int_0^{\infty} e^{-ax^2} dx$ where a > 0 is :

(A) $\frac{\sqrt{\pi}}{2}$ (B) $\frac{\sqrt{\pi}}{2a}$ (C) $2\frac{\sqrt{\pi}}{a}$ (D) $\frac{1}{2}\sqrt{\frac{\pi}{a}}$

A-15. The value of the definite integral $\int_0^{\pi/2} \sin x \sin 2x \sin 3x\, dx$ is equal to :

(A) $\frac{1}{3}$ (B) $-\frac{2}{3}$ (C) $-\frac{1}{3}$ (D) $\frac{1}{6}$

A-16. If f (x) = x sinx^2; g (x) = x cosx^2 for $x \in [-1, 2]$

A = $\int_{-1}^{2} f(x)\,dx$; B = $\int_{-1}^{2} g(x)\,dx$, then

(A) A > 0 ; B < 0 (B) A < 0 ; B > 0 (C) A > 0 ; B > 0 (D) A < 0 ; B < 0

A-17. If g (x) is the inverse of f (x) and f (x) has domain x $\in$ [1, 5], where f (1) = 2 and f (5) = 10 then the values of

$\int_{1}^{5} f(x)\,dx + \int_{2}^{10} g(y)\,dy$ equals -

(A) 48 (B) 64 (C) 71 (D) 52

Section (B) : Definite Integration using Properties

B-1. $\int_{0}^{\infty} f\left(x+\frac{1}{x}\right)\cdot\frac{\ln x}{x}\,dx =$

(A) 0 (B) 1 (C) $\frac{1}{2}$ (D) can not be evaluated

B-2. Suppose for every integer n, $\int_{n}^{n+1} f(x)dx = n^2$. The value of $\int_{-2}^{4} f(x)dx$ is :

(A) 16 (B) 14 (C) 19 (D) 21

B-3. Let f : R $\to$ R, g : R $\to$ R be continuous functions. Then the value of integeral

$$\int_{\ell n\lambda}^{\ell n 1/\lambda} \frac{f\left(\frac{x^2}{4}\right)[f(x)-f(-x)]}{g\left(\frac{x^2}{4}\right)[g(x)+g(-x)} dx \text{ is :}$$

(A) depend on λ (B) a non-zero constant (C) zero (D) 2

B-4. $\int_{-1}^{1} \cot^{-1}\left(\frac{x+x^3}{1+x^4}\right)dx$ is equal to

(A) 2π (B) $\frac{\pi}{2}$ (C) 0 (D) π

B-5. $\int_{-2}^{0} \{x^3+3x^2+3x+3+(x+1)\cos(x+1)\}$ dx is equal to

(A) – 4 (B) 0 (C) 4 (D) 6

B-6. $\int_{-1}^{1} x\ell n(1+e^x)dx$ =

(A) 0 (B) $\ell n(1 + e)$ (C) $\ell n(1 + e) - 1$ (D) 1/3

B-7. If $\int_{-1}^{3/2} |x\sin \pi x|\ dx = \frac{k}{\pi^2}$, then the value of k is :

(A) $3\pi + 1$ (B) $2\pi + 1$ (C) 1 (D) 4

B-8. The value of definite integral $\int_{0}^{\frac{\pi^2}{4}} \frac{dx}{1+\sin\sqrt{x}+\cos\sqrt{x}}$ is

(A) $\pi \ln 2$ (B) $\frac{\pi \ln 2}{2}$ (C) $\frac{\pi \ln 2}{4}$ (D) $2\pi \ln 2$

B-9. $\int_{2-\ell n 3}^{3+\ell n 3} \frac{\ell n(4+x)}{\ell n(4+x)+\ell n(9-x)}$ dx is equal to :

(A) cannot be evaluated (B) is equal to $\frac{5}{2}$

(C) is equal to $1 + 2\ \ell n\ 3$ (D) is equal to $\frac{1}{2} + \ell n\ 3$

B-10. The value of the definite integral I = $\int_{0}^{\pi} x\sqrt{1+|\cos x|}\,dx$ is equal to

(A) $2\sqrt{2}\,\pi$ (B) $\sqrt{2}\,\pi$ (C) 2π (D) 4π

B-11. The value of $\int_{0}^{\pi/2} \ell n\,|\tan x + \cot x|\ dx$ is equal to :

(A) $\pi\ \ell n\ 2$ (B) $-\pi\ \ell n\ 2$ (C) $\frac{\pi}{2}\ \ell n\ 2$ (D) $-\frac{\pi}{2}\ \ell n\ 2$

B-12. Let $I_1 = \int_{0}^{1} \frac{e^x dx}{1+x}$ and $I_2 = \int_{0}^{1} \frac{x^2 dx}{e^{x^3}(2-x^3)}$, then $\frac{I_1}{I_2}$ is

(A) 3/e (B) e/3 (C) 3e (D) 1/3e

B-13. The value of $\int_{0}^{[x]} \{x\}\,dx$ (where [.] and { . } denotes greatest integer and fraction part function respectively) is

(A) $\frac{1}{2}[x]$ (B) 2[x] (C) $\frac{1}{2[x]}$ (D) [x]

B-14. If $f(x) = \int_0^x (2\cos^2 3t + 3\sin^2 3t)\, dt$, $f(x + \pi)$ is equal to :

(A) $f(x) + 2f(\pi)$ (B) $f(x) + 2f\left(\frac{\pi}{2}\right)$ (C) $f(x) + 4f\left(\frac{\pi}{4}\right)$ (D) $2f(x)$

B-15. If $\int_0^{11} \frac{11^x}{11^{[x]}}\, dx = \frac{k}{\log 11}$, (where [ ] denotes greatest integer function) then value of k is

(A) 11 (B) 101 (C) 110 (D)121

B-16. If f(x) is a function satisfying $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$ for all non-zero x, then $\int_{\sin\theta}^{\operatorname{cosec}\theta} f(x)\, dx$ equals to :

(A) $\sin\theta + \operatorname{cosec}\theta$ (B) $\sin^2\theta$ (C) $\operatorname{cosec}^2\theta$ (D) none of these

B-17. If $\sum_{i=1}^{4}\left(\sin^{-1} x_i + \cos^{-1} y_i\right) = 6\pi$, then $\int_{\sum_{i=1}^{4} x_i}^{\sum_{i=1}^{4} y_i} x\ln(1+x^2)\left(\frac{e^x}{1+e^{2x}}\right) dx$ is equal to

(A) 0 (B) $e^4 + e^{-4}$ (C) $\ln\left(\frac{17}{12}\right)$ (D) $e^4 - e^{-4}$

B-18. If $f(x) = \begin{cases} 0 \ , & \text{where } x = \frac{n}{n+1}, n = 1, 2, 3..... \\ 1 \ , & \text{else where} \end{cases}$, then the value of $\int_0^2 f(x)\, dx$.

(A) 1 (B) 0 (C) 2 (D) ∞

B-19. If $g(x) = \int_0^x \cos^4 t\, dt$, then $g(x + \pi)$ equals

(A) $g(x) + g(\pi)$ (B) $g(x) - g(\pi)$ (C) $g(x)\, g(\pi)$ (D) $[g(x)/g(\pi)]$

B-20. Let $u = \int_0^1 \frac{l\text{n}(x+1)}{x^2+1} dx$ and $v = \int_0^{\pi/2} l\text{n}(\sin 2x) dx$ then -

(A) u = 4v (B) 4u + v = 0 (C) u + 4v = 0 (D) 2u + v = 0

B-21. Suppose f is continuous and satisfies $f(x) + f(-x) = x^2$ then the integral $\int_{-1}^{1} f(x) dx$ has the value equal to

(A) $\frac{2}{3}$ (B) $\frac{1}{3}$ (C) $\frac{4}{3}$ (D) zero

B-22. If $g(x) = \int_1^x e^{t^2} dt$ then the value of $\int_3^{x^3} e^{t^2} dt$ equals

(A) $g(x^3) - g(3)$ (B) $g(x^3) + g(3)$ (C) $g(x^3) - 3$ (D) $g(x^3) - 3g(x)$

B-23. Suppose that F(x) is an antiderivative of $f(x) = \frac{\sin x}{x}$, x > 0 then $\int_1^3 \frac{\sin 2x}{x}$ can be expressed as -

(A) F(6) – F(2) (B) $\frac{1}{2}(F(6) - F(2))$ (C) $\frac{1}{2}(F(3) - F(1))$ (D) 2(F(6) – F(2))

B-24. Let $f(x)$ be a continuous function on [0,4] satisfying $f(x)\, f(4 - x) = 1$.

The value of the definite integral $\int_0^4 \frac{1}{1+f(x)} dx$ equals-

(A) 0 (B) 1 (C) 2 (D) 4

Section (C) : Leibnitz formula and Wallis' formula

C-1. $f(x) = \int_x^{x^2} \frac{e^t}{t} dt$, then f ' (1) is equal to :

(A) e (B) 2e (C) $2e^2 - 2$ (D) $e^2 - e$

C-2. $f(x) = \int_0^x (t-1)(t-2)^2(t-3)^3(t-4)^5\, dt$ (x > 0) then number of points of extremum of f(x) is

(A) 4 (B) 3 (C) 2 (D) 1

C-3. $\underset{h\to 0}{\text{Limit}} \frac{\int_a^{x+h} \ell n^2 t dt - \int_a^x \ell n^2 t dt}{h}$ equals to :

(A) 0 (B) $\ell n^2 x$ (C) $\frac{2\ell nx}{x}$ (D) does not exist

C-4. The value of the function $f(x) = 1 + x + \int_1^x (\ell n^2 t + 2\,\ell nt)\, dt$, where f ′ (x) vanishes is:

(A) e^{-1} (B) 0 (C) $2e^{-1}$ (D) $1 + 2e^{-1}$

C-5. If $\int_a^y \cos t^2 dt = \int_a^{x^2} \frac{\sin t}{t} dt$, then the value of $\frac{dy}{dx}$ is

(A) $\frac{2\sin^2 x}{x\cos^2 y}$ (B) $\frac{2\sin x^2}{x\cos y^2}$ (C) $\frac{2\sin x^2}{x\left(1 - 2\sin\frac{y^2}{2}\right)}$ (D) $\frac{\sin x^2}{2y}$

C-6. If $\int_{\sin x}^{1} t^2 (f(t))\, dt = (1 - \sin x)$, then $f\left(\frac{1}{\sqrt{3}}\right)$ is

(A) 1/3 (B) $1/\sqrt{3}$ (C) 3 (D) $\sqrt{3}$

C-7. The value of $\lim_{a\to\infty} \frac{1}{a^2} \int_0^a \ln(1+e^x)\, dx$ equals

(A) 0 (B) 1 (C) $\frac{1}{2}$ (D) non-existent

C-8. $f(x) = \int_1^x \frac{\sin x \cos y}{y^2 + y^2 + 1} dy$, then

(A) $f'(x) = 0 \ \forall\ x = \frac{n\pi}{2}, n \in Z$ (B) $f'(x) = 0 \ \forall\ x = (2n + 1)\frac{\pi}{2}, n \in Z$

(C) $f'(x) = 0 \ \forall\ x = n\pi, n \in Z$ (D) $f'(x) \neq 0 \ \forall\ x \in R$

C-9. $\int_0^{\pi/2} \sin^4 x \cos^3 x\, dx$ is equal to :

(A) $\frac{6}{35}$ (B) $\frac{2}{21}$ (C) $\frac{2}{15}$ (D) $\frac{2}{35}$

C-10. $\int_0^1 x^2(1-x)^3\, dx$ is equal to :

(A) $\frac{1}{60}$ (B) $\frac{1}{30}$ (C) $\frac{2}{15}$ (D) $\frac{\pi}{120}$

C-11. $\lim_{x\to 0} \frac{\int_0^x \sin t^2 dt}{x(1-\cos x)}$ equals -

(A) $\frac{1}{3}$ (B) 2 (C) $\frac{1}{2}$ (D) $\frac{2}{3}$

C-12. Variable x and y are related by equation $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$. The value of $\frac{d^2y}{dx^2}$ is equal to

(A) $\frac{y}{\sqrt{1+y^2}}$ (B) y (C) $\frac{2y}{\sqrt{1+y^2}}$ (D) 4y

Section (D) : Estimation & Mean value theorem

D-1. Let $I = \int_1^3 \sqrt{x^4 + x^2}\, dx$, then

(A) $I > 6\sqrt{10}$ (B) $I < 2\sqrt{2}$ (C) $2\sqrt{2} < I < 6\sqrt{10}$ (D) $I < 1$

D-2. $I = \int_0^{2\pi} e^{\sin^2 x + \sin x + 1}\, dx$, then

(A) $\pi e^3 < I < 2\pi e^5$ (B) $2\pi e^{3/4} < I < 2\pi e^3$ (C) $2\pi e^3 < I < 2\pi e^4$ (D) $0 < I < 2\pi$

D-3. Let $f''(x) \geq 0$, $f'(x) > 0$, $f(0) = 3$ & $f(x)$ is defined in $[-2, 2]$. If $f(x)$ is non-negative, then

(A) $\int_{-1}^{0} f(x)dx > 6$ (B) $\int_{-2}^{2} f(x)dx > 12$ (C) $\int_{-2}^{2} f(x)dx \geq 12$ (D) $\int_{-1}^{1} f(x)dx > 12$

D-4. Let mean value of $f(x) = \frac{1}{x+c}$ over inteval $(0, 2)$ is $\frac{1}{2}\ell n3$ then positive value of c is

(A) 1 (B) $\frac{1}{2}$ (C) 2 (D) $\frac{3}{2}$

Section (E) : Integration as a limit of sum and reduction formula

E-1. $\lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{r^3}{r^4 + n^4}\right)$ equals to :

(A) $\ell n\, 2$ (B) $\frac{1}{2}\ell n\, 2$ (C) $\frac{1}{3}\ell n\, 2$ (D) $\frac{1}{4}\ell n\, 2$

E-2. $\underset{n\to\infty}{\text{Lt}} \sum_{r=2n+1}^{3n} \frac{n}{r^2 - n^2}$ is equal to :

(A) $\ell n\sqrt{\frac{2}{3}}$ (B) $\ell n\sqrt{\frac{3}{2}}$ (C) $\ell n\frac{2}{3}$ (D) $\ell n\frac{3}{2}$

E-3. $\lim_{n\to\infty} \left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)...\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$ is equal to :

(A) $\frac{e^{\pi/2}}{2e^2}$ (B) $2\, e^2\, e^{\pi/2}$ (C) $\frac{2}{e^2}\, e^{\pi/2}$ (D) $2\, e^{\pi}$

E-4. $\lim_{n\to\infty} \frac{\pi}{n}\left[\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + ... + \sin\frac{(n-1)\pi}{n}\right]$ is equals to :

(A) 0 (B) π (C) 2 (D) 3

E-5. Let $I_n = \int_0^1 (1-x^3)^n dx$, $(n \in N)$ then

(A) $3n\, I_n = (3n-1)\, I_{n-1}\ \forall\ n \geq 2$
(B) $(3n-1)I_n = 3n\, I_{n-1}\ \forall\ n \geq 2$
(C) $(3n-1)I_n = (3n+1)\, I_{n-1}\ \forall\ n \geq 2$
(D) $(3n+1)I_n = 3n\, I_{n-1}\ \forall\ n \geq 2$

Section (F) : Area Under Curve

F-1. The area of the figure bounded by right of the line y = x + 1, y = cos x and x–axis is:

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{5}{6}$ (D) $\frac{3}{2}$

F-2. The area of the region(s) enclosed by the curves $y = x^2$ and $y = \sqrt{|x|}$ is

(A) 1/3 (B) 2/3 (C) 1/6 (D) 1

F-3. The area bounded by $y = 2 - |2 - x|$ and $y = \frac{3}{|x|}$ is:

(A) $\frac{4+3\ell n3}{2}$ (B) $\frac{4-3\ell n3}{2}$ (C) $\frac{3}{2} + \ell n3$ (D) $\frac{1}{2} + \ell n3$

F-4. The area of the region bounded by x = 0, y = 0, x = 2, y = 2, $y \leq e^x$ and $y \geq \ell n\, x$, is

(A) $6 - 4\,\ell n\, 2$ (B) $4\,\ell n\, 2 - 2$ (C) $2\,\ell n\, 2 - 4$ (D) $6 - 2\,\ell n\, 2$

F-5. Consider the following statements :

S_1 **:** Area enclosed by the curve $|x-2| + |y+1| = 1$ is equal to 3 sq. unit

S_2 **:** Area of the region $R \equiv \{(x, y)\ ;\ x^2 \leq y \leq x\}$ is $\frac{1}{6}$

State, in order, whether S_1, S_2, S_3, S_4 are true or false

(A) TT (B) TF (C) FT (D) FF

F-6. The area bounded by the curves $y = x\, e^x$, $y = x\, e^{-x}$ and the line x = 1

(A) $\frac{2}{e}$ (B) $1 - \frac{2}{e}$ (C) $\frac{1}{e}$ (D) $1 - \frac{1}{e}$

F-7. The area enclosed between the curves

$y = \log_e(x + e)$, $x = \log_e\left(\frac{1}{y}\right)$ and the x-axis is

(A) 2 (B) 1 (C) 4 (D) 3

F-8. Let $f:[0, \infty) \rightarrow R$ be a continuous and strictly increasing function such that $f^3(x) = \int_0^x t f^2(t) dt$, $\forall\ x \geq 0$. The area enclosed by y = f(x), the x-axis and the ordinate at x = 3 is ————

(A) $\frac{3}{2}$ (B) $\frac{5}{2}$ (C) $\frac{7}{2}$ (D) $\frac{1}{2}$

F-9. Let 'a' be a positive constant number. Consider two curves C_1: $y = e^x$, C_2 : $y = e^{a-x}$. Let S be the area of the part surrounding by C_1, C_2 and the y-axis, then $\lim\limits_{a\to 0} \frac{S}{a^2}$ equals

(A) 4 (B) 1/2 (C) 0 (D) 1/4

F-10. Area enclosed by the graph of the function $y = \ln^2 x - 1$ lying in the 4^{th} quadrant is

(A) $\frac{2}{e}$ (B) $\frac{4}{e}$ (C) $2\left(e+\frac{1}{e}\right)$ (D) $4\left(e-\frac{1}{e}\right)$

F-11. The area bounded by the curve $y = f(x)$ (where $f(x) \geq 0$), the co-ordinate axes & the line $x = x_1$ is given by $x_1 \cdot e^{x_1}$. Therefore $f(x)$ equals :

(A) e^x (B) $x e^x$ (C) $xe^x - e^x$ (D) $x e^x + e^x$

F-12. The slope of the tangent to a curve $y = f(x)$ at $(x, f(x))$ is $2x + 1$. If the curve passes through the point (1, 2) then the area of the region bounded by the curve, the x-axis and the line $x = 1$ is

(A) $\frac{5}{6}$ (B) $\frac{6}{5}$ (C) $\frac{1}{6}$ (D) 1

PART-III : MATCH THE COLUMN

1. Let $\lim\limits_{T\to\infty} \frac{1}{T}\int_0^T (\sin x + \sin ax)^2 dx = L$ then

Column–I	**Column–II**
(A) for a = 0, the value of L is	(p) 0
(B) for a = 1 the value of L is	(q) 1/2
(C) for a = – 1 the value of L is	(r) 3/2
(D) $\forall a \in R - \{-1, 0, 1\}$ the value of L is	(s) 2
	(t) 1

2.

Column–I	**Column–II**
(A) Area bounded by region $0 \leq y \leq 4x - x^2 - 3$ is	(p) 32/3
(B) Area of the region enclosed by $y^2 = 8x$ and $y = 2x$ is	(q) 1/2
(C) The area bounded by $\|x\| + \|y\| \leq 1$ and $\|x\| \geq 1/2$ is	(r) 8/3
(D) Area bounded by $x \leq 4 - y^2$ and $x \geq 0$ is	(s) 4/3

Exercise # 2

PART-I : OBJECTIVE QUESTIONS

1. The value of the definite integral $\int_{0}^{\pi/3} \ln\left(1+\sqrt{3}\tan x\right)dx$ equals-

(A) $\frac{\pi}{3}\ln 2$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi^2}{6}\ln 2$ (D) $\frac{\pi}{2}\ln 2$

2. $\int_{0}^{\infty} [2e^{-x}]\, dx$, where [.] denotes the greatest integer function, is equal to :

(A) 0 (B) $\ell n\, 2$ (C) e^2 (D) $2e^{-1}$

3. $\int_{0}^{\pi/2} \sqrt{\frac{\sec x - \tan x}{\sec x + \tan x}}\, \frac{\text{cosec } x}{\sqrt{1+2\text{cosec } x}}\, dx$ has the value -

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $2\pi/3$

4. If $\int_{0}^{100} f(x)\, dx = a$, then $\sum_{r=1}^{100}\left(\int_{0}^{1} f(r-1+x)\, dx\right) =$

(A) 100 a (B) a (C) 0 (D) 10 a

5. $\lim_{t\to\left(\frac{\pi}{2}\right)^-} \int_{0}^{t} \tan\theta\sqrt{\cos\theta}\, \ell n(\cos\theta)d\theta$ is equal :

(A) –4 (B) 4 (C) –2 (D) Does not exists

6. The value of the defined integral $\int_{0}^{\pi/2} (\sin x + \cos x).\sqrt{\frac{e^x}{\sin x}}dx$ equals

(A) $2\sqrt{e^{\pi/2}}$ (B) $\sqrt{e^{\pi/2}}$ (C) $2\sqrt{e^{\pi/2}}.\cos 1$ (D) $\frac{1}{2}e^{\pi/4}$

7. The absolute value of $\frac{\int_{0}^{\pi/2}\left(x\cos x+1\right)e^{\sin x}dx}{\int_{0}^{\pi/2}\left(x\sin x-1\right)e^{\cos x}dx}$ is equal to -

(A) e (B) πe (C) e/2 (D) π/e

8. The tangent to the graph of the function y = f(x) at the point with abscissa x =1 form an angle of $\pi/6$ and at the point x = 2, an angle of $\pi/3$ and at the point x = 3, an angle of $\pi/4$ with positive x-axis. The value of $\int_1^3 f'(x)f''(x)dx + \int_2^3 f''(x)dx$ (f"(x) is supposed to be continuous) is :

(A) $\frac{4\sqrt{3}-1}{3\sqrt{3}}$ (B) $\frac{3\sqrt{3}-1}{2}$ (C) $\frac{4-\sqrt{3}}{3}$ (D) $\frac{4}{3}-\sqrt{3}$

9. Let A = $\int_0^1 \frac{e^t}{1+t}dt$, then $\int_{a-1}^{a} \frac{e^{-t}}{t-a-1}dt$ has the value :

(A) Ae^{-a} (B) $-Ae^{-a}$ (C) $-ae^{-a}$ (D) Ae^{a}

10. $\int_1^2 x^{2x^2+1}(1+2\ell n\,x)dx$ is equal to

(A) 256 (B) 255 (C) $\frac{255}{2}$ (D) 128

11. The value of $\int_{1/e}^{\tan x} \frac{t}{1+t^2}dt + \int_{1/e}^{\cot x} \frac{1}{t(1+t^2)}dt$, where x ∈ (π/6, π/3), is equal to :

(A) 0 (B) 2 (C) 1 (D) cannot be determined

12. If $\frac{C_0}{1}+\frac{C_1}{2}+\frac{C_2}{3}=0$, where C_0, C_1, C_2 are all real, the equation $C_2x^2 + C_1x + C_0 = 0$ has:

(A) atleast one root in (0, 1) (B) one root in (1, 2) & other in (3, 4)

(C) one root in (–1, 1) & the other in (–5, –2) (D) both roots imaginary

13. A continuous real function f satisfies f(2x) = 3 f(x) ∀ x ∈ R

If $\int_0^1 f(x)dx = 1$, then compute the value of definite integral $\int_1^2 f(x)dx$

(A) 2 (B) 3 (C) 4 (D) 5

14. Let f(x) be a function defined on R such that f '(x) = f '(3 – x) ∀ x ∈ [0,3] with f(0) = –32 and f(3) = 46. Then find the value of $\int_0^3 f(x)dx$.

(A) 20 (B) 21 (C) 22 (D) 23

15. Let f (x) = $\int_0^x \frac{dt}{\sqrt{1+t^3}}$ and g (x) be the inverse of f (x), then which one of the following holds good?

(A) $2g'' = g^2$ (B) $2g'' = 3g^2$ (C) $3g'' = 2g^2$ (D) $3g'' = g^2$

16. If $\int_0^x f(t)\,dt = x + \int_x^1 t^2 \cdot f(t)dt + \frac{\pi}{4} - 1$, then the value of the integral $\int_{-1}^{1} f(x)\,dx$ is equal to

(A) 0 (B) $\pi/4$ (C) $\pi/2$ (D) π

17. $\lim_{x\to\infty}\left(x^3 \int_{-1/x}^{1/x} \frac{\ln(1+t^2)}{1+e^t} dt\right)$ equals

(A) 1/3 (B) 2/3 (C) 1 (D) 0

18. Let $f(x) = \int_0^{g(x)} \frac{dt}{\sqrt{1+t^2}}$ where $g(x) = \int_0^{\cos x} (1+\sin t^2)\,dt$. Also $h(x) = e^{-|x|}$ and $l(x) = x^2 \sin\frac{1}{x}$ if $x \neq 0$ and $l(0) = 0$ then $f'\left(\frac{\pi}{2}\right)$ equals

(A) $l'(0)$ (B) $h'(0^-)$ (C) $h'(0^+)$ (D) $\lim_{x\to 0} \frac{1-\cos x}{x \sin x}$

19. Let f(x) is differentiable function satisfying $2\int_1^2 f(tx)\,dt = x + 2\ \forall x \in R$, then $\int_0^1 (8f(8x) - f(x) - 21x)dx$ is equal to

(A) 3 (B) 5 (C) 7 (D) 9

20. $\lim_{k\to 0} \frac{1}{k}\int_0^k (1+\sin 2x)^{\frac{1}{x}} dx$ -

(A) 2 (B) 1 (C) e^2 (D) non existent

21. Let $I_n = \int_0^1 x^n(\tan^{-1} x)dx$, $n \in N$, then

(A) $(n+1)I_n + (n-1)I_{n-2} = \frac{\pi}{4} + \frac{1}{n}\ \forall\ n \geq 3$

(B) $(n+1)I_n + (n-1)I_{n-2} = \frac{\pi}{2} - \frac{1}{n}\ \forall\ n \geq 3$

(C) $(n+1)I_n - (n-1)I_{n-2} = \frac{\pi}{4} + \frac{1}{n}\ \forall\ n \geq 3$

(D) $(n+1)I_n - (n-1)I_{n-2} = \frac{\pi}{2} - \frac{1}{n}\ \forall\ n \geq 3$

22. If $u_n = \int_0^{\pi/2} x^n \sin x\ dx$, then the value of $u_{10} + 90\,u_8$ is :

(A) $9\left(\frac{\pi}{2}\right)^8$ (B) $\left(\frac{\pi}{2}\right)^9$ (C) $10\left(\frac{\pi}{2}\right)^9$ (D) $9\left(\frac{\pi}{2}\right)^9$

23. $\lim_{n\to\infty}\left(\sin\frac{\pi}{2n}.\sin\frac{2\pi}{2n}.\sin\frac{3\pi}{2n}.......\sin\frac{(n-1)\pi}{n}\right)^{1/n}$ is equal to :

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{3}{4}$

24. For positive integers k = 1, 2, 3, ... n, let S_k denotes the area of ΔAOB_k (where 'O' is origin) such that that $\angle AOB_k = \frac{k\pi}{2n}$, $OA = 1$ and OB_k = k. The value of the $\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}S_k$ is

(A) $\frac{2}{\pi^2}$ (B) $\frac{4}{\pi^2}$ (C) $\frac{8}{\pi^2}$ (D) $\frac{1}{2\pi^2}$

25. If the value of the integral $\int_1^2 e^{x^2}dx$ is α, then the value of $\int_e^{e^4}\sqrt{\ell nx}\,dx$ is -

(A) $e^4 - e - \alpha$ (B) $2e^4 - e - \alpha$ (C) $2(e^4 - e) - \alpha$ (D) $2e^4 - 1 - \alpha$

26. The area bounded by the curve x = $a\cos^3 t$, y = $a\sin^3 t$ is

(A) $\frac{3\pi a^2}{8}$ (B) $\frac{3\pi a^2}{16}$ (C) $\frac{3\pi a^2}{32}$ (D) $3\pi a^2$

27. The area bounded by the curve f(x) = x + sin x and its inverse function between the ordinates x = 0 and x = 2π is

(A) 4π (B) 8π (C) 4 (D) 8

28. P(2, 2), Q(–2, 2), R(–2, –2) & S(2, –2) are vertices of a square. A parabola passes through P, S & its vertex lies on x-axis. If this parabola bisects the area of the square PQRS, then vertex of the parabola is

(A) (–2, 0) (B) (0, 0) (C) $\left(-\frac{3}{2}, 0\right)$ (D) (–1, 0)

29. The ratio in which the curve y = x^2 divides the region bounded by the curve; y = $\sin\left(\frac{\pi x}{2}\right)$ and the x–axis as x varies from 0 to 1, is :

(A) 2: π (B) 1: 3 (C) 3: π (D) $(6-\pi)$: π

30. If f(x) = sin x, $\forall\ x \in \left[0, \frac{\pi}{2}\right]$, f(x) + f($\pi$ – x) = 2. $\forall\ x \in \left(\frac{\pi}{2}, \pi\right]$ and f(x) = f(2π – x), $\forall\ x \in (\pi, 2\pi]$, then the area enclosed by y = f(x) and x-axis is

(A) π (B) 2π (C) 2 (D) 4

31. The area of the region on plane bounded by max (|x|, |y|) $\le$ 1 and xy $\le \frac{1}{2}$ is

(A) 1/2 + ℓn 2 (B) 3 + ℓn 2 (C) 31/4 (D) 1 + 2 ℓn 2

32. Suppose y = f (x) and y = g(x) are two functions whose graphs intersect at the three points (0, 4), (2, 2) and (4, 0) with f (x) > g (x) for 0 < x < 2 and f (x) < g (x) for 2 < x < 4. If $\int_0^4 [f(x)-g(x)]dx$ =10 and $\int_2^4 [g(x)-f(x)]dx$ =5, the area between two curves for 0 < x < 2, is

(A) 5 (B) 10 (C) 15 (D) 20

33. Area of the region enclosed between the curves $x = y^2 - 1$ and $x = |y|\sqrt{1-y^2}$ is

(A) 1 (B) 4/3 (C) 2/3 (D) 2

34. The area bounded by the curve $y = x e^{-x}$; xy = 0 and x = c where c is the x-coordinate of the curve's inflection point, is

(A) $1 - 3e^{-2}$ (B) $1 - 2e^{-2}$ (C) $1 - e^{-2}$ (D) 1

PART-II : NUMERICAL QUESTIONS

1. $\int_2^4 \frac{3x^2+1}{(x^2-1)^3}dx = \frac{\lambda}{n^2}$ where λ, n ∈ N and gcd(λ, n) = 1, then find the value of $\frac{3\lambda}{n}$

2. For a ≥ 2, if the value of the definite integral $\int_0^\infty \frac{dx}{a^2+(x-(1/x))^2}$ equals $\frac{\pi}{5050}$. Find the value of a.

3. Find the value of $\ell n\left(\int_0^1 e^{t^2+t}(2t^2+t+1)dt\right)$

4. If $\int_{-1}^0 \frac{x}{x+1+e^x}dx$ is equal to $-\ell nk$, then find the value of k.

5. Let $u = \int_0^{\pi/4}\left(\frac{\cos x}{\sin x+\cos x}\right)^2 dx$ and $v = \int_0^{\pi/4}\left(\frac{\sin x+\cos x}{\cos x}\right)^2 dx$. Find the value of $\frac{v}{u}$.

6. If f(π) = 2 and $\int_0^\pi (f(x)+f''(x))\sin x\,dx = 5$, then find the value of f(0)

(it is given that f(x) is continuous in [0, π])

7. Let U = $\int_{\pi/6}^{\pi/2} \min.(\sqrt{3}\sin x, \cos x)\,dx$ and V = $\int_{-3}^{5} x^2 \operatorname{sgn}(x-1)dx$. If V = λU, then find the value of λ.

[**Note :** sgn k denotes the signum function of k.]

8. Let f(x) be a function satisfying $f(x) = f\left(\frac{100}{x}\right) \forall x > 0$. If $\int_1^{10} \frac{f(x)}{x}dx = 5$ then find the value of $\int_1^{100} \frac{f(x)}{x}dx$

9. Let $I_n = \int_{-n}^{n} \left(\{x+1\}.\{x^2+2\} + \{x^2+2\}\{x^3+4\}\right)dx$, where {.} denotes the fractional part of x. Find I_1.

10. Evaluate

$$\frac{2005\int_0^{1002} \frac{dx}{\sqrt{1002^2 - x^2} + \sqrt{1003^2 - x^2}} + \int_{1002}^{1003} \sqrt{1003^2 - x^2}\,dx}{\int_0^1 \sqrt{1-x^2}\,dx}$$ = k, then find the sum of squares of digits of natural number k.

11. If $\int_0^{\pi/2} \sqrt{\sin 2\theta}.\sin\theta\, d\theta = \frac{\pi}{n}$ then find n

12. Evaluate: $3\int_0^{\pi} \frac{a^2\sin^2 x + b^2\cos^2 x}{a^4\sin^2 x + b^4\cos^2 x}dx$, where $a^2 + b^2 = \frac{3\pi}{4}$, $a^2 \neq b^2$ and $ab \neq 0$.

13. Let y = f(x) be a quadratic function with f '(2) = 1. Find the value of the integral $\int_{2-\pi}^{2+\pi} f(x).\sin\left(\frac{x-2}{2}\right)dx$.

14. Let $I_1 = \int_0^{\pi/4} (1+\tan x)^2 dx$, $I_2 = \int_0^1 \frac{dx}{(1+x)^2(1+x^2)}$

then find the value of $\frac{I_1}{I_2}$

15. If f, g, h be continuous functions on [0, a] such that f (a – x) = f (x), g (a – x) = – g (x)

and 3 h (x) – 4 h (a – x) = 5, then find the value of $\int_0^a f(x)\, g(x)\, h(x)\, dx$

16. If $f(x) = \frac{\sin x}{x}$ $\forall x \in (0, \pi]$, If $\frac{\pi}{k}\int_0^{\pi/2} f(x)\, f\left(\frac{\pi}{2} - x\right) dx = \int_0^{\pi} f(x)\, dx$ then find the value of k.

17. Evaluate : $\pi\int_0^{\pi} \frac{x^2 \sin 2x.\sin\left(\frac{\pi}{2}.\cos x\right)}{2x - \pi}dx$

18. $\int_0^{2\pi} |\sqrt{15}\sin x + \cos x|\, dx$

19. Let a be a real number in the interval [0, 314] such that $\int_{-\pi+a}^{3\pi+a} |x-a-\pi| \sin\left(\frac{x}{2}\right)dx = -16$, then determine number of such values of k.

20. $\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}-\frac{1}{4n-1}\right)=\frac{\pi}{n}$, find 'n' $\left(\text{Note that } \tan^{-1}x + c = \int \frac{1}{1+x^2}dx\right)$

21. If $f(x) = x + \int_0^1 t(x+t)f(t)\,dt$, then the value of the definite integral $\int_0^1 f(x)\,dx$ can be expressed in the form of rational as $\frac{p}{q}$ (where p and q are coprime). Find (p + q).

22. If f(x) = (ax + b) e^x satisfies the equation : $f(x) = \int_0^x e^{x-y} f'(y)dy - (x^2 - x + 1)e^x$, find $(a^2 + b^2)$

23. If the minimum of the following function f(x) defined at $0 < x < \frac{\pi}{2}$.

$f(x) = \int_0^x \frac{d\theta}{\cos\theta} + \int_x^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta}$ is equal to $\ell n(a+\sqrt{b})$ where a, b $\in$ N and b is not a perfect square then find the value of (a + b)

24. Let $F(x) = \int_{-1}^{x} \sqrt{4+t^2}dt$ and $G(x) = \int_x^1 \sqrt{4+t^2}dt$ then compute the value of (FG)' (0) where dash denotes the derivative.

25. A students forgot the product rule for differentiation and made the mistake of thinking that (f.g)' = f'g'. However he was lucky to get the correct answer. The function f that he used was $f(x) = e^{x^2}$. If the domain of g(x) was the interval $\left(\frac{1}{2}, \infty\right)$ with g(1) = e. Find the value of $\frac{g(5)}{e^5}$.

26. If $f(x) = 2x^3 - 15x^2 + 24x$ and $g(x) = \int_0^x f(t)dt + \int_0^{5-x} f(t)dt$ (0 < x < 5). Find the number of integers for which g(x) is increasing.

27. Let h(x) = (fog)(x) + K where K is any constant. If $\frac{d}{dx}(h(x)) = -\frac{\sin x}{\cos^2(\cos x)}$ then compute the value of j(0) + sec1 where $j(x) = \int_{g(x)}^{f(x)} \frac{f(t)}{g(t)}dt$, where f and g are trigonometric functions.

28. Evaluate : $\lim\limits_{n\to\infty}\frac{4}{\pi n^2}\sum\limits_{k=0}^{n-1}\left[k\int\limits_{k}^{k+1}\sqrt{(x-k)(k+1-x)}\,dx\right]$

29. Let $P_n=\sqrt[n]{\frac{(3n)!}{(2n)!}}$ (n= 1,2,3.......), then find $\lim\limits_{n\to\infty}\frac{e\times P_n}{n}$.

30. If $f(x)$ = x + sinx and I denotes the value of integral $\int\limits_{\pi}^{2\pi}\left(f^{-1}(x)+\sin x\right)dx$ then the value of $\left[\frac{2I}{3}\right]$

(where [.] denotes greatest integer function)

31. Find the value of m (m > 0) for which the area bounded by the line y = mx + 2 and x = 2y – y^2 is 9/2 square units.

32. Find area bounded by y = $f^{-1}(x)$, x = 10, x = 4 and x-axis

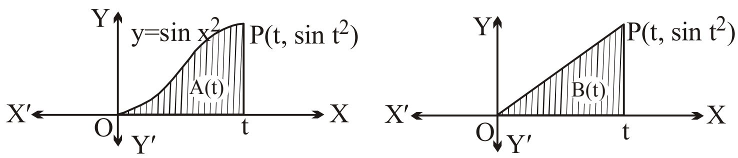
given that area bounded by y = f(x), x = 2, x = 6 and x-axis is 30 sq. units, where f(2) = 4 and f(6) = 10. (given f(x) is an invertible function)

33. Consider a line ℓ : 2x – $\sqrt{3}$y = 0 and a parameterized C : x = tan t, y = $\frac{1}{\cos t}\left(0\le t<\frac{\pi}{2}\right)$

If the area of the part bounded by ℓ, C and the y-axis is equal to $\frac{1}{4}\ell n(a+\sqrt{b})$, where a, b, $\in$ N, b, is not perfect square then find the value of (a + b)

34. A polynomial function f(x) satisfies the condition f(x + 1) = f(x) + 2x + 1 where f(0) = 1. compute the area enclosed by the curve and the pair of tangents from origin.

35. The figure shows two regions in the first quadrant.



A(t) is the area under the curve y = sin x^2 from 0 to t and B(t) is the area of the triangle with vertices O, P and M(t, 0). Find $\lim\limits_{t\to0}\frac{B(t)}{A(t)}$.

36. Let 'c' be the constant number such that c > 1. If the least area of the figure given by the line passing through the point (1, c) with gradient 'm' and the parabola y = x^2 is 36 sq. units find the value of (c^2 + m^2).

PART - III : ONE OR MORE THAN ONE CORRECT QUESTION

1. Let $u=\int_0^{\infty}\frac{dx}{x^4+7x^2+1}$ & $v=\int_0^{\infty}\frac{x^2dx}{x^4+7x^2+1}$ then -

(A) v > u (B) $6v = \pi$ (C) $3u + 2v = 5\pi/6$ (D) $u + v = \pi/3$

2. The value of integral $\int_a^b \frac{|x|}{x}\,dx$, a < b is :

(A) b – a if a > 0 (B) a – b if b < 0 (C) b + a if a < 0 < b (D) | b | – | a |

3. If $I_n=\int_0^1 \frac{dx}{\left(1+x^2\right)^n}$; $n \in N$, then which of the following statements hold good?

(A) $2n\, I_{n+1} = 2^{-n} + (2n-1)\, I_n$

(B) $I_2 = \frac{\pi}{8}+\frac{1}{4}$

(C) $I_2 = \frac{\pi}{8}-\frac{1}{4}$

(D) $I_3 = \frac{\pi}{16}-\frac{5}{48}$

4. If f(x) is integrable over [1, 2], then $\int_1^2 f(x)\,dx$ is equal to :

(A) $\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$

(B) $\lim_{n\to\infty}\frac{1}{n}\sum_{r=n+1}^{2n} f\left(\frac{r}{n}\right)$

(C) $\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r+n}{n}\right)$

(D) $\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n} f\left(\frac{r}{n}\right)$

5. The value of integral $\int_0^{\pi} x f(\sin x)\,dx$ is :

(A) $\frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$ (B) $\pi\int_0^{\pi/2} f(\sin x)\,dx$ (C) 0 (D) $2\pi\int_0^{\pi/4} f(\sin x)\,dx$

6. Given f is an odd function defined everywhere, periodic with period 2 and integrable on every interval. Let $g(x) = \int_0^x f(t)\,dt$. Then :

(A) g(2n) = 0 for every integer n

(B) g(x) is an even function

(C) g(x) and f(x) have the same period

(D) g(x) is an odd function

7. Let f(x) be a function satisfying f(x) + f(x + 2) = 10 $\forall x \in R$, then

(A) f(x) is a periodic function
(B) f(x) is aperiodic function
(C) $\int_{-1}^{7} f(x)dx = 20$
(D) $\int_{-1}^{7} f(x)dx = 40$

8. Let f : R $\to$ R be defined as f(x) = $\int_{-1}^{e^x} \frac{dt}{1+t^2} + \int_{1}^{e^{-x}} \frac{dt}{1+t^2}$, then

(A) f(x) is periodic
(B) f(f(x)) = f(x) $\forall x \in R$
(C) f(1) = f '(1) = $\frac{\pi}{2}$
(D) f(x) is unbounded

9. $I_1 = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$, $I_2 = \int_0^{\pi} \frac{x^3 \sin x}{(\pi^2 - 3\pi x + 3x^2)(1+\cos^2 x)} dx$, then

(A) $I_1 = \frac{\pi^2}{8}$
(B) $I_1 = \frac{\pi^2}{4}$
(C) $I_1 = I_2$
(D) $I_1 > I_2$

10. Let $f(x) = \begin{cases} x+1, & 0 \le x \le 1 \\ 2x^2 - 6x + 6, & 1 < x \le 2 \end{cases}$ and $g(t) = \int_{t-1}^{t} f(x)dx$ for t $\in$ [1,2]

Which of the following hold(s) good ?

(A) f(x) is continuous and differentiable in [0,2]
(B) g'(t) vanishes for t = 3/2 and 2
(C) g(t) is maximum at t = 3/2
(D) g(t) is minimum at t = 1

11. (a) Let g(x) = x^c. e^{2x} & let f(x) = $\int_0^x e^{2t}.(3t^2+1)^{1/2}dt$. For a certain value of 'c', the limit of $\frac{f'(x)}{g'(x)}$ as x $\to \infty$ is finite (equal to *l*) and non-zero, then

(A) c = 1
(B) c = 2
(C) $l = \frac{\sqrt{3}}{2}$
(D) $l = -\frac{\sqrt{3}}{2}$

12. Let f(x) = $\int_x^{x+\frac{\pi}{3}} |\sin\theta| d\theta$ (x $\in$ [0, π])

(A) f(x) is strictly increasing in this interval
(B) f(x) is differentiable in this interval
(C) Range of f(x) is $\left[2-\sqrt{3}, 1\right]$
(D) f(x) has a maxima at x = $\frac{\pi}{3}$

13. Let f(x) = $\int_0^x |2t-3| dt$, then f is

(A) continuous at x = 3/2
(B) continuous at x = 3
(C) differentiable at x = 3/2
(D) differentiable at x = 0

14. $f(x) = \int_0^1 f(tx)dt,$ where $f'(x)$ is a continuous function such that $f(1) = 2$, then

(A) $f(x)$ is a periodic function
(B) $f'(x) = 0$
(C) $f(x)$ is an even function
(D) $f(x)$ is an odd function

15. Let $I_n = \int_0^{\pi} (\sin x)^n dx,\ n \in N$, then

(A) I_n is rational if n is odd
(B) I_n is irrational if n is even
(C) I_n is an increasing sequence
(D) I_n is a decreasing sequence

16. Let $I_n = \int_0^{\pi} \frac{\sin^2(nx)}{\sin^2 x} dx$, $n \in N$, then

(A) $I_{n+2} + I_n = 2I_{n+1}$
(B) $I_n = I_{n+1}$
(C) $I_n = n\pi$
(D) $I_1, I_2, I_3, ...I_n$ are in Harmonic progression

17. Let $f(b) = \int_a^b (x^2 - 4x + 3)dx$, then

(A) If f(b) is least and a = 1 $\Rightarrow$ b = 3
(B) If a = 4, f(b) is an increasing function $\forall b \geq 4$
(C) If a = 0, f(b) is least for b = 2
(D) $f(b)_{min} = -\frac{4}{3}\ \forall a, b \in R$

18. Let $I = \int_2^{\infty} \left(\frac{\lambda x}{x^2+1} - \frac{1}{2x+1}\right) dx$ & I is a finite real number, then

(A) $\lambda = \frac{1}{2}$
(B) $\lambda = 1$
(C) $I = \frac{1}{2}\ell n\left(\frac{5}{2}\right)$
(D) $I = \frac{1}{4}\ell n\left(\frac{5}{4}\right)$

19. Let $L_1 = \lim_{x \to 0^+} \frac{\int_0^{x^2} \sin\sqrt{t}\, dt}{x - \sin x}$, $L_2 = \lim_{x \to 0^-} \frac{\int_0^{x^2} \sin\sqrt{t}\, dt}{x - \sin x}$, then identify the correct option(s).

(A) $L_1 = 4$
(B) $L_1 + L_2 = 8$
(C) $L_1 + L_2 = 0$
(D) $|L_2| = |L_1|$

20. If $a, b \in R^+$ then $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{n}{(k+an)(k+bn)}$ is equal to

(A) $\frac{1}{a-b} \ln \frac{b(b+1)}{a(a+1)}$ if $a \neq b$
(B) $\frac{1}{a-b} \ln \frac{a(b+1)}{b(a+1)}$ if $a \neq b$
(C) non existent if a = b
(D) $\frac{1}{a(1+a)}$ if a = b

21. $\lim\limits_{n\to\infty} \dfrac{(1^k + 2^k + 3^k + ... + n^k)}{(1^2 + 2^2 + ... + n^2)(1^3 + 2^3 + ... + n^3)}$ = F(k), then $(k \in N)$

(A) F(k) is finite for $k \le 6$ (B) F(5) = 0

(C) F(6) = $\dfrac{12}{7}$ (D) F(6) = $\dfrac{5}{7}$

22. Let $T_n = \sum\limits_{r=1}^{n} \dfrac{n}{r^2 - 2rn + 2n^2}$, $S_n = \sum\limits_{r=0}^{n-1} \dfrac{n}{r^2 - 2rn + 2n^2}$, then

(A) $T_n > S_n \ \forall \ n \in N$ (B) $T_n > \dfrac{\pi}{4}$ (C) $S_n < \dfrac{\pi}{4}$ (C) $\lim\limits_{n\to\infty} S_n = \dfrac{\pi}{4}$

23. Let f(x) be a strictly increasing, non-negative function such that $f''(x) < 0 \ \forall x \in R$ & $I = \int_{\alpha}^{\beta} f(x)dx \ (\beta > \alpha)$, then

(A) $I < f\left(\dfrac{\alpha+\beta}{2}\right)(\beta - \alpha)$ (B) $I > f\left(\dfrac{\alpha+\beta}{2}\right)(\beta - \alpha)$

(C) $I > \dfrac{1}{2}(f(\alpha) + f(\beta))(\beta - \alpha)$ (D) $I < \dfrac{1}{2}(f(\alpha) + f(\beta))(\beta - \alpha)$

24. Let $I_n = \int_0^{\frac{1}{2}} \dfrac{1}{\sqrt{1-x^n}} dx$ where n > 2 , then

(A) $I_n < \dfrac{\pi}{6}$ (B) $I_n > \dfrac{\pi}{6}$ (C) $I_n < \dfrac{1}{2}$ (D) $I_n > \dfrac{1}{2}$

25. Let f(x) be a continuous function and

$I = \int_1^9 \sqrt{x}\, f(x)\, dx$, then

(A) There exists some $c \in (1, 9)$ such that $I = 8\sqrt{c}\, f(c)$

(B) There exists some $p, q \in (1, 3)$ such that $I = 2[p^2 f(p^2) + q^2 f(q^2)]$

(C) There exists some $\alpha \in (1, 9)$ such that $I = 9\sqrt{\alpha}\, f(\alpha)$

(D) If $f(x) \ge 0 \ \forall x \in [1, 9]$ $\Rightarrow$ $I > 0$

26. Area bounded by $y = \sin^{-1}x$, $y = \cos^{-1}x$, $y = 0$ in first quadrant is equal to :

(A) $\int_0^{1/\sqrt{2}} (\sin^{-1} x)dx + \int_{1/\sqrt{2}}^{1} (\cos^{-1} x)dx$ (B) $\int_{\pi/4}^{\pi/2} (\sin y - \cos y)dy$

(C) $\int_0^{\pi/4} (\cos y - \sin y)dy$ (D) $(\sqrt{2} - 1)$ sq.unit

27. Let f(x) be a non-negative, continuous and even function such that area bounded by x-axis, y-axis & y = f(x) is equal to $(x^2 + x^3)$ sq. units $\forall x \geq 0$, then

(A) $\sum_{r=1}^{n} f'(r) = 3n^2 + 5n \ \forall \ n \in N$

(B) $\sum_{r=1}^{n} f'(r) = 6n^2 + 5n \ \forall \ n \in N$

(C) $f(x) = 3x^2 + 2x \ \forall \ x \leq 0$

(D) $f(x) = 3x^2 - 2x \ \forall \ x \leq 0$

28. Let 'c' be a positive real number such that area bounded by y = 0 y = $[\tan^{-1}x]$ from x = 0 to x = c is equal to area bounded by y = 0, y = $[\cot^{-1} x]$, from x = 0 to x = c (where [*] represents greatest integer function), then

(A) c = tan1 + cot1 (B) c = 2cosec2 (C) c = tan1 – cot1 (D) c = –2 cot2

29. Area bounded by the curve y = cotx, $x = \frac{\pi}{4}$ and y = 0 is-

(A) $\int_0^{\pi/4} \tan\left(\frac{\pi}{4} - x\right)dx$ (B) $\frac{\pi}{4} - \int_0^1 \tan^{-1} x dx$ (C) $1 - \int_0^1 \tan^{-1} x dx$ (D) $\int_0^{\pi.4} \tan^{-1} x dx$

30. Let T be the triangle with vertices (0, 0), (0, c^2) and (c, c^2) and let R be the region between y = cx and y = x^2 where c > 0 then

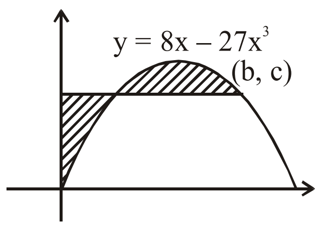
(A) Area (R) = $\frac{c^3}{6}$

(B) Area of R = $\frac{c^3}{3}$

(C) $\lim_{c \to 0^+} \frac{\text{Area (T)}}{\text{Area (R)}} = 3$

(D) $\lim_{c \to 0^+} \frac{\text{Area (T)}}{\text{Area (R)}} = \frac{3}{2}$

31. The figure shows a horizontal line y = c passing through (b, c) intersecting the curve y = 8x – 27x^3. If the shaded areas are equal, then



(A) $b = \frac{1}{9}$ (B) $b = \frac{4}{9}$ (C) $c = \frac{32}{27}$ (D) $c = \frac{23}{27}$

32. If A_1 denotes area of the region bounded by the curves C_1 : y = (x – 1)e^x, tangent to C_1 at (1,0) & y-axis and A_2 denotes the area of the region bounded by C_1 and co-ordinate axes in fourth quadrant, then -

(A) $A_1 > A_2$ (B) $A_1 < A_2$ (C) $2A_1 + A_2 = 2$ (D) $A_1 + 2A_2 = 4$

PART - IV : COMPREHENSION

Comprehension # 1 (Q.1 to Q.3)

Let the function *f* satisfies

$f(x) \cdot f'(-x) = f(-x) \cdot f'(x)$ for all x and $f(0) = 3$.

1. The value of $f(x) \cdot f(-x)$ for all x, is

(A) 4 (B) 9 (C) 12 (D) 16

2. $\int_{-51}^{51} \frac{dx}{3+f(x)}$ has the value equal to

(A) 17 (B) 34 (C) 102 (D) 0

3. Number of roots of $f(x) = 0$ in $[-2, 2]$ is

(A) 0 (B) 1 (C) 2 (D) 4

Comprehension # 2 (Q.4 to Q.6)

If $y = \int_{u(x)}^{v(x)} f(t)\,dt$, let us define $\frac{dy}{dx}$ in a different manner as $\frac{dy}{dx} = v'(x) f^2(v(x)) - u'(x) f^2(u(x))$ and the equation of the tangent at (a, b) as $y - b = \left(\frac{dy}{dx}\right)_{(a,b)} (x - a)$

4. If $y = \int_{x}^{x^2} t^2\,dt$, then equation of tangent at x = 1 is

(A) y = x + 1 (B) x + y = 1 (C) y = x – 1 (D) y = x

5. If $F(x) = \int_{1}^{x} e^{t^2/2}(1 - t^2)\,dt$, then $\frac{d}{dx}F(x)$ at x = 1 is

(A) 0 (B) 1 (C) 2 (D) – 1

6. If $y = \int_{x^3}^{x^4} \ell n t\,dt$, then $\lim_{x \to 0^+} \frac{dy}{dx}$ is

(A) 0 (B) 1 (C) 2 (D) – 1

Comprehension # 3 (Q.7 to Q.9)

Let $g(t) = \int_{x_1}^{x_2} f(t, x)\,dx$. Then $g'(t) = \int_{x_1}^{x_2} \frac{\partial}{\partial t}(f(t, x))\,dx$. Consider $f(x) = \int_0^{\pi} \frac{\ell n(1 + x\cos\theta)}{\cos\theta}\,d\theta$.

7. Range of $f(x)$ is

(A) $(0, \pi)$ (B) $(0, \pi^2)$ (C) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ (D) $\left(\frac{-\pi^2}{2}, \frac{\pi^2}{2}\right)$

8. The number of critical points of $f(x)$, in the interior of its domain, is

(A) 0 (B) 1 (C) 2 (D) infinitely many

9. $f(x)$ is

(A) discontinuous at x = 0 (B) continuous but not differentiable at x = 1

(C) continuous at x = 0 (D) differentiable at x = 1

Comprehension # 4 (Q.10 to Q.12)

If length of perpendicular drawn from points of a curve to a straight line approaches zero along an infinite branch of the curve, the line is said to be an asymptote to the curve. For example, y-axis is an asymptote to $y = \ell nx$ & x-axis is an asymptote to $y = e^{-x}$.

Asymptotes parallel to x-axis :

If $\lim_{x\to\infty} f(x) = e$ (a finite number) then y = e is an asymptote to y = f(x). Similarly if $\lim_{x\to-\infty} f(x) = \alpha$, then $y = \alpha$ is also an asymptote.

Asymptotes parallel to y-axis :

If $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$, then x = a is an asymptote to y = f(x).

10. Number of asymptotes parallel to co-ordinate axes for the function $f(x) = \frac{(x+1)(x+2)}{(x-1)(x-2)}$ is equal to :

(A) 1 (B) 2 (C) 3 (D) 4

11. Area bounded by $y = \frac{2x}{x^2+1}$, it's asymptote and ordinates at points of extremum is equal to (in square unit)

(A) $\ell n2$ (B) $2\ell n2$ (C) $\ell n3$ (D) $2\ell n3$

12. Area bounded by $y = x^2e^{-x}$ and it's asymptote in first quadrant is equal to (in square unit)

(A) 2e (B) e (C) 1 (D) 2

Exercise # 3

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

*** Marked Questions may have more than one correct option.**

1. The value of $\lim_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t\ell n(1+t)}{t^4+4}dt$ is **[IIT-JEE-2010, Paper-1 (3, –1)/84]**

(A) 0 (B) $\frac{1}{12}$ (C) $\frac{1}{24}$ (D) $\frac{1}{64}$

2. The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$ is (are) **[IIT-JEE-2010, Paper-1 (3, 0)/84]**

(A) $\frac{22}{7}-\pi$ (B) $\frac{2}{105}$ (C) 0 (D) $\frac{71}{15}-\frac{3\pi}{2}$

3*. Let f be a real-valued function defined on the interval $(0, \infty)$ by $f(x) = \ell n\, x + \int_0^x \sqrt{1+\sin t}\, dt$. Then which of the following statement(s) is (are) true? **[IIT-JEE-2010, Paper-1 (3, 0)/84]**

(A) $f''(x)$ exists for all $x \in (0, \infty)$

(B) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$

(C) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$

(D) there exists $\beta > 0$ such that $|f(x)| + |f'(x)| \le \beta$ for all $x \in (0, \infty)$

4. For any real number, let [x] denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval [–10, 10] by **[IIT-JEE-2010, Paper-1 (3, 0)/84]**

$$f(x) = \begin{cases} x-[x] & \text{if } [x] \text{ is odd,} \\ 1+[x]-x & \text{if } [x] \text{ is even} \end{cases}$$

Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\, dx$ is

5. Let f be a real-valued function defined on the interval (–1, 1) such that $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1}\, dt$, for all $x \in (-1, 1)$ and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to

[IIT-JEE-2010, Paper-2 (5, –2)/84]

(A) 1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{1}{e}$

Comprehension (Q.6 to Q.8)

Consider the polynomial

$f(x) = 1 + 2x + 3x^2 + 4x^3$

Let s be the sum of all distinct real roots of f(x) and let t = |s|

6. The real number s lies in the interval. **[IIT-JEE 2010, Paper-2, (3, –1), 79]**

(A) $\left(-\frac{1}{4}, 0\right)$ (B) $\left(-11, -\frac{3}{4}\right)$ (C) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (D) $\left(0, \frac{1}{4}\right)$

7. The area bounded by the curve y = f(x) and the lines x = 0, y = 0 and x = t, lies in the interval **[IIT-JEE 2010, Paper-2, (3, –1), 79]**

(A) $\left(\frac{3}{4}, 3\right)$ (B) $\left(\frac{21}{64}, \frac{11}{16}\right)$ (C) (9, 10) (D) $\left(0, \frac{21}{64}\right)$

8. The function f′(x) is **[IIT-JEE 2010, Paper-2, (3, –1), 79]**

(A) increasing in $\left(-t, \frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$

(B) decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$

(C) increasing in (–t, t)

(D) decreasing in (–t, t)

9. The value of $\int_{\sqrt{\ell n2}}^{\sqrt{\ell n3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ell n6 - x^2)} dx$ is **[IIT-JEE 2011, Paper-1, (3, –1), 80]**

(A) $\frac{1}{4} \ell n \frac{3}{2}$ (B) $\frac{1}{2} \ell n \frac{3}{2}$ (C) $\ell n \frac{3}{2}$ (D) $\frac{1}{6} \ell n \frac{3}{2}$

10. Let the straight line x = b divide the area enclosed by y = $(1 - x)^2$, y = 0, and x = 0 into two parts R_1 $(0 \le x \le b)$ and $R_2(b \le x \le 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then b equals **[IIT-JEE 2011, Paper-1, (3, –1), 80]**

(A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

11. Let f : [–1, 2] → [0, ∞) be a continuous function such that f(x) = f(1 – x) for all x ∈ [–1, 2].

Let $R_1 = \int_{-1}^{2} x f(x)\, dx$, and R_2 be the area of the region bounded by y = f(x), x = –1, x = 2, and the x-axis. Then

[IIT-JEE 2011, Paper-2, (3, –1), 80]

(A) $R_1 = 2R_2$ (B) $R_1 = 3R_2$ (C) $2R_1 = R_2$ (D) $3R_1 = R_2$

12.* If S be the area of the region enclosed by y = e^{-x^2}, y = 0, x = 0, and x = 1. Then

[IIT-JEE 2012, Paper-1, (4, 0), 70]

(A) $S \ge \frac{1}{e}$ (B) $S \ge 1 - \frac{1}{e}$ (C) $S \le \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$ (D) $S \le \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

13. The value of the integral $\int_{-\pi/2}^{\pi/2}\left(x^2+\ell n\frac{\pi+x}{\pi-x}\right)\cos x\,dx$ is **[IIT-JEE 2012, Paper-2, (3, –1), 66]**

(A) 0 (B) $\frac{\pi^2}{2}-4$ (C) $\frac{\pi^2}{2}+4$ (D) $\frac{\pi^2}{2}$

Comprehension # (Q.14 to Q.15)

Let $f(x) = (1-x)^2 \sin^2 x + x^2$ for all $x \in \mathbb{R}$ and let $g(x) = \int_1^x\left(\frac{2(t-1)}{t+1}-\ell nt\right)f(t)\,dt$ for all $x \in (1, \infty)$.

14. Which of the following is true ? **[IIT-JEE 2012, Paper-2, (3, –1), 66]**

(A) g is increasing on $(1, \infty)$

(B) g is decreasing on $(1, \infty)$

(C) g is increasing on (1, 2) and decreasing on $(2, \infty)$

(D) g is decreasing on (1, 2) and increasing on $(2, \infty)$

15. Consider the statements :

P : There exists some $x \in \mathbb{R}$ such that $f(x) + 2x = 2(1 + x^2)$

Q : There exists some $x \in \mathbb{R}$ such that $2f(x) + 1 = 2x(1 + x)$

Then

(A) both P and Q are true (B) P is true and Q is false

(C) P is false and Q is true (D) both P and Q are false

16.* If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)\,dt$ for all $x \in (0, \infty)$, then

(A) f has a local maximum at x = 2 **[IIT-JEE 2012, Paper-2, (4, 0), 66]**

(B) f is decreasing on (2, 3)

(C) there exists some $c \in (0, \infty)$ such that $f''(c) = 0$

(D) f has a local minimum at x = 3

17. The area enclosed by the curves y = sinx + cosx and y = |cosx – sinx| over the interval $\left[0, \frac{\pi}{2}\right]$ is

[JEE (Advanced) 2013, Paper-1, (2, 0)/60]

(A) $4\left(\sqrt{2}-1\right)$ (B) $2\sqrt{2}\left(\sqrt{2}-1\right)$ (C) $2\left(\sqrt{2}+1\right)$ (D) $2\sqrt{2}\left(\sqrt{2}+1\right)$

18. Let $f:\left[\frac{1}{2}, 1\right] \to R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then the value of $\int_{1/2}^{1} f(x)dx$ lies in the interval

[JEE (Advanced) 2013, Paper-1, (2, 0)/60]

(A) (2e – 1, 2e) (B) (e – 1, 2e – 1) (C) $\left(\frac{e-1}{2}, e-1\right)$ (D) $\left(0, \frac{e-1}{2}\right)$

19.* For $a \in R$ (the set of all real numbers), $a \neq -1$, $\lim_{n\to\infty} \frac{(1^a + 2^a + ... + n^a)}{(n+1)^{a-1}[(na+1)+(na+2)+...+(na+n)]} = \frac{1}{60}$. Then a =

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

(A) 5 (B) 7 (C) $\frac{-15}{2}$ (D) $\frac{-17}{2}$

20.* Let $f:[a, b] \to [1, \infty)$ be a continuous function and let $g : R \to R$ be defined as

$$g(x) = \begin{cases} 0 & \text{if} \quad x < a, \\ \int_a^x f(t)dt & \text{if} \quad a \le x \le b, \\ \int_a^b f(t)dt & \text{if} \quad x > b. \end{cases}$$, Then

[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

(A) g(x) is continuous but not differentiable at a
(B) g (x) is differentiable on R
(C) g(x) is continuous but not differentiable at b
(D) g(x) is continuous and differentiable at either a or b but not both

21.* Let $f: (0, \infty) \to R$ be given by $f(x) = \int_{\frac{1}{x}}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$. Then

[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

(A) f(x) is monotonically increasing on $[1, \infty)$ (B) f(x) is monotonically decreasing on (0, 1)

(C) $f(x) + f\left(\frac{1}{x}\right) = 0$, for all $x \in (0, \infty)$ (D) $f(2^x)$ is an odd function of x on R

22. The value of $\int_0^1 4x^3 \left\{\frac{d^2}{dx^2}(1-x^2)^5\right\} dx$ is **[JEE (Advanced) 2014, Paper-1, (3, 0)/60]**

23. The following integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\operatorname{cosec}x)^{17}\, dx$ is equal to **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**

(A) $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16} du$ (B) $\int_0^{\log(1+\sqrt{2})} (e^u + e^{-u})^{17} du$

(C) $\int_0^{\log(1+\sqrt{2})} (e^u - e^{-u})^{17} du$ (D) $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16} du$

Comprehension # (Q.24 to Q.25)

Given that for each $a \in (0, 1)$

$$\lim_{h \to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}\,dt$$

exists. Let this limit be g(a). In addition, it is given that the function g(a) is differentiable on (0, 1).

$$\lim_{h \to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}\,dt$$ **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**

24. The value of $g\left(\frac{1}{2}\right)$ is

(A) π (B) 2π (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

25. The value of $g'\left(\frac{1}{2}\right)$ is

(A) $\frac{\pi}{2}$ (B) π (C) $-\frac{\pi}{2}$ (D) 0

26. **List I** **List II**

[JEE (Advanced) 2014, Paper-2, (3, –1)/60]

P. The number of polynomials f(x) with non-negative integer coefficients of degree ≤ 2, satisfying f(0) = 0 and $\int_0^1 f(x)dx = 1$, is — **1.** 8

Q. The number of points in the interval $\left[-\sqrt{13}, \sqrt{13}\right]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is — **2.** 2

R. $\int_{-2}^{2} \frac{3x^2}{(1+e^x)} dx$ equals — **3.** 4

S. $\dfrac{\left(\int_{-1/2}^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx\right)}{\left(\int_{0}^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx\right)}$ equals — **4.** 0

	P	Q	R	S		P	Q	R	S
(A)	3	2	4	1	(B)	2	3	4	1
(C)	3	2	1	4	(D)	2	3	1	4

27. Let f: R $\to$ R be a function defined by f(x) = $\begin{cases} [x], & x \le 2 \\ 0, & x > 2 \end{cases}$ where [x] is the greatest integer less than or equal to x. If $I = \int_{-1}^{2} \frac{xf(x^2)}{2+f(x+1)}dx$, then the value of (4I–1) is **[JEE (Advanced) 2015, P-1 (4, 0) /88]**

28. If $\alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right)dx$ where $\tan^{-1} x$ takes only principal values, then the value of $\left(\log_e |1+\alpha| - \frac{3\pi}{4}\right)$ is **[JEE (Advanced) 2015, P-2 (4, 0) / 80]**

29. Let f: R $\rightarrow$ R be a continuous odd function, which vanishes exactly at one point and f(1) = $\frac{1}{2}$. Suppose that $F(x) = \int_{-1}^{x} f(t)dt$ for all $x \in [-1, 2]$ and $G(x) = \int_{-1}^{x} t\,|f(f(t))|dt$ for all $x \in [-1, 2]$. If $\lim_{x\to 1}\frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is. **[JEE (Advanced) 2015, P-2 (4, 0) / 80]**

30*. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is (are)

[JEE (Advanced) 2015, P-2 (4, –2)/ 80]

(A) $\int_0^{\pi/4} xf(x)\,dx = \frac{1}{12}$ (B) $\int_0^{\pi/4} f(x)\,dx = 0$ (C) $\int_0^{\pi/4} xf(x)\,dx = \frac{1}{6}$ (D) $\int_0^{\pi/4} f(x)\,dx = 1$

31*. Let $f'(x) = \frac{192x^3}{2+\sin^4 \pi x}$ for all $x \in R$ with $f\left(\frac{1}{2}\right) = 0$. If $m \le \int_{1/2}^{1} f(x)\,dx \le M$, then the possible values of m and M are **[JEE (Advanced) 2015, P-2 (4, –2)/ 80]**

(A) m = 13, M = 24 (B) m = $\frac{1}{4}$, M = $\frac{1}{2}$ (C) m = – 11, M = 0 (D) m = 1, M = 12

32*. The option(s) with the values of a and L that satisfy the following equation is(are)

$$\frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt} = L\ ?$$

[JEE (Advanced) 2015, P-2 (4, –2)/ 80]

(A) a = 2, L = $\frac{e^{4\pi}-1}{e^{\pi}-1}$ (B) a = 2, L = $\frac{e^{4\pi}+1}{e^{\pi}+1}$

(C) a = 4, L = $\frac{e^{4\pi}-1}{e^{\pi}-1}$ (D) a = 4, L = $\frac{e^{4\pi}+1}{e^{\pi}+1}$

Comprehension # (Q.33 to Q.34)

Let F : R → R be a thrice differentiable function. Supose that F(1) = 0, F(3) = –4 and F'(x) < 0 for all x ∈ (1/2, 3). Let f(x) = xF(x) for all x ∈ R. **[JEE (Advanced) 2015, P-2 (4, –2)/ 80]**

33*. The correct statement(s) is(are)

(A) f '(1) < 0 (B) f (2) < 0

(C) f '(x) ≠ 0 for any x ∈ (1, 3) (D) f '(x) = 0 for some x ∈ (1, 3)

34*. If $\int_1^3 x^2 F'(x)dx = -12$ and $\int_1^3 x^3 F''(x)dx = 40$, then the correct expression(s) is(are)

(A) 9f '(3) + f '(1) – 32 = 0 (B) $\int_1^3 f(x)dx = 12$

(C) 9f '(3) – f '(1) + 32 = 0 (D) $\int_1^3 f(x)dx = -12$

35. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \cos x}{1+e^x}dx$ is equal to **[JEE(Advanced)-2016, 3(–1)]**

(A) $\frac{\pi^2}{4}-2$ (B) $\frac{\pi^2}{4}+2$ (C) $\pi^2 - e^{\frac{\pi}{2}}$ (D) $\pi^2 + e^{\frac{\pi}{2}}$

36. The total number of distinct x ∈ [0, 1] for which $\int_0^x \frac{t^2}{1+t^4}dt$ = 2x – 1 is **[JEE(Advanced)-2016]**

37. Area of the region $\left\{(x,y)\in\mathbb{R}^2 : y \ge \sqrt{|x+3|}, 5y \le x+9 \le 15\right\}$ is equal to - **[JEE(Advanced)-2016]**

(A) $\frac{1}{6}$ (B) $\frac{4}{3}$ (C) $\frac{3}{2}$ (D) $\frac{5}{3}$

38. Let $f(x) = \lim_{n\to\infty}\left(\frac{n^n(x+n)\left(x+\frac{n}{2}\right).....\left(x+\frac{n}{n}\right)}{n!(x^2+n^2)\left(x^2+\frac{n^2}{4}\right).....\left(x^2+\frac{n^2}{n^2}\right)}\right)^{x/n}$, for all x > 0. Then

[JEE(Advanced)-2016]

(A) $f\left(\frac{1}{2}\right) \ge f(1)$ (B) $f\left(\frac{1}{3}\right) \le f\left(\frac{2}{3}\right)$ (C) $f'(2) \le 0$ (D) $\frac{f'(3)}{f(3)} \ge \frac{f'(2)}{f(2)}$

39. Let f : R → R be a differentiable function such that f(0) = 0, $f\left(\frac{\pi}{2}\right)$ = 3 and f'(0) = 1. If

$$g(x) = \int_x^{\frac{\pi}{2}} \left[f'(t)\,\text{cosec}\,t - \cot t\,\text{cosec}\,t\,f(t)\right] dt$$

for $x \in \left(0, \frac{\pi}{2}\right]$, then $\lim_{x \to 0} g(x)$ = **[JEE(Advanced)-2017, 3]**

40. If $I = \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} dx$, then **[JEE(Advanced)-2017, 4]**

(A) $I < \frac{49}{50}$ (B) $I < \log_e 99$ (C) $I > \frac{49}{50}$ (D) $I > \log_e 99$

41. If $g(x) = \int_{\sin x}^{\sin(2x)} \sin^{-1}(t)\, dt$, then **[JEE(Advanced)-2017, 4]**

(A) $g'\left(\frac{\pi}{2}\right) = -2\pi$ (B) $g'\left(-\frac{\pi}{2}\right) = 2\pi$ (C) $g'\left(\frac{\pi}{2}\right) = 2\pi$ (D) $g'\left(-\frac{\pi}{2}\right) = -2\pi$

42. If the line x = α divides the area of region $R = \{(x, y) \in \mathbb{R}^2 : x^3 \le y \le x, 0 \le x \le 1\}$ into two equal parts, then **[JEE(Advanced)-2017, 3(–2)]**

(A) $\frac{1}{2} < \alpha < 1$ (B) $\alpha^4 + 4\alpha^2 - 1 = 0$ (C) $0 < \alpha \le \frac{1}{2}$ (D) $2\alpha^4 - 4\alpha^2 + 1 = 0$

43. Let f : R → (0, 1) be a continuous function. Then, which of the following function(s) has (have) the value zero at some point in the interval (0, 1)? **[JEE(Advanced)-2017, 4(–2)]**

(A) $e^x - \int_0^x f(t)\sin t\, dt$ (B) $f(x) + \int_0^{\frac{\pi}{2}} f(t)\sin t\, dt$ (C) $x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\, dt$ (D) $x^9 - f(x)$

44. For each positive integer n, let

$$y_n = \frac{1}{n}\left((n+1)(n+2)...(n+n)\right)^{1/n}$$

For $x \in \mathbb{R}$, let [x] be the greatest integer less than or equal to x. If $\lim_{n\to\infty} y_n = L$, then the value of [L] is ______ **[JEE(Advanced)-2018, 3(0)]**

45. The value of the integral

$$\int_0^{\frac{1}{2}} \frac{1+\sqrt{3}}{\left((x+1)^2(1-x)^6\right)^{\frac{1}{4}}} dx$$

is _____ . **[JEE(Advanced)-2018, 3(0)]**

46. Let f : [0, ∞) → $\mathbb{R}$ be a continuous function such that $f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)dt$ for all x ∈ [0, ∞). Then, which of the following statement(s) is (are) TRUE ?

(A) The curve y = f(x) passes through the point (1, 2) **[JEE(Advanced)-2018, 4(–2)]**

(B) The curve y = f(x) passes through the point (2, –1)

(C) The area of the region {(x, y) ∈ [0, 1] × $\mathbb{R}$: $f(x) \le y \le \sqrt{1-x^2}$ } is $\frac{\pi - 2}{4}$

(D) The area of the region {(x, y) ∈ [0, 1] × $\mathbb{R}$: $f(x) \le y \le \sqrt{1-x^2}$ } is $\frac{\pi - 1}{4}$

47. A farmer F_1 has a land in the shape of a triangle with vertices at P(0, 0), Q(1, 1) and R(2, 0). From this land, a neighbouring farmer F_2 takes away the region which lies between the side PQ and a curve of the form $y = x^n (n > 1)$. If the area of the region taken away by the farmer F_2 is exactly 30% of the area of ΔPQR, then the value of n is _____. **[JEE(Advanced)-2018, 3(0)]**

48. If $I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$ then $27I^2$ equals _____ **[JEE(Advanced)-2019, 3(0)]**

49. For a ∈ $\mathbb{R}$, |a| > 1, let $\lim_{n\to\infty}\left(\frac{1+\sqrt[3]{2}+.....+\sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(an+1)^2}+\frac{1}{(an+2)^2}+.......+\frac{1}{(an+n)^2}\right)}\right) = 54$. Then the possible value(s) of a is/are : **[JEE(Advanced)-2019, 4(–1)]**

(A) 8 (B) –9 (C) –6 (D) 7

50. The value of the integral $\int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5} d\theta$ equals **[JEE(Advanced)-2019, 3(0)]**

51. The area of the region {(x, y) : xy ≤ 8, 1 ≤ y ≤ x^2} is **[JEE(Advanced)-2019, 3(–1)]**

(A) $8\log_e 2 - \frac{14}{3}$ (B) $16\log_e 2 - \frac{14}{3}$ (C) $16\log_e 2 - 6$ (D) $8\log_e 2 - \frac{7}{3}$

PART - II : JEE(MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. Let p(x) be a function defined on **R** such that p′(x) = p′(1 – x), for all x ∈ [0, 1], p(0) = 1 and p(1) = 41. Then $\int_0^1 p(x)dx$ equals **[AIEEE 2010 (8, –2), 144]**

(1) 21 (2) 41 (3) 42 (4) $\sqrt{41}$

2. The area bounded by the curves y = cos x and y = sinx between the ordinates x = 0 and x = $\frac{3\pi}{2}$ is **[AIEEE 2010 (4, –1), 144]**

(1) $4\sqrt{2}+2$ (2) $4\sqrt{2}-1$ (3) $4\sqrt{2}+1$ (4) $4\sqrt{2}-2$

3. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t}\ \sin t\, dt$. Then f has : **[AIEEE 2011, I, (4, –1), 120]**

(1) local maximum at π and 2π.
(2) local minimum at π and 2π
(3) local minimum at π and local maximum at 2π.
(4) local maximum at π and local minimum at 2π.

4. Let [.] denote the greatest integer function then the value of $\int_0^{1.5} x[x^2]\ dx$ is : . **[AIEEE 2011, II, (4, –1), 120]**

(1) 0 (2) $\frac{3}{2}$ (3) $\frac{3}{4}$ (4) $\frac{5}{4}$

5. The area of the region enclosed by the curves y = x, x = e, $y = \frac{1}{x}$ and the positive x-axis is

[AIEEE 2011, I, (4, –1), 120]

(1) $\frac{1}{2}$ square units (2) 1 square units (3) $\frac{3}{2}$ square units (4) $\frac{5}{2}$ square units

6. The area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$ is : **[AIEEE 2011, II, (4, –1), 120]**

(1) $\frac{32}{3}$ (2) $\frac{16}{3}$ (3) $\frac{8}{3}$ (4) 0

7. The area bounded between the parabolas $x^2 = \frac{y}{4}$ and $x^2 = 9y$ and the straight line y = 2 is :

[AIEEE-2012, (4, –1)/120]

(1) $20\sqrt{2}$ (2) $\frac{10\sqrt{2}}{3}$ (3) $\frac{20\sqrt{2}}{3}$ (4) $10\sqrt{2}$

8.* If $g(x) = \int_0^x \cos 4t\ dt$, then $g(x + \pi)$ equals **[AIEEE-2012, (4, –1)/120]**

(1) $\frac{g(x)}{g(\pi)}$ (2) $g(x) + g(\pi)$ (3) $g(x) - g(\pi)$ (4) $g(x)\ .\ g(\pi)$

9. Statement-I : The value of the integral $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ is equal to $\pi/6$. **[AIEEE - 2013, (4, –¼),360]**

Statement-II : $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$. **[AIEEE - 2013, (4, –¼),360]**

(1) Statement-I is true; Statement-II is true; Statement-II is a **correct** explanation for Statement-I.
(2) Statement-I is true; Statement-II is true; Statement-II is **not** a correct explanation for Statement-I.
(3) Statement-I is true; Statement-II is false.
(4) Statement-I is false; Statement-II is true.

10. The area (in square units) bounded by the curves $y = \sqrt{x}$, $2y - x + 3 = 0$, x-axis, and lying in the first quadrant is : **[AIEEE - 2013, (4, –¼),360]**

(1) 9 (2) 36 (3) 18 (4) $\frac{27}{4}$

11. The integral $\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ equals : **[JEE(Main) 2014, (4, – ¼), 120]**

(1) $4\sqrt{3}-4$ (2) $4\sqrt{3}-4-\frac{\pi}{3}$ (3) $\pi - 4$ (4) $\frac{2\pi}{3}-4-4\sqrt{3}$

12. The area of the region described by $A = \{(x, y) : x^2 + y^2 \le 1 \text{ and } y^2 \le 1 - x\}$ is :

[JEE(Main) 2014, (4, – ¼), 120]

(1) $\frac{\pi}{2}-\frac{2}{3}$ (2) $\frac{\pi}{2}+\frac{2}{3}$ (3) $\frac{\pi}{2}+\frac{4}{3}$ (4) $\frac{\pi}{2}-\frac{4}{3}$

13. The integral $\int_2^4 \frac{\log x^2}{\log x^2 + \log(36-12x+x^2)}dx$ is equal to **[JEE(Main) 2015, (4, – ¼), 120]**

(1) 2 (2) 4 (3) 1 (4) 6

14. The area (in sq. units) of the region described by $\{(x, y); y^2 \le 2x \text{ and } y \ge 4x - 1\}$ is

[JEE(Main) 2015, (4, – ¼), 120]

(1) $\frac{7}{32}$ (2) $\frac{5}{64}$ (3) $\frac{15}{64}$ (4) $\frac{9}{32}$

15. $\lim_{n\to\infty}\left(\frac{(n+1)(n+2)....3n}{n^{2n}}\right)^{1/n}$ is equal to :- **[JEE-MAIN-2016]**

(1) 3 log3 – 2 (2) $\frac{18}{e^4}$ (3) $\frac{27}{e^2}$ (4) $\frac{9}{e^2}$

16. The area (in sq.units) of the region $\{(x, y) : y^2 \ge 2x \text{ and } x^2 + y^2 \le 4x,\ x \ge 0,\ y \ge 0\}$ is :-

[JEE(Main)-2016]

(1) $\frac{\pi}{2}-\frac{2\sqrt{2}}{3}$ (2) $\pi-\frac{4}{3}$ (3) $\pi-\frac{8}{3}$ (4) $\pi-\frac{4\sqrt{2}}{3}$

17. The integral $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{dx}{1+\cos x}$ is equal to :- **[JEE-MAIN-2017]**

(1) –1 (2) –2 (3) 2 (4) 4

18. The area (in sq. units) of the region $\{(x, y) : x \geq 0, x + y \leq 3, x^2 \leq 4y \text{ and } y \leq 1 + \sqrt{x}\}$ is : **[JEE(Main)-2017]**

(1) $\frac{5}{2}$ (2) $\frac{59}{12}$ (3) $\frac{3}{2}$ (4) $\frac{7}{3}$

19. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x} dx$ is : **[JEE-MAIN-2018]**

(1) $\frac{\pi}{2}$ (2) 4π (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{8}$

20. Let $g(x) = \cos x^2$, $f(x) = \sqrt{x}$ and α, β $(\alpha < \beta)$ be the roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$. Then the area (in sq. units) bounded by the curve y = (gof) (x) and the lines $x = \alpha$, $x = \beta$ and y = 0 is- **[JEE(Main)-2018]**

(1) $\frac{1}{2}\left(\sqrt{3}+1\right)$ (2) $\frac{1}{2}\left(\sqrt{3}-\sqrt{2}\right)$ (3) $\frac{1}{2}\left(\sqrt{2}-1\right)$ (4) $\frac{1}{2}\left(\sqrt{3}-1\right)$

21. The value of $\int_0^{\pi} |\cos x|^3 dx$ **[JEE(Main)-JAN2019]**

(1) 2/3 (2) 0 (3) –4/3 (4) 4/3

22. The area (in sq. units) bounded by the parabola $y = x^2 - 1$, the tangent at the point (2, 3) to it and the y-axis is : **[JEE(Main)-JAN2019]**

(1) $\frac{14}{3}$ (2) $\frac{56}{3}$ (3) $\frac{8}{3}$ (4) $\frac{32}{3}$

23. If $\int_0^{\frac{\pi}{3}} \frac{\tan\theta}{\sqrt{2k\sec\theta}} d\theta = 1 - \frac{1}{\sqrt{2}}, (k > 0)$, then the value of k is : **[JEE(Main)-JAN2019]**

(1) 2 (2) $\frac{1}{2}$ (3) 4 (4) 1

24. The area of the region $A = \left[(x, y) : 0 \leq y \leq x|x| + 1 \text{ and } -1 \leq x \leq 1\right]$ in sq. units, is : **[JEE(Main)-JAN2019]**

(1) $\frac{2}{3}$ (2) $\frac{1}{3}$ (3) 2 (4) $\frac{4}{3}$

25. The area of the region, enclosed by the circle $x^2 + y^2 = 2$ which is not common to the region bounded by the parabola $y^2 = x$ and the straight line $y = x$, is : **[JEE(Main)-JAN2020]**

(1) $\frac{1}{3}(12\pi - 1)$ (2) $\frac{1}{6}(12\pi - 1)$ (3) $\frac{1}{6}(24\pi - 1)$ (4) $\frac{1}{3}(6\pi - 1)$

26. If $f(a + b + 1 - x) = f(x)$, for all x, where a and b are fixed positive real numbers, then $\frac{1}{a+b}\int_a^b x(f(x) + f(x+1))dx$ is equal to : **[JEE(Main)-JAN2020]**

(1) $\int_{a+1}^{b+1} f(x)dx$ (2) $\int_{a+1}^{b+1} f(x+1)dx$ (3) $\int_{a-1}^{b-1} f(x+1)dx$ (4) $\int_{a-1}^{b-1} f(x)dx$

27. The area (in sq. units) of the region **[JEE(Main)-JAN2020]**
$\{(x, y) \in R^2 | 4x^2 \le y \le 8x + 12)$ is :

(1) $\frac{127}{3}$ (2) $\frac{125}{3}$ (3) $\frac{124}{3}$ (4) $\frac{128}{3}$

28. The value of α for which $4\alpha\int_{-1}^{2} e^{-\alpha|x|}dx = 5$, is : **[JEE(Main)-JAN2020]**

(1) $\log_e\left(\frac{3}{2}\right)$ (2) $\log_e\left(\frac{4}{3}\right)$ (3) $\log_e 2$ (4) $\log_e \sqrt{2}$

Answers

Exercise # 1

PART-I

SECTION-(A)

A-1. (i) $\frac{104}{5}$ (ii) $-\ell n\, 4$

(iii) $-\frac{10}{21}$ (iv) $\sqrt{2}-1$

(v) $-\frac{\pi}{8}(b-a)^2$ (vi) π (vii) $\frac{\pi}{4}$

(viii) $\frac{8}{21}$ (ix) $4+\ell n\, 5$

A-2. (i) $\frac{\pi-2}{2}$ (ii) $\frac{1}{2}\ell n\left(\frac{e}{2}\right)$

(iii) 1 (iv) $\frac{\pi}{6}-\frac{2}{9}$

(v) $\pi/2$ (vi) $\frac{\pi}{2}-\ell n\, 2$

(vii) $\frac{4-\pi}{4\sqrt{2}}$

A-3. (i) $\frac{\pi}{4}$ (ii) $\frac{5}{3}-2\,\ell n2$

(iii) $\ell n\left(\frac{9}{8}\right)$ (iv) $\frac{\pi}{2}$

(v) $\frac{1}{20}\ell n\, 3$ (vi) (1/2) ln 3

(vii) $\frac{\pi}{4}$ (viii) $\frac{\pi}{8}$

A-4. (i) $\ell n\, 11$ (ii) $\frac{1}{3}$

A-6. 2/3

SECTION-(B)

B-2. (i) $5-\sqrt{2}-\sqrt{3}$ (ii) $2\sqrt{2}$

(iii) 9 (iv) 4

(v) cot 1

(vi) $2(\sqrt{2}-1)$

(vii) cos 1 + cos 2 + cos 3 + 3

B-3. (i) $\ell n(\sqrt{3})$

(ii) 7/6

(iii) $\frac{3}{8}$

B-4 0

B-5. (i) 2e – 2 (ii) $2-\sqrt{2}$

(iii) $\frac{\pi^2}{6\sqrt{3}}$ (iv) 0

(v) 0

B-6. (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{4}$

(iii) $\frac{a}{2}$ (iv) $(a+b)\frac{\pi}{4}$

(v) 0

B-7. (i) 0 (ii) $\frac{\pi}{3}$

(iii) 0 (iv) $\pi\ell n2$

B-8 π^2

B-9. (i) $\frac{3}{2}$ (ii) 40

(iii) n – 1 (iv) 4n

(v) $100\log_2 e$ (vi) 2

C-1. (i) $4\sqrt{2}$ (ii) 12

(iii) $\left(\frac{\sqrt[4]{8}}{3}-\frac{1}{4}\right)\pi$

C-2. (ii) 1, 3

C-3. 5/2

C-4. 13.5

C-5. (1 + e)

C-6. (i) $\frac{4}{15}$ (ii) $\frac{8\pi}{15}$

(iii) $\frac{\pi}{2}$ (iv) $\frac{\pi^2}{4}$

SECTION-(E)

E-1. (i) $\frac{\pi}{2}$ (ii) 2

(iii) 12 (iv) –1/2

E-4. 90

SECTION-(F)

F-1. $\frac{51}{4}$ sq. unit

F-2. (i) $\frac{\pi}{2}-\frac{4}{\pi}$

(ii) $\frac{7}{120}$

(iii) 9π

F-3. (i) 4/3 sq. units

(ii) $\frac{(e+1)\pi}{1+\pi^2}$

F-4. (i) $\frac{3\pi+2}{\pi-2}$

(ii) $\frac{\pi}{2}$; $\frac{\pi-1}{\pi+1}$

(iii) $\frac{1}{8}$

F-5. **(i)** 4 sq. units. (ii) 23/6

F-9. **e**

F-10. a = – 3/4

F-11. a = 8 or $\frac{2}{5}\left(6-\sqrt{21}\right)$

PART-II

SECTION-(A)

A-1.	(A)	**A-2.**	(C)
A-3.	(C)	**A-4.**	(C)
A-5.	(D)	**A-6.**	(A)
A-7.	(A)	**A-8.**	(A)
A-9.	(B)	**A-10.**	(D)
A-11.	(C)	**A-12.**	(C)
A-13.	(A)	**A-14.**	(D)
A-15.	(D)	**A-16.**	(A)
A-17.	(A)		

SECTION-(B)

B-1.	(A)	**B-2.**	(C)
B-3.	(C)	**B-4.**	(D)
B-5.	(C)	**B-6.**	(D)
B-7.	(A)	**B-8.**	(B)
B-9.	(D)	**B-10.**	(C)
B-11.	(A)	**B-12.**	(C)
B-13.	(A)	**B-14.**	(B)
B-15.	(C)	**B-16.**	(D)
B-17.	(A)	**B-18.**	(C)
B-19.	(A)	**B-20.**	(B)
B-21.	(B)	**B-22.**	(A)
B-23.	(A)	**B-24.**	(C)

SECTION-(C)

C-1.	(A)	**C-2.**	(B)
C-3.	(B)	**C-4.**	(D)
C-5.	(B)	**C-6.**	(C)
C-7.	(C)	**C-8.**	(B)
C-9.	(D)	**C-10.**	(A)
C-11.	(D)	**C-12.**	(B)

SECTION-(D)

D-1.	(C)	**D-2.**	(B)
D-3.	(C)	**D-4.**	(A)

SECTION-(E)

E-1.	(D)	**E-2.**	(B)
E-3.	(C)	**E-4.**	(C)
E-5.	(D)		

SECTION-(F)

F-1.	(D)	**F-2.**	(B)
F-3.	(B)	**F-4.**	(A)
F-5.	(C)	**F-6.**	(A)
F-7.	(A)	**F-8.**	(A)
F-9.	(D)	**F-10.**	(B)
F-11.	(D)	**F-12.**	(A)

PART-III

1. A - q, B - s, C - p, D - t

2. (A) $\rightarrow$ (s), (B) $\rightarrow$ (s), (C) $\rightarrow$ (q), (D) $\rightarrow$ (p)

Exercise # 2

PART-I

1.	(A)	**2.**	(B)
3.	(C)	**4.**	(B)
5.	(A)	**6.**	(A)
7.	(A)	**8.**	(D)
9.	(B)	**10.**	(C)
11.	(C)	**12.**	(A)
13.	(D)	**14.**	(B)
15.	(B)	**16.**	(C)
17.	(A)	**18.**	(C)
19.	(C)	**20.**	(C)
21.	(B)	**22.**	(C)
23.	(C)	**24.**	(A)
25.	(B)	**26.**	(A)
27.	(D)	**28.**	(D)
29.	(D)	**30.**	(B)
31.	(B)	**32.**	(C)
33.	(D)	**34.**	(A)

PART-II

1.	9.2	**2.**	2525
3.	2	**4.**	2
5.	4	**6.**	3
7.	64	**8.**	10
9.	2/3	**10.**	29
11.	4	**12.**	8
13.	8	**14.**	4
15.	0	**16.**	2
17.	8	**18.**	16

19.	25	**20.**	4
21.	65	**22.**	5
23.	11	**24.**	0
25.	3	**26.**	2
27.	1	**28.**	0.25
29.	(b) 6.75	**30.**	9
31.	1	**32.**	22 sq. units
33.	55	**34.**	0.66 or 0.67
35.	1.5	**36.**	104

PART - III

1. (B), (C), (D)

2. (A), (B), (C), (D)

3. (A), (B)

4. (B), (C)

5. (A), (B)

6. (A), (B), (C)

7. (A), (D)

8. (A), (B)

9. (B), (C)

10. (B), (C), (D)

11. (A), (C)

12. (B), (C), (D)

13. (A), (B), (C), (D)

14. (A), (B), (C)

15. (A), (B), (D)

16. (A), (C)

17. (A), (B), (D)

18. (A), (D)

19. (A), (C), (D)

20. (B), (D)

21. (A), (B), (C)

22. (A), (B), (C), (D)

23. (A), (C)

24. (A), (D)

25. (A), (B)

26. (A), (B), (C), (D)

27. (A), (D)

28. (A), (B)

29. (A), (B)

30. (A), (C)

31. (B), (C)

32. (B), (C)

PART - IV

1.	(B)	**2.**	(A)
3.	(A)	**4.**	(C)
5.	(A)	**6.**	(A)
7.	(D)	**8.**	(A)
9.	(B), (C)	**10.**	(C)
11.	(B)	**12.**	(D)

Exercise # 3

PART - I

1.	(B)	**2.**	(A)
3.	(B), (C)	**4.**	4
5.	(B)	**6.**	(C)
7.	(A)	**8.**	(B)
9.	(A)	**10.**	(B)
11.	(C)	**12.**	(A), (B), (D)
13.	(B)	**14.**	(B)

15.	(C)	**16.**	(A), (B), (C), (D)
17.	(B)	**18.**	(D)
19.	(B)	**20.**	(A), (C)
21.	(A), (C), (D)	**22.**	(2)
23.	(A)	**24.**	(A)
25.	(D)	**26.**	(D)
27.	0	**28.**	9
29.	7	**30.**	(A), (B)
31.	(D)	**32.**	(A), (C)
33.	(A), (B), (C)	**34.**	(C), (D)
35.	(A)	**36.**	1
37.	(C)	**38.**	(B), (C)
39.	2	**40.**	(B), (C)
41.	**Bouns**	**42.**	(A), (D)
43.	(C), (D)	**44.**	1
45.	2	**46.**	(B), (C)
47.	**4**	**48.**	4.00
49.	(A), (B)	**50.**	0.50
51.	(B)		

PART - II

1.	(1)	**2.**	(4)
3.	(4)	**4.**	(3)
5.	(3)	**6.**	(2)
7.	(3)	**8.**	(2), (3)
9.	(4)	**10.**	(1)
11.	(2)	**12.**	(3)
13.	(3)	**14.**	(4)
15.	(3)	**16.**	(3)
17.	(3)	**18.**	(1)
19.	(3)	**20.**	(4)
21.	(4)	**22.**	(3)
23.	(1)	**24.**	(3)
25.	(2)	**26.**	(1), (3)
27.	(4)	**28.**	(3)

Reliable Ranker Problems

SUBJECTIVE QUESTIONS

1. Find the integral value of a for which $\int_0^{\frac{\pi}{2}}(\sin x + a\cos x)^3 dx - \frac{4a}{\pi-2}\int_0^{\frac{\pi}{2}} x\cos x dx = 2$

2. Evaluate :

$$\int_0^{\pi}\sqrt{(\cos x+\cos 2x+\cos 3x)^2+(\sin x+\sin 2x+\sin 3x)^2}\,dx$$

3. Let $I=\int_0^{\pi/2}\frac{\cos x+4}{3\sin x+4\cos x+25}dx$ and $J=\int_0^{\pi/2}\frac{\sin x+3}{3\sin x+4\cos x+25}dx$. If 25I = a$\pi$ + bℓn$\frac{c}{d}$ where a,b,c and d $\in$N and $\frac{c}{d}$ is not a perfect square of a rational then find the value of (a + b + c + d).

4. Let α & β be distinct positive roots of the equation tanx = 2x, then evaluate $\int_0^1 \sin(\alpha x).\sin(\beta x)dx$

5. Comment upon the nature of roots of the quadratic equation $x^2 + 2x = k + \int_0^1 |t+k|dt$ depending on the value of k $\in$ R.

6. Evaluate : $\lim_{a\to\left(\frac{\pi}{2}\right)^-}\int_0^a(\cos x)\ln(\cos x)dx$

7. Show that $\int_0^{\infty} f\left(\frac{a}{x}+\frac{x}{a}\right).\frac{\ell nx}{x}dx = \ell na.\int_0^{\infty} f\left(\frac{a}{x}+\frac{x}{a}\right).\frac{dx}{x}$

8. Let $f(x)=\begin{cases}-1 & \text{if } -2\le x\le 0\\ |x-1| & \text{if } 0<x\le 2\end{cases}$ and $g(x)=\int_{-2}^{x} f(t)dt$. Define g(x) as a function of x and test the continuity and differentiability of g(x) in (–2,2).

9. Evaluate $\lim_{n\to\infty} n^2\int_{-\frac{1}{n}}^{\frac{1}{n}}(2014\sin x+2015\cos x)\,|x|\,dx$

10. Find f(x) if it satisfies the relation $f(x) = e^x + \int_0^1 (x + ye^x)\, f(y)\, dy$.

11. Prove that $\int_0^x \left(\int_0^u f(t)\, dt\right) du = \int_0^x f(u).(x-u)\, du$.

12. Evaluate : $\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \cos^{-1}\left(\frac{2x}{1+x^2}\right) dx$.

13. Evaluate $\int_0^1 \frac{1}{\left(5+2x-2x^2\right)\left(1+e^{(2-4x)}\right)} dx$

14. Prove that for any positive integer k ;

$\frac{\sin 2kx}{\sin x}$ = 2 [cos x + cos 3x +..... + cos (2k – 1)x]. Hence prove that;

$$\int_0^{\pi/2} \sin(2kx).\cot x\, dx = \frac{\pi}{2}.$$

15. If n > 1, evaluate $\int_0^{\infty} \frac{dx}{\left(x+\sqrt{1+x^2}\right)^n}$

16. Let f(x) be a continuous function ∀ x ∈ R, except at x = 0 such that $\int_0^a f(x)dx$, a ∈ R^+ exists. If

$g(x) = \int_x^a \frac{f(t)}{t} dt$, prove that $\int_0^a g(x)\, dx = \int_0^a f(x)\, dx$

17. Evaluate : $\lim_{m\to\infty} \int_{-\infty}^{\infty} \frac{dx}{1+x^2+x^4+....+x^{2m}}$; m ∈ N

18. Find the value of a(0 < a < 1) for which the following definite integral is minimized.

$$\int_0^{\pi} |\sin x - ax|\, dx$$

19. If $\phi(x) = \cos x - \int_0^x (x-t)\,\phi(t)dt$. Then find the value of $\phi''(x) + \phi(x)$.

20. Find the range of the function, $f(x) = \int_{-1}^{1} \frac{\sin x dt}{1 - 2t\cos x + t^2}$

21. Given that $\lim_{x \to 0} \frac{\int_0^x \frac{t^2}{\sqrt{a+t}} dt}{bx - \sin x} = 1$, then find the values of a and b

22. For a natural number n, let $a_n = \int_0^{\pi/4} (\tan x)^{2n} dx$

Now answer the following questions :

(1) Express a_{n+1} in terms of a_n

(2) Find $\lim_{n \to \infty} a_n$

(3) Find $\lim_{n \to \infty} \sum_{k=1}^{n} (-1)^{k-1} (a_k + a_{k-1})$

23. If $U_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x} dx$, then show that $U_1, U_2, U_3, \ldots\ldots, U_n$ constitute an AP. Hence or otherwise find the value of U_n.

24. Let sequence $\{a_n\}$ be defined as $a_1 = \frac{\pi}{4}$, $a_n = \int_0^{\frac{1}{2}} (\cos(\pi x) + a_{n-1}) \cos \pi x \, dx$, (n = 2, 3, 4, ...) then evaluate $\lim_{n \to \infty} a_n$

25. Prove that $m \sin x + \int_0^x \sec^m t \; dt > (m+1)x \quad \forall x \in \left(0, \frac{\pi}{2}\right) \; m \in N$

26. f(x) is differentiable function: g(x) is double differentiable function such that $|f(x)| \le 1$ and $g(x) = f'(x)$. If $f^2(0) + g^2(0) = 9$ then show that there exists some $C \in (-3, 3)$ such that $g(c)\, g''(c) < 0$

27. Given that $\lim_{n \to \infty} \sum_{r=1}^{n} \frac{\log_e(n^2 + r^2) - 2\log_e n}{n} = \log_e 2 + \frac{\pi}{2} - 2$, then evaluate : $\lim_{n \to \infty} \frac{1}{n^{2m}} [(n^2 + 1^2)^m (n^2 + 2^2)^m \ldots\ldots (2n^2)^m]^{1/n}$.

28. Prove that $\underset{n \to \infty}{\text{Limit}} \frac{1}{n} \left[\cos^{2p} \frac{\pi}{2n} + \cos^{2p} \frac{2\pi}{2n} + \cos^{2p} \frac{3\pi}{2n} + \ldots\ldots + \cos^{2p} \frac{\pi}{2} \right] = \prod_{r=1}^{p} \frac{2r-1}{2r}$,

where Π denotes the continued product and $p \in N$.

29. Find the $\lim_{n\to\infty}\left(\frac{{}^{3n}C_n}{{}^{2n}C_n}\right)^{\frac{1}{n}}$ where ${}^iC_j$ is a binomial coefficient which means $\frac{i.(i-1)....(i-j+1)}{j.(j-1)....2.1}$

30. Consider a square with vertices at (1, 1), (– 1, 1), (–1, –1) and (1, –1). Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region S and find its area.

31. If [x] denotes the greatest integer function. Draw a rough sketch of the portions of the curves $x^2 = 4\left[\sqrt{x}\right]\ y$ and $y^2 = 4\left[\sqrt{y}\right]\ x$ that lie within the square {(x, y) | $1 \le x \le 4$, $1 \le y < 4$} Find the area of the part of the square that is enclosed by the two curves and the line x + y = 3

32. Let f(x) = $\begin{cases} -2 & , \ -3 \le x \le 0 \\ x-2 & , \ 0 < x \le 3 \end{cases}$, where g(x) = min {f(|x|) + |f(x)|, f(|x|) – |f(x)|}

Find the area bounded by the curve g(x) and the x-axis between the ordinates x = 3 and x = –3.

33. Find the area of region $\{(x, y) : 0 \le y \le x^2 + 1,\ 0 \le y \le x + 1, 0 \le x \le 2\}$.

34. A curve y = f(x) passes through the point P(1, 1), the normal to the curve at P is a(y – 1) + (x – 1) = 0. If the slope of the tangent at any point on the curve is proportional to the ordinate of that point, determine the equation of the curve. Also obtain the area bounded by the y-axis, the curve and the normal to the curve at P.

35. Find the area bounded by $y = [-0.01x^4 - 0.02x^2]$, (where [.] G.I.F.) and curve $3x^2 + 4y^2 = 12$, which lies below y = – 1.

36. Let ABC be a triangle with vertices A(6, 2($\sqrt{3}$ + 1)), B(4, 2) and C(8, 2). If R be the region consisting of all these points and point P inside ΔABC which satisfy d(P, BC) $\ge$ max. {d(P, AB), d(P, AC)}

where d(P, L) denotes the distance of the point P from the line L. Sketch the region R and find its area.

37. Find the area of the region which contains all the points satisfying the condition |x – 2y| + |x + 2y| $\le$ 8 and xy $\ge$ 2.

38. Consider the curve C: y = sin 2x – $\sqrt{3}$ |sin x|, C cuts the x – axis at (a, 0), a $\in$ (– π, π).

A_1 : The area bounded by the curve C and the positive x – axis between the origin and the line x = a.

A_2 : The area bounded by the curve C and the negative x – axis between the line x = a and the origin.

Prove that $A_1 + A_2 + 8A_1A_2 = 4$.

39. Area bounded by the line y = x, curve y = f(x), (f(x) > x ∀ x > 1) and the lines x = 1, x = t is

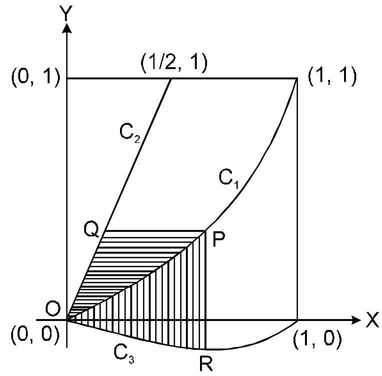
(t + $\sqrt{1+t^2}$ – (1 + $\sqrt{2}$) ∀ t > 1. Find f(x) for x > 1.

40. Consider the two curves $y = 1/x^2$ and y = 1/[4 (x – 1)].

(i) At what value of 'a' (a > 2) is the reciprocal of the area of the figure bounded by the curves, the lines x = 2 and x = a equal to 'a' itself ?

(ii) At what value of 'b' (1 < b < 2) the area of the figure bounded by these curves, the lines x = b and x = 2 equal to 1 – 1/b.

41. Let C_1 and C_2 be the graphs of the functions $y = x^2$ and $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graph of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 and C_3 at Q and R respectively (see figure). If for every position of P (on C_1), the areas of the shaded regions OPQ and ORP are equal, determine the function f(x).



42. Given the parabola C : $y = x^2$. If the circle centred at y axis with radius 1 has common tangent lines with C at distinct two points, then find the coordinate of the center of the circle K and the area of the figure surrounded by C and K.

43. If $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$, f(x) is a quadratic function and its maximum value occurs at a point V. A is a point of intersection of y = f(x) with x-axis and point B is such that chord AB subtends a right angle at V. Find the area enclosed by f(x) and chord AB.

44. A figure is bounded by the curves $y = \left|\sqrt{2}\sin\frac{\pi x}{4}\right|$, y = 0, x = 2 & x = 4. At what angles to the positive x-axis straight lines must be drawn through (4 , 0) so that these lines partition the figure into three parts of the same area.

45. For the curve $f(x) = \frac{1}{1+x^2}$, let two points on it are $A(\alpha, f(\alpha))$, $B\left(-\frac{1}{\alpha}, f\left(-\frac{1}{\alpha}\right)\right)$ $(\alpha > 0)$. Find the minimum area bounded by the line segments OA, OB and *f* (x), where 'O' is the origin.

46. Consider the two curves $C_1 : y = 1 + \cos x$ & $C_2 : y = 1 + \cos(x - \alpha)$ for $\alpha \in (0, \pi/2)$; $x \in [0, \pi]$. Find the value of α, for which the area of the figure bounded by the curves C_1, C_2 & x = 0 is same as that of the figure bounded by C_2, y = 1 & x = π. For this value of α, find the ratio in which the line y = 1 divides the area of the figure by the curves C_1, C_2 & x = π.

Answers

1. -1 **2.** $\frac{\pi}{3}+2\sqrt{3}$ **3.** 62

4. 0 **5.** real & distinct $\forall$ k$\in$R **6.** $\ell n2-1$

8. g(x) is cont. in(–2,2); g(x) is der. at x = 1 & not der. at x = 0. Note that : $g(x)=\begin{cases} -(x+2) & \text{for } -2\le x\le 0 \\ -2+x-\frac{x^2}{2} & \text{for } 0<x<1 \\ \frac{x^2}{2}-x-1 & \text{for } 1\le x\le 2\end{cases}$

9. 2015 **10.** $\frac{-3e^x}{2(e-1)}-3x$ **12.** $\frac{\pi}{4}\ \ell n\left(2+\sqrt{3}\right)+\frac{\pi^2}{12}-\frac{\pi}{\sqrt{3}}$

13. $\frac{1}{\sqrt{11}}\ \ell n\ \frac{\sqrt{11}+1}{\sqrt{11}-1}$ **15.** $\frac{n}{n^2-1}$ **17.** $\frac{4}{3}$

18. $a=\frac{\sqrt{2}}{\pi}\sin\left(\frac{\pi}{\sqrt{2}}\right)$ **19.** –cosx **20.** $\left\{-\frac{\pi}{2},\frac{\pi}{2}\right\}$

21. a = 4, b = 1

22. **(1)** $\frac{1}{2n+1}-a_n$ **(2) 0** **(3)** $\frac{\pi}{4}$ **23.** $\frac{n\pi}{2}$ **24.** $\frac{\pi^2}{4(\pi-1)}$

27. $\left(\frac{2\sqrt{e^{\pi}}}{e^2}\right)^m$ **29.** $\frac{27}{16}$ **30.** $\frac{1}{3}\left(16\sqrt{2}-20\right)$

31. $\frac{19}{6}$ **32.** $\frac{23}{2}$ **33.** $\frac{23}{6}$

34. $y=e^{a(x-1)}$, $\left(1+\frac{e^{-a}}{a}-\frac{1}{2a}\right)$ **35.** $2\sqrt{3}\sin^{-1}\sqrt{\frac{2}{3}}-\frac{2\sqrt{2}}{\sqrt{3}}$ **36.** $\frac{4\sqrt{3}}{3}$

37. 2(6 – 2 log 4) **38.** **39.** $1+x+\frac{x}{\sqrt{1+x^2}}$.

40. (i) $a=1+e^2$ (ii) $b=1+e^{-2}$ **41.** $f(x)=x^3-x^2$ **42.** centre $\left(0,\frac{5}{4}\right)$ and area $=\frac{3\sqrt{3}}{4}-\frac{\pi}{3}$

43. $\frac{125}{3}$ square units. **44.** $\pi-\tan^{-1}\frac{2\sqrt{2}}{3\pi}$; $\pi-\tan^{-1}\frac{4\sqrt{2}}{3\pi}$

45. $\frac{(\pi-1)}{2}$ **46.** $\alpha=\pi/3$, ratio = $2:\sqrt{3}$

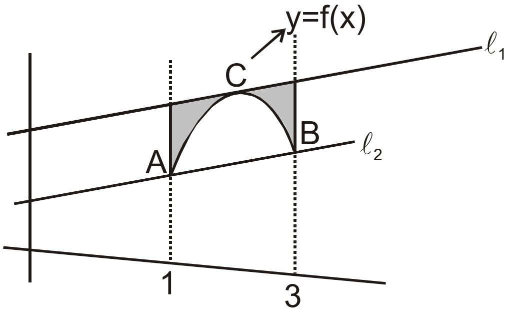
Self Assessment Test

JEE ADVANCED

Maximum Marks : 62 **Total Time : 1:00 Hr**

SECTION-1 : ONE OPTION CORRECT (Marks - 12)

1. The following figure shows the graph of continuous function y = ƒ (x) on the interval [1, 3]. The points A, B, C have coordinates (1, 1), (3, 2), (2, 3) respectively, and the lines ℓ_1 and ℓ_2 are parallel, with .1 being tangent to the curve at C. If the area under the graph of y = ƒ (x) from x = 1 to x = 3 is 4 square units, then the area of the shaded region is :



(A) 2 (B) 3 (C) 4 (D) 5

2. Let $I_n = \int_1^e (\log x)^n\, dx$, where n is a non-negative integer. Then $I_{2011} + 2011\, I_{2010}$ is equal to

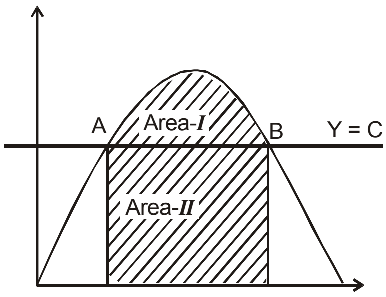
(A) $I_{1000} + 999\, I_{998}$ (B) I889 + 890 I_{891} (C) $I_{100} + 100\, I_{99}$ (D) $I_{53} + 54\, I_{52}$

3. Consider $L = \sqrt[3]{2012} + \sqrt[3]{2013} + ... + \sqrt[3]{3011}$ $R = \sqrt[3]{2013} + \sqrt[3]{2014} + ... + \sqrt[3]{3012}$

and $I = \int_{2012}^{3012} 3\sqrt{x}\, dx.$

(A) L + R < 21 (B) L + R = 21 (C) L + R > 21 (D) $\sqrt{LR} = 1$

4. The f igure shows a por tions of the graph y = 2x – 4x^3. The line y = c is such that the areas of the regions marked I and II are equal. If a,b are the x-coordinates of A,B respectively, then a + b equals



(A) $\frac{2}{\sqrt{7}}$ (B) $\frac{3}{\sqrt{7}}$ (C) $\frac{4}{\sqrt{7}}$ (D) $\frac{5}{\sqrt{7}}$

SECTION-2 : ONE OR MORE THAN ONE CORRECT (Marks - 32)

5. On the real line R, we define two functions f and g as follows : f(x) = min {x – [x], 1 – x + [x]}, g(x) = max {x – [x], 1 – x + [x]}, Where [x] denotes the largest integer not exceeding x. The positive n for which

$\int_0^n (g(x) - f(x))dx = 100$ is

(A) 100 (B) 198 (C) 200 (D) 202

6. Lety I = $\int_0^{\sqrt{\pi}} \frac{\sin x^2}{x} dx$ then

(A) $I < \frac{6}{5}$ (B) $I > 1$ (C) $I < \frac{4+\pi}{8}$ (D) $I > \frac{\pi+4}{8}$

7. Let f(x) is cubic polynomial such that its leading coefficient equal to 1 and f(1) = 0. If g(x) = $\int_x^{2x} f(t)\,dt$ has two of the extremum at 1 and 2 then

(A) Extremum of g(x) are x = 1, 2, $\frac{8}{15}$

(B) Point of inflection of f(x) is x = $\frac{7}{3}$

(C) g(6) = 2f(2x) – f(x)

(D) Point of inflection of f(x) is x = $\frac{7}{6}$

8. Let f(x) = 1 + $\frac{1^2x^2}{2!} + \frac{1^2 3^2 x^2}{4!} + \frac{1^2 3^2 5^2 x^6}{6!}$ + ...upto ∞ terms where x Î (–1, 1) and g(x) $\int f(x)dx$ and g(0) = 0, then which of the following statements is/are correct?

(A) $f\left(\frac{4}{5}\right) = \frac{5}{3}$

(B) $f\left(\frac{4}{5}\right) = \frac{3}{5}$

(C) $\int_0^1 g(x)dx = \frac{\pi}{2} - 1$

(D) $\int_0^1 g(x)dx = \frac{\pi}{2}$

9. If f(x) = a cos (πx) + b, $f'\left(\frac{1}{2}\right) = \pi$ and $\int_{1/2}^{3/2} f(x)dx = \frac{2}{\pi} + 1$, then find the value of $-\frac{12}{\pi}(\sin^{-1}a + \cos^{-1}b)$.

(A) a = – 1

(B) b = 1

(C) $-\frac{12}{\pi}(\sin^{-1}a + \cos^{-1}b) = 6$

(D) a + b = 0

10. Consider the collection of all curves of the form y = a – bx^2 that passes the point (2, 1) where a and b are positive real numbers, if the minimum area of the region bounded by y = a – bx^2 and the x-axis is $\sqrt{A}$, then which of the following statement(s) is/are correct ?

(A) 2a = 3 (B) 8b =1 (C) A = 48 (D) a + b = 4

11. Let $I_n = \int_{-2n\pi}^{2n\pi} |\cos x|[\cos x]dx$ (where [.] denotes greatest integer function), then

(A) $\sum_{k=1}^{10} I_k = -220$

(B) $\sum_{k=0}^{10} I_k = -220$

(C) $\sum_{k=1}^{100} I_k = -20200$

(D) $\sum_{k=0}^{100} I_k = -20200$

12. If $f(x + y) = f(x)f(y)\ \forall$ x, y $\in$ R and $f(0) \neq 0$ and $g(x) = \frac{f(x)}{1+f^2(x)}$, then

(A) g(x) is an even function

(B) $\int_{-2010}^{2011} g(x)dx = \int_{0}^{2010} g(x)dx + \int_{0}^{2011} g(x)dx$

(C) $\int_{-2010}^{2010} g(x)dx = 0$

(D) $\int_{-2010}^{0} g(x)dx = \int_{0}^{2010} g(x)dx$

SECTION-3 : NUMERICAL VALUE TYPE (Marks - 18)

13. Let S = $\left\{(x,y);\ \frac{y(3x-1)}{x(3x-2)} < 0\right\}$ and S' = {(x, y) $\in$ A × B; $-1 \le A \le 1$ and $-1 \le B \le 1$}. Then area of S $\cap$ S' is

14. Let the function $f : [2, \infty) \rightarrow [1, \infty)$ defined by $f(x) = x^2 - 4x + 5$. If g(x) is inverse of $f(x)$ and area bounded by g(x), x-axis, x = 2 and x = 5 is equal to $\frac{8k}{3}$, then value of k is

15. The value of integral $\int_0^4 \min\{|x-1|, |x-2|, |x-3|\}\,dx = \frac{a}{b}$ where a and b are co-prime, then the value of (a + b) is

16. If $I_1 = \int_0^1 \prod_{r=1}^{2018}(r-x)\,dx$ and $I_2 = \int_0^1 \prod_{r=0}^{2017}(r+x)\,dx$, then the value of $2\left(\frac{7I_1 - 3I_2}{I_1 + I_2}\right)$ is

17. Value of $\frac{29\int_0^1 (1-x^4)^7\,dx}{4\int_0^1 (1-x^4)^6\,dx}$ is equal to

18. If $L = \lim_{n\to\infty} \frac{1}{n}\left\{\sum_{r=1}^{n}\left(\sin\frac{r\pi}{2n} - r\right)\left(\sin\frac{r\pi}{2n} + r\right) + \int_0^n \left(x^2 + x + \frac{1}{6}\right)dx\right\}$, then 2L is equal to

Answers

1.	(A)	**2.**	(C)	**3.**	(A)	**4.**	(A)
5.	(C)	**6.**	(A), (D)	**7.**	(C)	**8.**	(A), (C)
9.	(A), (B), (C)	**10.**	(A), (B), (C)	**11.**	(A), (B), (C), (D)	**12.**	(A), (B), (D)
13.	2	**14.**	4	**15.**	5	**16.**	4
17.	7	**18.**	(1)				