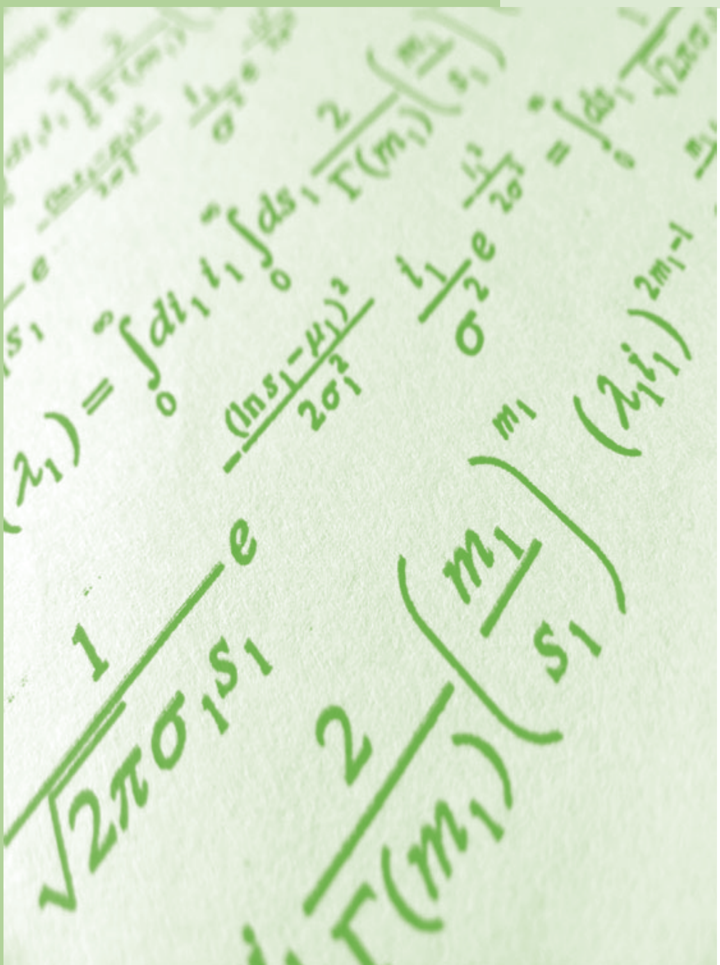
Chapter

27

Logarithms



REMEMBER

Before beginning this chapter, you should be able to:

- Define logarithms
- Know properties of logarithms

KEY IDEAS

After completing this chapter, you would be able to:

- Study system of logarithms
- Study use of signs for log values
- Find the log of a number to base 10
- Use log tables
- Find the antilog

INTRODUCTION

In earlier classes, we have learnt about indices. One of the results we have learnt is that, if $2^x = 2^3$, then $x = 3$ and if $4^x = 4^y$, then $x = y$, i.e., if two powers of the same base are equal and the base is not equal to -1, 0 or 1, then the indices are equal. But when $3^x = 5^2$, just by using the knowledge of indices, we cannot find the numerical value of x. The necessity of the concept of logarithms arises here. Logarithms are useful in long calculations involving multiplication and division. Logarithms also useful in the study of Probability, Statistics, Psychology, etc.

Definition

The logarithm of any positive number to a given base (a positive number not equal to 1) is the index of the power of the base which is equal to that number. If N and a ($\neq 1$) are any two positive real numbers and for some real x, $a^x = N$, then x is said to be the logarithm of N to the base a. It is written as $\log_a N = x$, i.e., if $a^x = N$, then $x = \log_a N$.

Examples:

1. $3^4 = 81 \quad \Rightarrow \quad 4 = \log_3 81$
2. $7^3 = 343 \quad \Rightarrow \quad 3 = \log_7 343$
3. $2^5 = 32 \quad \Rightarrow \quad 5 = \log_2 32$
4. $5^4 = 625 \quad \Rightarrow \quad 4 = \log_5 625$
5. $10^2 = 100 \quad \Rightarrow \quad 2 = \log_{10} 100$

If in a particular relation, all the log expressions are to the same base, we normally do not specify the base.

From the definition of logs, we get the following results:
When $a > 0$, $b > 0$ and $a \neq 1$,

1. $\log_a a^n = n$ e.g., $\log_6 6^3 = 3$
2. $a^{\log_a b} = b$ e.g., $9^{\log_9 5} = 5$

SYSTEM OF LOGARITHMS

Though we can talk of the logarithm of a number to any positive base as not equal to 1, there are two systems of logarithms, viz., natural logarithms and common logarithms, which are used most often.

1. **Natural logarithms:** These were discovered by Napier. They are calculated to the base 'e' which is approximately equal to 2.7828. These are used in higher mathematics.
2. **Common logarithms:** Logarithms to the base 10 are known as common logarithms. This system was introduced by Briggs, a contemporary of Napier. In the rest of this chapter, we shall use the short form 'log' instead of 'logarithm'.

Properties

1. Logs are defined only for positive real numbers.
2. Logs are defined only for positive bases (other then 1).
3. In $\mathbf{log}_{\boldsymbol{b}}\boldsymbol{a}$ neither a is negative nor b is negative but the value of $\mathbf{log}_{\boldsymbol{b}}\boldsymbol{a}$ can be negative.

Example: As $10^{-2} = 0.01$, $\log_{10}0.01 = -2$.

4. Logs of different numbers to the same base are different, i.e., if $a \neq b$, then $\log_m a \neq \log_m b$. In other words, if $\log_m a = \log_m b$, then $a = b$.

   Example:

   $\log_{10}2 \neq \log_{10}3$

   $\log_{10}2 = \log_{10}y \quad \Rightarrow \quad y = 2.$

5. Logs of the same number to different bases have different values, i.e., if $m \neq n$, then $\log_m a \neq \log_n a$. In other words, if $\log_m a = \log_n a$, then $m = n$.

   Example:

   $\log_2 16 \neq \log_4 16$

   $\log_2 16 = \log_n 16 \quad \Rightarrow \quad n = 2.$

6. Log of 1 to any base is 0.

   Example: $\log_2 1 = 0$ ($\because \quad 2^\circ = 1$)

7. Log of a number to the same base is 1.

   Example: $\log_4 4 = 1$.

8. Log of 0 is not defined.

Laws

1. $\log_m(ab) = \log_m a + \log_m b$

   Example: $\log 56 = \log(7 \times 8) = \log 7 + \log 8$

2. $\log_m\left(\frac{a}{b}\right) = \log_m a - \log_m b$

   Example: $\log\left(\frac{81}{23}\right) = \log 81 - \log 23$

3. $\log a^m = m \log a$

   Example: $\log 216 = \log 6^3 = 3\log 6$

4. $\log_b a \log_c b = \log_c a$ (Chain Rule)

   Example: $\log_2 3 \times \log_8 2 \times \log_5 8 = \log_8 3 \times \log_5 8 = \log_5 3$

5. $\log_b a = \frac{\log_c a}{\log_4 b}$ (Change of Base Rule)

   Example: $\log_9 25 = \frac{\log_4 25}{\log_4 9}$

Note In this relation, if we replace c by a, then we get the following result:

$$\log_b a = \frac{1}{\log_a b}$$

Variation of $\log_a x$ with x

For $1 < a$, and $0 < p < q$, $\log_a p < \log_0 q$.
For $0 < a < 1$ and $0 < p < q$, $\log_a p > \log_a q$.

Example: $\log_{10}2 < \log_{10}3$ and $\log_{0.1}2 > \log_{0.1}3$

Bases which are greater than 1 are called **strong bases** and bases which are less than 1 are called **weak bases**. Therefore, for strong bases, the log increases with the number and for weak bases, the log decreases with the number.

Sign of $\log_a x$ for Different Values of x and a

Strong Bases ($a > 1$)

1. If $x > 1$, $\log_a x$ is positive.

   For example, $\log_2 8$, $\log_4 81$ are positive.

2. If $0 < x < 1$, then $\log_a x$ is negative.

   For example, $\log_4 0.02 = \dfrac{\log 0.02}{\log 4} = \dfrac{\log 2 - \log 100}{\log 4}$

   $\log 2 < \log 100$ and $0 < \log 4$ for strong bases

   $\therefore \dfrac{\log 2 - \log 100}{\log 4} < 0$

   $\Rightarrow \log_4 0.02$ is negative.

Weak Bases ($0 < a < 1$)

1. If $x > 1$, then $\log_a x$ is negative.

   For example $\log_{0.3}15$ and $\log_{0.4}16$ are negative.

   Consider $\log_{0.3} 15 = \dfrac{\log 15}{\log 0.3} = \dfrac{\log 15}{\log 3 - \log 10}$

   $\log 3 < \log 10$ (for any strong base)

   $\Rightarrow \quad \dfrac{\log 15}{\log 3 - \log 10} < 0.$

2. If $0 < x < 1$, then $\log_a x$ is positive.

   For example, $\log_{0.1}0.2$, $\log_{0.4}0.3$ are positive.

Note

Logs of numbers (> 1) to strong bases and numbers (< 1) to weak bases are positive.

EXAMPLE 27.1

If $p = \log_{2a} a$, $q = \log_{3a} 2a$ and $r = \log_{4a} 3a$, then find the value of $qr(2 - p)$.

(a) 1 **(b)** 0 **(c)** 2 **(d)** 3

SOLUTION

$qr = \log_{3a} 2a \log_{4a} 3a = \log_{4a} 2a$

$pqr = \log_{2a} a \log_{4a} 2a = \log_{4a} a$

Now $qr\,(2 - p) = 2qr - pqr$

$= 2\log_{4a} 2a - \log_{4a} a$

$= \log_{4a} 4a^2 - \log_{4a} a$

$= \log_{4a} \dfrac{4a^2}{a} = \log_{4a} 4a = 1.$

EXAMPLE 27.2

If $3^x = (0.3)^y = 10000$, then find the value of $\dfrac{1}{x} - \dfrac{1}{y}$.

(a) 1 **(b)** $\dfrac{1}{2}$ **(c)** $\dfrac{1}{4}$ **(d)** $\dfrac{1}{3}$

SOLUTION

Given,, $3^x = (0.3)^y = 10000$

$3^x = 10^4 \; (0.3)^y = 10^4$

$x = 4 \log_3 10 \qquad \Rightarrow \qquad y = 4 \log_{0.3} 10$

$\dfrac{1}{x} = \dfrac{1}{4\log_3 10} \qquad \dfrac{1}{y} = \dfrac{1}{4\log_{0.3} 10}$

$= \left(\dfrac{1}{4}\right)\log_{10} 3 \qquad = \left(\dfrac{1}{4}\right)\log_{10} 0.3$

$\dfrac{1}{x} - \dfrac{1}{y} = \dfrac{1}{4}[\log_{10} 3 - \log_{10} 0.3]$

$= \dfrac{1}{4}\left[\log_{10} \dfrac{3}{0.3}\right]$

$= \dfrac{1}{4}[\log_{10} 10]$

$\Rightarrow \quad \dfrac{1}{x} - \dfrac{1}{y} = \dfrac{1}{4}.$

EXAMPLE 27.3

If $(2x)^{\log_{\sqrt{x}} x} = 16$, then find the value of x.

(a) 2 **(b)** –2 **(c)** Both (a) and (b) **(d)** None of these

SOLUTION

$(2x)^{\log_{\sqrt{x}} x} = 16$

$$
\begin{aligned}
(2x)^{2\log_x x} &= 16 \\
(2x)^2 &= 16 \\
4x^2 &= 16 \\
x^2 &= 4 \\
x &= \pm 2.
\end{aligned}
$$

When $x = -2, \log_{\sqrt{x}} x$ is not defined.

$\therefore\ x = 2.$

EXAMPLE 27.4

Find the value of $3^{\frac{4}{\log_2 9}} + 27^{\frac{1}{\log_{49} 9}} + 81^{\frac{1}{\log_4 3}}$.

(a) 603 **(b)** 585 **(c)** 676 **(d)** 524

SOLUTION

$3^{\frac{4}{\log_2 9}} + 27^{\frac{1}{\log_{49} 9}} + 81^{\frac{1}{\log_4 3}}$

$= 3^{4\log_9 2} + 27^{\log_9 49} + 81^{\log_3 4}$

$= 3^{2\log_3 2} + 3^{3\log_3 7} + 3^{4\log_3 4}$

$= 3^{\log_3 4} + 3^{\log_3 343} + 3^{4\log_3 256}$

$= 4 + 343 + 256 = 603.$

EXAMPLE 27.5

If $p \in R$ and $q = \log_x\left(p + \sqrt{p^2 + 1}\right)$, then find the value of p in terms of x and q.

(a) $\dfrac{x^q + x^{-q}}{2}$ **(b)** $\dfrac{x^q - x^{-q}}{2}$ **(c)** $x^q + x^{-q}$ **(d)** $x^q - x^{-q}$

SOLUTION

Given, $q = \log_x \left(p + \sqrt{p^2+1}\right) x^q$

$= p + \sqrt{p^2+1}$

$$\Rightarrow \quad x^{-q} = \frac{1}{x^q} = \frac{1}{p+\sqrt{p^2+1}}$$

$$= \frac{p-\sqrt{p^2+1}}{\left(p+\sqrt{p^2+1}\right)\left(p-\sqrt{p^2+1}\right)}$$

$$= \frac{p-\sqrt{p^2+1}}{p^2-(p^2+1)} = -p+\sqrt{p^2+1}$$

$$\therefore\ x^q - x^{-q} = p + \sqrt{p^2+1} - \left(\sqrt{p^2+1} - p\right)$$

$$x^q - x^{-q} = 2p$$

$$\frac{x^q - x^{-q}}{2} = p.$$

To Find the log of a Number to the Base 10

Consider the following numbers:

2, 20, 200, 0.2 and 0.02.

We see that 20 = 10(2) and 200 = 100(2)

∴ log 20 = 1 + log 2 and log 200 = 2 + log 2.

Similarly, log 0.2 = –1 + log 2 and log 0.02 = –2 + log 2.

From the tables, we see that log 2 = 0.3010. ('Using the tables' is explained in greater detail in later examples).

∴ log 20 = 1.3010, log 200 = 2.3010, log 0.2 = –1 + 0.3010 and log 0.02 = –2 + 0.3010.

Notes

1. Multiplying or dividing by a power of 10 changes only the integral part of the log, not the fractional part.
2. For numbers less than 1, (for example log 0.2) it is more convenient to leave the log value as –1 + 0.3010 instead of changing it to –0.6090. We refer to the first form (in which the fraction is positive) as the standard form and the second form as the normal form. Both the forms represent the same number.

For numbers less than 1, it is convenient to express the log in the standard form. As the negative sign refers only to the integral part, it is written above the integral part, rather than in front, i.e., $\log 0.2 = \bar{1}.3010$ and not –1.3010.

The convenience of the standard form will be clear when we learn how to take the antilog, which will later be explained in detail.

antilog (–0.6090) = antilog (–1 + 0.3010) = antilog $\bar{1}.3010 = 0.2$.

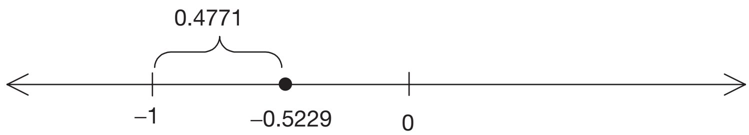
When the logs of numbers are expressed in the standard form (for numbers greater than 1, the standard form of the log is the same as the normal form), the integral part is called the characteristic and the fractional part (which is always positive) is called the mantissa.

EXAMPLE 27.6

Express –0.5229 in the standard form and locate it on the number line.

SOLUTION

$-0.5229 = -1 + 1 - 0.5229 = \bar{1}.4771.$



The Rule to Obtain the Characteristic of log *x*

1. If $x > 1$ and there are n digits in x, then the characteristic is $n - 1$.
2. If $x < 1$ and there are m zeroes between the decimal point and the first non-zero digit of x, then the characteristic is $(-m + 1)$ more commonly written as $(\overline{m+1})$.

Note

$-4 = \bar{4}$ but $-4.01 \neq \bar{4}.01$

To Find the log of a Number from the log Tables

EXAMPLE 27.7

Find the values of log36, log3600 and log0.0036.

SOLUTION

In log tables we find the number 36 in the first column. In this row in the next column (under zero), we find .5563 (the decimal point is dropped in other columns). This gives 5563 as the mantissa for the log of all numbers whose significant digits are 3 and 6.

$\therefore$ Prefixing characteristic, we have $\log 36 = 1.5563$

Similarly $\log 3600 = 3.5563$ and $\log 0.0036 = \bar{3}.5563$.

EXAMPLE 27.8

Find the values of log3.74, log374000 and log0.3740.

SOLUTION

In the log table we locate 37 in the first column. In this row, in the column under 4, we find 5729. As in the earlier example, the same line as before gives the mantissa of logarithms of all numbers which begin with 37. From this line, we select the mantissa which is located in the column number 4. This gives 5729 as the mantissa for all numbers whose significant digits are 3, 7 and 4.

$\therefore \log 3.74 = 0.5729$

$\log 374000 = 5.5729$ and

$\log 0.3740 = \bar{1}.5729$.

EXAMPLE 27.9

Find the values of log5.342 and log 0.05342.

SOLUTION

As found in the above example, we can find the mantissa for the sequence of digits 534 as 7275. Since there are four significant digits in 5342, in the same row where we found 7275 under the column 2 in the mean difference column, we can find the number 2.

$\therefore$ The mantissa of the logarithm of 5342 is $7275 + 2 = 7277$.

Thus, $\log 5.342 = 0.7277$

Similarly, $\log 0.05342 = \bar{2}.7277$.

ANTILOG

As $\log_2 8 = 3$, 8 is the **antilogarithm** of 3 to the base 2, i.e., **antilog** of b to base m is $\boldsymbol{m^b}$.

In the above example we have seen that $\log 5.342 = 0.7277$.

$\therefore$ antilog$0.7277 = 5.342$.

To Find the Antilog

EXAMPLE 27.10

Find the antilog of 2.421.

SOLUTION

Step 1: In the antilog table we find the number.42 in the first column. In that row in the column under 1, we find 2636.

Step 2: As the characteristic is 2, we place the decimal after three digits from the left, i.e., antilog $2.421 = 263.6$.

Note If the characteristic is n (a non-negative integer), then we would place the decimal after $(n + 1)$ digits from the left.

EXAMPLE 27.11

Find the antilog of 1.4215.

SOLUTION

We have to locate .42 in the first column and scan along the horizontal line and pick out the number in the column headed by 1. We see that the number is 2636. The mean difference for 5 in the same line is 3.

$\therefore$ The sum of these numbers is $2636 + 3 = 2639$.

As the characteristic is 1, the required antilog is 26.39.

EXAMPLE 27.12

Find the value of $\dfrac{7.211 \times 0.084}{16.52 \times 0.016}$.

SOLUTION

log of a fraction = (log of numerator) − (log of denominator)

log of numerator = log7.211 + log0.084 = 0.8580 + $\bar{2}.9243 = \bar{1}.7823$

log of denominator = log16.52 + log0.016 = 1.2180 + $\bar{2}.2041 = \bar{1}.4221$

log of the given fraction = $\bar{1}.7823 - \bar{1}.4221 = 0.3602$

Value of the fraction = antilog (0.3602) = 2.292. (As the characteristic is 0, the decimal is kept after one digit from the left)

EXAMPLE 27.13

If $\log_{10}4$ = 0.6021 and $\log_{10}5$ = 0.6990, then find the value of $\log_{10}1600$.

SOLUTION

$$\log_{10} 1600 = \log_{10}(64 \times 25) = \log_{10}(4^3 \times 5^2)$$

$$= \log_{10} 4^3 + \log_{10} 5^2 \rightarrow$$

$$= 3\log_{10} 4 + 2\log_{10} 5$$

$$= 3(0.6021) + 2(0.6990)$$

$$= 1.8063 + 1.3980$$

$$\log_{10} 1600 = 3.2043.$$

EXAMPLE 27.14

Find the value of $\sqrt[3]{16.51}$ approximately.

SOLUTION

Let $P = \sqrt[3]{16.51}$

$$\log P = \log(16.51)^{\frac{1}{3}}$$

$$= \frac{1}{3}\log 16.51$$

$$= \frac{1}{3}(1.2178) = 0.4059$$

$$\log P = 0.4059$$

$$P = \text{antilog}(0.4059)$$

$\therefore$ P = 2.546.

TEST YOUR CONCEPT

Very Short Answer Type Questions

1. $\frac{1}{5}\log_2 32 + 3\log_{64} 4 =$ ________.
2. The characteristic of the logarithm of 3.6275 is ________.
3. If $4\log_x 8 = 3$, then $x =$ ________.
4. If $\log x - \frac{2}{3}\log x = 1$, then $x =$ ________.
5. If $a = \log\frac{3}{2}, b = \frac{4}{25}$ and $c = \log\frac{5}{9}$, then $a + b + c$ = ________.
6. The number of digits in the integral part of the number whose logarithm is 4.8345 is ________.
7. If $\log x = 32.756$, then $\log 10x =$ ________.
8. The characteristic of the logarithm of 0.0062 is ________.
9. If $\log_a x$ (where $a > 1$) is positive, then the range of x is ________.
10. If $\log 27.91 = 1.4458$, then $\log 2.791 =$ ________.
11. $\frac{\log 15 - \log 6}{\log 20 - \log 8} =$ ________.
12. If $\log 2 = 0.3010$, then $\log 5 =$ ________.
13. The value of $\log_{16}\sqrt[5]{64} =$ ________.
14. $\frac{\log 216}{\log 6} =$ ________.
15. If $\log_4 3 = x$, then $\log_{\sqrt[4]{3}}\sqrt[4]{64} =$ ________.
16. If $\log_x\left(\frac{1}{243}\right) = -5$, then find the value of x.
17. $7^{\log_{343} 27} =$ ________.
18. If $3^{\log_9 x} = 2$, then $x =$ ________.
19. If $\log_{xyz}x + \log_{xyz}y + \log_{xyz}z = \log_{10p}$, then $p =$ ________.
20. If $\log_{10}4 + \log_{10}m = 2$, then $m =$ ________.
21. Simplify: $3\log_3 5 + \log_3 10 - \log_3 625$.
22. If $\log(a + 1) + \log(a - 1) = \log 15$, then $a =$ ________.
23. The value of $\log 10 + \log 100 + \log 1000 + \cdots + \log 10000000000 =$ ________.
24. If the number of zeroes between the decimal point and the first non-zero digit of a number is 2, then the characteristic of logarithm of that number is ________.
25. The value of $\log(\tan 10°) + \log(\tan 20°) + \log(\tan 45°) + \log(\tan 70°) + \log(\tan 80°) =$ ________.

Short Answer Type Questions

26. Simplify: $\log\left(\frac{3}{18}\right) + \log\left(\frac{45}{8}\right) - \log\left(\frac{15}{16}\right)$.
27. Show that $\frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc} = 1$.
28. Solve for real value of x: $\log(x - 1) + \log(x^2 + x + 1) = \log 999$.
29. If $\frac{1}{1 + \log_a 10} = \frac{3}{2}$, then find the value of a.
30. If $x^2 + y^2 = 23xy$, then show that $2\log(x + y) = 2\log 5 + \log x + \log y$.
31. If $\log_{10}2 = 0.3010$ and $\log_{10}3 = 0.4771$, then find the value $\log_{10}135$.
32. If $\log_{10}2 = x$ and $\log_{10}3 = y$, then find $\log_{10}21.6$.
33. If $\log_{10}2 = 0.3010$, then find the number of digits in $(64)^{10}$.
34. Simplify $\frac{1}{\log_2 \log_2 \log_2 256}$.
35. Prove that $\log_3 810 = 4 + \log_3 10$.

Essay Type Questions

36. Solve: $x^{\log_4 3} + 3^{\log_4 x} = 18$.

37. If $p^2 + q^2 = 14pq$, then prove that
$\log\left(\dfrac{p+q}{4}\right) = \dfrac{1}{2}[\log p + \log q]$.

38. Without using tables, find the value of
$4\log_{10} 5 + 5\log_{10} 2 - \dfrac{1}{2}\log_{10} 4$.

39. If $\dfrac{\log a}{b-c} = \dfrac{\log b}{c-a} = \dfrac{\log c}{a-b}$, then prove that $a^a b^b c^c = 1$.

40. Arrange the following numbers in the increasing order of their magnitude. $\log_7 9$, $\log_{18} 16$, $\log_6 41$, $\log_2 10$.

CONCEPT APPLICATION

Level 1

1. If $\log_{16x} = 2.5$, then $x =$
(a) 40 (b) 256
(c) 1024 (d) None of these

2. If $\log 5 = 0.699$ and $(1000)^x = 5$, then find the value of x.
(a) 0.0699 (b) 0.0233
(c) 0.233 (d) 10

3. The value of $\log\left(\dfrac{18}{14}\right) + \log\left(\dfrac{35}{48}\right) - \log\left(\dfrac{15}{16}\right) =$
(a) 0 (b) 1
(c) 2 (d) $\log_{16} 15$

4. If $\log_3 a + \log_9 a + \log_{81} a = \dfrac{35}{4}$, then $a =$
(a) 27 (b) 243
(c) 81 (d) None of these

5. If $\log_9[(\log_8 x)] < 0$, then x belongs to _______.
(a) (1, 8) (b) $(-\infty, 8)$
(c) $(8, \infty)$ (d) None of these

6. If $\log_3 \dfrac{x^3}{3} - 2\log_3 3x^3 = a - b\log_3 x$, then find the value of $a + b$.
(a) 6 (b) −6
(c) 0 (d) −3

7. The value of $\log_{40} 5$ lies between
(a) $\dfrac{1}{3}$ and $\dfrac{1}{2}$ (b) $\dfrac{1}{4}$ and $\dfrac{1}{3}$
(c) $\dfrac{1}{2}$ and 1 (d) None of these

8. If $x = \log_{\frac{1}{2}} \dfrac{4}{3} \cdot \log_2 \dfrac{1}{3} \cdot \log_{\frac{2}{3}} 0.8$, then
(a) $x > 0$ (b) $x < 0$
(c) $x = 0$ (d) $x \geq 0$

9. If $\log_{144} 729 = x$, then the value of $\log_{36} 256$ is
(a) $\dfrac{4(3-x)}{(3+x)}$ (b) $\dfrac{4(3+x)}{(3-x)}$
(c) $\dfrac{(3+x)}{4(3-x)}$ (d) $\dfrac{(3-x)}{4(3+x)}$

10. The solution set of the equation $\log(2x - 5) - \log 3 = \log 4 - \log(x + 9)$ is
(a) $\left\{\dfrac{-19}{2}, 3\right\}$ (b) $\left\{-3, \dfrac{19}{2}\right\}$
(c) $\left\{3, \dfrac{19}{2}\right\}$ (d) $\{3\}$

11. If $\log_{10}\tan 19° + \log_{10}\tan 21° + \log_{10}\tan 37° + \log_{10}\tan 45° + \text{long}_{10}\tan 69° + \log_{10}\tan 71° + \log_{10}\tan 53° = \log_{10} \dfrac{x}{2}$, then $x =$
(a) 0 (b) 1
(c) 2 (d) 4

12. The solution set of the equation $\log(x + 6) - \log 8 = \log 9 - \log(x + 7)$ is
(a) $\{-15, 2\}$ (b) $\{2\}$
(c) $\{-15, 0, 2\}$ (d) $\{0, 2\}$

13. If log40 $4 = x$ and $\log_{40} 5 = y$, then express $\log_{40} 32$ in terms of x and y.

(a) $5(1 + x + y)$ (b) $5(1 - x + y)$

(c) $5(1 - x - y)$ (d) $5(1 + x - y)$

14. If $\log_{10} 11 = p$, then $\log_{10}\left(\dfrac{1}{110}\right) =$

(a) $(1 + p)^{-1}$ (b) $-(1 + p)$

(c) $1 - p$ (d) $\dfrac{1}{10p}$

15. If $\log_4 \dfrac{x^4}{4} + 3\log_4 4x^4 = p + q\log_4 x$, then the value of $\log_p(q)$ is ________.

(a) 4 (b) −4

(c) 3 (d) 2

16. If $\log_{4x} + \log_{8x}{}^2 + \log_{16x}{}^3 = \dfrac{23}{2}$, then $\log_x 8 =$

(a) 2 (b) $\dfrac{1}{2}$

(c) 3 (d) $\dfrac{3}{4}$

17. If $\log_{(x+y)}(x - y) = 7$, then the value of $\log_{(x^2-y^2)}(x^2 + 2xy + y^2)$ is ________.

(a) 14 (b) $\dfrac{2}{7}$

(c) $\dfrac{7}{2}$ (d) $\dfrac{1}{4}$

18. The value of $\log_{35}3$ lies between

(a) $\dfrac{1}{4}$ and $\dfrac{1}{3}$ (b) $\dfrac{1}{3}$ and $\dfrac{1}{2}$

(c) $\dfrac{1}{2}$ and 1 (d) None of these

19. If $\log\left(\dfrac{a+b}{6}\right) = \dfrac{1}{2}(\log a + \log b)$, then $\dfrac{a}{b} + \dfrac{b}{a} =$

(a) 30 (b) 31

(c) 32 (d) 34

20. If $\log_p q = x$, then $\log_{\frac{1}{p}}\left(\dfrac{1}{q}\right) =$

(a) $\dfrac{1}{x}$ (b) $-x$

(c) x (d) x^2

21. If $\log_{(x-y)}(x + y) = 5$, then what is the value of $\log_{x^2-y^2}(x^2 - 2xy + y^2)$?

(a) 1 (b) $\dfrac{\sqrt{5}}{3}$

(c) $\dfrac{1}{3}$ (d) 0

22. The value of $\log_a 1 + \log_2 2^2 + \log_3 3^3 \log_a 1 + \log_2 2^2 + \log_3 3^3$ (where a is a positive number and $a \neq 1$) is

(a) 210 (b) 209

(c) 145 (d) 89

23. $\log_{\frac{1}{2}} \dfrac{2}{3}$ ______ $\log_{\frac{2}{3}} \dfrac{1}{2}$. The appropriate symbol in the blank is

(a) $>$ (b) $<$

(c) $=$ (d) Cannot be determined

24. The value of $\log_3[\log_2\{\log_4(\log_5 625^4)\}]$ is

(a) 0 (b) 1

(c) 2 (d) 3 log 4

25. If $\log(x - 3) + \log(x + 2) = \log(x^2 + x - 6)$, then the real value of x, which satisfies the above equation is

(a) is any value of x.

(b) is any value of x except $x = 0$.

(c) is any values of x except $x = 3$.

(d) Does not exist

Level 2

26. If $\log_{\left(\sqrt{b\sqrt{b\sqrt{b\sqrt{b}}}}\right)}\left(\sqrt{a\sqrt{a\sqrt{a\sqrt{a\sqrt{a}}}}}\right) = x\log_b a$, then $x =$

(a) $\dfrac{32}{16}$ (b) $\dfrac{31}{15}$

(c) $\dfrac{31}{30}$ (d) $\dfrac{1}{2}$

27. If $7^{\log x} + x^{\log 7} = 98$, then $\log_{10}\sqrt{x}$ then $\dfrac{a}{b} + \dfrac{b}{a} =$

(a) 47 (b) 51

(c) 14 (d) 49

28. If $7^{\log x} + x^{\log 7} = 98$, then $\log_{10}\sqrt{x} =$

(a) 1 (b) $\frac{1}{2}$

(c) 2 (d) Cannot be determined

29. The value of $\log_b a + \log_{b^2} a^2 + \log_{b^3} a^3 + \cdots + \log_{b^n} a^n$ is

(a) n (b) $\log_b{}^a$

(c) $\frac{n(n+1)}{2}\log_b a$ (d) $\log_b a^n$

30. If $(\log_2 x) + \log_2(\log_4 x) = 2$, then find $\log_x 4$.

(a) 2 (b) $\frac{1}{2}$

(c) 1 (d) Cannot be determined

31. If $pqr = 1$ then find the value of $\log_{rq} p + \log_{rp} q + \log_{pq} r$.

(a) 0 (b) -1

(c) -3 (d) 1

32. If $\log_3[\log_2\{\log_x(\log_6 216^3)\}] = 0$, then $\log_3(3x) =$

(a) $\log_3 12$ (b) 1

(c) 2 (d) $\log_3 6$

33. If a^x, b^x and c^x are in GP, then which of the following is/are true?

(A) a, b, c are in GP

(B) $\log a$, $\log b$, $\log c$ are in GP

(C) $\log a$, $\log b$, $\log c$ are in AP

(D) a, b, c are in AP

(a) A and B (b) A and C

(c) B and D (d) Only A

34. The value of $\frac{1}{\log_3 n} + \frac{1}{\log_4 n} + \frac{1}{\log_5 n} + \cdots + \frac{1}{\log_8 n}$ is ________.

(a) $\log_n 8!$ (b) $\log_{n!} 8$

(c) $\log_n\left(\frac{8!}{2}\right)$ (d) $\log_{n!} 8!$

35. If $\frac{\log a}{y-z} = \frac{\log b}{z-x} = \frac{\log c}{x-y}$, then $abc =$

(a) $a^x b^y c^z$ (b) $a^{y+z}\, b^{z+x}\, c^{x+y}$

(c) 1 (d) All of these

36. If $\frac{1}{\log_x 10} = \frac{3}{\log_p 10} - 3$, then $x =$ ________.

(a) $100p^2$ (b) $\frac{p^2}{100}$

(c) $1000p^3$ (d) $\frac{p^3}{1000}$

37. If $\log_8 m = 3.5$ and $\log_2 n = 7$, then the value of m in terms of n is ________.

(a) $n\sqrt{n}$ (b) $2n$

(c) n^2 (d) $\sqrt[3]{n}$

38. If $\log_{12}(\log_7 x) < 0$, then x belongs to ______.

(a) $(1, \infty)$ (b) $(1, 7)$

(c) $(1, \infty)$ (d) $(1, 7)$

39. The value of $\log_{381} 7$ lies between ________.

(a) $\frac{1}{3}$ and $\frac{1}{2}$ (b) $\frac{1}{4}$ and $\frac{1}{3}$

(c) $\frac{1}{5}$ and $\frac{1}{4}$ (d) $\frac{1}{6}$ and $\frac{1}{5}$

40. If $\log_{10}\tan 31° \cdot \log_{10}\tan 32° \ldots \log_{10}\tan 60° = \log 10a$, then $a =$ ________.

(a) 10 (b) 1

(c) 4 (d) 2

Level 3

41. The solution set for $|1-x|^{\log_{10}(x^2-5x+5)} = 1$, is

(a) $\{0, 1, 4\}$ (b) $\{1, 4\}$

(c) $\{0, 4\}$ (d) $\{0, 2, 4\}$

42. The value of $\log\sqrt{2\sqrt{2\sqrt{2\ldots\infty \text{ times}}}} + \log\sqrt{3\sqrt{3\sqrt{3\ldots\infty \text{ times}}}}$ is

(a) 1 (b) 2

(c) log5 (d) log6

43. The least positive integral value of the expression $\frac{1}{2}\log_{10} m - \log_{m^{-2}} 10$ is

(a) 0 (b) 1

(c) 2 (d) -1

44. The domain of $\log(3-5x)$ is

(a) $\left(\frac{3}{5},\infty\right)$ (b) $\left(0,\frac{3}{5}\right)$

(c) $\left(-\infty,\frac{3}{5}\right)$ (d) $\left(-\frac{3}{5},0\right)$

45. If $\log_7 x + \log_7 y \ge 2$, then the smallest possible integral value of $x + y$ (given $x \ne y$) is ________.

(a) 7 (b) 14

(c) 15 (d) 20

46. If p, q, r are in GP and $a^p = b^q = c^r$, then which of the following is true?

(a) $\log_c b = \log_a c$ (b) $\log_c b = \log_b a$

(c) $\log_c a = \log_b c$ (d) None of these

47. The value of $\log_5 \sqrt{5\sqrt{5\sqrt{5\ldots^\circ}}}$

$+\log\left(\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots\infty\right)$ is ________.

(a) 1 (b) 25

(c) 10 (d) 20

48. The solution set of $|x+2|^{\log_{10}(x^2+6x+9)} = 1$ is ________.

(a) $\{-3, -4\}$ (b) $\{0, -3\}$

(c) $\{-4, -1\}$ (d) $\{-3, -1\}$

49. If $\log_p pq = x$, then $\log_q pq =$ ______.

(a) $\frac{x}{x-1}$ (b) $\frac{x-1}{x}$

(c) $\frac{x}{x+1}$ (d) $\frac{x+1}{x}$

50. If $\log_{\frac{1}{8}}\left(\log_4(x^2-5)\right) > 0$, then

(a) $x \in (-\infty,-3)\cup(3,\infty)$

(b) $x \in \left(-\infty,-\sqrt{6}\right)\cup\left(\sqrt{6},\infty\right)$

(c) $x \in \left(-3,\sqrt{6}\right)\cup\left(\sqrt{6},\infty\right)$

(d) $x \in \left(-3,-\sqrt{6}\right)\cup\left(\sqrt{6},3\right)$

51. If $p = \log_a bc$, $q = \log_b ca$ and $r = \log_c ab$, then which of the following is true?

(a) $p + q + r + 2 = pqr$ (b) $pqr = 2$

(c) $p + q + r = pqr$ (d) $pqr = 1$

52. If $\log_2 p + \log_8 p + \log_{32} p = \frac{46}{5}$, then $p =$ ________.

(a) 128 (b) 64

(c) 32 (d) 256

53. If p and q are positive numbers other than 1, then the least value of $|\log_q p + \log_p q|$ is ______.

(a) 3 (b) 1

(c) 2 (d) 4

54. If $\log_{48} 81 = x$, then $\log_{12} 3 =$ ________.

(a) $\frac{x+4}{2x}$ (b) $\frac{x+4}{x}$

(c) $\frac{x}{x+4}$ (d) $\frac{2x}{x+4}$

55. If $\log_l p$, $\log_m p$ and $\log_n p$ are in AP, then $(ln)^{\log_l m} =$ ________.

(a) n^2 (b) m^2

(c) l^2 (d) p^2

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. 2
2. 0
3. 16
4. 10^3
5. $\log \frac{2}{15}$
6. 5
7. 33.756
8. $\bar{3}$
9. $(1, \infty)$
10. 0.4458
11. 1
12. 0.6990
13. $\frac{3}{10}$
14. 3
15. $\frac{3}{x}$
16. 3
17. 3
18. 4
19. 10
20. 25
21. $\log_3 2$
22. 4
23. 55
24. $\bar{3}$
25. 0

Short Answer Type Questions

26. 0.
28. $x = 10$
29. 10^{-3}
31. 2.1303
32. $3(x + y) - 1$.
33. 19.
34. $\log_3 2$

Essay Type Questions

36. 16.
38. 4
40. $\log_{18}16, \log_7 9, \log_6 41, \log_2 10$.

CONCEPT APPLICATION

Level 1

1. (c)	**2.** (c)	**3.** (a)	**4.** (b)	**5.** (a)	**6.** (c)	**7.** (a)	**8.** (a)	**9.** (a)	**10.** (d)
11. (c)	**12.** (b)	**13.** (c)	**14.** (b)	**15.** (a)	**16.** (b)	**17.** (d)	**18.** (a)	**19.** (d)	**20.** (c)
21. (c)	**22.** (b)	**23.** (b)	**24.** (a)	**25.** (d)					

Level 2

26. (c)	**27.** (b)	**28.** (a)	**29.** (d)	**30.** (b)	**31.** (c)	**32.** (c)	**33.** (b)	**34.** (c)	**35.** (d)
36. (d)	**37.** (a)	**38.** (b)	**39.** (b)	**40.** (b)					

Level 3

41. (c)	**42.** (d)	**43.** (b)	**44.** (c)	**45.** (c)	**46.** (b)	**47.** (a)	**48.** (c)	**49.** (a)	**50.** (d)
51. (a)	**52.** (b)	**53.** (c)	**54.** (d)	**55.** (a)					

CONCEPT APPLICATION

Level 1

1. $\log_b a = n \Rightarrow a = b^n$.
2. Taking logarithms for $10^{3x} = 5$ and substituting in the given equation we get the value of x.
3. $\log a + \log b - \log c = \log\left(\frac{ab}{c}\right)$ and $\log 1 = 0$.
4. $\log_{b^n} a = \frac{1}{n}\log_b a$.
5. Given, $\log_9(\log_8 x) < 0$
   $\log_8 x < 9^0$
   $\log_7 x < 1$
   $x < 7^1$
   There fore $x \in (1, 7)$.
6. Use $\log a - \log b = \log\frac{a}{b}$ and $\log a + \log b = \log ab$.
7. Consider $5^2 < 40 < 5^3$ take logarithm with base 5.
8. $\log b\ a > 0$; when $a > 1$ and $b < 1$
   $\log_b a < 0$ when $a < 1$ and $b > 1$
   $\log_b a > 0$ when $a < 1$ and $b < 1$.
9. Find the values of $\log_{12}4$, $\log_{12}36$ and using $\log_{12} 27 = x$.
10. $\log a - \log b = \log\left(\frac{a}{b}\right)$.
11. $\tan\theta \cdot \tan(90 - \theta) = 1$.
12. $\log a - \log b = \log\frac{a}{b}$.
13. Express $\log_{40}32$ in terms of x and y.
14. $\log\frac{1}{a} = \log a^{-1}; \log mn = \log m + \log n$ and $\log_a a = 1$.
15. $\log a + \log b = \log ab$
    $\log a - \log b = \log\frac{a}{b}$.
16. $\log_{b^n} a^m = \frac{m}{n}\log_b a$.
17. Adding '1' on both sides and the '1' on the left side is expressed as $\log_{x+y}(x + y)$.
18. Consider the inequality and $3^3 < 35 < 3^4$ taking logarithm with base 3.
19. (i) Use, $\log a + \log b = \log ab$. remove the logarithms on both sides and evaluate $(a + b)^2$.
    (ii) Use $m(\log a + \log b) = \log(ab)^m$.
    (iii) Eliminate logarithms on both sides and obtain equations in terms of a and b.
    (iv) Divide both sides of the equation with ab and obtain the required answer.
20. $\log_{b^n} a^m = \frac{m}{n}\log_b a$.
21. Adding '1' on both sides, the '1' on the left side is expressed as $\log_{x-y}(x - y)$.
22. (i) $\log_b a^m = m\log_b a$ and $\sum n = \frac{n(n+1)}{2}$.
    (ii) $\log_a 1 = 0$, $\log_2 2^2 = 2$, $\log_3 3^3 = 3$, and so on.
    (iii) The required answer is the sum of first 20 natural numbers except 1.
23. (i) Use $a > b \Rightarrow \log_b a > 1$ and $a < b \Rightarrow \log_b a < 1$.
24. (i) $\log a_m = m\log a$.
    (ii) Express 625 in terms of base 5 and simplify from the extreme right logarithm.
25. $\log a + \log b = \log ab$.

Level 2

26. (i) Use, $\sqrt{a\sqrt{a\sqrt{a\cdots n \text{ terms}}}} = a^{\frac{2^n-1}{2^n}}$ and then $\log_{b^n} a^m = \frac{m}{n}\log_b a$ and simplify LHS.
    (ii) Compare LHS and RHS and find the value of x.
27. (i) Use $\log a + \log b = \log ab$ and remove logarithms on both the sides and evaluate $(a - b)^2$.
    (ii) Express RHS into single logarithm with coefficient 1.
    (iii) Apply antilog and cancel the logarithms on both sides.
    (iv) Divide LHS and RHS by ab and obtain the required value.
28. (i) $x^{\log y} = y^{\log x}$.
    (ii) $7^{\log x} = x^{\log 7}$.
    (iii) Convert LHS into $7^{\log x}$ (or) $x^{\log 7}$ and solve for x.
29. (i) $\log_{b^n} a^m = \frac{m}{n}\log_b a$.
    $\log a + \log b = \log ab$.

(ii) Each term of the given expression is equal to $\log_b a$.
(iii) There are n terms in the expression.
(iv) Use the above information and find the required sum.

30. (i) Assume $\log_2 x = a$ then $\log_4 x = \frac{a}{2}$.
(ii) Take $\log_2 (\log_{4x})$ as $2 \log_4 (\log_{4x})$ and convert LHS into single logarithm.
(iii) Express the result in terms of $\log_2 x$ and solve for $\log_2 x$.
(iv) Find $\log_x 2$ and then $\log_x 4$.

31. (i) Use $\log_a a = 1$.
(ii) Replace $rq = p^{-1}$, $rp = q^{-1}$ and $pq = r^{-1}$ in the given expression.
(iii) Simplify and eliminate logarithms.

32. (i) Remove logarithm one by one by using

$\log_a^x = b \Rightarrow x = a^b$.

(ii) Now substitute the value of x in $\log_3 3x$ and simplify.

33. (i) Verify from options.
(ii) Use if a, b and c are in GP, then $b^2 = ac$.
(iii) Substitute a^x, b^x and c^x in the above equation and simplify.
(iv) Apply logarithm for the above result and proceed.

34. (i) $\text{Log} \log_b a = \dfrac{1}{\log_a b}$.

(ii) Take all the logarithms to the numerators by using the formula $\dfrac{1}{\log_b a} = \log_a b$.

(iii) Use, $\log a + \log b + \cdots + \log n = \log(abc\ldots n)$ and simplify.

35. (i) Equate the given ratios to k and get the values of $\log a$, $\log b$ and $\log c$.
(ii) Add $\log a$, $\log b$ and $\log c$ and solve for abc.

36. Given, $\dfrac{1}{\log_x 10} = \dfrac{3}{\log_p 10} - 3$

$$\log_{10} x = 3\log_{10} p - 3\left(\therefore \log_b a = \frac{1}{\log_a b}\right)$$

$$\log_{10} x = 3(\log_{10} p - 1)$$
$$= 3(\log_{10} p - \log_{10} 10)$$
$$\log_{10} x = \log_{10}\left(\frac{p}{10}\right)^3$$
$$x = \left(\frac{p}{10}\right)^3 = \frac{p^3}{1000}.$$

37. Given, $\log_8 m = 3.5 \log_2 n = 7$

$\Rightarrow \quad m = 8^{3.5}$ (1)

and $n = 2^7$ (2)

$$m = 8^{\frac{35}{10}}$$
$$m = (2^3)^{\frac{7}{2}}$$
$$= (2^7)^{\frac{3}{2}}$$
$$= n^{\frac{3}{2}} \quad (\because \text{from Eq. (1)})$$
$$\Rightarrow \quad m = n\sqrt{n}.$$

38. Given, $\log_{12}(\log_7 x) < 0$ is defined when $\log_7 x > 0$

$x > 7^0 \quad \Rightarrow \quad x > 1$ (1)

$\log_{12}(\log_7 x) < 0 \quad \Rightarrow \quad \log_7 x < 12^\circ$

$\Rightarrow \quad \log_7 x < 1 = x < 7^1 \quad \Rightarrow \quad x < 7$ (2)

From Eqs. (1) and (2), we get $x \in (1, 7)$.

39. We know that $7^3 < 381 < 7^4$

$\Rightarrow \quad \log_7 7^3 < \log_7 381 < \log_7 7^4$

$\Rightarrow \quad 3 < \log_7 381 < 4$

$\Rightarrow \quad \dfrac{1}{4} < \dfrac{1}{\log_7 381} < \dfrac{1}{3}$

$\dfrac{1}{4} < \log_{381} 7 < \dfrac{1}{3}\left(\because \log_b a = \dfrac{1}{\log_a b}\right).$

40. Given, $\log_{10} a = \log_{10} \tan 31° \log_{10} \tan 32° \ldots \log_{10} \tan 45° \ldots \log_{10} \tan 60°$

$= \log_{10} \tan 31° \ldots \log_{10} 1 \ldots \log_{10} \tan 60°$

$= 0 \ (\because \log_{10} 1 = 0)$

$\log_{10} a = 0 \Rightarrow a = 1.$

Level 3

41. (i) Use $a^0 = 1$ and $1^m = 1$.
(ii) Consider RHS, i.e., 1 as $|1 - x|^0$ and equate the exponents.
(iii) Convert the logarithm in the exponential form by using $\log_b a = N \Rightarrow a = b^N$.
(iv) Solve the quadratic equation for x.

42. (i) $\log a + \log b = \log ab$ and $\log\sqrt{x\sqrt{x\ldots\infty}} = x$.

(ii) Use, $\sqrt{a\sqrt{a\sqrt{a\sqrt{a\ldots\infty}}}} = a$ and then use $\log a + \log b = \log ab$.

43. (i) The least positive integral value of $x + \dfrac{1}{x} = 2$.

(ii) Let $\log_m 10 = x$ then $\log_m 10 = \dfrac{1}{x}$.

(iii) The given expression becomes $\dfrac{1}{2}\left(x + \dfrac{1}{x}\right)$.

(iv) Now the least positive value is obtained if $x + \dfrac{1}{x}$ is minimum.

44. (i) $\log f(x)$ is defined only when $f(x) > 0$.

(ii) logarithms take only positive values. That is, $(3 - 5x) > 0$.

(iii) Solve the above inequation for x.

45. $\log_7 x + \log_7 y \geq 2$

$\log_7 xy \geq 2$

$xy \geq 7^2$

$xy \geq 49$

$\therefore$ The possible values of (x, y) are (1, 49), (2, 25), (3, 17), (4, 14), (5, 10), (6, 9), (7, 8), …

$\therefore$ Smallest possible value of $5 + 10 = 6 + 9 = 7 + 8 = 15$

$\therefore$ Hence, option (c) is the correct answer.

46. Let $ap = bq = cr = k$

Then $p = \log_a k$, $q = \log_b k$ and $r = \log_c k$

Also given $q^2 = pr$

$(\log_b k)^2 = (\log_a k)(\log_c k)$

$(\log_b k)(\log_b k) = (\log_a k)(\log_c k)$

$$\frac{\log_b k}{\log_a k} = \frac{\log_c k}{\log_b k}$$

$\log_b a = \log_c b$.

47. Let $x = \sqrt{5\sqrt{5\sqrt{5\ldots\infty}}}$

$x = \sqrt{5x}$

$x^2 = 5x$

$\Rightarrow \quad x = 5 (\because x \neq 0)$

$\therefore \log_5 \sqrt{5\sqrt{5\ldots\infty}} = \log_5 5 = 1$.

Consider $\dfrac{1}{2} + \left(\dfrac{1}{2}\right)^2 + \left(\dfrac{1}{2}\right)^3 + \cdots\infty$

Clearly it is a GP, $s_\infty = \dfrac{a}{1-r}$

Here $a = \dfrac{1}{2}$ and $r = \dfrac{1}{2}$

$$= \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

$\therefore \log\left(\dfrac{1}{2} + \left(\dfrac{1}{2}\right)^2 + \cdots\right) = \log 1 = 0$.

$\therefore$ The required value $= 1 + 0 = 1$.

48. Given, $|x+2|^{\log_{10} x^2+6x+9} = 1$

As the RHS = 1

The exponent of $x + 2$ should be 0 ($a^0 = 1$)

or $|x + 2| = 1$

$\Rightarrow \log_{10} x^2 + 6x + 9 = 0$ or $x + 2 = \pm 1$

$x^2 + 6x + 9 = 1$ or $x = -1$ or -3

$x = -2$ or -4

$x = -1$ or -3

when $x = -3$, $\log_{10}(x^2 + 6x + 9)$ is not defined.

When $x = -2$, $|x + 2| = 0$ which is not possible.

$x = -4$, or -1.

49. Given, $\log_p pq = x$

$\log_p p + \log_p q = x$

$1 + \log_p q = x$

$\log_p q = x - 1$

$\log_q pq = \log_q p + \log_q q$

$$= \frac{1}{x-1} + 1 = \frac{1+x-1}{x-1} = \frac{x}{x-1}.$$

50. Given, $\log_{\frac{1}{8}}(\log_4(x^2 - 5)) > 0$

Here $\log_4 x^2 - 5 > 0$

$x^2 - 5 > 4^0 \quad \Rightarrow \quad x^2 - 5 > 1$

$\Rightarrow \quad x^2 - 6 > 0 \quad \Rightarrow \quad x^2 - \left(\sqrt{6}\right)^2 > 0$

$\left(x - \sqrt{6}\right)\left(x + \sqrt{6}\right) > 0$

$\therefore x \in \left(-\infty, -\sqrt{6}\right) \cup \left(\sqrt{6}, \infty\right)$ (1)

And $\log_{\frac{1}{8}}(\log_4 x^2 - 5) > 0$

$$\Rightarrow \quad \log_4(x^2-5) < \left(\frac{1}{8}\right)^0$$

$x^2 - 5 < 4^1$

$\Rightarrow \quad x^2 - 9 < 0$

$(x-3)(x+3) < 0$

$\therefore x \in (-3, 3)$ (2)

From Eqs. (1) and (2)

$x \in (-3, -\sqrt{6}) \cup (\sqrt{6}, 3)$.

51. $pqr = (\log_a b + \log_a c)(\log_b c + \log_b a)(\log_c a + \log_c b)$

$= (\log_a c + 1 + \log_a c \cdot \log_b c + \log_b c)(\log_c a + \log_c b)$

$= 1 + \log_a b + \log_c a + \log_c b + \log_b c + \log_a c + \log_b a + 1$

$= 2 + (\log_a b + \log_a c) + (\log_c a + \log_c b) + (\log_b c + \log_b a)$

$= 2 + \log_a bc + \log_b ac + \log_c ab$

$= 2 + p + q + r$.

52. Given, $\log_2 p + \log_8 p + \log_{32} p = \frac{46}{5}$

$$\log_2 p + \left(\frac{1}{3}\right)\log_2 p + \left(\frac{1}{5}\right)\log_2 p = \frac{46}{5}$$

$$\log_2 p\left(1 + \frac{1}{3} + \frac{1}{5}\right) = \frac{46}{5}$$

$$\left(\frac{15+5+3}{15}\right)\log_2 p = \frac{46}{5}$$

$$\left(\frac{23}{15}\right)\log_2 p = \frac{46}{5}$$

$$\log_2 p = \frac{46}{5} \times \frac{15}{23}$$

$\log_2 p = 6$

$p = 2^6 = 64$.

53. We know that AM $(a, b) \geq$ GM (a, b)

AM $(\log_q p, \log_p q) \geq$ GM $(\log_q p, \log_p q)$

$$\frac{|\log_q p + \log_p q|}{2} \geq \sqrt{\log_p q \cdot \log_q p}$$

$\Rightarrow \quad |\log_q p + \log_p q| \geq 2$.

$\therefore$ The least value is 2.

54. Given, $\log_{48} 81 = x$

$\log_{48} 3^4 = x$

$4 \log_{48} 3 = x$

$$\log_3 48 = \frac{4}{x}$$

$$\log_3 3(16) = \frac{4}{x}$$

$$\log_3 3 + \log_3 16 = \frac{4}{x}$$

$$1 + \log_3 4^2 = \frac{4}{x}$$

$$2\log_3 4 = \left(\frac{4}{x}\right) - 1$$

$$\log_3 4 = \frac{4-x}{2x}.$$

Consider $\log_3 12 = \log_3 3(4)$

$= \log_3 3 + \log_3 4$

$$\Rightarrow \quad 1 + \frac{4-x}{2x}$$

$$\Rightarrow \quad \log_3 12 = \frac{x+4}{2x}$$

$$\log_{12} 3 = \frac{2x}{x+4}.$$

55. Given, $\log_l p$, $\log_m p$, $\log_n p$ are in AP.

$\Rightarrow \log_p l$, $\log_p m$ and $\log_p n$ are in HP.

$$\log_p m = \frac{2\log_p l \cdot \log_p n}{\log_p l + \log_p n}$$

$$\frac{\log_p m}{\log_p l} = \frac{2\log_p n}{\log_p ln}$$

$\Rightarrow \quad \log_l m = \log_{ln} n^2$

$\Rightarrow \quad (ln)^{\log_l m} = n^2$.