Trigonometric Ratios And Identities

The word trigonon means a triangle and the word metron means a measurement. Hence trigonometry means the science of measuring triangles.

SYSTEMS OF MEASUREMENT OF ANGLES

There are three systems for measuring angles

1. Sexagesimal or English system
2. Centesimal or French system
3. Circular system

Sexagesimal system : The principal unit in this system is degree (°). One right angle is divided into 90 equal part and each part is called one degree (1°) . One degree is divided into 60 equal parts and each part is called one minute and is denoted by (1'). One minute is equally divided into 60 equal parts and each part is called one second (1'').

In Mathematical form :

One right angle = 90° (Read as 90 degrees)

1° = 60' (Read as 60 minutes)

1' = 60'' (Read as 60 seconds)

Centesimal system : The principal unit in this system is grade and is denoted by (g). One right angle is divided into 100 equal parts, called grades, and each grade is subdivided into 100 minutes, and each minute into 100 seconds.

In Mathematical form :

One right angles = 100^g (Read as 100 grades)

1^g = 100' (Read as 100 minutes)

1' = 100'' (Read as 100 seconds)

Circular system : In circular system the unit of measurement is radian. One radian, written as 1^C, is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.

Relation between systems of measurement of angles

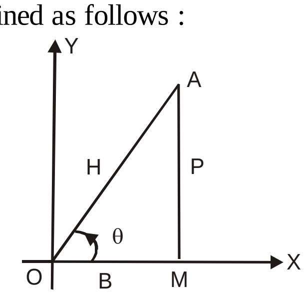
$$\frac{D}{90} = \frac{G}{100} = \frac{2C}{\pi}$$

TRIGONOMETRICAL RATIOS OR FUNCTIONS

Let a line OA make θ angle with a fixed line OX and AM is perpendicular from A on OX. Then in right-angled triangle AMO, trigonometrical ratios (functions) with respect to θ are defined as follows :

$\sin\theta = \frac{P}{H}$, $\cos\theta = \frac{B}{H}$, $\tan\theta = \frac{P}{B}$

$\text{cosec}\ \theta = \frac{H}{P}$. $\sec\theta = \frac{H}{B}$, $\cot\theta = \frac{B}{P}$



Note :

(i) Since t-ratios are ratios between two sides of a right angled triangle with respect to an angle, so they are real numbers.

(ii) θ may be acute angle or obtuse angle or right angle.

SIGN OF TRIGONOMETRIC RATIOS

(i) All ratios $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$ are positive in Ist quadrant.

(ii) $\sin\theta$ (or $\operatorname{cosec}\theta$) positive in IInd quadrant, rest are negative.

(iii) $\tan\theta$ (or $\cot\theta$) positive in IIIrd quadrant, rest are negative.

(iv) $\cos\theta$ (or $\sec\theta$) positive in IVth quadrant, rest are negative.

DOMAIN AND RANGE OF A TRIGONOMETRICAL FUNCTION

If $f : X \to Y$ is a function, defined on the set X, then the domain of the function f, written as Domain is the set of all independent variables x, for which the image f(x) is well defined element of Y, called the co-domain of f.

Range of f : $X \to Y$ is the set of all images f(x) which belongs to Y, i.e.

Range f = $\{f(x) \in Y : x \in X\} \subseteq Y$

The domain and range of trigonometrical functions are tabulated as follows

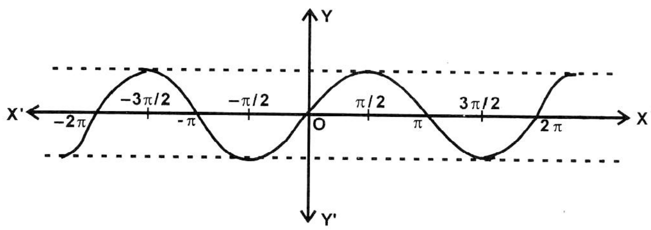
Trigonometric function	Domain	Range
sin x	R, the set of all the real number	[–1, 1]
cosx	R	$-1 \le \cos x \le 1$
tan x	$R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$	R
cosecx	$R - \{n\pi, n \in I\}$	$R - \{x : -1 < x < 1\}$
sec x	$R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$	$R - \{x : -1 < x < 1\}$
cot x	$R - \{n\pi, n \in I\}$	R

TRIGONOMETRICAL RATIOS OF STANDARD ANGLES

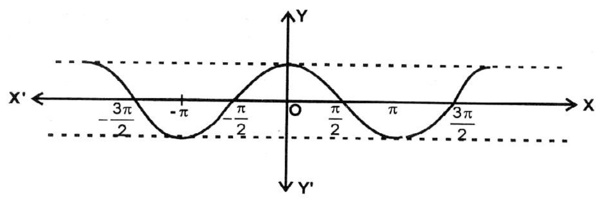
Angle → Ratio ↓	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
cot	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	not defined
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1
cosec	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined

GRAPH OF DIFFERENT TRIGONOMETRICAL RATIOS

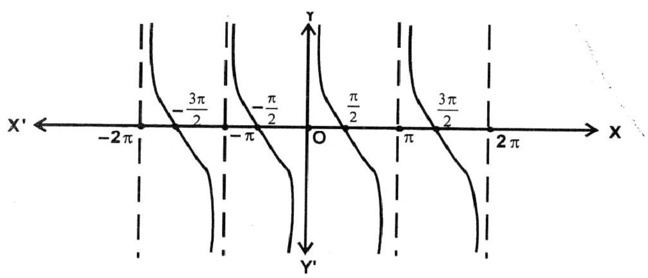
sin θ



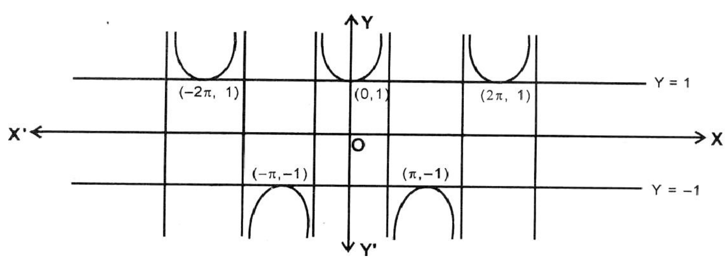
cos θ



cot θ

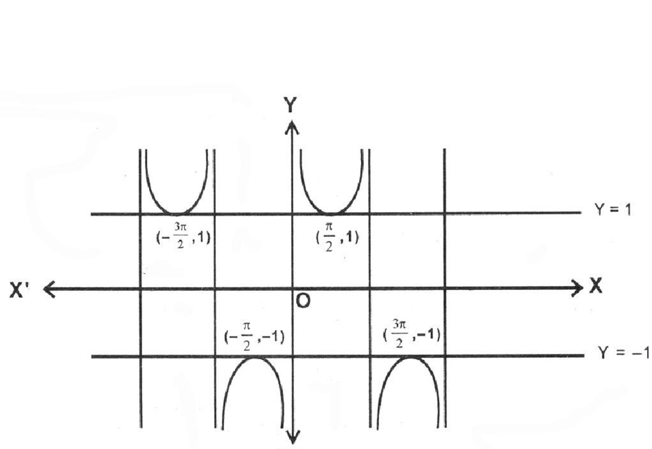


sec θ



tan x

cosec θ



SUM AND DIFFERENCE FORMULAE

(i) sin(A + B) = sinA cosB + cosA sin B

(ii) sin(A – B) = sinA cosB – cosA sin B

(iii) cos(A + B) = cosA cosB – sinA sinB

(iv) cos(A – B) = cosA cosB + sinA sinB

(v) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

(vi) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

(vii) $\tan\left(\frac{\pi}{4} + \theta\right) = \frac{1 + \tan\theta}{1 - \tan\theta}$

(viii) $\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1 - \tan\theta}{1 + \tan\theta}$

(ix) $\cot(A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}$

(xi) $\cot(A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$

(xii) $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(xiii) $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(xiv) $\sin 2\theta = 2\sin\theta \cos\theta = \frac{2\tan\theta}{(1 + \tan^2\theta)}$

(xv) $(\cos A \pm \sin A)^2 = 1 \pm \sin 2A$

(xvi) $\cos 2\theta = \frac{(1 - \tan^2\theta)}{(1 + \tan^2\theta)} = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

(xvii) $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

(xviii) $\sin\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}}$, $\cos\frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}}$

(xix) $\tan\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{1 + \mathrm{co}A}}$

(xx) $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$ $(A \neq n\pi + \pi/6)$

FORMULAE FOR TRANSFORMATION OF SUM OR DIFFERENCE INTO PRODUCT

(i) $\sin C + \sin D = 2\sin\left\{\frac{(C + D)}{2}\right\}\cos\left\{\frac{(C - D)}{2}\right\}$

(ii) $\sin C - \sin D = 2\cos\left\{\frac{(C + D)}{2}\right\}\sin\left\{\frac{(C - D)}{2}\right\}$

(iii) $\cos C + \cos D = 2\cos\left\{\frac{(C + D)}{2}\right\}\cos\left\{\frac{(C - D)}{2}\right\}$

(iv) $\cos C - \cos D = 2\sin\left\{\frac{(C + D)}{2}\right\}\sin\left\{\frac{(D - C)}{2}\right\}$

(v) $\tan A \pm \tan B = \frac{\sin A}{\cos A} \pm \frac{\sin B}{\cos B} = \frac{\sin A \cos B \pm \cos A \sin B}{\cos A \cos B} = \frac{\sin(A \pm B)}{\cos A \cos B}\left(A \neq n\pi + \frac{\pi}{2}, B \neq m\pi\right)$

(vi) $\cot A \pm \cot B = \dfrac{\sin(B \pm A)}{\sin A \sin B}\left(A \neq n\pi, B \neq m\pi + \dfrac{\pi}{2}\right)$

(vii) $\cos A \pm \sin A = \sqrt{2}\sin\left(\dfrac{\pi}{4} \pm A\right) = \sqrt{2}\cos\left(\dfrac{\pi}{4} \pm A\right)$ · $\tan A + \cot A = \dfrac{1}{(\sin A \cos A)}$

(viii) $1 + \tan A \tan B = \dfrac{\cos(A - B)}{\cos A \cos B}$ · $1 - \tan A \tan B = \dfrac{\cos(A + B)}{\cos A \cos B}$

(ix) $\cot A - \tan A = 2\cot 2A$ · $\tan A + \cot A = 2\operatorname{cosec} 2A$

(x) $\sin\dfrac{A}{2} + \cos\dfrac{A}{2} = \pm\sqrt{1 + \sin A}$ · $\sin\dfrac{A}{2} - \cos\dfrac{A}{2} = \pm\sqrt{1 - \sin A}$

FORMULAE FOR TRANSFORMATION OF PRODUCT INTO SUM OR DIFFERENCE

(i) $2\sin A \cos B = \sin(A + B) + \sin(A - B)$

(ii) $2\cos A \sin B = \sin(A + B) - \sin(A - B)$

(iii) $2\cos A \cos B = \cos(A + B) + \cos(A - B)$

(iv) $2\sin A \sin B = \cos(A - B) - \cos(A + B)$

TRIGONOMETRICAL RATIOS OF SOME IMPORTANT ANGLES

(i) $\sin 7\dfrac{1}{2}^\circ = \dfrac{\sqrt{4 - \sqrt{2} - \sqrt{6}}}{2\sqrt{2}}$

(ii) $\cos 7\dfrac{1}{2}^\circ = \dfrac{\sqrt{4 + \sqrt{2} + \sqrt{6}}}{2\sqrt{2}}$

(iii) $\tan 7\dfrac{1}{2}^\circ = \left(\sqrt{3} - \sqrt{2}\right)\left(\sqrt{2} - 1\right)$

(iv) $\sin 15^\circ = \dfrac{(\sqrt{3} - 1)}{2\sqrt{2}} = \cos 75^\circ$

(v) $\cos 15^\circ = \dfrac{(\sqrt{3} + 1)}{2\sqrt{2}} = \sin 75^\circ$

(vi) $\tan 15^\circ = 2 - \sqrt{3} = \cot 75^\circ$

(vii) $\cot 15^\circ = 2 + \sqrt{3} = \tan 75^\circ$

(viii) $\sin 22\dfrac{1}{2}^\circ = \dfrac{1}{2}\sqrt{2 - \sqrt{2}}$

(ix) $\cos 22\dfrac{1}{2}^\circ = \dfrac{1}{2}\sqrt{2 + \sqrt{2}}$

(x) $\tan 22\dfrac{1}{2}^\circ = \sqrt{2} - 1$

(xi) $\cot 22\dfrac{1}{2}^\circ = \sqrt{2} + 1$

(xii) $\sin 18^\circ = \dfrac{1}{4}(\sqrt{5} - 1) = \cos 72^\circ$

(xiii) $\cos 18^\circ = \dfrac{1}{4}\sqrt{10 + 2\sqrt{5}} = \sin 72^\circ$

(xiv) $\sin 36^\circ = \dfrac{1}{4}\sqrt{10 - 2\sqrt{2}} = \cos 54^\circ$

(xv) $\cos 36^\circ = \dfrac{1}{4}(\sqrt{5} + 1) = \sin 54^\circ$

FORMULAE FOR SUM OF THREE ANGLES

(i) $\sin(A + B + C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$
$= \cos A \cos B \cos C\,(\tan A + \tan B + \tan C - \tan A \tan B \tan C)$

(ii) $\cos(A + B + C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C$
$= \cos A \cos B \cos C\,(1 - \tan A \tan B - \tan B \tan C - \tan C \tan A)$

(iii) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

(iv) $4\sin(60^\circ - A)\sin A \sin(60^\circ + A) = \sin 3A$

$4\cos(60^\circ - A)\cos A \cos(60^\circ + A) = \cos 3A$

$\tan(60^\circ - A)\tan A \tan(60^\circ + A) = \tan 3A$

CONDITIONAL IDENTITIES

(1) If $A + B + C = 180^\circ$, then

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(ii) $\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C$

(iii) $\sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C) = 4 \sin A \sin B \sin C$

(iv) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(v) $\cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A \sin B \cos C$

(2) If $A + B + C = 180^\circ$, then

(i) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(ii) $\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

(iii) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(iv) $\cos A + \cos B - \cos C = -1 + 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

(v) $\dfrac{\cos A}{\sin B \sin C} + \dfrac{\cos B}{\sin C \sin A} + \dfrac{\cos C}{\sin A \sin B} = 2$

(3) If $A + B + C = \pi$, then

(i) $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C$

(ii) $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$

(iii) $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$

(iv) $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \cos C$

(4) If $A + B + C = \pi$, then

(i) $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(ii) $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2 + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(iii) $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

(iv) $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} - \cos^2\frac{C}{2} = 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

(5) If $x + y + z = \frac{\pi}{2}$, then

(i) $\sin^2 x + \sin^2 y + \sin^2 z = 1 - 2 \sin x \sin y \sin z$

(ii) $\cos^2 x + \cos^2 y + \cos^2 z = 2 + 2 \sin x \sin y \sin z$

(iii) sin2x + sin2y + sin 2z = 4 cos x cosy cos z

(6) If A + B + C = π , then

(i) tanA + tan B + tan c = tan A tan B tan C

(ii) cotB cot C + cot C cot A + cot A cot B = 1

(iii) $\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{A}{2}\tan\frac{B}{2}=1$

(iv) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

MOTHOD OF COMPONENDO AND DIVIDENDO

If $\frac{p}{q}=\frac{a}{b}$, then by componendo an dividendo we can write

$\frac{p-q}{q+q}=\frac{a-b}{a+b}$ or $\frac{q-p}{q+p}=\frac{b-a}{b+a}$

or $\frac{p+q}{p-q}=\frac{a+b}{a-b}$ or $\frac{q+p}{q-p}=\frac{b+a}{b-a}$

Note :- Reference of the above formulae will be given in the solutions of problems.

SOME IMPORTANT RESULTS

(i) $-\sqrt{a^2+b^2}\le a\sin x+b\cos x\le\sqrt{a^2+b^2}$

(ii) $\sin^2 x + \operatorname{cosec}^2 x \ge 2$

(iii) $\cos^2 x + \sec^2 x \ge 2$

(iv) $\tan^2 x + \cot^2 x \ge 2$

(v) $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\sec\theta+\tan\theta$

(vi) $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}=\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=\sec\theta-\tan\theta$

(vii) $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}=\cot\frac{\theta}{2}=\operatorname{cosec}\theta+\cot\theta$

(viii) $\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=\tan\frac{\theta}{2}=\operatorname{cosec}\theta-\cot\theta$

(ix) $\cos\theta \,.\, \cos 2\theta \,.\, \cos 2^2\theta$ .... $\cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n \sin\theta}$; $(\theta \ne n\pi)$

(x) cosA + cos (A +B) + cos (A + 2B) + ... + cos { A + (n –1) B } = $\frac{\sin nB/2}{\sin B/2}\cos\left\{A+(n-1)\frac{B}{2}\right\}$

MISCELLANEOUS POINTS

(i) Some useful identities :

(a) $\tan(A+B+C)=\frac{\Sigma\tan A-\tan A\tan B\tan C}{1-\Sigma\tan A\tan B}$

(b) $\tan\theta = \cot\theta - 2\cot 2\theta$

(c) $\tan 3\theta = \tan\theta \,.\, \tan(60° - \theta) \,.\tan(60° + \theta)$ (d) tan (A+B) – tanA – tanB = tanA. tanB.tan(A+B)

(e) $\sin\theta \sin(60° - \theta)\sin(60° + \theta) = \frac{1}{4}\sin 3\theta$ (f) $\cos\theta \cos(60° - \theta)\cos(60° + \theta) = \frac{1}{4}\cos 3\theta$

(ii) Some useful series :

(a) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta)$ .... + to n terms = $\dfrac{\sin\left[\alpha+\left(\frac{n-1}{2}\right)\beta\right]\left[\sin\left(\frac{n\beta}{2}\right)\right]}{\sin\left(\frac{\beta}{2}\right)}$; $\beta\neq 2n\pi$

(b) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta)+$ .... + to n terms = $\dfrac{\cos\left[\alpha+\left(\frac{n-1}{2}\right)\beta\right]\left[\sin\left(\frac{n\beta}{2}\right)\right]}{\sin\left(\frac{\beta}{2}\right)}$; $\beta\neq 2n\pi$

(iii) Least value of a sinx + b cos x + c is $c-\sqrt{a^2+b^2}$ and greatest value is $c+\sqrt{a^2+b^2}$