

INVERSE TRIGONOMETRIC FUNCTIONS

1. Principal Values & Domains of Inverse Trigonometric/Circular Functions:

	Function	Domain	Range
(i)	$y = \sin^{-1} x$ where	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
(ii)	$y = \cos^{-1} x$ where	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
(iii)	$y = \tan^{-1} x$ where	$x \in \mathbb{R}$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
(iv)	$y = \operatorname{cosec}^{-1} x$ where	$x \leq -1$ or $x \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$
(v)	$y = \sec^{-1} x$ where	$x \leq -1$ or $x \geq 1$	$0 \leq y \leq \pi; y \neq \frac{\pi}{2}$
(vi)	$y = \cot^{-1} x$ where	$x \in \mathbb{R}$	$0 < y < \pi$

P - 2	(i)	$\sin^{-1}(\sin x) = x,$	$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
	(ii)	$\cos^{-1}(\cos x) = x;$	$0 \leq x \leq \pi$
	(iii)	$\tan^{-1}(\tan x) = x;$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
	(iv)	$\cot^{-1}(\cot x) = x;$	$0 < x < \pi$
	(v)	$\sec^{-1}(\sec x) = x;$	$0 \leq x \leq \pi, x \neq \frac{\pi}{2}$
	(vi)	$\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x;$	$x \neq 0, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

P - 3	(i)	$\sin^{-1}(-x) = -\sin^{-1} x,$	$-1 \leq x \leq 1$
	(ii)	$\tan^{-1}(-x) = -\tan^{-1} x,$	$x \in \mathbb{R}$
	(iii)	$\cos^{-1}(-x) = \pi - \cos^{-1} x,$	$-1 \leq x \leq 1$
	(iv)	$\cot^{-1}(-x) = \pi - \cot^{-1} x,$	$x \in \mathbb{R}$

P - 5	(i)	$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2},$	$-1 \leq x \leq 1$
	(ii)	$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2},$	$x \in \mathbb{R}$
	(iii)	$\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2},$	$ x \geq 1$

2. Identities of Addition and Substraction:

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(i) $\sin^{-1} x + \sin^{-1} y$

$$= \sin^{-1} \left[x \sqrt{1-y^2} + y \sqrt{1-x^2} \right], x \geq 0, y \geq 0 \text{ \& } (x^2 + y^2) \leq 1$$

$$= \pi - \sin^{-1} \left[x \sqrt{1-y^2} + y \sqrt{1-x^2} \right], x \geq 0, y \geq 0 \text{ \& } x^2 + y^2 > 1$$

(ii) $\cos^{-1} x + \cos^{-1} y = \cos^{-1} \left[xy - \sqrt{1-x^2} \sqrt{1-y^2} \right], x \geq 0, y \geq 0$

(iii) $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}, x > 0, y > 0 \text{ \& } xy < 1$

$$= \pi + \tan^{-1} \frac{x+y}{1-xy}, x > 0, y > 0 \text{ \& } xy > 1$$

$$= \frac{\pi}{2}, x > 0, y > 0 \text{ \& } xy = 1$$

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(i) $\sin^{-1} x - \sin^{-1} y = \sin^{-1} \left[x \sqrt{1-y^2} - y \sqrt{1-x^2} \right], x \geq 0, y \geq 0$

(ii) $\cos^{-1} x - \cos^{-1} y = \cos^{-1} \left[xy + \sqrt{1-x^2} \sqrt{1-y^2} \right],$
 $x \geq 0, y \geq 0, x \leq y$

(iii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}, x \geq 0, y \geq 0$

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(i) $\sin^{-1} \left(2x \sqrt{1-x^2} \right) = \begin{cases} 2 \sin^{-1} x & \text{if } |x| \leq \frac{1}{\sqrt{2}} \\ \pi - 2 \sin^{-1} x & \text{if } x > \frac{1}{\sqrt{2}} \\ -(\pi + 2 \sin^{-1} x) & \text{if } x < -\frac{1}{\sqrt{2}} \end{cases}$

(ii) $\cos^{-1} (2x^2 - 1) = \begin{cases} 2 \cos^{-1} x & \text{if } 0 \leq x \leq 1 \\ 2\pi - 2 \cos^{-1} x & \text{if } -1 \leq x < 0 \end{cases}$

(iii) $\tan^{-1} \frac{2x}{1-x^2} = \begin{cases} 2 \tan^{-1} x & \text{if } |x| < 1 \\ \pi + 2 \tan^{-1} x & \text{if } x < -1 \\ -(\pi - 2 \tan^{-1} x) & \text{if } x > 1 \end{cases}$

$$(iv) \quad \sin^{-1} \frac{2x}{1+x^2} = \begin{cases} 2 \tan^{-1} x & \text{if } |x| \leq 1 \\ \pi - 2 \tan^{-1} x & \text{if } x > 1 \\ -(\pi + 2 \tan^{-1} x) & \text{if } x < -1 \end{cases}$$

$$(v) \quad \cos^{-1} \frac{1-x^2}{1+x^2} = \begin{cases} 2 \tan^{-1} x & \text{if } x \geq 0 \\ -2 \tan^{-1} x & \text{if } x < 0 \end{cases}$$

If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1} \left[\frac{x+y+z-xyz}{1-xy-yz-zx} \right]$ if, $x > 0, y > 0, z > 0$ & $(xy + yz + zx) < 1$

NOTE:

(i) If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$ then $x + y + z = xyz$

(ii) If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{\pi}{2}$ then $xy + yz + zx = 1$

(iii) $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$