Chapter 1

Relations and Functions

Solutions

SECTION - A

Objective Type Questions (One option is correct)

[Range]

1. If $f(x)=\ln(x^2+7|x|+10)$ is a single valued real function then the range of $f(x)$ in its natural domain will be

(1) $[0, +\infty)$ (2) $[\ln 10, +\infty)$ (3) $[0, 10]$ (4) R

Sol. Answer (2)

$f(x)=\ln(x^2+7|x|+10)$

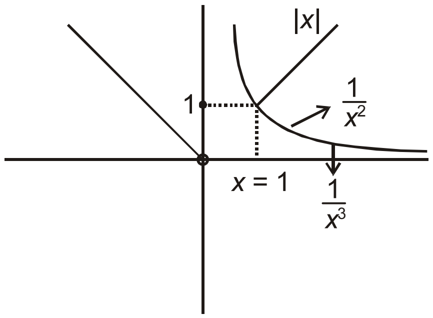
$\because \quad 10 \le x^2+7|x|+10<\infty$ hence range is $[\ln 10, \infty)$

2. If $f(x)$ is a real valued function defined as $f(x)=\begin{cases}\min\left\{|x|,\dfrac{1}{x^2},\dfrac{1}{x^3}\right\} & , \quad x\neq 0\\ 1 & , \quad x=0\end{cases}$ then range of $f(x)$ is

(1) $(-\infty, 1]$ (2) $(-\infty, 1]-\{0\}$ (3) $[1, \infty)$ (4) R

Sol. Answer (2)

$$f(x)=\begin{cases}\min\left\{|x|,\dfrac{1}{x^2},\dfrac{1}{x^3}\right\} & , \quad x\neq 0\\ 1 & , \quad x=0\end{cases},$$



Redefine the function = $\begin{cases}\dfrac{1}{x^3} & \text{if } x<0\\ 1 & x=0\\ |x| & 0<x<1\\ \dfrac{1}{x^3} & 1<x\end{cases}$

Hence range is = $(-\infty, 1]-\{0\}$

[Mapping]

3. Let $f:\left(-1,\frac{-1}{\sqrt{3}}\right)\to B$, be a function defined by $f(x)=\tan^{-1}\frac{3x-x^3}{1-3x^2}$, then f is both one-one and onto when B is the interval

(1) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (2) $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ (3) $\left(0,\frac{3\pi}{4}\right)$ (4) $\left(-\frac{3\pi}{4},0\right)$

Sol. Answer (2)

In question $f(x)$ must be 3 $\tan^{-1}x$

Now, $-1 < x < 1$

$\Rightarrow \quad -\frac{\pi}{4}<\theta<\frac{\pi}{4}\Rightarrow\frac{-3\pi}{4}<3\theta<\frac{3\pi}{4}$

$\therefore \quad f(x) = 3\theta = 3\tan^{-1}x$ and $-\frac{3\pi}{4}<f(x)<\frac{3\pi}{4}$

For which $f(x)$ is one-one as well as onto.

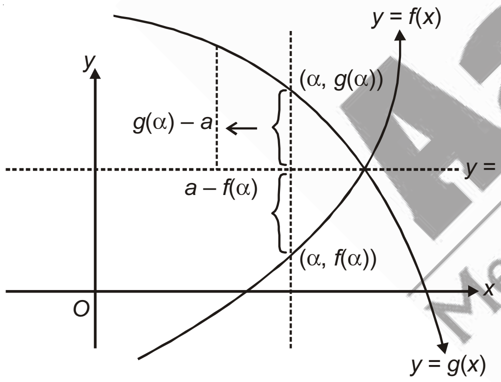
So, $B=\left(-\frac{3\pi}{4},\frac{3\pi}{4}\right)$

4. Let $f: R\to R$ and $g: R\to R$ be two one-one and onto function such that they are the mirror images of each other about the line $y = a$ if, $h(x) = f(x) + g(x)$, then $h(x)$ is

(1) One-one and onto (2) One-one only

(3) Onto only (4) Neither one-one nor onto

Sol. Answer (4)



$g(\alpha) - a = a - f(\alpha)$

$f(\alpha) + g(\alpha) = 2a$

In general $f(x) + g(x) = 2a$

$\therefore \quad h(x) = 2a$ = constant

$h(x)$ is neither one-one and onto.

5. The function $f: [0, 3] \to [1, 29]$, defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is

(1) One-one and onto (2) Onto but not one-one

(3) One-one but not onto (4) Neither one-one nor onto

Sol. Answer (2)

$f(x) = 2x^3 - 15x^2 + 36x + 1$

$f'(x) = 6x^2 - 30x + 36$

$= 6(x^2 - 5x + 6)$

$= 6(x - 2)(x - 3)$

Clearly the derivative changes sign in [0, 3] so, f is NOT one-one.

Now, the function is increasing in [0, 2] and decreasing in [2, 3].

Also, $f(0) = 1$

$f(2) = 29$

$f(3) = 8$

Hence the range is [1, 29] and so, the function is onto.

[Composition of Functions]

6. If $f(x) = \frac{x}{1-x}; x \neq 1$, then $f^{100}(x)$ is [where $(fof)(x) = f^2(x)$]

(1) $\frac{x}{1-100x}$ (2) $\frac{100x}{1-100x}$ (3) $\frac{x}{1-x}$ (4) $\frac{x}{100-x}$

Sol. Answer (1)

We have,

$f(x) = \frac{x}{1-x}$

$\Rightarrow f^2(x) = f(f(x)) = \frac{\frac{x}{1-x}}{1-\frac{x}{1-x}} = \frac{x}{1-2x}$

Also, $f^3(x) = \frac{\frac{x}{1-x}}{1-2\left(\frac{x}{1-x}\right)} = \frac{x}{1-3x}$

Similarly, $f^{100}(x) = \frac{x}{1-100x}$

7. If $g(f(x)) = |\sin x|$ and $f(g(x)) = (\sin\sqrt{x})^2$, then

(1) $f(x) = \sin^2 x$, $g(x) = \sqrt{x}$ (2) $f(x) = \sin x$, $g(x) = |x|$

(3) $f(x) = x^2$, $g(x) = \sin\sqrt{x}$ (4) f and g cannot be determined

Sol. Answer (1)

Obviously $f(x) = \sin^2 x$, $g(x) = \sqrt{x}$

8. Let $f(x) = x^2$ and $g(x) = 2^x$ then the solution set of $(fog)(x) = (gof)(x)$ is

(1) R (2) {0} (3) {0, 2} (4) {2, 3}

Sol. Answer (3)

$f(x) = x^2$ $g(x) = 2^x$

$\Rightarrow$ Domain of $f(x) = R$ Domain of $g(x) = R$

and range of $f(x) = R^+ \cup \{0\}$ Range of $g(x) = R^+$

$\Rightarrow$ *fog* is defined for R^+ whereas *gof* is defined $R^+ \cup \{0\}$

If $(fog)(x) = (gof)(x)$

$\Rightarrow \quad 2^{2x} = 2^{x^2} \quad \Rightarrow \qquad x^2 - 2x = 0$; $x = 0, 2$

$\Rightarrow$ Thus the required solution set is {0, 2}

9. Let $f(x) = \tan x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $g(x) = \sqrt{1-x^2}$ then $g(f(x))$ is

(1) $\frac{(\sqrt{\cos 2x})}{\cos x}$ (2) $-\frac{(\sqrt{\cos 2x})}{\cos x}$ (3) $\frac{(\sqrt{\cos 2x})}{|\cos x|}$ (4) Not defined

Sol. Answer (4)

Range of $f(x) = R$

Domain of $f(x) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Range of $g(x) = [0, 1]$

Domain of $g(x) = [-1, 1]$

For $g[f(x)]$ to be defined Range of $f(x) \subseteq$ domain of $g(x)$

But range of $f(x) \not\subseteq$ domain of $g(x)$

Hence $g[f(x)]$ is not defined.

10. Let *f*, *g* and *h* be real valued functions defined from *R* to *R* by

$f(x) = x^2 - 1, \ \forall x \in R$

$g(x) = \sqrt{1+x^2} \ \forall x \in R$

$h(x) = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$

then the composite function $(hofog)(x)$ is given by

(1) x (2) x^2 (3) $x^2 + 1$ (4) $x^2 - 1$

Sol. Answer (2)

$f(x) = x^2 - 1 \ \forall x \in R$

$g(x) = \sqrt{1+x^2} \ \forall x \in R$, $h(x) = \begin{cases} 0 & , \ x < 0 \\ x & , \ x \geq 0 \end{cases}$

$fog(x) = 1 + x^2 - 1 = x^2$

$h(fog(x)) = x^2$ as $x^2 > 0$

11. Let *g* be a real valued function defined on the interval (–1, 1) such that $e^{-x}(g(x) - 2e^x) = \int_0^x \sqrt{y^4+1}\, dy$ for all $x \in (-1, 1)$ and *f* be an another function such $f(g(x)) = g(f(x)) = x$. Then the value of $f'(2)$ is

(1) $\frac{1}{2}$ (2) $\frac{1}{4}$ (3) $\frac{1}{5}$ (4) $\frac{1}{3}$

Sol. Answer (4)

The given equation can be written as

$$g'(x) = 2e^x + e^x\int_0^x \sqrt{1+y^4}\,dy$$

$$\Rightarrow g'(x) = 2 + e^x\sqrt{1+x^4} + \left(\int_0^x \sqrt{1+y^4}\,dy\right)e^x$$

But $g(f(x)) = x$

$\Rightarrow g'(f(x)) = (f(x))$

$= f'(x) = 1$

$$f'(x) = \frac{1}{g'(f(x))}$$

$$f'(2) = \frac{1}{g'(f(2))}$$

But $f(2)$ is the value of x for which $g(x) = 2$, as they are inverse of each other hence $f(2) = 0$

$$\Rightarrow f'(2) = \frac{1}{g'(0)} = \frac{1}{3}$$

12. Let $g(x) = x - [x] - 1$ and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$, where [·] represents the greatest integer function, then for all x, $f(g(x)) =$

(1) 2 (2) 1 (3) 0 (4) –1

Sol. Answer (4)

$g(x) = x - [x] - 1 = \{x\} - 1 < 0$

$f(g(x)) = -1$

13. Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in \mathbb{R}$. Then the set of all x satisfying $(fogogof)(x) = (gogof)(x)$, where $(fog)(x) = f(g(x))$, is

(1) $\pm\sqrt{n\pi}, n \in \{0, 1, 2, ...\}$ (2) $\pm\sqrt{n\pi}, n \in \{1, 2, ...\}$

(3) $\frac{\pi}{2} + 2n\pi, n \in \{..., -2, -1, 0, 1, 2, ...\}$ (4) $2n\pi, n \in \{..., -2, -1, 0, 1, 2, ...\}$

Sol. Answer (1)

We have,

$f(x) = x^2$ and $g(x) = \sin x$, $\forall x$

$\Rightarrow f(g(g(f(x)))) = g(g(f(x)))$

$\Rightarrow g(f(x)) = g(x^2) = \sin x^2$

$\Rightarrow g(g(f(x))) = g(\sin x^2) = \sin(\sin x^2)$

$\Rightarrow$ $f(g(g(f(x)))) = (\sin(\sin x^2))^2$

$\Rightarrow$ $(\sin \sin x^2)^2 = \sin(\sin x^2)$

$\Rightarrow$ $\sin(\sin x^2) = 0$ or $\sin(\sin x^2) = 1$

But $\sin(\sin x^2) = 1$ is not possible hence $\sin x^2 = 0$

$\Rightarrow$ $x^2 = n\pi$

$\Rightarrow$ $x = \pm\sqrt{n\pi}, n \in \{0, 1, 2, 3......\}$

[Odd and Even Functions]

14. Let $f(x) = 3\sin^3 x + 4x - \sin|x| + \log(1 + |x|)$ be defined on the interval [0, 1]. The even extension of $f(x)$ to the interval [–1, 0] is

(1) $3\sin^3 x + 4x - \sin|x| + \log(1 + |x|)$

(2) $-3\sin^3 x - 4x + \sin|x| - \log(1+|x|)$

(3) $-3\sin^3 x - 4x - \sin|x| + \log(1+|x|)$

(4) $3\sin^3 x + 4x + \sin|x| - \log(1+|x|)$

Sol. Answer (3)

Even extension of $f(x)$ is obtained by changing sign of odd terms present in the function while keeping sign of even terms same.

Hence, required extension of $f(x) = 3\sin^3 x + 4x - \sin|x| + \log(1 + |x|)$ will be

$= -3\sin^3 x - 4x - \sin|x| + \log(1+|x|)$

[Periodic Function]

15. Let $f(x) = |\sin x| + |\cos x|$, $g(x) = \cos(\cos x) + \cos(\sin x)$

$h(x) = \left\{-\frac{x}{2}\right\} + \sin\pi x$, where { } represents the fractional function, then the period of

(1) $f(x) + g(x)$ is π

(2) $f(x) - g(x)$ is π

(3) $f(x) + g(x) + h(x)$ is 2π

(4) $f(x) + g(x) + h(x)$ is non-existent

Sol. Answer (4)

Period of $f(x)$, $g(x)$, $h(x)$ is $\frac{\pi}{2}, \frac{\pi}{2}, 2$ respectively.

16. Identify the correct statement

(1) The period of $f(x) = \sin\cos\left(\frac{x}{2}\right) + \cos(\sin x)$ is 2π

(2) The period of $f(x) = \cos x \cos 2x \cos 3x$ is 2π

(3) Let $n \in Z$ and the period of $f(x) = \dfrac{\sin nx}{\sin\left(\frac{x}{n}\right)}$ is 4π then $n = 2$

(4) If the period of $f(x) = \cos\sqrt{(a)}x$ is π and ( ) denotes the least integer function then $a \in [2, 4)$

Sol. Answer (3)

(1) $f(x) = \sin\left(\cos\frac{x}{2}\right) + \cos(\sin x)$

Period of $\cos\frac{x}{2}$ is 4π hence period of $\sin\left(\cos\frac{x}{2}\right)$ is 4π period of sin x is 2π but cos x is even hence period of cos (sin x) will be π.

Hence period of complete function will be LCM of $(4\pi, \pi) \Rightarrow 4\pi$.

(2) $f(x) = \cos x \cos 2x \cos 3x$

We have $f(x + \pi) = \cos(\pi + x) \cos(2\pi + 2x) \cos(3\pi + 3x) = \cos x \cos 2x \cos 3x$

Hence period = π

(3) The fundamental period of $\frac{\sin nx}{\sin\left(\frac{x}{n}\right)}$ = L.C.M. of $\left\{\frac{2\pi}{n}, 2\pi n\right\} = 2\pi n$

$\Rightarrow 2\pi n = 4\pi \quad \Rightarrow \quad n = 2$

(4) $f(x) = \cos\sqrt{(a)}x$

Period of $f(x)$ is $\frac{2\pi}{\sqrt{(a)}} = \pi$

$\therefore \quad \sqrt{(a)} = 2 \Rightarrow (a) = 4$

$\Rightarrow \quad a \in (3, 4]$

17. Consider that the graph of $y = f(x)$ is symmetric about the lines $x = 2$ and $x = 4$ then the period of $f(x)$ is

(1) 1 (2) 2 (3) 3 (4) 4

Sol. Answer (4)

$f(2 + x) = f(2 - x)$ ... (i)

$f(4 + x) = f(4 - x)$ ... (ii)

By (i), (ii) we get $f(x) = f(x + 4)$

Hence period = 4

18. If $f(x)$ is a real valued function defined as $f(x) = \ln(1 - \sin x)$ then graph of $f(x)$ is

(1) Symmetric about line $x = \pi$ (2) Symmetric about y axis.

(3) Symmetric about line $x = \frac{\pi}{2}$ (4) Symmetric about origin

Sol. Answer (3)

$\because \quad f(x) = \ln(1 - \sin x)$

$\therefore \quad \sin\left(\frac{\pi}{2} + x\right) = \sin\left(\frac{\pi}{2} - x\right) \forall x \in R$

$\therefore$ Hence graph will be symmetric about line $x = \frac{\pi}{2}$

19. If tanax + cotax and |tan$x$| + |cot$x$| are periodic functions of the same fundamental period then a equals

(1) 4 (2) 2 (3) 1 (4) 3

Sol. Answer (2)

Fundamental period of |tan$x$| + |cot$x$| is $\frac{\pi}{2}$

Fundamental period of tanax + cotax is $\frac{\pi}{a}$

$\therefore$ $a = 2$

[Miscellaneous]

20. Identify the correct option

(1) If $f(x)=\frac{1}{1-x}$, $x > 0$, then the graph of $y = f(f(f(x)))$ is a parabola

(2) If $g(f(x)) = |\sin x|$ and $f(g(x)) = (\sin\sqrt{x})^2$, then $f(x) = \sin^2 x$ and $g(x)=\sqrt{x}+1$

(3) Let $f(x) = (a-x^n)^{\frac{1}{n}}, x>0, n\in N$ and $fof(x) = f^2(x)$, then $f^{2006}(x) = x$

(4) Even functions are one-one

Sol. Answer (3)

(1) $f(x) = \frac{1}{1-x}$

$y = f\{f(f(x))\}$

$$y = f\left\{f\left(\frac{1}{1-x}\right)\right\} = f\left\{\frac{1}{1-\frac{1}{1-x}}\right\} = f\left\{\frac{x-1}{x}\right\} = \frac{1}{1-\frac{x-1}{x}}$$

$y = x$, which represents a straight line.

(2) $g(f(x)) = |\sin x|$

$f(g(x)) = (\sin\sqrt{x})^2$

$\Rightarrow f(x) = \sin^2 x;\ g(x) = \sqrt{x}$

(3) $f(x) = (a - x^n)^{1/n}$

$\Rightarrow f\{f(x)\} = x$

$\Rightarrow f^2(x) = x$

Hence $f^{2006}(x) = x$

(4) Even function is symmetrical about y axis hence it is many one.

21. If $2^{f(x)} = \frac{2+x}{2-x}, x\in(-2, 2)$ and $f(x)=\lambda f\left(\frac{8x}{4+x^2}\right)$ then value of 'λ' will be

(1) 2 (2) $\frac{1}{2}$ (3) 1 (4) –1

Sol. Answer (2)

$2^{f(x)} = \frac{2+x}{2-x}$

$f(x) = \log_2\left(\frac{2+x}{2-x}\right)$

Now $f\left(\dfrac{8x}{4+x^2}\right) = \log_2\left[\dfrac{2+\dfrac{8x}{4+x^2}}{2-\dfrac{8x}{4+x^2}}\right] = \log_2\left[\dfrac{8+2x^2+8x}{8+2x^2-8x}\right] = \log_2\left[\dfrac{4+x^2+4x}{4+x^2-4x}\right]$

$= \log_2\left(\dfrac{2+x}{2-x}\right)^2 = 2.\log\left(\dfrac{2+x}{2-x}\right) = 2.f(x)$

$\therefore\quad f\left(\dfrac{8x}{4+x^2}\right) = 2.f(x)$

$\Rightarrow\quad \boxed{\lambda = \dfrac{1}{2}}$

22. If {x} and [x] represent fractional and integral part of x, then $[x]+\sum\limits_{r=1}^{1090}\dfrac{\{x+r\}}{1090} =$

(1) x (2) $1090x$ (3) $\dfrac{x}{1090}$ (4) 1090

Sol. Answer (1)

$\because\quad \{x + r\} = \{x\}$, as $r \in$ integer

$[x]+\sum\limits_{r=1}^{1090}\dfrac{\{x\}}{1090} = [x]+\dfrac{1090\{x\}}{1090} = x$

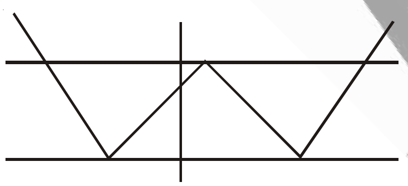
23. Let $f(x) = ||x - 1| + a| - 4$, if $f(x) = 0$ has three real solution, then the values of a lies in

(1) $a \in \{-4\}$ (2) $a \in (-\infty, -4)$ (3) $a \in [4, \infty)$ (4) $a \in [4, 10)$

Sol. Answer (1)

Given $f(x) = 0$

$\Rightarrow ||x - 1| + a| = 4$



Clearly, it has 3 solutions, from the graph

$\therefore a = -4$

SECTION - B

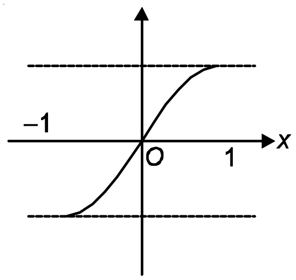
Objective Type Questions (More than one options are correct)

1. If $f : \{x : -1 \le x \le 1\} \to \{x : -1 \le x \le 1\}$, then which is/are bijective?

(1) $f(x) = [x]$ (2) $f(x) = \sin\dfrac{\pi x}{2}$ (3) $f(x) = |x|$ (4) $f(x) = x|x|$

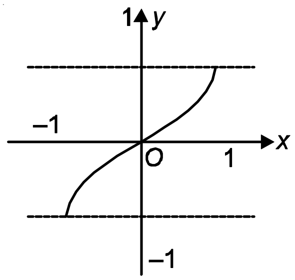
Sol. Answer (2, 4)

$f(x) = \sin\dfrac{\pi x}{2}$



Clearly, it is one-one and onto *i.e.*, bijective.

$f(x) = x|x|$



Clearly, it is bijective

2. Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ be given by $f(x) = (\log(\sec x + \tan x))^3$. Then

(1) $f(x)$ is an odd function

(2) $f(x)$ is a one-one function

(3) $f(x)$ is an onto function

(4) $f(x)$ is an even function

Sol. Answer (1, 2, 3)

$f(x) = (\log(\sec x + \tan x))^3$

$f(-x) = (\log(\sec x - \tan x))^3 = \log\left(\frac{1}{\sec x + \tan x}\right)^3 = -f(x)$

$\therefore$ f is odd.

Also $f'(x) = 3(\log(\sec x + \tan x))^2 \cdot \frac{(\sec x \tan x + \sec^2 x)}{\sec x + \tan x} = 3 \sec x \cdot (\log(\sec x + \tan x))^2 > 0$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore$ f is increasing on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore$ f is one-one

$\lim_{x \to \frac{\pi}{2}^-} (\log(\sec x + \tan x))^3 \to \infty$ and $\lim_{x \to -\frac{\pi}{2}^+} (\log(\sec x + \tan x))^3 \to -\infty$

$\therefore$ Range is $\mathbb{R}$.

3. Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{\sqrt{x}}$, then

(1) $f[g(x)]$ and $g[f(x)]$ have different domain

(2) $f[g(x)]$ and $g[f(x)]$ have the same range

(3) $f[g(x)]$ is a one-one

(4) $g[f(x)]$ is neither odd nor even

Sol. Answer (2, 3, 4)

$f(x) = \frac{1}{x}$ Dom $(f(x)) = R_0$

$g(x)=\frac{1}{\sqrt{x}}$ Dom$(g(x)) = R^+$

$$\left.\begin{aligned} f(g(x)) &= \frac{1}{\frac{1}{\sqrt{x}}} = \sqrt{x}, \quad x \ge 0 \\ g(f(x) &= \sqrt{x} \qquad\qquad x \ge 0 \end{aligned}\right] \text{domain is same.}$$

$\because \quad f(g(x)) = g(f(x)$ hence range is $[0, \infty)$ for both.

4. Let $f(x)=\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$ for all $x \in \mathbb{R}$ and $g(x)=\frac{\pi}{2}\sin x$ for all $x \in \mathbb{R}$. Let $(fog)(x)$ denote $f(g(x)$ and $(gof)(x)$ denote $g(f(x))$. Then which of the following is (are) true?

(1) Range of f is $\left[-\frac{1}{2},\frac{1}{2}\right]$

(2) Range of fog is $\left[-\frac{1}{2},\frac{1}{2}\right]$

(3) $\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\frac{\pi}{6}$

(4) There is an $x \in \mathbb{R}$ such that $(gof)(x) = 1$

Sol. Answer (1, 2, 3)

$$f(x)=\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$$

$$\frac{-\pi}{2} \le \frac{\pi}{2}\sin x \le \frac{\pi}{2}$$

$$-1 \le \sin\left(\frac{\pi}{2}\sin x\right) \le 1$$

$$\sin\left(\frac{-\pi}{6}\right) \le \sin\left(\frac{\pi}{6}\left(\sin\left(\frac{\pi}{2}\sin x\right)\right)\right) \le \sin\frac{\pi}{6}$$

$$\frac{-1}{2} \le \sin\left(\frac{\pi}{6}\left(\sin\left(\frac{\lambda}{2}\sin x\right)\right)\right) \le \frac{1}{2}$$

$$f(g(x)) = \sin\left(\frac{\pi}{6}\left(\sin\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}\sin x\right)\right)\right)\right)$$

$$-1 \le \sin\left(\frac{\pi}{2}\sin x\right) \le 1$$

$$\frac{-\pi}{2} \le \frac{\pi}{2}\sin\left(\frac{\pi}{2}\sin x\right) \le \frac{\pi}{2}$$

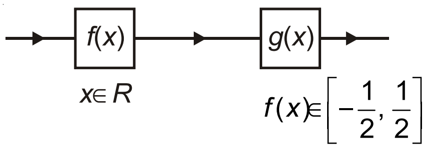
$$-1 \le \sin\left(\frac{\pi}{2}\left(\sin\left(\frac{\pi}{2}\sin x\right)\right)\right) \le 1$$

$$-\frac{\pi}{6} \le \frac{\pi}{6}\sin\left(\frac{\pi}{2}\left(\sin\left(\frac{\pi}{2}\sin x\right)\right)\right) \le \frac{\pi}{6}$$

$$\frac{-1}{2} \le \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}\sin x\right)\right)\right) \le \frac{1}{2}$$

$$\lim_{x\to 0} \frac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{2}\sin x} \times \frac{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}$$

$$\Rightarrow \lim_{x\to 0} \frac{\pi}{6} \frac{\sin\left(\frac{\pi}{2}\sin x\right)}{\frac{\pi}{2}\sin x} = \frac{\pi}{6}$$



Range of $g(f(x))$ is $\left[\frac{\pi}{2}\sin\left(\frac{-1}{2}\right), \frac{\pi}{2}\sin\frac{1}{2}\right]$

$$\left[-\frac{\pi}{2}\sin\frac{1}{2}, \frac{\pi}{2}\sin\frac{1}{2}\right]$$

Hence, 1 does not belong to this range.

5. If $f(x) = \sin\theta.\ x + a$ and the equation $f(x) = f^{-1}(x)$ is satisfied by every real value of x, then

(1) $\theta = \frac{\pi}{2}$ (2) $\theta = \frac{3\pi}{2}$ (3) $a \in R$ (4) $a = 1, \theta = \frac{\pi}{2}$

Sol. Answer (2, 3)

$$f(x) = \sin\theta.x + a \Rightarrow f^{-1}(x) = \frac{x}{\sin\theta} = \frac{a}{\sin\theta}$$

Since, $f(x) = f^{-1}(x)$, $\forall\ x \in R$

$$\Rightarrow \frac{1}{\sin\theta} = \sin\theta \text{ and } a = \frac{-a}{\sin\theta}$$

$$\Rightarrow \theta = \frac{3\pi}{2} \text{ and } a \in R$$

6. Let $f : (0, 1) \to \mathbb{R}$ be defined by

$$f(x) = \frac{b - x}{1 - bx}$$

where b is a constant such that $0 < b < 1$. Then

(1) f is not invertible on (0, 1)

(2) $f \ne f^{-1}$ on (0, 1) and $f'(b) = \frac{1}{f'(0)}$

(3) $f = f^{-1}$ on (0, 1) and $f'(b) = \frac{1}{f'(0)}$

(4) f^{-1} is differentiable on (0, 1)

Sol. Answer (1)

Let $f : (0, 1) \to \mathbb{R}$ defined by

$$f(x) = \frac{b - x}{1 - bx}, \text{ where } 0 < b < 1$$

We observe that

$$f'(x) = \frac{1 + b^2}{(1 - bx)^2} > 0$$

$\Rightarrow$ $f(x)$ is strictly increasing $\forall x \in (0, 1)$

It is obvious that $f(x)$ does not take all real values for $0 < b < 1$

$\Rightarrow$ $f : (0, 1) \to \mathbb{R}$ is into function, and hence its increase does not exist.

7. Let $f(x) = \left[x^2\left[\frac{1}{x^2}\right]\right]$, $x \in R - \{0\}$. [.] represents greatest integral function. Then

(1) $f(x)$ is an even function (2) $f(1) = 0$

(3) $f(x) = 0$ $\forall x \in (1, \infty) \cup (-\infty, -1)$ (4) $f(1) = 1$

Sol. Answer (1, 3, 4)

Clearly f(x) is even function.

For $x \in (-\infty, -1) \cup (1, \infty)$

$$0 < \frac{1}{x^2} < 1$$

Hence $f(x) = 0$

Also $f(1) = 1$

8. Let $f(x) = \sin ax + \cos bx$ be a periodic function, then

(1) $a = \frac{3\pi}{2}$, $b = \pi$ (2) $a = \sqrt{3},\ b = 5\sqrt{3}$ (3) $a = 3\sqrt{2},\ b = 2\sqrt{3}$ (4) $a, b \in R$

Sol. Answer (1, 2)

$f(x) = \sin ax + \cos bx$ is periodic with fundamental period = LCM $\left(\frac{2\pi}{|a|}, \frac{2\pi}{|b|}\right)$

From (1), if $a = \frac{3\pi}{2}$, $b = x$

$$\text{Period} = \text{LCM}\left(\frac{4}{3}, 2\right) = 4$$

From (2), if $a = \sqrt{3}$, $b = 5\sqrt{3}$

$$\text{Period} = \text{LCM}\left(\frac{2\pi}{\sqrt{3}}, \frac{2\pi}{5\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}$$

From (3), if $a = 3\sqrt{2}$, $b = 2\sqrt{3}$

$$\text{Period} = \text{LCM}\left(\frac{2\pi}{3\sqrt{2}}, \frac{2\pi}{2\sqrt{3}}\right) = \frac{\pi\sqrt{6}}{3} \text{ but } f\left(x + \frac{\pi\sqrt{6}}{3}\right) \neq f(x)$$

From (4) if $a, b \in R$ such period of one of $\sin ax$ and $\cos bx$ is rational and other is irrational then LCM is not possible. Hence options (1) and (2) are correct.

9. Which of the following function is periodic?

(1) $\text{sgn}(e^{-x})$ $x > 0$

(2) $|\sin x| + \sin x$

(3) $\min(4\cos x, |x|)$

(4) $\left[x+\frac{1}{2}\right]+\left[x-\frac{1}{2}\right]+2[-x]$, [.] represents greatest integral function

Sol. Answer (1, 2, 4)

(1) $e^{-x} > 0$

$\Rightarrow$ $\text{sgn}(e^{-x}) = 1$ for all $x \forall R$

Hence, $\text{sgn}(e^{-x})$ is periodic.

(2) $f(x) = |\sin x| + \sin x$

$f(x)$ is periodic with fundamental period 2π.

10. If $f(x) = \sin^2 x$ and $g(x) = \{x\}$ are two real valued function then

(1) Period of $f[g(x)]$ will be '1'

(2) Period of $g[f(x)]$ will be π

(3) Period of $f[g(x)] + g[f(x)]$ will be π

(4) Period of $f(g(x)) + g(f(x))$ will be 1

Sol. Answer (1, 2)

Period of $f(x)$ is π, period of $g(x)$ is 1

Hence period of $f[g(x)]$ = $\sin^2\{x\}$ will be 1.

Period of $g[f(x)] = \{\sin^2 x\}$ will be π

$\because$ $\{\sin^2 x\} = \sin^2 x, \forall x \in R - \left\{(2n+1)\frac{\pi}{2}\right\}$

But period of $f(g(x)) + g(f(x))$ will not be defined since LCM of rational and irrational is not defined.

11. Let us consider a function $f(x) = \sin[x]$, where $[x]$ denotes the greatest integer function. Then

(1) $f(x)$ is non-periodic

(2) There does not exist x such that $\sin[x] = \cos[x]$

(3) There exist infinitely many x for which $\sin[x] \neq \cos[x]$

(4) There exist infinitely many x for which $\sin[x] = \tan[x]$

Sol. Answer (1, 2, 3, 4)

$f(x) = \sin[x]$ is non-periodic

For sin (x) = [0] [x]

$[x] = 2n\pi + \frac{\pi}{4}$ which is not possible.

Also, for $0 \le x < 1$

$\sin[x] = \tan[x] = 0$

12. Which of the following functions are periodic?

(1) $f(x) = \begin{cases} |\operatorname{sgn}(x)|, & x \neq 0 \\ 1, & x = 0 \end{cases}$

(2) $g(x) = \sin^{-1}(\sin x)$

(3) $h(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

(4) $w(x) = \dfrac{e^x + e^{-x}}{2}$, $\forall x \in R$

Sol. Answer (1, 2)

(1) $f(x) = 1$, $\forall x \in R$ hence constant hence periodic.

(2) Periodic with period 2π.

(3) Graph of function oscillates but is not periodic.

(4) $f(x + 2\pi i) = f(x)$ function is not periodic since time period is imaginary.

13. Which of the following statements is/are true?

(1) $f : R \to R$, $f(x) = \log\left(x + \sqrt{1 + x^2}\right)$ is an odd function

(2) $f : R \to R$, $f(0) = 3$, then $f(x)$ must not be an odd function

(3) $f : R \to R$, $f(x) = x\left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right)$ is an onto function

(4) Graph of $f(x) = x \sin x$ is bounded between lines $y = x$, $y = -x$

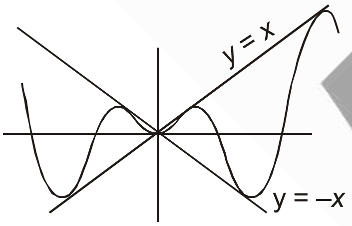
Sol. Answer (1, 2, 4)

(1) $f(-x) = -f(x)$

(2) A function can be odd only if its value at $x = 0$ is zero or undefined.

(3) Given function is even, continuous hence range cannot be 'R' hence into in $R \to R$.

(4) True, see graph



14. Which of the following functions are bounded in the interval as indicated?

(1) $f(x) = \sin x$, $x \in R$

(2) $g(x) = x\cos\dfrac{1}{x}$ on $(-\infty, \infty)$

(3) $h(x) = xe^{-x}$ on $(0, \infty)$

(4) $l(x) = \text{arc}\tan 2^x$ on $(-\infty, \infty)$

Sol. Answer (1, 3, 4)

A function is called bounded if $|f|x|| \leq M$ (M is finite number)

(1) $f(x) = \sin x$, $|\sin x| \leq 1$, hence bounded.

(2) At $x = 0$ $x\cos\dfrac{1}{x}$ is undefined also as $x \to \infty$ $x \cos x \to \infty$ hence unbounded.

(3) If $x \in (0, \infty)$ $\quad xe^{-x} \in (0, e^{-1})$ hence bound

(4) $x \in (-\infty, \infty)$ $\quad \text{arc }\tan 2^x \in \left(0, \dfrac{\pi}{2}\right)$

15. Let $f(x)=\begin{cases}2 & , \quad x\in Q\\ -2 & , \quad x\notin Q\end{cases}$, then

(1) $f\left(f\left(\sqrt{2}\right)\right)=2$

(2) $f\left(f\left(\pi\right)\right)=2$

(3) $f(x)$ is non-periodic

(4) $f(x)$ is periodic but fundamental period does not exist

Sol. Answer (1, 2, 4)

$$f(x)=\begin{cases}2 & , \quad x\in Q\\ -2 & , \quad x\notin Q\end{cases}$$

Between two rational numbers at least one irrational number exists and between two irrational numbers at least one rational number exists. So, function is periodic but fundamental period does not exist.

SECTION - C

Linked Comprehension Type Questions

Comprehension-I

Let $f(x)$ and $g(x)$ be a function defined on [–2, 2] such that $f(x) = -1$, $-2 \le x \le 0$; $f(x) = x - 1$, $0 < x \le 2$ and $g(x) = |x|$. Let $h(x)$ be a function defined as $h(x) = fog(x) + gof(x)$

1. The range of $h(x)$ is

(1) [0, 1] (2) [–2, 2] (3) [0, 2] (4) [1, 2]

2. The function $h(x)$ is

(1) One-one (2) One-one on [–1, 1]

(3) A linear function on [–2, 1] (4) A linear function on [1, 2]

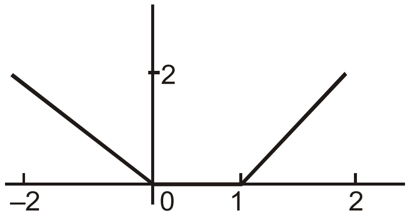
3. The function $h(x)$

(1) Decrease in [–2, 2] (2) Decreases strictly in [–2, 1]

(3) Increases in [–2, 2] (4) Increases in [1, 2]

Solution of Comprehension-I

$$f(x) = \begin{cases}-1, & -2\le x\le 0\\ x-1, & 0\le x\le 2\end{cases}$$



$$g(x) = |\,x\,| = \begin{cases}x, & x\ge 0\\ -x, & x<0\end{cases}$$

$$(fog)(x) = \{\,|x|-1 \quad 0\le |x|\le 2$$

$$\Rightarrow \quad (fog)(x) = \begin{cases}x-1, & 0\le x\le 2\\ -x-1, & -2\le x\le 0\end{cases}$$

$$\text{Also } (gof)(x) = \begin{cases}|-1|, & -2\le x\le 0\\ |x-1|, & 0\le x\le 2\end{cases}$$

$$\Rightarrow \quad (gof)(x) = \begin{cases} 1, & -2 \le x \le 0 \\ -(x-1), & 0 \le x \le 1 \\ x-1, & 1 \le x \le 2 \end{cases}$$

$$h(x) = fog + gof = \begin{cases} -x, & -2 \le x \le 0 \\ 0, & 0 \le x \le 1 \\ 2(x-1), & 1 \le x \le 2 \end{cases}$$

Graph of $h(x)$

1. Answer (3)

   From graph range [0, 2]

2. Answer (4)

   Linear in [1, 2]

3. Answer (4)

   From graph $f(x)$ increases in [1, 2]

Comprehension-II

Consider that $f : A \to B$

(i) If $f(x)$ is one-one $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $f'(x) \ge 0$ or $f'(x) \le 0$.

(ii) If $f(x)$ is onto the range of $f(x) = B$.

(iii) If $f(x)$ and $g(x)$ are inverse of each other then $f(g(x)) = g(f(x)) = x$.

Now, consider the answer of the following questions.

1. If $f(x)$ and $g(x)$ are mirror image of each other through $y = x$ and such that $f(x) = e^x + x$ then the value of $g'(1)$ is

(1) $\frac{1}{2}$ (2) 2 (3) 1 (4) 3

Sol. Answer (1)

$fg(x) = gf(x) = x \quad \Rightarrow f'(g(x)) \cdot g'(x) = g'f(x) \cdot f'(x) = 1$

$\Rightarrow \quad g'(x) = \frac{1}{f'(g(x))} \quad \Rightarrow g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)} = \frac{1}{2}$

2. Let f be one-one function with domain $\{x, y, z\}$ and range {1, 2, 3}. It is given that exactly one of the following statement is true and the remaining two are false. $f(x) = 1$, $f(y) \ne 1$, $f(z) \ne 2$, then the value of $f^{-1}(1)$ is

(1) x (2) z (3) y (4) Does not exist

Sol. Answer (3)

$f^{-1}(1) = y$

3. Let $f(x) = \frac{kx}{x+1}$ then the value of k such that $f(x)$ is inverse of itself is

(1) 0 (2) –1 (3) 1 (4) 2

Sol. Answer (2)

$f(f(x)) = x$

$\Rightarrow \quad \frac{\alpha^2 x}{\alpha x + x + 1} = x$

$\Rightarrow \quad (\alpha^2 - 1)x = x^2(\alpha + 1) \quad \Rightarrow \alpha = -1$

Comprehension-III

Let $f : [-3, 3] \rightarrow R$ defined by $f(x) = \left[\frac{x^2}{a}\right] \tan ax + \sec ax$. then

1. If $f(x)$ is an even function, then

(1) $a > 3$ (2) $a < 3$ (3) $a > 9$ (4) $a < 9$

Sol. Answer (3)

$$f(x) = \left[\frac{x^2}{a}\right]\tan ax + \sec ax$$

$$f(-x) = -\left[\frac{x^2}{a}\right]\tan ax + \sec ax$$

For even function $f(x) = f(-x)$

$\Rightarrow \quad 2\left[\frac{x^2}{a}\right]\tan ax = 0$

$\Rightarrow \quad \left[\frac{x^2}{a}\right] = 0$

$\Rightarrow \quad 0 \le \frac{x^2}{a} < 1$

$\Rightarrow \quad a > 9$

2. If $f(x)$ is an odd function, then

(1) $a > 3$

(2) $a < 3$

(3) $a > 9$

(4) $f(x)$ can't be an odd function for any real value of a

Sol. Answer (4)

For odd function $f(x) = -f(-x)$

$$\left[\frac{x^2}{a}\right]\tan ax + \sec ax = -\left(-\left[\frac{x^2}{a}\right]\tan ax + \sec ax\right)$$

$\Rightarrow \quad 2\sec ax = 0$

$\Rightarrow$ No real value of a exist

3. The fundamental period of $f(x)$ when $a = 10$

(1) π (2) 2π (3) $\frac{\pi}{10}$ (4) $\frac{\pi}{5}$

Sol. Answer (4)

For $a = 10$

$F(x) = \sec 10x$

Fundamental period of $f(x) = \frac{2\pi}{10} = \frac{\pi}{5}$

Comprehension-IV

Let $f(x) = f_1(x) - 2f_2(x)$

where $f_1(x) = \min(x^2, |x|)$ for $-1 \le x \le 1$

$= \max(x^2, |x|)$ for $|x| > 1$

$f_2(x) = \max(x^2, |x|)$ for $-1 \le x \le +1$

$= \min(x^2, |x|)$ for $|x| > 1$

$g(x) = \min(f(t) : -3 \le t \le x, -3 \le x \le 0)$

$= \max(f(t) : 0 \le t \le x, 0 \le x \le 3)$

1. For $-3 \le x \le -1$, range of $g(x)$ is

(1) [–1, 3] (2) [–1, –15] (3) [–1, 9] (4) {–1}

2. Number of critical points of $f(x)$ is

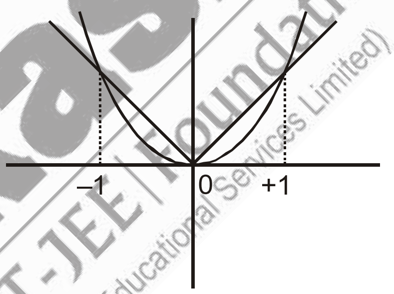
(1) 1 (2) 2 (3) 3 (4) 4

3. For $x \in (-1, 0)$; $f(x) - g(x)$ is

(1) $x^2 - 2x + 1$ (2) $x^2 + 2x - 1$ (3) $x^2 + 2x + 1$ (4) $x^2 - 2x - 1$

Solution of Comprehension-IV

$$f_1(x) = \begin{cases} \min(x^2, |x|) \text{ for } -1 \le x \le 1 \\ \max(x^2, |x|) \text{ for } |x| 1 > 1 \end{cases}$$



Hence $f_1(x) = x^2$

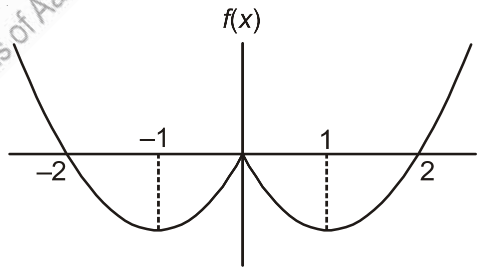
$$f_2(x) = \begin{cases} \max(x^2, |x|), -1 \le x \le +1 \\ \min(x^2, |x|) \text{ if } |x| > 1 \end{cases}$$

$f_2(x) = |x|$

Hence $f(x) = f_1(x) - 2f_2(x)$

$= x^2 - 2|x|$

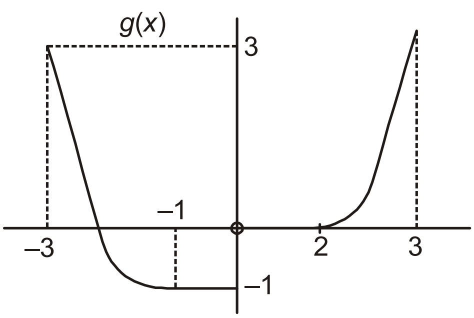
$$= \begin{cases} x^2 + 2x, & x \le 0 \\ x^2 - 2x, & x \ge 0 \end{cases}$$



$$g(x) = \begin{cases} \min. f(x); & -3 \le t \le x, -3 \le x \le 0 \\ \max. f(x); & 0 \le t \le x, 0 \le x \le 3 \end{cases}$$

$$= \begin{cases} f(x) & ; -3 \le x \le -1 \\ f(-1) & ; -1 \le x \le 0 \\ f(0) & ; 0 < x \le 2 \\ f(x) & ; 2 < x \le 3 \end{cases}$$

$$= \begin{cases} x^2 + 2x & ; -3 \le x \le -1 \\ -1 & ; -1 < x \le 0 \\ 0 & ; 0 < x \le 2 \\ x^2 - 2x & ; 2 < x \le 3 \end{cases}$$



1. Answer (1)

   If $-3 \le x \le -1$, then

   Range = [–1, 3]

2. Answer (3)

   Critical point = 3

3. Answer (3)

   $x \in (-1, 0)$, $f(x) = x^2 - 2|x| = x^2 + 2x$

   $x \in (-1, 0)$, $g(x) = -1$

   $f(x) - g(x) = x^2 + 2x + 1$

Comprehension-V

Three students *A*, *B*, *C* applied for admission in three universities *P*, *Q*, *R* where eligibility criteria is min 60%. Form processing software of university. *P*, *Q* and *R* use three functions [*x*], (*x*) and {*x*} respectively for conversion of percentage to nearest integer. Percentage marks of *A*, *B* and *C* are respectively 59.4, 59.5, 60.1. Hence due to rounding of all the three qualified for university *P*. Only '*C*' qualified for university '*Q*' but *B* and *C* both qualified for university '*R*' since software '*R*' rounds of as per normal calculator.

1. Domain of the function $f(x) = \sqrt{2\{\sin x\} - 1}$, if $n \in I$

   (1) $\left[2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right]$ (2) $\left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right)$

   (3) $R - \left[2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right]$ (4) $R - \left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right)$

2. Solution of equation {*x*} = [*x*] will lie in the interval. (*I* is an integer)

   (1) $\left[I, I + \frac{1}{2}\right]$ (2) $\left(I, I + \frac{1}{2}\right]$ (3) $\left[I - \frac{1}{2}, I\right]$ (4) R

3. Which of the following statement will be true for all real value of *x*?

   (1) Solution of equation {*x*} = *x* will be all integers (2) $\{x\} \le x,\ \forall\, x \in R$

   (3) Both (1) & (2) (4) $[x] < (x),\ \forall x \in R - I$

Solution of Comprehension-V

Symbols [*x*], (*x*), {*x*} do not have there usual meaning and have been redefined as follow.

[*x*] $\rightarrow$ represent least integer function.

(*x*) $\rightarrow$ represent greatest integer function

{*x*} $\rightarrow$ represent round of function which will convert any number to next integer if decimal part of it is 0.5 or more otherwise to previous integer.

1. Answer (1)

   $f(x) = \sqrt{\{\sin x\} - 1}$

   $\Rightarrow \{\sin x\} - 1 \ge 0 \Rightarrow \{\sin x\} \ge 1$

   $\Rightarrow \sin x \ge \frac{1}{2}$

   $\Rightarrow x \in \left[2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right]$

2. Answer (3)

If $x \in \left[I - \frac{1}{2}, I + \frac{1}{2}\right) \rightarrow \{x\} = I$

$x \in (I - 1, I] \rightarrow [x] = I$

Taking intersection of intervals

$x \in \left[I - \frac{1}{2} \cdot I\right]$

3. Answer (1)

$\{x\} \in I$, hence $\{x\} = x$ only for $x \in I$.

SECTION - D

Matrix-Match Type Questions

1. Let us consider a real valued function f defined as $f(x) = \frac{1-x}{1+x}$, $x \neq -1$. Then match the following.

Column-I	Column-II
(A) $f(x) + f\left(\frac{1}{x}\right); x \neq 0$ equals	(p) 1
(B) $\frac{1}{x}\ f(f(x))$; $x \neq 0$ equals	(q) 0
(C) $f(f(x)) + f\left(f\left(\frac{1}{x}\right)\right)$ when $x > 0$ or $x < 0$ may be	(r) 2
(D) $\frac{1}{x} f(f(f(f(...f(x)))))...; x \neq 0$ 2008 times equals	(s) – 2

Sol. Answer A(q), B(p), C(r), D(p)

$f(x) = \frac{1-x}{1+x}, x \neq -1$

(A) $f(x) + f\left(\frac{1}{x}\right) = \frac{1-x}{1+x} + \frac{x-1}{x+1} = 0$

(B) $\frac{1}{x} f(f(x)) = \frac{1}{x} \times \frac{1 - \frac{1-x}{1+x}}{1 + \frac{1-x}{1+x}} = \frac{1}{x}\frac{1+x-1+x}{1+x+1-x} = \frac{2x}{2x} = 1$

(C) $f(f(x)) + f\left(f\left(\frac{1}{x}\right)\right) = x + f\left(\frac{x-1}{x+1}\right) = x + \frac{1 - \frac{x-1}{x+1}}{1 + \frac{x-1}{x+1}} = x + \frac{1}{x}$

$x + \frac{1}{x} \geq 2$ for $x > 0$ or $x + \frac{1}{x} \leq -2$ for $x < 0$

(D) $\frac{1}{x} f(f(f(f...f(x)))).... = 1$

As $\frac{1}{x} f(f(x)) = 1 f$

2. Match the following

Column-I	**Column-II**
(A) $y = x - \frac{x^3}{6} + \frac{x^5}{120}$	(p) Even function
(B) $y = \frac{x}{a^x - 1}$	(q) Odd function
(C) $y = x\frac{a^x - 1}{a^x + 1}$	(r) $\lim_{x \to 0} f(x) = \frac{1}{\ln a}$
(D) $y = \frac{a^x + 1}{a^x - 1}$	(s) Neither even nor odd

Sol. Answer A(q), B(r, s), C(p), D(q)

(A) $y = x - \frac{x^3}{6} + \frac{x^5}{120}$

$f(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$

$f(-x) = -x + \frac{x^3}{6} - \frac{x^5}{120} = -\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right) = -f(x)$

which is an odd function

(B) $y = \frac{x}{a^x - 1}$

$f(x) = \frac{x}{a^x - 1}$

$f(-x) = \frac{-x}{a^{-x} - 1} = \frac{-x}{\frac{1}{a^x} - 1} \Rightarrow \frac{-x}{1 - a^x}(a^x)$

which is neither even nor odd

$f(0) = \lim_{x \to 0} \frac{x}{a^x - 1} = \frac{1}{\ell n\, a}$

(C) $f(x) = x\frac{a^x - 1}{a^x + 1}$

$f(-x) = (-x)\frac{\frac{1}{a^x} - 1}{\frac{1}{a^x} + 1} = (-x)\frac{1 - a^x}{1 + a^x}$

$f(-x) = x\frac{(a^x - 1)}{a^x + 1} = f(x)$

which is an even function.

(D) $f(x) = \frac{a^x + 1}{a^x - 1}$

$f(-x) = \frac{a^{-x} + 1}{a^{-x} - 1} = \frac{1 + a^x}{1 - a^x} = -\left(\frac{a^x + 1}{a^x - 1}\right)$

$f(-x) = -f(x) \Rightarrow f(x)$ is an odd function

3. Match the following with their fundamental periods (where [.] denotes greatest integer function and {.} denotes fractional part function)

Column-I	Column-II
(A) $f(x) = e^{\cos\{x\}} + \sin\pi[x]$	(p) 2
(B) $f(x) = \sin^4\pi x + \cos^3\pi x + \tan^2\pi x + \sin^2\pi x$	(q) 4
(C) $f(x) = e^{\sin\{x\}} + \sin\left(\frac{\pi}{2}[x]\right)$	(r) 24
(D) $f(x) = e^{x-[x]} + \cos 2\pi x$	(s) 1

Sol. Answer A(s), B(p), C(q), D(s)

(A) $f(x) = e^{\cos\{x\}} + 0 \quad (\sin\pi[x] = 0)$

Fundamental period is 1

(B) $f(x) = \sin^4\pi x + \cos^3\pi x + \tan^2\pi x + \sin^2\pi x$

Fundamental period of $\sin^4\pi x = 1$

Fundamental period of $\cos^3\pi x = 2$

Fundamental period of $\tan^2\pi x = 1$

Fundamental period of $\sin^2\pi x = 1$

$\therefore$ L.C.M. of 1, 2, 1, 1 is 2.

(C) $f(x) = e^{\sin\{x\}} + \sin\left(\frac{\pi}{2}[x]\right)$

$f(x) = e^{\sin\{x\}}$ if (x) is even.

$f(x) = e^{\sin\{x\}} \pm 1$ if (x) is odd

$\therefore$ Fundamental period is 1.

(D) $f(x) = e^{x-[x]} + \cos 2\pi x$

$= e^{\{x\}} + \cos 2\pi x$

Fundamental period of $e^{\{x\}}$ is 1

Fundamental period of $\cos 2\pi x$ is 1

$\therefore$ Fundamental period of $f(x)$ is 1

4. Match the following

Column-I	Column-II
(A) Number of solution of $2[x] = x + 2\{x\}$	(p) 0
(B) Number of solution of $\{x\} = e^{x^2}$	(q) 3
(C) Number of solution of $\sin^{-1} x = \text{Sgn}(x)$	(r) 1
(D) Fundamental period of function $f(x) = \left\{\frac{2}{3}x\right\} + \sin 6\pi x$ where {.} represents fractional part function	(s) Not defined

Sol. Answer A(q), B(p), C(p), D(q)

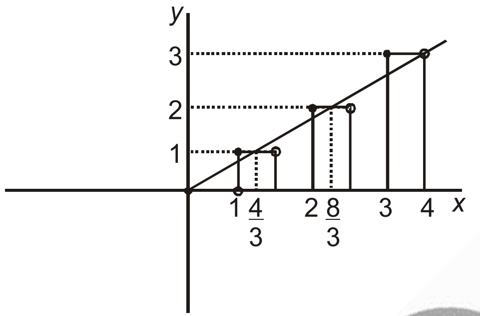
(A) $2[x] = x + 2\{x\}$

$2[x] = x + 2[x - [x]]$

$4[x] = 3x$

$[x] = \frac{3}{4}x$

Hence three solution $x = 0, \frac{4}{3}, \frac{8}{3}$



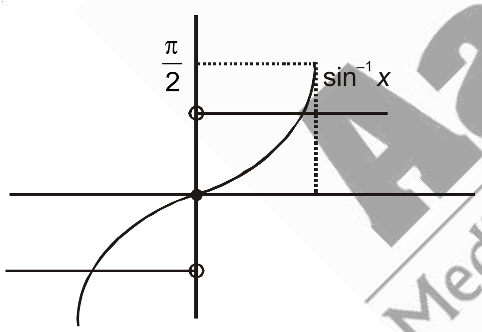
(B) $\because\ 0 \le x^2$ and $0 < \{x\} < 1$

$\therefore\ e^0 \le e^{x^2}$

$1 \le e^{x^2}$

Hence equation $\{x\} = e^{x^2}$ will have zero solution.

(C) See graph



No. of solution are '3'

(D) $f(x) = \left\{\frac{2}{3}x\right\} + \sin 6\pi x$

Fundamental period of $\left\{\frac{2}{3}x\right\} \to \frac{1}{\frac{2}{3}} \Rightarrow \frac{3}{2}$

Fundamental period of $\sin 6\pi x \to \frac{2\pi}{6\pi} \Rightarrow \frac{1}{3}$

Fundamental period of $f(x)$ = LCM of $\left(\frac{3}{2}, \frac{1}{3}\right)$ = 3

5. Match the following

Column-I	**Column-II**
(A) Continuous domain of $f(x) = \sqrt{x^x + \sqrt{x^x + \sqrt{x^x}}}$	(p) R^+
(B) Range of $f(x) = \sqrt{2 - [\sin x] - [\sin x]^2}$	(q) $\{0, \sqrt{2}\}$
(C) Solution of the equation $1 + \sin\frac{\pi}{\sqrt{2}}x = x^2 - \sqrt{2}x + 1$	(r) (1, 2)
(D) Domain of $f(x) = \sqrt{\log_{\{x\}}[x]}$	(s) R

Sol. Answer A(p), B(q), C(q), D(r)

(A) $f(x) = \sqrt{x^x + \sqrt{x^x + \sqrt{x^x}}}$

x^x is defined for $x > 0$ and I^- and some other negative value of x. But continuous domain will be only R^+

(B) $f(x) = \sqrt{2 - [\sin x] - [\sin x]^2}$

$\because \ -1 \le \sin x \le +1$

$[\sin x] = \{-1, 0, +1\}$

Hence range = $\{\sqrt{2}, 0\}$

(C) $1 + \sin\frac{\pi}{\sqrt{2}}x = x^2 - \sqrt{2}x + 1$

$\sin\frac{\pi}{\sqrt{2}}x = x(x - \sqrt{2})$

By observation at $x = 0$, $x = \sqrt{2}$

(D) $f(x) = \sqrt{\log_{\{x\}}[x]}$

For $\log_{\{x\}}[x]$ to be defined $[x] > 0$ $0 < \{x\} < 1$ $\{x\} \ne 1$

$\Rightarrow$ $[x] \ge 1$ $x \notin I$ $x \in R$

$\Rightarrow$ $x \in [1, \infty)$ ...(i)

Also $\log_{\{x\}}[x] \ge 0$ $\Rightarrow$ $[x] \le 1$ ...(ii)

From (i) and (ii)

$[x] = 1 \Rightarrow 1 \le x < 2$ $x \notin I$

Hence $x \in (1, 2)$

6. Let us consider two functions $f(x) = \ln(2x - x^2) + \sin\frac{\pi x}{2}$ and $g(x) = \log_{[x-1]}\frac{|x|}{x}$, where [.] denotes G.I.F. Match the items of Column I with those of Column II.

Column I	**Column II**
(A) Graph of f is symmetrical about the line	(p) $x = 1$
(B) Maximum value of f occurs at	(q) $x = 2$
(C) Domain of g is not equal to	(r) $[3, \infty)$
(D) Range of g is not equal to	(s) {0}
	(t) {0, 1}

Sol. Answer A(p), B(p), C(p, q, s, t), D(p, q, r, t)

$f(1 + \alpha) = f(1 - \alpha), \forall \alpha \in (0, 1)$

Domain of f is (0, 2)

Maxima of $\ln(2x - x^2)$ as well as that of $\sin\left(\frac{\pi x}{2}\right)$ occurs at $x = 1$

$g(x) = \log_{[x-1]} \text{sgn}(x)$

Domain $(g) = [3, \infty)$

Range $(g) = \{0\}$

SECTION - E

Assertion-Reason Type Questions

1. STATEMENT-1 : $f(x) = x^7 + x^6 - x^5 + 3$ is an onto function.

and

STATEMENT-2 : $f(x)$ is a continuous function.

Sol. Answer (2)

Statement 1 : Range of $f(x)$ is R.

Statement 2 : $f(x)$ is polynomial function of degree 7 hence it is continuous, but it is not necessary that continuous function is onto hence

Statement 1 is true statement 2 is true but statement 2 is not correct explanation.

2. STATEMENT-1 : If $f(x)$ and $g(x)$ are one-one functions then $f(g(x))$ and $g(f(x))$ is also a one-one function.

and

STATEMENT-2 : The composite function of two one-one function may or may not be one-one.

Sol. Answer (3)

$f(x)$ and $g(x)$ are one-one functions

Thus $f[g(x_1)] = f[g(x_2)]$

$\Rightarrow$ $g(x_1) = g(x_2)$ as f is one-one

$\Rightarrow$ $x_1 = x_2$ i.e, g is one-one

$\Rightarrow$ $f(g(x)$ is also one-one

Now $g[f(x_1)] = g[f(x_2)]$

$\Rightarrow$ $f(x_1) = f(x_2)$

$\Rightarrow$ $x_1 = x_2, \forall x_1, x_2,$

Hence fog and gof are one-one functions

$\Rightarrow$ Statement 1 is true.

Statement 2 is false as composite function of two one-one function is one-one.

3. STATEMENT-1 : Let $f : [1, \infty) \to [1, \infty)$ be a function such that $f(x) = x^x$ then the function is an invertible function.

and

STATEMENT-2 : The bijective functions are always invertible.

Sol. Answer (1)

Statement 1 : $f(x) = x^x$

$f'(x) = x^x [1 + \ell n\, x]$

$f'(x) > 0$ hence $f(x)$ is one-one.

Range of $f(x) = [1, \infty)$

Hence $f(x)$ is onto.

i.e., $f(x)$ is invertible.

Statement 2 is true and statement 2 is a correct explanation of statement 1.

4. STATEMENT-1 : $fog = gof \Rightarrow (fog)(x) = (gof)(x) = x$.

and

STATEMENT-2 : $fog = gof \Rightarrow$ either $f^{-1} = g$ or $g^{-1} = f$.

Sol. Answer (1)

Statement 1 : Let $f : A \to B$ and $g : B \to A$

$gof = I_A$ and $fog = I_B$

If $gof = fog$

$\Rightarrow A = B$

Hence $gof(x) = fog(x) = x$

Statement 1 is true.

Statement 2 :

Let $f : A \to B$ and $g : B \to A$

$gof = I_A$ and $fog = I_B$

Then f and g are bijection $g = f^{-1}$ or $f = g^{-1}$

Statement 2 is true and is a correct explanation of statement 1.

5. STATEMENT-1 : $f(x) = [\{x\}]$ is a periodic function with no fundamental period.

and

STATEMENT-2 : $f(g(x))$ is periodic if $g(x)$ is periodic.

Sol. Answer (2)

If $g(x)$ is periodic then $g(x + T) = g(x)$

Also $f[g(x+T)] = f(g(x)]$ hence $f(g(x))$ is periodic.

$f(x) = [\{x\}] = 0$ is a constant function, hence periodic but its fundamental period cannot be defined.

6. STATEMENT-1 : $f(x) = \log_{10}(\log_{1/x} x)$ will not be defined for any value of x.

and

STATEMENT-2 : $\log_{1/x} x = -1, \forall\, x > 0,\ x \neq 1$

Sol. Answer (1)

Statement-2 is true since $\log_{1/x} x = -1$ for $x > 0, x \neq 1$ hence $\log_{10}(\log_{1/x} x)$ will not be defined for any value of x.

7. STATEMENT-1 : $y = \log_e[x^{1001} - x^{101} + x - 1]$ is an onto function in $R \to R$.

and

STATEMENT-2 : If $f : R \to R^+$ is an onto function then $y = \log(f(x))$ will also be an onto function in $R \to R$.

Sol. Answer (4)

Since $f(x)$ is an onto function in $R \to R^+$, hence range of $f(x)$ is R^+ whereas log ($f(x)$) need only $f(x) > 0$ to show its complete range 'R' hence range of log($f(x)$) will be 'R' hence onto.

Statement-1 is false since domain of given function is not R.

8. STATEMENT-1 : $f : R \to R$, $f(x) = x^2 \log(|x| + 1)$ is an into function.

and

STATEMENT-2 : $f(x) = x^2 \log(|x| + 1)$ is a continuous even function.

Sol. Answer (1)

$f(x) = x^2 \log(|x| + 1)$

$f(-x) = (-x)^2 \log(|-x| + 1) = f(x)$ hence even domain of $f(x)$ is 'R' and it is continuous function.

Hence statement-2 is true.

$\because$ Range of any even continuous function cannot be 'R' hence any even continuous function cannot be onto in co-domain R.

9. STATEMENT-1 : $f(x) = x^4 - 3x^2 + 4x - 1$ is many one into in $R \to R$.

and

STATEMENT-2 : If $f : R \to R$ is a polynomial of even degree it will neither be injective nor surjective.

Sol. Answer (1)

Statement-1: $f'(x) = 4x^3 - 6x + 1$, may be positive as well as negative, hence $f(x)$ is many one

Statement-2 is clearly true

10. STATEMENT-1 : $f(x) = \tan 3x + \{2x\}$, where $\{x\}$ is fractional part of x, $f(x)$ is a periodic function.

and

STATEMENT-2 : LCM of a rational and irrational number is not possible.

Sol. Answer (4)

Statement-1 is false but Statement-2 is correct.

SECTION - F

Integer Answer Type Questions

1. Period of the function $f(x) = \cos(\cos\pi x) + e^{\{4x\}}$, where {.} denotes the fractional part of x, is_______.

Sol. Answer (1)

Period of $\cos(\cos\pi x)$ is $\left|\frac{\pi}{\pi}\right| = 1$ and period of $e^{\{4x\}}$ is $\frac{1}{4}$

$\therefore$ Period of $f(x)$ = LCM of $\left\{1 \frac{1}{4}\right\} = 1$

2. The function $f(x)$ satisfies the equation $f(x + 1) + f(x - 1) = \sqrt{3}\, f(x)\ \forall\ x \in R$, then the period of $f(x)$ is......

Sol. Answer (8)

We have

$f(x + 1) + f(x - 1) = \sqrt{3}\, f(x),\ \forall\ x \in R.$

$f(x + 1) = \sqrt{3}\, f(x) - f(x - 1)$

Let $x + 1 = r$

$f(r) = \sqrt{3}\, f(r - 1) - f(r - 2)$

$\Rightarrow f(r) = \sqrt{3}\,[\sqrt{3}\, f(r - 2) - f(r - 3)] - f(r - 2) \Rightarrow f(r) = 3f(r - 2) - \sqrt{3}\, f(r - 3) - f(r - 2)$

$\Rightarrow f(r) = 2f(r - 2) - \sqrt{3}\, f(r - 3)$

$\Rightarrow f(r) = 2[\sqrt{3}\, f(r - 3) - f(r - 4)] - \sqrt{3}\, f(r - 3)$

$\Rightarrow f(r) = \sqrt{3}\, f(r - 3) - 2f(r - 4)]$

$\Rightarrow f(r) = \sqrt{3}\,[\sqrt{3}\, f(r - 4) - f(r - 5)] - 2f(r - 4)$

$\Rightarrow f(r) = f(r - 4) - \sqrt{3}\, f(r - 5)$

$\Rightarrow f(r) = \sqrt{3}\,[\sqrt{3}\, f(r - 5) - f(r - 6)] - \sqrt{3}\, f(r - 5)$

$\Rightarrow f(r) + f(r - 6) = 0$

$\Rightarrow f(r) = -f(r + 6) = f(r + 12)$

Period of the given function is = 12

3. Let $g : R \to R$ be given by $g(x) = 3 + 4x$. If $g^n(x) = gogo........og(x)$ n times, then $g^4(1)$ equals......

Sol. Answer (7)

$g : R \to R\ g(x) = 3 + 4x.$

$g(x) = 3 + 4x$

$g^2(x) = g(g(x)) = 3 + 4g(x) = 3 + 4(3 + 4x) = 4^2x + 4.3 + 3$

$g^4(x) = g[g\{g\{g(x)\}\}] = 255 + 256x$

$g^4(1) = 511.$

4. Let $f(x)$ satisfies the relation $f(x + y) = f(x) + f(y)\ \forall\ x, y \in R$ and $f(1) = 2$ then the value of $\sum_{r=1}^{50} f(r)$ is.......

Sol. Answer (3)

$f(x + y) = f(x) + f(y)$

$\Rightarrow f(1) = 2,\ f(2) = 4,\ f(3) = 6\ ...,\ f(50) = 100$

$\sum_{r=1}^{50} f(r) = f(1) + f(2) + f(3) + ... + f(50)$

$= 2 + 4 + 6 + 8 + ... + 100$

$= 25[4 + 98] = 25 \times 102 = 2550.$

5. Let $f(x) = \dfrac{e^x - e^{-x}}{2}$ and if $g(f(x)) = x$, then $g\left(\dfrac{e^{1002}-1}{2e^{501}}\right)$ equals..........

Sol. Answer (5)

$$f(x) = \frac{e^x - e^{-x}}{2} = \frac{e^x - \frac{1}{e^x}}{2} = \frac{e^{2x}-1}{2e^x}$$

$g\{f(x)\} = x$

$$\therefore \quad g\left(\frac{e^{2x}-1}{2e^x}\right) = x \Rightarrow g\left(\frac{e^{1002}-1}{2e^{501}}\right) = 501$$

6. Let $f(x)$ be a function such that $f(x + y) = f(x) + f(y)\ \forall\ x, y \in N$ and $f(1) = 4$. If $\sum_{K=1}^{n} f(a+K) = 2n(33 + n)$, then '$a$' equals.......

Sol. Answer (4)

$f(x + y) = f(x) + f(y)$

$f(1) = 4$

$\Rightarrow\ f(2) = f(1) + f(1) = 4 + 4 = 8$

$\Rightarrow\ f(3) = 12$

. . . .

. . . .

$\Rightarrow\ f(a) = 4a$

Now $\sum_{k=1}^{n} f(a+k) = 2n(33+n) \Rightarrow \sum_{k=1}^{n} [f(a)+f(k)] = 2n(33+n)$

$\Rightarrow\ 4an + [f(1) + f(2) + \ldots + f(n)] = 2n(33 + n) \Rightarrow\ 4an + 2n(n + 1) = 2n(33 + n)$

$\Rightarrow\ 2a + n + 1 = 33 + n \quad \Rightarrow a = 16.$

7. Let $g(x) = f^{-1}(x)$, where $f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \le x \le 4 \\ 8\sqrt{x}, & 4 < x \end{cases}$ and $g(256) = \lambda$, then the sum of the digits of λ is...........

Sol. Answer (7)

Clearly $g(x) = f^{-1}(x) = \dfrac{x^2}{64}$ for $x > 4$

$$g(256) = \frac{(256)^2}{64} = 1024.$$

8. Let $f(x) = \dfrac{1}{2} + \sqrt{x - \dfrac{3}{4}}$ and $\phi(x) = f^{-1}(x)$ then $\phi(30)$ equals................

Sol. Answer (7)

$$f(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$$

$$f(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}} = y$$

$\Rightarrow \sqrt{x - \frac{3}{4}} = y - \frac{1}{2}$

$\Rightarrow x - \frac{3}{4} = \left(y - \frac{1}{2}\right)^2$

$\Rightarrow x = y^2 + \frac{1}{4} - y + \frac{3}{4}$

$\Rightarrow 4x = 4y^2 + 1 - 4y + 3$

$\Rightarrow 4x = 4y^2 - 4y + 4$

$\Rightarrow x = y^2 - y + 1$

$\Rightarrow f^{-1}(x) = x^2 - x + 1$

$\Rightarrow f^{-1}(30) = (30)^2 - 30 + 1 \Rightarrow 871$

$\therefore f^{-1}(30) =$ **871**.

$\Rightarrow \phi(30) = 871$

9. Let $f : R \to R$ and $g : R \to R$ be two given functions defined as $f(x) = 3x^2 + 2$ and $g(x) = 3x - 1$, $\forall x \in R$

   Then $\frac{1}{2}\sqrt{[(gof)(x)][fog(x)]}$ at $x = 1$ is......

Sol. Answer (7)

$f(x) = 3x^2 + 2$

$g(x) = 3x - 1 \quad \forall x \in R$

$fog(x) = 3(3x - 1)^2 + 2$

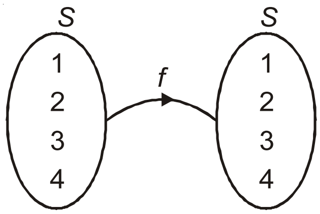
$gof(x) = 3(3x^2 + 2) - 1$

$fog(x).gof(x) = (3(3x - 1)^2 + 2)(9x^2 + 5) = 196$ and $x = 1$

10. Let $S = \{1, 2, 3, 4\}$. The number of functions $f : S \to S$. Such that $f(i) \le 2i$ for all $i \in S$ is equal to 2^k where k is equal to.....

Sol. Answer (7)

Given $S = \{1, 2, 3, 4\}$



$\therefore f(i) \le 2i, \ \forall \ i \in S$

For $i = 1$, $f(1)$ can be 1 or 2

$i = 2$, $f(2)$ can be 1, 2, 3 or 4.

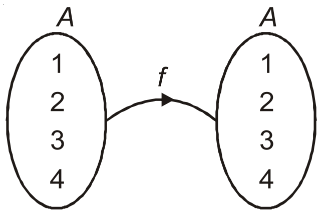
$i = 3$, $f(3)$ can be 1, 2, 3 or 4

$i = 4$, $f(4)$ can be 1, 2, 3 or 4

$\therefore$ Total number of such functions = 2 × 4 × 4 × 4 = 128

11. Let $A = \{1, 2, 3, 4\}$. The number of functions $f : A \rightarrow A$ satisfying $f(f(i)) = 1$ for all $1 \le i \le 4$ is

Sol. Answer (1)



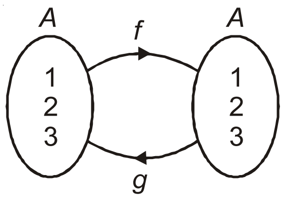
$\therefore \quad f(f(i)) = 1$

$\Rightarrow \quad f(\text{i}) = f^{-1}(1)$

Clearly, number of function = 10

12. Let f : {1, 2, 3} → {1, 2, 3} be a function. If the number of functions g : {1, 2, 3} → {1, 2, 3}. Such that $f(x) = g(x)$ for atleast one $x \in \{1, 2, 3\}$ is k, then $(k - 10)$ is equal to

Sol. Answer (9)



Number of function $= 3^3 - 2^3 = 27 - 8 = 19$

13. Let $f : (-2, 2) \rightarrow (-2, 2)$ be a continuous function such that $f(x) = f(x^2)\ \forall\ x \in d_f$ and $f(0) = \frac{1}{2}$, then the value of $4f\left(\frac{1}{4}\right)$ is equal to

Sol. Answer (2)

Given $f : (-2, 2) \rightarrow (-2, 2)$

$\therefore \quad f(x) = f(x^2)\ \forall\ x \in D_f$ and $f(0) = \frac{1}{2}$

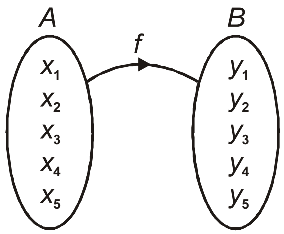
$\Rightarrow \quad f(0)$ is a rational number

$\Rightarrow \quad f(x)$ is a constant function

$\therefore \quad 4f\left(\frac{1}{4}\right) = 4 \times \frac{1}{2} = 2$

14. Let $A = \{x_1, x_2, x_3, x_4, x_5\}$, $B = \{y_1, y_2, y_3, y_4\}$. A function '$f$' is defined from A to B, such that $f(x_1) = y_1$ and $f(x_2) = y_2$, if the number of onto functions from A to B is n, then $(n - 10)$ is

Sol. Answer (8)



$\therefore \quad f(x_1) = y_1,\ f(x_2) = y_2$

Case-I :

When x_3, x_4, x_5 are not related to y_1 and y_2.

The number of onto functions

$$= \frac{3!}{1!\,2!} \times 2!$$

$= 6$

Case-II :

When x_3, x_4, x_5 are related to any one of y_1 and y_2.

The number of onto functions

$$= \frac{3!}{3!} \times 3! \times 2!$$

$= 12$

Thus, total number of onto functions

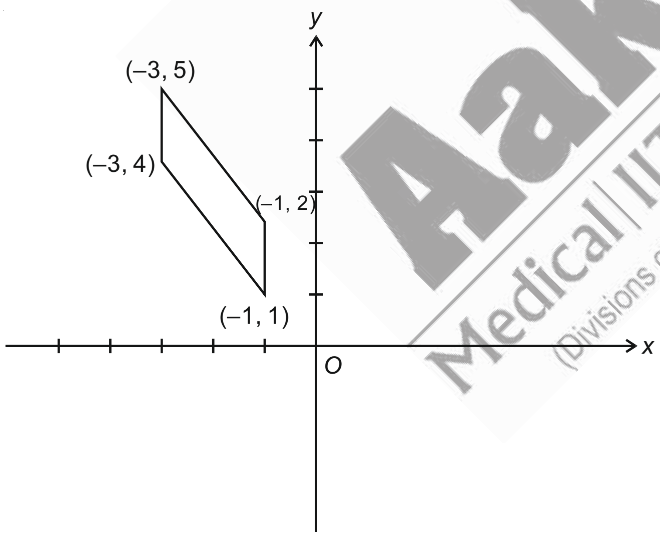
$= 6 + 12$

$= 18$

15. The area enclosed by the curve

$|x + y - 1| + |2x + y + 1| = 1$ in square units is

Sol. Answer (2)



Area = 2 sq. units

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