# Statics

## Chapter 36

**DEFINITIONS**

**Matter**

Anything that occupies space and is perceived by our senses is matter. Table, cup, water, etc. are example of matter.

**Body**

A body is a portion of matter occupying finite space. It has, therefore, a definite volume and a definite mass.

**Particle**

A particle is a body indefinitely small in size, so that the distance between its different parts are negligible. It may be regarded as a mathematical point associated with mass

**Rigid Body**

A body is said to be rigid when it does not change its shape and size when subjected to external forces *i.e.* a rigid body is a body the distance between any two points of which always remains the same .

**Force**

Force is an agent which changes or tends to change the state of rest or uniform motion of a body.

**Note :**

Force is a vector quantity

**REPRESENTATION OF A FORCE**

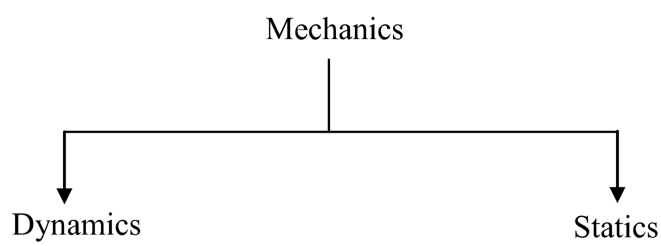
A force is completely known if we know the following data about it :

(i) its magnitude
(ii) its direction
(iii) its point of application.

Thus, we can completely represent a force by a straight line $AB$ drawn through the point of application along the line of action of the force, the length of the line $AB$ representing the magnitude of the force and the order of the letters $A$, $B$ specifying the direction.

**MECHANICS**

It is the science which deals with moving bodies or bodies at rest under the action of some forces.




**DYNAMICS**

It is that branch of mechanics which deals with the action of forces on bodies in motion.

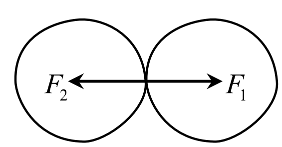
**STATICS**

It is that branch of mechanics which deals with the action of forces on bodies, the forces being so arranged that the bodies are at rest.

**FORCES IN STATICS**

1. **Action and Reaction**

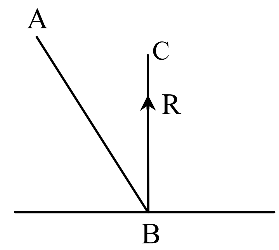
Whenever one body is in contact with another body, they apply equal and opposite forces at the point of contact. Such pair forces are called action and reaction pairs



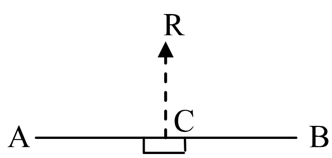

$\vec{F}_2 = -\vec{F}_1$

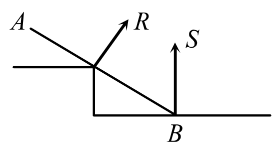
**Some Important Cases**

(i) When a rod AB rests with one end B upon a smooth plane, the reaction is along the normal to the plane at the point of contact.

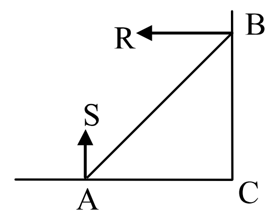



(ii) When a rod rests over a smooth peg, the reaction at the point of contact is $\perp$ to the rod.

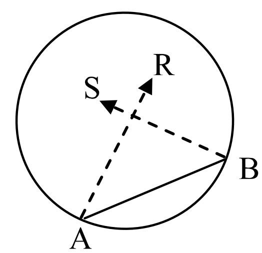



(iii) If one point of a body is in contact with the surface of another body, the reaction at the point of contact is $\perp$ to the surface, *e.g.* the equilibrium of a ladder in contact with the ground and a wall [both being smooth].




(iv) when a rod rests completely within a hollow sphere, the reactions at the extremities of the rod are along the normals at those points and will pass through centre of the hollow sphere .




2. **Weight**

Everybody is attracted towards the centre of the earth with a force proportional to its mass (the quantity of matter in the body).This force is called the **weight** of the body. If m is the mass of the body and g, is the acceleration due to gravity, then its weight $W = mg$ .

3. **Tension or Thrust**

Whenever a string is used to support a weight or drag a body, there is a force of pull along the string. This force is called tension. Similarly, if some rod be compressed, a force will be exerted. This type of force is called thrust.

**Note :**

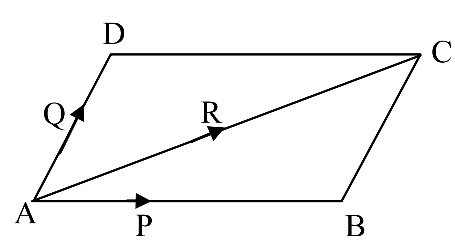
(i) The tension in a string is the same throughout. When two string are knotted together, the tensions in the two portions are different.

(ii) When a weight W hangs by a string, the tension in the string must be equal to the weight suspended. *i.e.* $T = W$ .

(iii) The tension of a string always acts in a direction diverging away from the body under consideration and acts along the string.

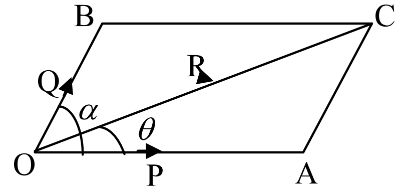
**PARALLELOGRAM LAW OF FORCES**

If two forces acting at a point , be represented in magnitude and direction by the sides of a parallelogram drawn from the point, their resultant is represented both in magnitude and direction, by the diagonal of the parallelogram drawn through that point. Let P and Q be the forces represented in magnitude and direction by the sides



AB and AD of a parallelogram ABCD, then their resultant R is represented in magnitude and direction by the diagonal AC.

**RESULTANT OF TWO FORCES**




If two concurrent forces P and Q are inclined at an angle $\alpha$ to each other, then the magnitude R of their resultant is given by $R=\sqrt{P^2+Q^2+2PQ\cos\alpha}.$ If R makes an angle $\theta$ with the direction of $P$, then $\tan\theta=\dfrac{Q\sin\alpha}{P+Q\cos\alpha}$

**PARTICULAR CASES**

1. When P and Q are at right angles to each other *i.e.* $\alpha=90°$

   In this case $R=\sqrt{P^2+Q^2}$ and $\tan\theta=\dfrac{Q}{P}.$

2. When $P=Q$. In this case, $R=2P\cos\dfrac{\alpha}{2}$ *and* $\theta=\dfrac{\alpha}{2}.$

3. When P and Q are in the same direction $i.e. \alpha=0$

   In this case, $R$ is in the same direction as P and Q and $R=P+Q$.

   This is called the greatest resultant of the two forces.

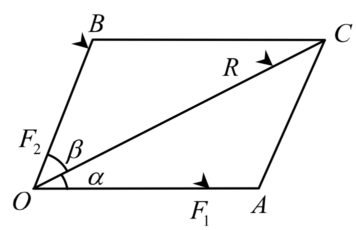
4. When $P$ and $Q$ are in the opposite direction $\left(i.e. \alpha=180°\right)$ and $P>Q$.

   In this case, $R$ is in the direction of $P$ and $R=P-Q$.

   This is called the least resultant of the two forces.

**COMPONENT OF A FORCE IN TWO DIRECTIONS :**

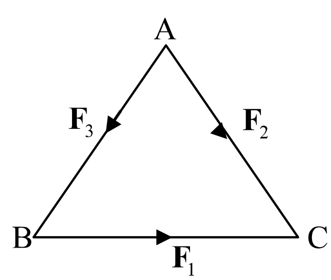
The component of a force $R$ in two directions making angles $\alpha$ and $\beta$ with the line of action of $R$ on and opposite sides of it are




$$F_1=\frac{OC.\sin\beta}{\sin(\alpha+\beta)}=\frac{R\sin\beta}{\sin(\alpha+\beta)}$$

and $$F_2=\frac{OC.\sin\alpha}{\sin(\alpha+\beta)}=\frac{R\sin\alpha}{\sin(\alpha+\beta)}$$

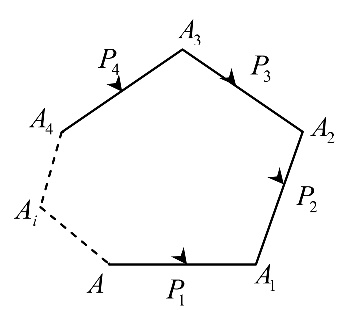
**TRIANGLE LAW OF FORCES**




If three forces, acting at a point, be represented in magnitude and direction by the three sides of triangle, taken in order, they are in equilibrium.

$$\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3=0$$

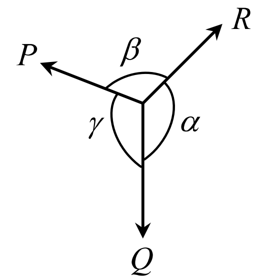
**POLYGON LAW OF FORCES**




If any number of forces acting on a particle be represented in magnitude and direction by the sides of a polygon taken in order, the forces shall be in equilibrium.

$$\mathbf{P}_1+\mathbf{P}_2+......+\mathbf{P}_n=0$$

**LAMI'S THEOREM**

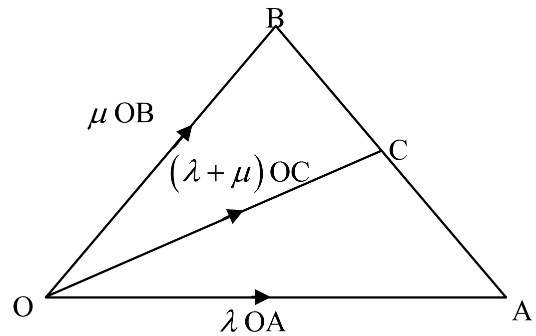



If three forces acting at a point be in equilibrium, each force is proportional to the sine of the angle between the other two. Thus if the forces are $P, Q$ and $R$; $\alpha, \beta, \gamma$ be the angle between $Q$ and $R$, $R$ and $P$, $P$ and $Q$ respectively, also the forces are in equilibrium, we have,

$$\frac{P}{\sin\alpha}=\frac{Q}{\sin\beta}=\frac{R}{\sin\gamma}.$$

The converse of this theorem is also true.

**$\lambda-\mu$ THEOREM**




The resultant of two forces, acting at a point $O$ along $OA$ and $OB$ and represented in magnitude $\lambda$ . $OA$ and $\mu$ . $OB$, is represented by a force $(\lambda+\mu).OC$, where $C$ is a point on $AB$ such that $\lambda\, CA=\mu CB$ *i.e.* $C$ divides AB in the ratio $\mu:\lambda$. In vector notation the above statement can be written as : $\lambda.\overrightarrow{OA}+\mu.\overrightarrow{OB}=(\lambda+\mu).\overrightarrow{OC}$, where C is a point on AB dividing it in the ratio $\mu:\lambda$.

**Note :**

In the above theorem, if $\lambda=\mu=1$, then $\overrightarrow{OA}+\overrightarrow{OB}=2\overrightarrow{OC}$, where C is the mid point of AB ,*i.e.* the resultant of two forces $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is $2\overrightarrow{OC}$, where C is the mid point of AB.

**EQUILIBRIUM OF FORCES**

A system of forces acting on a body is said to be in equilibrium if it produces no change in the motion of the body i.e.

(i) Vector sum of all forces is equal to zero and

(ii) Vector sum of all the moments of these forces about any point is zero.

**EQUILIBRIUM OF TWO FORCES**

Two forces acting at a point are in equilibrium if and only if they,

(i) are equal in magnitude

(ii) act along the same line

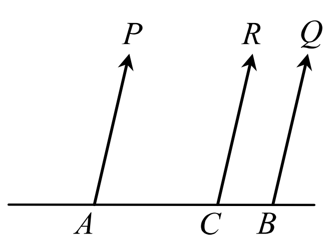
(iii) have opposite directions.

**CONDITION OF EQUILIBRIUM OF A NUMBER OF COPLANAR CONCURRENT FORCES**

A given number of forces acting at a point are in equilibrium if and only if the algebraic sum of their resolved parts in each of the two perpendicular directions OX and OY vanish separately.

**PARALLEL FORCES**

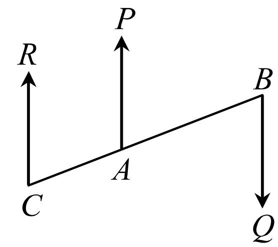
1. **Like parallel forces :** Two parallel forces are said to be like parallel forces when they act in the same direction.




The resultant $R$ of two like parallel forces $P$ and $Q$ is equal in magnitude of the sum of the magnitudes of forces and $R$ acts in the same direction as the forces $P$ and $Q$ and at the point on the line segment joining the point of action $P$ and $Q$, which divides it in the ratio $Q : P$ internally.

2. **Two unlike parallel forces**

Two parallel forces are said to be unlike if they act in opposite directions.

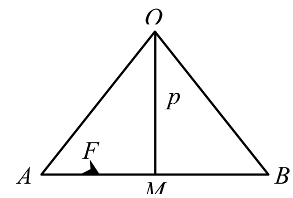



If $P$ and $Q$ be two unlike parallel forces acting at $A$ and $B$ and $P$ is greater in magnitude than $Q$. Then their resultant $R$ acts in the same direction as $P$ and acts at a point $C$ on $BA$ produced. Such that $R = P - Q$ and $P.CA = Q.CB$

Then in this case $C$ divides $BA$ externally in the inverse ratio of the forces,

$$\frac{P}{CB} = \frac{Q}{CA} = \frac{P-Q}{CB-CA} = \frac{R}{AB}.$$

**MOMENT**




The moment of a force about a point $O$ is given in magnitude by the product of the forces and the perpendicular distance of $O$ from the line of action of the force. If $F$ be a force acting at point $A$ of a rigid body along the line $AB$ and $OM\left(=P\right)$ be the perpendicular distance of the fixed point $O$ from $AB$, then the moment of force about $O$.

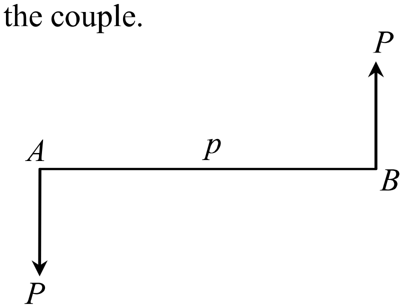
$$= F.p = AB \times OM = 2\left[\frac{1}{2}\left(AB \times OM\right)\right] = 2\left(\text{Area of } \Delta\, AOB\right)$$

The S.I. unit of moment is Newton-meter *(N-m)*.

**COUPLES**

Two equal unlike parallel forces which do not have the same line of action, are said to form a couple.

1. **Arm of the couple :** The perpendicular distance between the lines of action of the forces forming the couple is known as the arm of the couple.

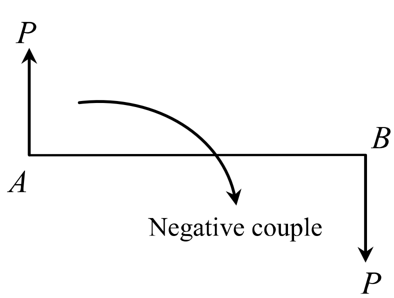


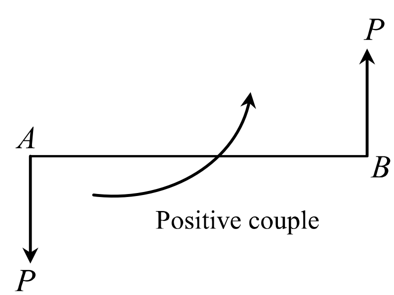

2. **Moment of couple**

The moment of a couple is obtained in magnitude by multiplying the magnitude of one of the forces forming the couple and perpendicular distance between the lines of action of the force. The perpendicular distance between the forces is called the arm of the couple. The moment of the couple is regarded as positive or negative according as it has a tendency to turn the body in the anticlockwise or clockwise direction.

Moment of a couple $=$ Force $\times$ Arm of the couple $= P.p$

3. **Sign of the moment of a couple :** The moment of a couple is taken with positive or negative sign according as it has a tendency to turn the body in the anticlockwise or clockwise direction.

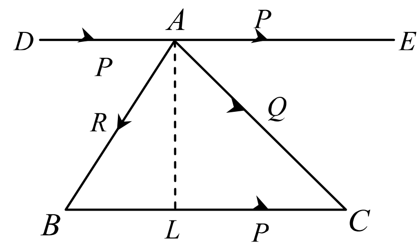


**Note :** A couple can not be balanced by a single force, but can be balanced by a couple of opposite sign.

**TRIANGLE THEOREM OF COUPLES**

If three forces acting on a body be represented in magnitude, direction and line of action by the sides of triangle taken in order, then they are equivalent to a couple whose moment is represented by twice the area of triangle.



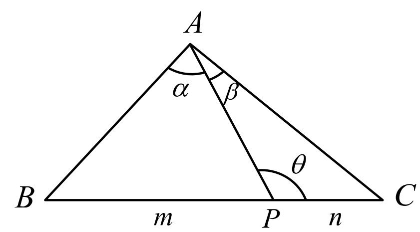

Consider the force $P$ along $AE, Q$ along $CA$ and $R$ along $AB$. These forces are three concurrent forces acting at $A$ and represented in magnitude and direction by the sides $BC$, $CA$ and $AB$ of $\Delta ABC$. So, by the triangle law of forces, they are in equilibrium.

The remaining two forces $P$ along $AD$ and $P$ along $BC$ form a couple, whose moment is, $m = P.AL = BC.AL$

Since, $\frac{1}{2}(BC.AL) =$ Area of the $\Delta ABC$

$\therefore$ Moment $= BC.AL = 2(\text{Area of } \Delta ABC)$.

**EQUILIBRIUM OF COPLANAR FORCES**




1. If three forces keep a body in equilibrium, they must be coplanar.
2. If three forces acting in one plane upon a rigid body keep it in equilibrium, they must either meet in a point or be parallel.
3. When more than three forces acting on a rigid body, keep it in equilibrium, then it is not necessary that they meet at a point. A system of coplanar forces acting upon a rigid body will be in equilibrium if the algebraic sum of their resolved parts in any two mutually perpendicular directions vanish separately, and if the algebraic sum of their moments about any point in their plane is zero.
   i.e., $X = 0, Y = 0, G = 0$ or $R = 0, G = 0$.
4. A system of coplanar forces acting upon a rigid body will be in equilibrium if the algebraic sum of the moments of the forces about each of three non-collinear points is zero.
5. Trigonometrical theorem : If $P$ is any point on the base $BC$ of $\Delta ABC$ such that $BP : CP = m : n$.
   Then, (i) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$,
   where $\angle BAP = \alpha$, $\angle CAP = \beta$.
   (ii) $(n+m)\cot\theta = n\cot B - m\cot C$.

**VARIGNON'S THEOREM OF MOMENTS**

The algebraic sum of moments of any number of coplanar forces about any point in their plane is equal to the moment of their resultant about the same point.

**FRICTION AND FORCE OF FRICTION**

The property by virtue of which a resisting force is created between two rough bodies which prevents the sliding of one body over the other is called the friction and this force which always acts in the direction opposite to that in which the body has a tendency to slide or move is called forces of friction.

**LIMITING FRICTION**

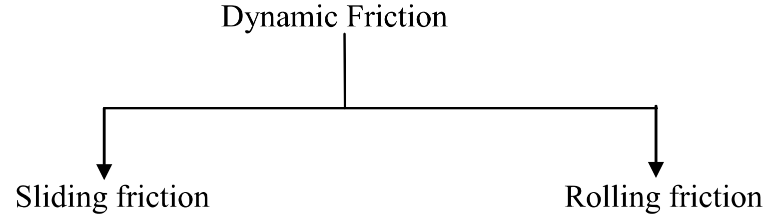
When one body is just on the point of sliding on another body, the force of friction called into play attains its maximum value and is called limiting friction and the equilibrium then is said to be limiting equilibrium.

**STATIC FRICTION**

When a body in contact with another body is in any position of equilibrium but not limiting equilibrium , then the friction exerted is called static friction. Thus, **static friction is less than the limiting friction.**

**DYNAMIC FRICTION**

When motion ensues by one body sliding on the other, the friction exerted between the bodies is called dynamic friction.




**SLIDING FRICTION**

If the body is sliding, the force of friction that comes into play is called sliding friction.

**ROLLING FRICTION**

If the body is rolling , the force of friction that comes into play is called rolling friction.

**LAWS OF FRICTION**

The following laws govern the different kinds of friction *i.e.* static, limiting and dynamic friction.

**LAWS OF STATIC FRICTION**

1. The direction of friction is opposite to the direction in which the body tends to move.
2. The magnitude of the force of friction is just sufficient to prevent the body from moving.

**LAWS OF LIMITING FRICTION**

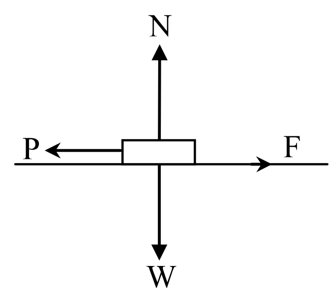
1. Limiting friction is equal in magnitude and opposite in direction to the force which tends to produce motion.
2. The magnitude of limiting friction at the point of contact between two bodies bears a constant ratio to the normal reaction at the point.
3. The constant ratio depends entirely on the nature of the material of which the surfaces in contact are composed of and is independent of their extent and shape.

**LAWS OF DYNAMIC FRICTION**

1. The direction of dynamic friction is opposite to that in which the body is moving.
2. The magnitude of dynamic friction bears a constant ratio to the normal reaction on the body but this ratio is slightly less than the coefficient of friction in the case of limiting friction.
3. The dynamic friction is independent of the velocity of motion.

**COEFFICIENT OF FRICTION**

When a rough body is on the verge of sliding on another, the friction exerted bears a constant ratio to the normal reaction. This ratio of the limiting friction to the normal reaction is called the coefficient of friction. It is usually denoted by $\mu$ .



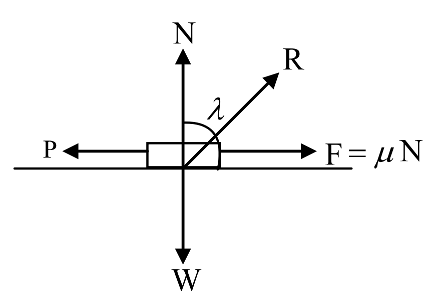

If F be the limiting friction and N, the normal reaction between the two bodies, then for the equilibrium to be limiting , we have

$$\frac{F}{N} = \mu \text{ or } = F = \mu N.$$

As friction is maximum when the equilibrium is limiting , $\mu N$ is the maximum value of friction.

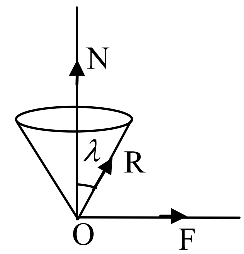
**ANGLE OF FRICTION**

When one body, placed on another body is in limiting equilibrium, the friction exerted is the limiting friction. In this case , the angle which the resultant of the force of friction and the normal reaction makes with the normal reaction at the point of contact is called the angle of friction and is usually denoted by $\lambda$.




Now, N and $\mu N\left(= F\right)$ being the resolved parts of R, we have $R\cos\lambda = N$ and $R\sin\lambda = \mu N \Rightarrow \tan\lambda = \mu$ Hence, the coefficient of friction is equal to the tangent of the angle of friction.

**CONE OF FRICTION**




The right cone described with its vertex at the point of contact of two rough bodies and having the common normal at the point of contact as axis and the angle of friction as the semi–vertical angle , is called the cone of fraction.

**LEAST FORCE ON HORIZONTAL PLANE**

The least force required to pull a body of weight W on the rough horizontal plane is W $\sin\lambda$.

**LEAST FORCE ON INCLINED PLANE**

Let $\alpha$ be the inclination of rough inclined plane to the horizontal and $\lambda$, the angle of friction.

1. If $\alpha = \lambda$, then the body is in limiting equilibrium and is just on the point of moving downwards.
2. If $\alpha < \lambda$, then the least force required to pull a body of weight W down the plane is $W \sin(\lambda - \alpha)$.
3. If $\alpha > \lambda$, then the body cannot rest on the plane under its own weight and reaction of the plane. So, the question of finding the least force does not arise.

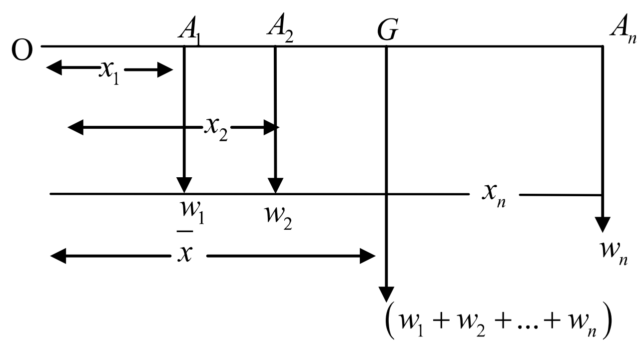
**Note :** The least force required to pull a body of weight W up an inclined rough plane is W $\sin(\alpha + \lambda)$.

**CENTRE OF GRAVITY**

The centre of gravity of a body or system of particles rigidly connected together, is that point through which the line of action of the weight of the body always passes.

**CENTRE OF GRAVITY OF A NUMBER OF PARTICLES ARRANGED IN A STRAIGHT LINE**

If $n$ particles of weights $w_1, w_2, w_3, ..., w_n$ be placed at points $A_1, A_2, A_3, ..., A_n$ on the straight line $OA_n$ such that the distance of these points from $O$ are $x_1, x_2, ..., x_n$ respectively, Then, the distance of their centre of gravity $G$ (say) from $O$ is given by

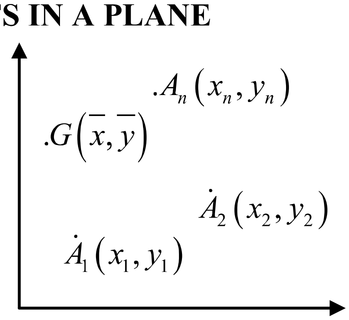



$$OG = \bar{x} = \frac{\sum w_i x_i}{\sum w_i}.$$

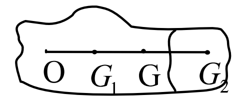
**CENTRE OF GRAVITY OF A NUMBER OF WEIGHTS PLACED AT POINTS IN A PLANE**

If $w_1, w_2, ..., w_n$ be the weights of the particles placed at the points $A_1(x_1, y_1), A_2(x_2, y_2) ..., A_n(x_n, y_n)$ respectively, then the centre of gravity $G(\bar{x}, \bar{y})$ of these particles is given by

$$\bar{x} = \frac{\sum w_i x_i}{\sum w_i}, \bar{y} = \frac{\sum w_i y_i}{\sum w_i}.$$



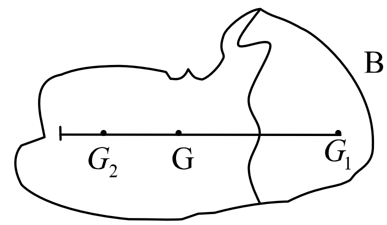
**CENTRE OF GRAVITY OF COMPOUND BODY**




Let $G_1, G_2$, be the centres of gravity of the two parts of a body and let $w_1, w_2$ be their weights. Let G be the centre of gravity of the whole body .Then at G, acts the whole weight $(w_1 + w_2)$ of the body. Join $G_1G_2$; then G must lie on $G_1G_2$.

Let O be any fixed point on $G_1G_2$. Let $OG_1 = x_1$, $OG_2 = x_2$ and $OG = \bar{x}$.Taking moments about O, we have $(w_1 + w_2)\bar{x} = w_1x_1 + w_2x_2 \quad \therefore \bar{x} = \dfrac{w_1x_1 + w_2x_2}{w_1 + w_2}$

**CENTRE OF GRAVITY OF THE RAMAINDER**




Let w be the weight of the whole body .Let a part B of the body of weight $w_1$ be removed so that a part A of weight $w - w_1$ is left behind .Let G be the centre of gravity of whole body and $G_1$, the C.G. of portion B which is removed. Let $G_2$ be the C .G. of the remaining portion A. Let O be a point on $G_1G_2$ and let it be regarded as origin. Let $OG_1 = x_1, OG = x, OG_2 = x_2$ Taking moments about O, $(w - w_1)x_2 + w_1x_1 = wx \therefore x_2 = \dfrac{wx - w_1x_1}{w - w_1}$.

**POSITION OF CENTRE OF GRAVITY IN SOME SPECIAL CASES:**

1. **UNIFORM ROD :** At its mid point.
2. **PARALLELOGRAM, RECTANGLE OR SQUARE :** At the intersection of the diagonals.
3. **TRIANGULAR LAMINA :** At the centre.
4. **CIRCULAR ARC**

   At a distance $\dfrac{a \sin\alpha}{\alpha}$ form the centre on the symmetrical radius. Where a = radius and $2\alpha$ = angle subtended by the arc C at the centre.
5. **SECTOR OF A CIRCLE**

   At a distance $\dfrac{2a}{3} \cdot \dfrac{\sin\alpha}{\alpha}$ form the centre on the symmetrical radius. Where a = radius and $2\alpha$ = angle subtended at th centre.
6. **SEMI-CIRCULAR ARC**

   At a distance $\dfrac{4a}{3\pi}$ from the centre on the symmetrical radius, where $\alpha$ is the radius.
7. **HEMISPHERE**

   At a distance $\dfrac{3a}{8}$ from the centre on the symmetrical radius, where $\alpha$ is the radius.
8. **HEMISPHERICAL SHELL**

   At a distance $\dfrac{a}{2}$ from the centre on the symmetrical radius, where $\alpha$ is the radius.
9. **SOLID CONE**

   At a distance $\dfrac{h}{4}$ from the base on the axis, where $h$ is the height of the cone.
10. **CONICAL SHELL**

    At a distance $\dfrac{h}{3}$ from the base on the axis, where $h$ is the height of the cone.