23 Three Dimensional Geometry

QUICK LOOK

The planes *XOY, YOZ* and *ZOX* are known as *xy*-plane, *yz*-plane and *zx*-plane respectively. Let *P* be a point in space and distances of *P* from *yz*, *zx* and *xy*-planes be *x, y, z* respectively (with proper signs), then we say that co-ordinates of *P* are (*x*,*y*,*z*). Also $OA = x, OB = y, OC = z$. The three co-ordinate planes (*XOY*, *YOZ* and *ZOX*) divide space into eight parts and these parts are called octants.

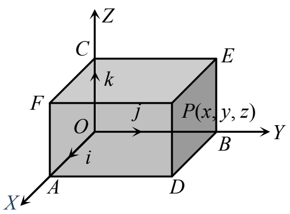


Figure: 23.1

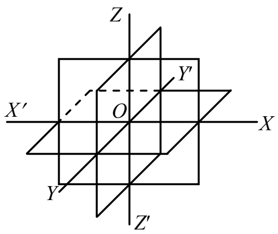


Figure: 23.2

The following table shows the signs of co-ordinates of points in various octants:

Table :23.1

Octant co-ordinate	*OXYZ*	*OX′YZ*	*OXY′Z*	*OX′Y′Z*
x	+	–	+	–
y	+	+	–	–
z	+	+	+	+
Octant co-ordinate	*OXYZ′*	*OX′YZ′*	*OXY′Z′*	*OX′Y′Z′*
x	+	–	+	–
y	+	+	–	–
z	–	–	–	–

Other Methods of Defining the Position of any Point *P* in Space

- **Cylindrical co-ordinates**

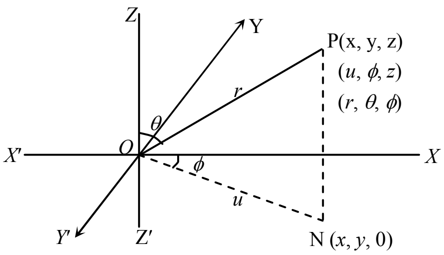


Figure: 23.3

- If the rectangular cartesian co-ordinates of *P* are (x, y, z), then those of *N* are $(x, y, 0)$ and we can easily have the following relations : $x = u\cos\phi$, $y = u\sin\phi$ and $z = z$.

  Hence, $u^2 = x^2 + y^2$ and $\phi = \tan^{-1}(y/x)$.

  Cylindrical co-ordinates of $P \equiv (u, \phi, u)$

- **Spherical polar co-ordinates:** The measures of quantities r, θ, ϕ are known as spherical or three dimensional polar co-ordinates of the point *P*. If the rectangular cartesian co-ordinates of *P* are (x, y, z) then $z = r\cos\theta$, $u = r\sin\theta$

$\therefore$ $x = u\cos\phi = r\sin\theta\cos\phi$, $y = u\sin\phi = r\sin\theta\sin\phi$ and $z = r\cos\theta$

Also $r^2 = x^2 + y^2 + z^2$ and $\tan\theta = \frac{u}{z} = \frac{\sqrt{x^2+y^2}}{z}$; $\tan\phi = \frac{y}{x}$

Note

- The co-ordinates of a point on *xy*-plane is $(x, y, 0)$, on *yz*-plane is $(0, y, z)$ and on *zx*-plane is $(x, 0, z)$
- The co-ordinates of a point on *x*-axis is $(x, 0, 0)$, on *y*-axis is $(0, y, 0)$ and on *z*-axis is $(0, 0, z)$
- **Position vector of a point:** Let $i, j, \hat{k}$ be unit vectors along *OX*, *OY* and *OZ* respectively. Then position vector of a point $P(x, y, z)$ is $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$.

Distance Formula

- **Distance formula:** The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

  $$AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$$

- **Distance from origin:** Let *O* be the origin and $P(x,y,z)$ be any point, then $OP = \sqrt{(x^2 + y^2 + z^2)}$.

Distance of a point from co-ordinate axes

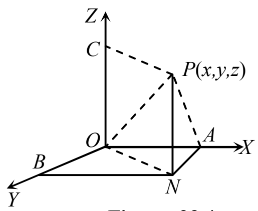


Figure: 23.4

Let $P(x,y,z)$ be any point in the space. Let *PA*, *PB* and *PC* be the perpendiculars drawn from *P* to the axes *OX*, *OY* and *OZ* respectively. Then, $PA = \sqrt{(y^2 + z^2)}$

$$PB = \sqrt{(z^2 + z^2)}; PC = \sqrt{(x^2 + y^2)}$$

Section Formulas

- **Section formula for internal division**

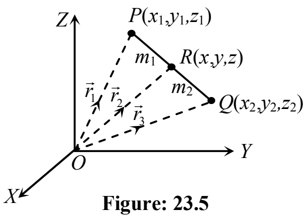


Figure: 23.5

Let $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ be two points. Let R be a point on the line segment joining P and Q such that it divides the join of P and Q internally in the ratio $m_1:m_2$. Then the co-ordinates of R are $\left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}, \frac{m_1z_2+m_2z_1}{m_1+m_2}\right)$

- **Section formula for external division**

Let $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ be two points, and let R be a point on PQ produced, dividing it externally in the ratio $m_1 : m_2$ $(m_1 \neq m_2)$. Then the co-ordinates of R are

$$\left(\frac{m_1x_2-m_2x_1}{m_1-m_2}, \frac{m_1y_2-m_2y_1}{m_1-m_2}, \frac{m_1z_2-m_2z_1}{m_1-m_2}\right).$$

Triangle

Co-ordinates of the centroid

- If $(x_1,y_1,z_1),(x_2,y_2,z_2)$ and (x_3,y_3,z_3) are the vertices of a triangle, then co-ordinates of its centroid are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$.
- If (x_r,y_r,z_r) ; r = 1, 2, 3, 4, are vertices of a tetrahedron, then co-ordinates of its centroid are $\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$.
- If G (α, β, γ) is the centroid of ΔABC, where A is (x_1,y_1,z_1), B is (x_2,y_2,z_2), then C is $(3\alpha - x_1 - x_2, 3\beta - y_1 - y_2,.$ $3\gamma - z_1 - z_2)$

Area of triangle

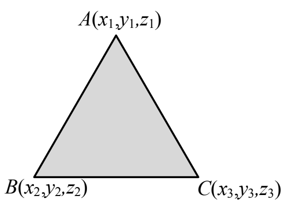


Figure: 23.6

- Let $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$ and $C(x_3,y_3,z_3)$ be the vertices of a triangle, then $\Delta_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}$, $\Delta_y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}$,

$$\Delta_z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Now, area of ΔABC is given by the relation $\Delta = \sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}$.

Also, $\Delta = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{matrix}\right\|$

- **Condition of collinearity:** Points $A(x_1,y_1,z_1), B(x_2,y_2,z_2)$, $C(x_3,y_3,z_3)$ are collinear; If $\frac{x_1-x_2}{x_2-x_3} = \frac{y_1-y_2}{y_2-y_3} = \frac{z_1-z_2}{z_2-z_3}$

Volume of Tetrahedron

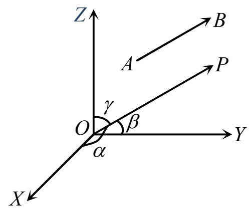


Figure: 23.7

Volume of tetrahedron with vertices (x_r, y_r, z_r); r = 1, 2, 3, 4, is $V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$

Direction cosines and Direction ratio

Direction cosines: The cosines of the angle made by a line in anticlockwise direction with positive direction of co-ordinate axes are called the direction cosines of that line. If α, β, γ be the angles which a given directed line makes with the positive direction of the x, y, z co-ordinate axes respectively, then cosα, cosβ, cosγ are called the direction cosines of the given line and are generally denoted by l, m, n respectively.

Thus, $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.

By definition, it follows that the direction cosine of the axis of x are respectively $\cos 0°, \cos 90°, \cos 90°$ *i.e.* (1, 0, 0). Similarly direction cosines of the axes of y and z are respectively (0, 1, 0) and (0, 0, 1).

- **Relation between the direction cosines**

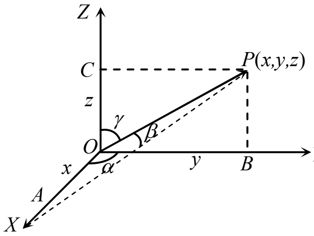


Figure: 23.8

Let OP be any line through the origin O which has direction cosines l, m, n. Let $P = (x, y, z)$ and $OP = r$.

Then $OP^2 = x^2 + y^2 + z^2 = r^2$ . . . (i)

From P draw PA, PB, PC perpendicular on the co-ordinate axes, so that $OA = x$, $OB = y$, $OC = z$.

Also, $\angle POA = \alpha$, $\angle POB = \beta$ and $\angle POC = \gamma$.

From triangle AOP, $l = \cos\alpha = \frac{x}{r} \Rightarrow x = lr$

Similarly $y = mr$ and $z = nr$.

Hence from (i), $r^2(l^2 + m^2 + n^2) = x^2 + y^2 + z^2 = r^2$

$\Rightarrow \quad l^2 + m^2 + n^2 = 1$

or, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

or, $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Note

- If $OP = r$ and the co-ordinates of point P be (x, y, z) then d.c.'s of line OP are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$.
- Direction cosines of $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ are $\frac{a}{|\vec{r}|}, \frac{b}{|\vec{r}|}, \frac{c}{|\vec{r}|}$.
- Since $-1 \le \cos x \le 1, \forall x \in R$, hence values of l, m, n are such real numbers which are not less than –1and not greater than 1. Hence d.c.'s $\in [-1, 1]$.
- The direction cosines of a line parallel to any co-ordinate axis are equal to the direction cosines of the co-ordinate axis.
- The number of lines which are equally inclined to the co-ordinate axes is 4.
- If l, m, n are the d.c.'s of a line, then the maximum value of $lmn = \frac{1}{3\sqrt{3}}$.
- The angles α,β,γ are called the direction angles of line AB.
- The d.c.'s of line BA are $\cos(\pi - \alpha), \cos(\pi - \beta)$, and $\cos(\pi - \gamma)$ i.e., $-\cos\alpha, -\cos\beta, -\cos\gamma$
- Angles α,β,γ are not coplanar.
- $\alpha + \beta + \gamma$ is not equal to $360°$ as these angles do not lie in same plane.

- If $P(x, y, z)$ be a point in space such that $r = OP$ has d.c.'s l, m, n then $x = l|\vec{r}|$, $y = m|\vec{r}|, z = n|\vec{r}|$.
- Projection of a vector $\vec{r}$ on the co-ordinate axes are $l|\vec{r}|, m|\vec{r}|, n|\vec{r}|$.
- $\vec{r} = |\vec{r}|(l\hat{i} + m\hat{j} + n\hat{k})$ and $\vec{r} = l\hat{i} + m\hat{j} + n\hat{k}$

Direction ratio: Three numbers which are proportional to the direction cosines of a line are called the direction ratio of that line. If a, b, c are three numbers proportional to direction cosines l, m, n of a line, then a, b, c are called its direction ratios. They are also called direction numbers or direction components.

Hence by definition, we have $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$ (say)

$\Rightarrow \quad l = ak, m = bk, n = ck$

$\Rightarrow \quad l^2 + m^2 + n^2 = (a^2 + b^2 + c^2) = k^2$

$\Rightarrow \quad k = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$

$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$

where the sign should be taken all positive or all negative.

Note

Direction ratios are not uniques, whereas d.c.'s are unique. *i.e.*, $a^2 + b^2 + c^2 \ne 1$

- Let $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ be a vector. Then its d.r.'s are a, b, c If a vector **r** has d.r.'s a, b, c then $\vec{r} = \frac{|\vec{r}|}{\sqrt{a^2 + b^2 + c^2}}(a\hat{i} + b\hat{j} + c\hat{k})$

D.c.'s and d.r.'s of a line joining two points

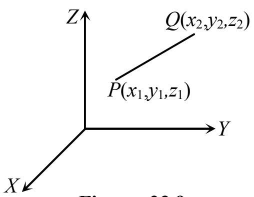


Figure: 23.9

The direction ratios of line PQ joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1 = a, y_2 - y_1 = b$ and $z_2 - z_1 = c$ (say). Then direction cosines are,

$l = \frac{(x_2 - x_1)}{\sqrt{\Sigma(x_2 - x_1)^2}}, m = \frac{(y_2 - y_1)}{\sqrt{\Sigma(x_2 - x_1)^2}}, n = \frac{(z_2 - z_1)}{\sqrt{\Sigma(x_2 - x_1)^2}}$

i.e., $l = \frac{x_2 - x_1}{PQ}, m = \frac{y_2 - y_1}{PQ}, n = \frac{z_2 - z_1}{PQ}$.

Projection

- **Projection of a point on a line**

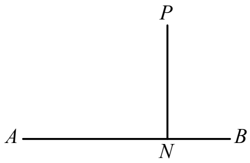


Figure: 23.10

The projection of a point P on a line AB is the foot N of the perpendicular PN from P on the line AB.N is also the same point where the line AB meets the plane through P and perpendicular to AB.

- **Projection of a segment of a line on another line and its length**

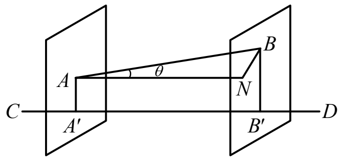


Figure: 23.11

The projection of the segment AB of a given line on another line CD is the segment $A'B'$ of CD where A' and B' are the projections of the points A and B on the line CD. The length of the projection $A'B'$. $A'B' = AN = AB \cos\theta$

Projection of a line joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line whose direction cosines are l, m and n :

Let PQ be a line segment where $P \equiv (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ and AB be a given line with d.c.'s as l, m, n. If the line segment PQ makes angle θ with the line AB, then

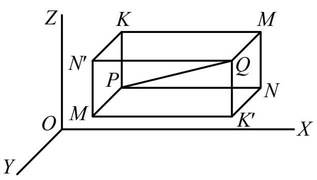


Figure: 23.12

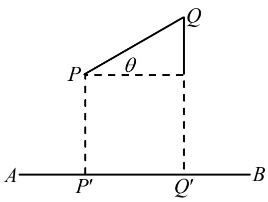


Figure: 23.13

Projection of PQ is $P'Q' = PQ\cos\theta$

$$= (x_2 - x_1)\cos\alpha + (y_2 - y_1)\cos\beta + (z_2 - z_1)\cos\gamma$$

$$= (x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$$

Note

- For x-axis, $l = 1$, $m = 0$, $n = 0$

  Hence, projection of PQ on x-axis $= x_2 - x_1$, Projection of PQ on y-axis $= y_2 - y_1$ and Projection of PQ on z-axis $= z_2 - z_1$

- If P is a point (x_1, y_1, z_1) then projection of OP on a line whose direction cosines are l, m, n is $l_1x_1 + m_1y_1 + n_1z_1$, where O is the origin.
- If l_1, m_1, n_1 and l_2, m_2, n_2 are the d.c.'s of two concurrent lines, then the d.c.'s of the lines bisecting the angles between them are proportional to $l_1 \pm l_2, m_1 \pm m_2, n_1 \pm n_2$.

Angle between Two Lines

Cartesian form

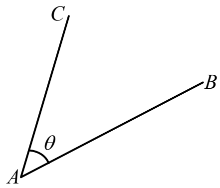


Figure: 23.14

Let θ be the angle between two straight lines AB and AC whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 respectively, is given by $\cos\theta = l_1l_2 + m_1m_2 = n_1n_2$, If direction ratios of two lines a_1, b_1, c_1 and a_2, b_2, c_2 are given, then angle between two lines is given by $\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}.\sqrt{a_2^2 + b_2^2 + c_2^2}}$.

- **Condition of perpendicularity :** If the given lines are perpendicular, then $\theta = 90°$ *i.e.* $\cos\theta = 0$

$\Rightarrow$ $l_1l_2 + m_1m_2 + n_1n_2 = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$

- **Condition of parallelism :** If the given lines are parallel, then $\theta = 0°$ *i.e.* $\sin\theta = 0$

$\Rightarrow$ $(l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1) = 0$, which is true, only when $l_1m_2 - l_2m_1 = 0$, $m_1n_2 - m_2n_1 = 0$ and $n_1l_2 - n_2l_1 = 0$

$\Rightarrow$ $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$.

Similarly, $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

The Straight Line

Straight line in Space: Every equation of the first degree represents a plane. Two equations of the first degree are satisfied by the co-ordinates of every point on the line of intersection of the planes represented by them. Therefore, the two equations together represent that line. Therefore $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ together represent a straight line.

Equation of a Line Passing Through a Given Point

- **Cartesian form or symmetrical form**

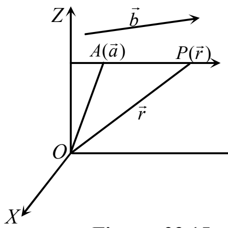


Figure: 23.15

Cartesian equation of a straight line passing through a fixed point (x_1, y_1, z_1) and having direction ratios a, b, c is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}.$$

- **Vector form :** Vector equation of a straight line passing through a fixed point with position vector a and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$.

Equation of Line Passing Through Two Given Points.

- **Cartesian form :** If $A(x_1, y_1, z_1), B(x_2, y_2, z_2)$ be two given points, the equations to the line AB are

  $$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

  The co-ordinates of a variable point on AB can be expressed in terms of a parameter λ in the form

  $$x = \frac{\lambda x_2 + x_1}{\lambda+1}, y = \frac{\lambda y_2 + y_1}{\lambda+1}, z = \frac{\lambda z_2 + z_1}{\lambda+1}$$

  λ being any real number different from –1. In fact, (x, y, z) are the co-ordinates of the point which divides the join of A and B in the ratio $\lambda : 1$.
- **Vector form :** The vector equation of a line passing through two points with position vectors $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

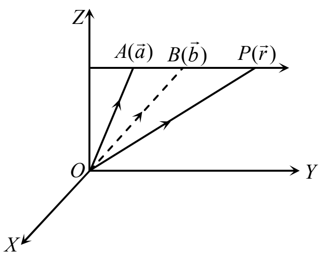


Figure: 23.16

Changing Unsymmetrical form to Symmetrical form: The unsymmetrical form of a line $ax + by + cz + d$

$= 0, a'x + b'y + c'z + d' = 0$

Can be changed to symmetrical form as follows:

$$\frac{x - \frac{bd'-b'd}{ab'-a'b}}{bc'-b'c} = \frac{y - \frac{da'-d'a}{ab'-a'b}}{ca'-c'a} = \frac{z}{ab'-a'b}$$

Angle between Two lines: Let the cartesian equations of the two lines be $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ . . .(*i*)

and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ . . .(*ii*)

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

- **Condition of perpendicularity:** If the lines are perpendicular, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$
- **Condition of parallelism :** If the lines are parallel, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Reduction of Cartesian form of the Equation of a line to Vector form and Vice Versa.

- **Cartesian to vector:** Let the Cartesian equation of a line be

  $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ . . .(*i*)

  This is the equation of a line passing through the point $A(x_1, y_1, z_1)$ and having direction ratios a, b, c. In vector form this means that the line passes through point having position vector $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and is parallel to the vector $\vec{m} = a\hat{i} + b\hat{j} + c\hat{k}$. Thus, the vector form of (*i*) is $\vec{r} = \vec{a} + \lambda\vec{m}$ or $\vec{r} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$, where λ is a parameter.
- **Vector to cartesian:** Let the vector equation of a line be

  $\vec{r} = \vec{a} + \lambda\vec{m}$ . . .(*ii*)

  Where $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, m = a\hat{i} + b\hat{j} + c\hat{k}$ and λ is a parameter.

  To reduce (*ii*) to Cartesian form we put $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and equate the coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$ as discussed below.

  Putting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, a = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and

  $\vec{m} = a\hat{i} + b\hat{j} + c\hat{k}$ in (*ii*), we obtain

  $x\hat{i} + y\hat{j} + z\hat{k} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + bj + c\hat{k})$

  Equating coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$, we get

  $x = x_1 + a\lambda, y = y_1 + b\lambda, z = z_1 + c\lambda$

or $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = \lambda$

Intersection of Two lines: Determine whether two lines intersect or not. In case they intersect, the following algorithm is used to find their point of intersection.

- **Algorithm for cartesian form:** Let the two lines be

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \quad \ldots(i)$$

and $$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2} \quad \ldots(ii)$$

Step (i): Write the co-ordinates of general points on (*i*) and (*ii*). The co-ordinates of general points on (*i*) and (*ii*) are given by $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}=\lambda$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}=\mu$ respectively. *i.e.*, $(a_1\lambda+x_1, b_1\lambda+y_1+c_1\lambda+z_1)$ and $(a_2\mu+x_2, b_2\mu+y_2, c_2\mu+z_2)$

Step (ii): If the lines (*i*) and (*ii*) intersect, then they have a common point. $a_1\lambda+x_1=a_2\mu+x_2, b_1\lambda+y_1=b_2\mu+y_2$ and $c_1\lambda+z_1=c_2\mu+z_2$.

Step (iii): Solve any two of the equations in λ and μ obtained in step (ii). If the values of λ and μ satisfy the third equation, then the lines (*i*) and (*ii*) intersect, otherwise they do not intersect.

Step (iv): To obtain the co-ordinates of the point of intersection, substitute the value of λ (or μ) in the co-ordinates of general point (*s*) obtained in step (*i*).

Foot of perpendicular from a point $A(\alpha, \beta, \gamma)$ to the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$.

Cartesian form

- **Foot of perpendicular from a point $A(\alpha, \beta, \gamma)$ to the line** $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

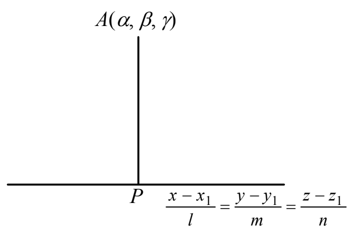


Figure: 23.17

If P be the foot of perpendicular, then P is $(lr+x_1, mr+y_1, nr+z_1)$. Find the direction ratios of AP and apply the condition of perpendicularity of AP and the given line. This will give the value of r and hence the point P which is foot of perpendicular.

- **Length and equation of perpendicular:** The length of the perpendicular is the distance AP and its equation is the line joining two known points A and P.

Note

The length of the perpendicular is the perpendicular distance of given point from that line.

- **Reflection or image of a point in a straight line**

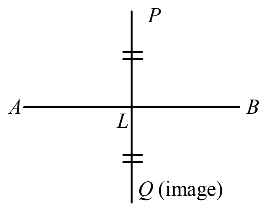


Figure: 23.18

If the perpendicular PL from point P on the given line be produced to Q such that $PL = QL$, then Q is known as the image or reflection of P in the given line. Also, L is the foot of the perpendicular or the projection of P on the line.

Vector Form

- **Perpendicular distance of a point from a line**

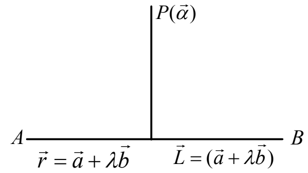


Figure: 23.19

Let L is the foot of perpendicular drawn from $P(\vec{\alpha})$ on the line $\vec{r}=\vec{a}+\lambda\vec{b}$. Since $\vec{r}$ denotes the position vector of any point on the line $\vec{r}=\vec{a}+\lambda\vec{b}$. So, let the position vector of L be $\vec{a}+\lambda\vec{b}$.

Then $\overrightarrow{PL}=\vec{a}-\vec{\alpha}+\lambda\vec{b}=(\vec{a}-\vec{\alpha})-\left(\frac{(\vec{a}-\vec{\alpha})\vec{b}}{|\vec{b}|^2}\right)\vec{b}$

The length PL, is the magnitude of $\overrightarrow{PL}$, and required length of perpendicular.

- **Image of a point in a straight line**

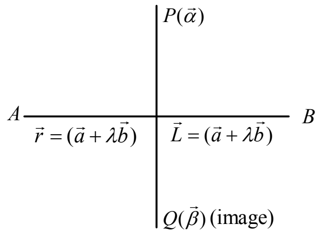


Figure: 23.20

- Let $Q(\vec{\beta})$ is the image of P in $\vec{r}=\vec{a}+\lambda\vec{b}$ Then, $\vec{\beta}=2\vec{a}-\left(\frac{2(\vec{a}-\vec{\alpha})\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}\cdot\vec{\alpha}$

Shortest Distance between two Straight Lines

- **Skew lines :** Two straight lines in space which are neither parallel nor intersecting are called skew lines. Thus, the skew lines are those lines which do not lie in the same plane.

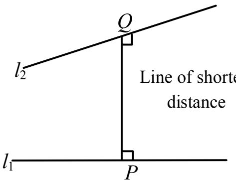


Figure: 23.21

- **Line of shortest distance :** If l_1 and l_2 are two skew lines, then the straight line which is perpendicular to each of these two non-intersecting lines is called the "line of shortest distance."

Note

There is one and only one line perpendicular to each of lines l_1 and l_2.

- **Shortest distance between two skew lines**

  Cartesian form: Let two skew lines be $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$

  Therefore, the shortest distance between the lines is given by

$$d=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-l_1n_2)^2+(l_1m_2-l_2m_1)^2}}$$

Vector form : Let l_1 and l_2 be two lines whose equations are $\vec{l_1}:\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{l_2}:\vec{r}=\vec{a}_2+\mu\vec{b}_2$ respectively. Then, Shortest distance $PQ=\left|\frac{(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|=\frac{|[\vec{b}_1\vec{b}_2(\vec{a}_2-\vec{a}_2)]|}{|\vec{b}_1\times\vec{b}_2|}$

- **Shortest distance between two parallel lines:** The shortest distance between the parallel lines $\vec{r}=\vec{a}_1+\lambda\vec{b}$ and $\vec{r}=\vec{a}_2+\mu\vec{b}$ is given by $d=\frac{|(\vec{a}_2-\vec{a}_2)\times\vec{b}|}{|\vec{b}|}$.

- **Condition for two lines to be intersecting *i.e.* coplanar**

  Cartesian form : If the lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ intersect, then

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

Vector form : If the lines $r=a_1+\lambda b_1$ and $r=a_2+\lambda b_2$ intersect, then the shortest distance between them is zero. Therefore, $[\vec{b}_1\vec{b}_2(\vec{a}_2-\vec{a}_1)]=0\Rightarrow[(\vec{a}_2-\vec{a}_1)\,\vec{b}_1\vec{b}_2]=0\Rightarrow(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=0$

To determine the equation of line of shortest distance : To find the equation of line of shortest distance, we use the following procedure :

- From the given equations of the straight lines,

  i.e. $\frac{x-a_1}{l_1}=\frac{y-b_1}{m_1}=\frac{z-c_1}{n_1}=\lambda$ (say) . . .(*i*)

  and $\frac{x-a_2}{l_2}=\frac{y-b_2}{m_2}=\frac{z-c_2}{n_2}=\mu$ (say) . . .(*ii*)

  Find the co-ordinates of general points on straight lines (*i*) and (*ii*) as $(a_1+\lambda l_1, b_1+\lambda m_1, c_1+\lambda n_1)$ and $(a_2+\lambda l_2, b_2+\mu m_2, c_2+\mu n_2)$.
- Let these be the co-ordinates of P and Q, the two extremities of the length of shortest distance. Hence, find the direction ratios of PQ as $(a_2+l_2\mu)-(a_1+l_1\lambda)$, $(b_2+m_2\mu)-(b_1+m_1\lambda),(c_2+m_2\mu)-(c_1+n_1\lambda)$.
- Apply the condition of PQ being perpendicular to straight lines (*i*) and (*ii*) in succession and get two equations connecting λ and μ. Solve these equations to get the values of λ and μ.
- Put these values of λ and μ in the co-ordinates of P and Q to determine points P and Q.
- Find out the equation of the line passing through P and Q, which will be the line of shortest distance.

The Plane

Definition of plane and its equations

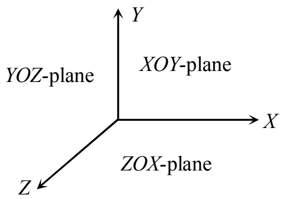


Figure: 23.22

If point $P(x, y, z)$ moves according to certain rule, then it may lie in a 3-D region on a surface or on a line or it may simply be a point. Whatever we get, as the region of P after applying the rule, is called locus of P. Let us discuss about the plane or curved surface. If Q be any other point on it's locus and all points of the straight line PQ lie on it, it is a plane. In other words if the straight line PQ, however small and in whatever direction it may be, lies completely on the locus, it is a plane, otherwise any curved surface.

- **General equation of plane:** Every equation of first degree of the form $Ax + By + Cz + D = 0$ represents the equation of a plane. The coefficients of x, y and z *i.e.* A, B, C are the direction ratios of the normal to the plane.
- **Equation of co-ordinate planes:** *XOY*-plane: $z = 0$ *YOZ* -plane: $x = 0$ *ZOX*-plane: $y = 0$
- **Vector equation of plane**

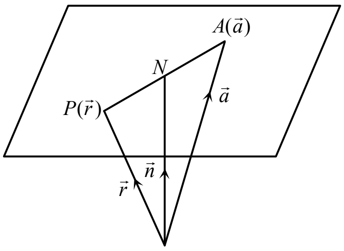


Figure: 23.23

- Vector equation of a plane through the point $A(\vec{a})$ and perpendicular to the vector $\vec{n}$ is $(\vec{r}-\vec{a}).\vec{n}=0$ or $\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$
- **Normal form:** Vector equation of a plane normal to unit vector $\hat{n}$ and at a distance d from the origin is $\vec{r}\cdot\hat{n}=d$.

Note

If $\vec{n}$ is not a unit vector, then to reduce the equation $\vec{r}\cdot\vec{n}=d$ to normal form we divide both sides by $|\vec{n}|$ to obtain $\vec{r}\cdot\frac{\vec{n}}{|\vec{n}|}=\frac{\vec{d}}{|\vec{n}|}$ or $\vec{r}\cdot\hat{n}=\frac{\vec{d}}{|\vec{n}|}$.

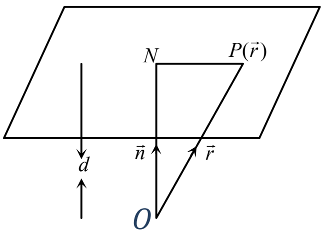


Figure: 23.24

- **Equation of a plane passing through a given point and parallel to two given vectors :** The equation of the plane passing through a point having position vector $\vec{a}$ and parallel to $\vec{b}$ and $\vec{c}$ is $\vec{r}=\vec{a}+\lambda\vec{b}+\mu\vec{c}$, where λ and μ are scalars.
- **Equation of plane in various forms**

  Intercept form

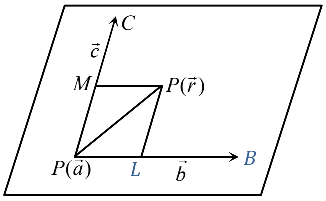


Figure: 23.25

If the plane cuts the intercepts of length a, b, c on co-ordinate axes, then its equation is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Normal form: Normal form of the equation of plane is $lx+my+nz=p,$ where l, m, n are the d.c.'s of the normal to the plane and p is the length of perpendicular from the origin.

- **Equation of plane in particular cases:** Equation of plane through the origin is given by $Ax + By + Cz = 0$. *i.e.* if $D = 0$, then the plane passes through the origin.
- **Equation of plane parallel to co-ordinate planes or perpendicular to co-ordinate axes**

  Equation of plane parallel to *YOZ* -plane (or perpendicular to x-axis) and at a distance 'a' from it is $x = a$.

  Equation of plane parallel to *ZOX* -plane (or perpendicular to y-axis) and at a distance 'b' from it is $y = b$.

  Equation of plane parallel to *XOY* -plane (or perpendicular to z-axis) and at a distance 'c' from it is $z = c$.

Note

- Any plane perpendicular to co-ordinate axis is evidently parallel to co-ordinate plane and vice versa.
- A unit vector perpendicular to the plane containing three points A, B, C is $\frac{\overrightarrow{AB}\times\overrightarrow{AC}}{|\overrightarrow{AB}\times\overrightarrow{AC}|}$

- **Equation of plane perpendicular to co-ordinate planes or parallel to co-ordinate axes**

  Equation of plane perpendicular to *YOZ*-plane or parallel to x-axis is $By + Cz + D = 0$.

  Equation of plane perpendicular to *ZOX*-plane or parallel to y axis is $Ax + Cz + D = 0$.

  Equation of plane perpendicular to *XOY*-plane or parallel to z-axis is $Ax + By + D = 0$.
- **Equation of plane passing through the intersection of two planes**

  Cartesian form: Equation of plane through the intersection of two planes $P=a_1x+b_1y+c_1z+d_1=0$ and $Q=a_2x+b_2y+c_2z+d_2=0$ is $P+\lambda Q=0,$ where λ is the parameter.

  Vector form: The equation of any plane through the intersection of planes $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$ is $\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2)=d_1+\lambda d_2$, where λ is an arbitrary constant.
- **Equation of plane parallel to a given plane**

  Cartesian form: Plane parallel to a given plane $ax+by+cz+d=0$ is $ax+by+cz+d'=0,$ *i.e.* only constant term is changed.

Vector form: Since parallel planes have the common normal, therefore equation of plane parallel to plane $\vec{r}\cdot\vec{n}=d_1$ is $\vec{r}\cdot\vec{n}=d_2$, where d_2 is a constant determined by the given condition.

Equation of plane passing through the given point.

- **Equation of plane passing through a given point:** Equation of plane passing through the point (x_1,y_1,z_1) is $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$,where A, B and C are d.r.'s of normal to the plane.
- **Equation of plane through three points:** The equation of plane passing through three non-collinear points (x_1,y_1,z_1), (x_2,y_2,z_2) and (x_3,y_3,z_3) is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0 \text{ or } \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0.$$

Foot of perpendicular from a point $A(\alpha,\beta,\gamma)$ to a given plane $ax+by+cz+d=0$: If AP be the perpendicular from A to the given plane, then it is parallel to the normal, so that its equation is

$$\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{c}=r \text{ (say)}$$

Any point P on it is $(ar+\alpha, br+\beta, cr+\gamma)$. It lies on the given plane and we find the value of r and hence the point P.

- **Perpendicular distance**

Cartesian form: The length of the perpendicular from the point $P(x_1,y_1,z_1)$ to the plane $ax+by+cz+d=0$ is

$$\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|.$$

Vector form: The perpendicular distance of a point having position vector $\vec{a}$ from the plane $\vec{r}\cdot\vec{n}=d$ is given by

$$p=\frac{|\vec{a}\cdot\vec{n}-d|}{|\vec{n}|}$$

Note

- The distance between two parallel planes is the algebraic difference of perpendicular distances on the planes from origin.
- Distance between two parallel planes $Ax+By+Cz+D_1=0$ and $Ax+By+Cz+D_2=0$ is $\dfrac{D_2\sim D_1}{\sqrt{A^2+B^2+C^2}}$.

- **Position of two points *w.r.t.* a plane :** Two points $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ lie on the same or opposite sides of a plane $ax+by+cz+d=0$ according to $ax_1+by_1+cz_1+d$ and $ax_2+by_2+cz_2+d$ are of same or opposite signs. The plane divides the line joining the points P and Q externally or internally according to P and Q are lying on same or opposite sides of the plane.

Angle between two planes.

- **Cartesian form:** Angle between the planes is defined as angle between normals to the planes drawn from any point. Angle between the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is

$$\cos^{-1}\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)}}\right)$$

Note

- If $a_1a_2+b_1b_2+c_1c_2=0$, then the planes are perpendicular to each other.
- If $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$, then the planes are parallel to each other.
- **Vector form :** An angle θ between the planes $\vec{r}_1\cdot\vec{n}_1=d_1$ and $\vec{r}_2\cdot\vec{n}_2=d_2$ is given by $\cos\theta=\pm\dfrac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$.

Equation of planes bisecting angle between two given planes

- **Cartesian form:** Equations of planes bisecting angles between the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d=0$ are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{(a_2^2+b_2^2+c_2^2)}}$$

- **Vector form:** The equation of the planes bisecting the angles between the planes $\vec{r}_1\cdot\vec{n}_1=d_1$ and $\vec{r}_2\cdot\vec{n}_2=d_2$ are

$$\frac{|\vec{r}\cdot\vec{n}_1-d_1|}{|\vec{n}_1|}=\frac{|\vec{r}\cdot\vec{n}_2-d_2|}{|\vec{n}_2|}$$

or $$\frac{|\vec{r}\cdot\vec{n}_1-d_1|}{|\vec{n}_1|}=\pm\frac{|\vec{r}\cdot\vec{n}_2-d_2|}{|\vec{n}_2|}$$

or $$\vec{r}\cdot(\hat{n}_1\pm\hat{n}_2)=\frac{d_1}{|\vec{n}_1|}\pm\frac{d_2}{|\vec{n}_2|}.$$

Image of a point in a plane: Let P and Q be two points and let π be a plane such that

- Line PQ is perpendicular to the plane π, and
- Mid-point of PQ lies on the plane π.

  Then either of the point is the image of the other in the plane π.

To find the image of a point in a given plane, we proceed as follows

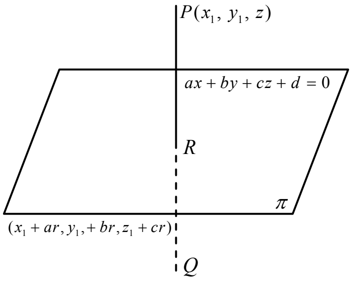


Figure: 23.26

- Write the equations of the line passing through P and normal to the given plane as $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$.
- Write the co-ordinates of image Q as $(x_1 + ar, y_1, + br, z_1 + cr)$.
- Find the co-ordinates of the mid-point R of PQ.
- Obtain the value of r by putting the co-ordinates of R in the equation of the plane.
- Put the value of r in the co-ordinates of Q.

Coplanar lines: Lines are said to be coplanar if they lie in the same plane or a plane can be made to pass through them.

Cartesian form: If the lines $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ are coplanar

Then $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$.

The equation of the plane containing them is

$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$ or $\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$.

- **Vector form :** If the lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$ are coplanar, then $[\vec{a}_1 \vec{b}_1 \vec{b}_2] = [\vec{a}_2 \vec{b}_1 \vec{b}_2]$ and the equation of the plane containing them is $[\vec{r}\, \vec{b}_1 \vec{b}_2] = [\vec{a}_1 \vec{b}_1 \vec{b}_2]$ or $[\vec{r}\, \vec{b}_1 \vec{b}_2] = [\vec{a}_2 \vec{b}_1 \vec{b}_2]$.

Sphere

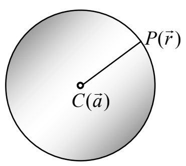


Figure:23.27

A sphere is the locus of a point which moves in space in such a way that its distance from a fixed point always remains constant. The fixed point is called the centre and the constant distance is called the radius of the sphere.

General equation of sphere: The general equation of a sphere is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ with centre $(-u, -v, -w)$. *i.e.* (–(1/2) coeff. of x, –(1/2) coeff. of y, –(1/2) coeff. of z) and, radius $= \sqrt{u^2 + v^2 + w^2 - d}$

From the above equation, we note the following characteristics of the equation of a sphere :

- It is a second degree equation in x, y, z;
- The coefficients of x^2, y^2, z^2 are all equal;
- The terms containing the products xy, yz and zx are absent.

Note

The equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents,

- A real sphere, if $u^2 + v^2 + w^2 - d > 0$.
- A point sphere, if $u^2 + v^2 + w^2 - d = 0$.
- An imaginary sphere, if $u^2 + v^2 + w^2 - d < 0$.
- If $u^2 + v^2 + w^2 - d < 0$, then the radius of sphere is imaginary, whereas the centre is real. Such a sphere is called "pseudo-sphere" or a "virtual sphere.
- The equation of the sphere contains four unknown constants u, v, w and d and therefore a sphere can be found to satisfy four conditions.

Equation in sphere in various forms.

- **Equation of sphere with given centre and radius**

  Cartesian form: The equation of a sphere with centre (a, b, c) and radius R is $(x - a)^2 + (y - b)^2 + (z - c)^2 = R^2 \ldots (i)$

  If the centre is at the origin, then equation (i) takes the form $x^2 + y^2 + z^2 = R^2$, which is known as the standard form of the equation of the sphere.

Vector form: The equation of sphere with centre at $C(\vec{c})$ and radius 'a' is $|\vec{r} - \vec{c}| = a$

- **Diameter form of the equation of a sphere**

  Cartesian form: If (x_1, y_1, z_1) and (x_2, y_2, z_2) are the co-ordinates of the extremities of a diameter of a sphere, then its equation is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

  Vector form: If the position vectors of the extremities of a diameter of a sphere are $\vec{a}$ and $\vec{b}$, then its equation is $(\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 0$ or $|\vec{r}|^2 - \vec{r} \cdot (\vec{a} - \vec{b}) + \vec{a} \cdot \vec{b} = 0$

Section of a sphere by a plane

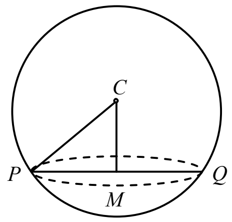


Figure: 23.28

Consider a sphere intersected by a plane. The set of points common to both sphere and plane is called a plane section of a sphere. The plane section of a sphere is always a circle. The equations of the sphere and the plane taken together represent the plane section.

Let C be the centre of the sphere and M be the foot of the perpendicular from C on the plane. Then M is the centre of the circle and radius of the circle is given by $PM = \sqrt{CP^2 - CM^2}$

The centre M of the circle is the point of intersection of the plane and line CM which passes through C and is perpendicular to the given plane.

- **Centre:** The foot of the perpendicular from the centre of the sphere to the plane is the centre of the circle. (radius of circle)^2 = (radius of sphere)^2 – (perpendicular from centre of spheres on the plane)^2
- **Great circle:** The section of a sphere by a plane through the centre of the sphere is a great circle. Its centre and radius are the same as those of the given sphere.

Condition of tangency of a plane to a sphere: A plane touches a given sphere if the perpendicular distance from the centre of the sphere to the plane is equal to the radius of the sphere.

- **Cartesian form:** The plane $lx + my + nz = p$ touches the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, if $(ul + vm + wn - p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$
- **Vector form:** The plane $\vec{r} \cdot \vec{n} = d$ touches the sphere $|\vec{r} - \vec{a}| = R$ if $\frac{|\vec{a} \cdot \vec{n} - d|}{|\vec{n}|} = R$.

Note

Two spheres S_1 and S_2 with centres C_1 and C_2 and radii r_1 and r_2 respectively

- Do not meet and lies farther apart iff $|C_1C_2| > r_1 + r_2$
- Touch internally iff $|C_1C_2| = |r_1 - r_2|$
- Touch externally iff $|C_1C_2| = r_1 + r_2$
- Cut in a circle iff $|r_1 - r_2| < |C_1C_2| < r_1 + r_2$
- One lies within the other if $|C_1C_2| < |r_1 - r_2|$.

When two spheres touch each other the common tangent plane is $S_1 - S_2 = 0$ and when they cut in a circle, the plane of the circle is $S_1 - S_2 = 0$; coefficients of x^2, y^2, z^2 being unity in both the cases. Let p be the length of perpendicular drawn from the centre of the sphere $x^2 + y^2 + z^2 = r^2$ to the plane $Ax + By + Cz + D = 0$ then

- The plane cuts the sphere in a circle iff $p < r$ and in this case, the radius of circle is $\sqrt{r^2 - p^2}$.
- The plane touches the sphere iff $p = r$.
- The plane does not meet the sphere iff $p < r$.

Equation of concentric sphere: Any sphere concentric with the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + \lambda = 0$, where λ is some real which makes it a sphere.

Intersection of straight line and a sphere: Let the equations of the sphere and the straight line be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \ldots(i)$$

and $\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} = r$ (say) $\quad \ldots(ii)$

Any point on the line (ii) is $(\alpha + lr, \beta + mr, \gamma + nr)$.

If this point lies on the sphere (i) then we have,

$$(\alpha + lr)^2 + (\beta + mr)^2 + (\gamma + nr)^2 + 2u(\alpha + lr) + 2v(\beta + mr) + 2w(\gamma + nr) + d = 0$$

or

$$r^2[l^2 + m^2 + n^2] + 2r[l(u + \alpha) + m(v + \beta)] + n(w + \gamma)] + (\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d) = 0 \quad \ldots(iii)$$

This is a quadratic equation in r and so gives two values of r and therefore the line (ii) meets the sphere (i) in two points which may be real, coincident and imaginary, according as root of (iii) are so.

Angle of intersection of two spheres: The angle of intersection of two spheres is the angle between the tangent planes to them at their point of intersection. As the radii of the spheres at this common point are normal to the tangent planes so this angle is also equal to the angle between the radii of the spheres at their point of intersection. If the angle of intersection of two spheres is a right angle, the spheres are said to be orthogonal.

- **Condition for orthogonality of two spheres:** Let the equation of the two spheres be

  $$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \ldots(i)$$

  and $x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0 \quad \ldots(ii)$

  If the sphere (i) and (ii) cut orthogonally, then $2uu' + 2vv' + 2ww' = d + d'$, which is the required condition.

MULTIPLE CHOICE QUESTIONS

Distance and Section Formula

1. The distance of the point (4, 3, 5) from the y-axis is:

a. $\sqrt{34}$ **b.** 5

c. $\sqrt{41}$ **d.** $\sqrt{15}$

2. If the x-co-ordinate of a point P on the join of Q (2, 2, 1) and R (5, 1, –2) is 4, then its z-co-ordinate is:

a. 2 **b.** 1

c. –1 **d.** –2

Triangle and Tetrahedron

3. If centroid of tetrahedron $OABC$, where A, B, C are given by (a, 2, 3), (1, b, 2) and (2, 1, c) respectively be (1, 2, –1), then distance of $P(a, b, c)$ from origin is equal to:

a. $\sqrt{107}$ **b.** $\sqrt{14}$

c. $\sqrt{107/14}$ **d.** None of these

Direction Cosines and Direction Ratio

4. A line which makes angle 60° with y-axis and z-axis, then the angle which it makes with x-axis is:

a. 45° **b.** 60°

c. 75° **d.** 30°

5. A line passes through the points (6, –7, –1) and (2,–3, 1). The direction cosines of line, so directed that the angle made by it with the positive direction of x-axis is acute, are:

a. $\frac{2}{3}, \frac{-2}{3}, \frac{-1}{3}$ **b.** $\frac{-2}{3}, \frac{2}{3}, \frac{1}{3}$

c. $\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$ **d.** $\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}$

6. If the direction cosines of a line are $\left(\frac{1}{c}, \frac{1}{c}, \frac{1}{c}\right)$, then:

a. $c > 0$ **b.** $c = \pm\sqrt{3}$

c. $0 < c < 1$ **d.** $c > 2$

7. If **r** is a vector of magnitude 21 and has d.r.'s 2, –3, 6. Then **r** is equal to:

a. $6\mathbf{i} - 9\mathbf{j} + 18\mathbf{k}$ **b.** $6\mathbf{i} + 9\mathbf{j} + 18\mathbf{k}$

c. $6\mathbf{i} - 9\mathbf{j} - 18\mathbf{k}$ **d.** $6\mathbf{i} + 9\mathbf{j} - 18\mathbf{k}$

Projection

8. The projection of a line on co-ordinate axes are 2, 3, 6. Then the length of the line is:

a. 7 **b.** 5 **c.** 1 **d.** 11

Line , Changing Unsymmetrical form to Symmetrical form and Angle between Two lines

9. If the direction ratio of two lines are given by $3lm - 4ln + mn = 0$ and $l + 2m + 3n = 0$, then the angle between the lines is:

a. $\frac{\pi}{2}$ **b.** $\frac{\pi}{3}$ **c.** $\frac{\pi}{4}$ **d.** $\frac{\pi}{6}$

10. If a line makes angles α, β, γ, δ with four diagonals of a cube, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta$ is:

a. $\frac{4}{3}$ **b.** 1 **c.** $\frac{8}{3}$ **d.** $\frac{7}{3}$

11. The d.c.'s of the line $6x - 2 = 3y + 1 = 2z - 2$ are:

a. $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ **b.** $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

c. 1, 2, 3 **d.** None of these

12. The vector equation of line through the point A(3, 4, –7) and B(1, –1, 6) is

a. $\mathbf{r} = (3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + 6\mathbf{k})$

b. $\mathbf{r} = (\mathbf{i} - \mathbf{j} + 6\mathbf{k}) + \lambda(3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k})$

c. $\mathbf{r} = (3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}) + \lambda(-2\mathbf{i} - 5\mathbf{j} + 13\mathbf{k})$

d. $\mathbf{r} = (\mathbf{i} - \mathbf{j} + 6\mathbf{k}) + \lambda(4\mathbf{i} + 3\mathbf{j} - \mathbf{k})$

13. If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5}$ are at right angles, then k = ?

a. –10 **b.** 10/7 **c.** –10/7 **d.** –7/10

14. Distance of the point (x_1, y_1, z_1) from the line $\frac{x - x_2}{l} = \frac{y - y_2}{m} = \frac{z - z_2}{n}$, where l, m and n are the direction cosines of line is:

a. $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - [l(x_1 - x_2) + m(y_1 - y_2) + n(z_1 - z_2)]^2}$

b. $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

c. $\sqrt{(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n}$

d. None of these

15. The angle between the lines $\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$ and $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ is:

a. $\cos^{-1}\frac{1}{5}$ **b.** $\cos^{-1}\frac{1}{3}$

c. $\cos^{-1}\frac{1}{2}$ **d.** $\cos^{-1}\frac{1}{4}$

16. The angle between the lines whose direction cosines are proportional to (1, 2, 1) and (2, –3, 6) is:

a. $\cos^{-1}\left(\frac{2}{7\sqrt{6}}\right)$ **b.** $\cos^{-1}\left(\frac{1}{7\sqrt{6}}\right)$

c. $\cos^{-1}\left(\frac{3}{7\sqrt{6}}\right)$ **d.** $\cos^{-1}\left(\frac{5}{7\sqrt{6}}\right)$

17. The angle between the lines whose direction cosines satisfy the equations $l+m+n=0$, $l^2+m^2-n^2=0$ is given by:

a. $\frac{2\pi}{3}$ **b.** $\frac{\pi}{6}$ **c.** $\frac{5\pi}{6}$ **d.** $\frac{\pi}{3}$

18. The angle between two lines $\frac{x+1}{2}=\frac{y+3}{2}=\frac{z-4}{-1}$ and $\frac{x-4}{1}=\frac{y+4}{2}=\frac{z+1}{2}$ is:

a. $\cos^{-1}\left(\frac{1}{9}\right)$ **b.** $\cos^{-1}\left(\frac{2}{9}\right)$

c. $\cos^{-1}\left(\frac{3}{9}\right)$ **d.** $\cos^{-1}\left(\frac{4}{9}\right)$

19. The angle between the pair of lines with direction ratios (1, 1, 2) and $(\sqrt{3}-1, -\sqrt{3}-1, 4)$ is:

a. 30° **b.** 45°

c. 60° **d.** 90°

20. If direction ratios of two lines are $5, -12, 13$ and $-3, 4, 5$ then the angle between them is:

a. $\cos^{-1}(1/65)$ **b.** $\cos^{-1}(2/65)$

c. $\cos^{-1}(3/65)$ **d.** $\pi/2$

21. The point of intersection of the lines $\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1}$, $\frac{x+3}{-36}=\frac{y-3}{2}=\frac{z-6}{4}$ is:

a. $21, \frac{5}{3}, \frac{10}{3}$ **b.** $(2, 10, 4)$

c. $(-3, 3, 6)$ **d.** $(5, 7, -2)$

Reduction of Cartesian form of the Equation of a line to Vector form and Vice Versa

22. The cartesian equations of a line are $6x-2$. $=3y+1=2z-2$ The vector equation of the line is:

a. $\mathbf{r}=\left(\frac{1}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}+\mathbf{k}\right)+\lambda(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$

b. $\mathbf{r}=(3\mathbf{i}-3\mathbf{j}+\mathbf{k})+\lambda(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$

c. $\mathbf{r}=(\mathbf{i}+\mathbf{j}+\mathbf{k})+\lambda(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$

d. None of these

Intersection of Two Lines and Foot of Perpendicular

23. A line with direction cosines proportional to 2, 1, 2 meets each of the lines $x = y + a = z$ and $x + a = 2y = 2z$. The co-ordinates of each of the points of intersection are given by:

a. $(2a, 3a, 3a)$ $(2a, a, a)$ **b.** $(3a, 2a, 3a)$ (a, a, a)

c. $(3a, 2a, 3a)$ $(a, a, 2a)$ **d.** $(3a, 3a, 3a)$ (a, a, a)

24. If the line $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect, then k = ?

a. 2/9 **b.** 9/2

c. 0 **d.** –1

25. The co-ordinates of the foot of the perpendicular drawn from the point $A(1, 0, 3)$ to the join of the points $B(4, 7, 1)$ and $C(3, 5, 3)$ are:

a. (5/3, 7/3, 17/3) **b.** (5, 7, 17)

c. (5/3, –7/3, 17/3) **d.** (–5/3, 7/3, –17/3)

26. The length of the perpendicular from the origin to line $\mathbf{r}=(4\mathbf{i}+2\mathbf{j}+4\mathbf{k})+\lambda(3\mathbf{i}+4\mathbf{j}-5\mathbf{k})$ is:

a. $2\sqrt{5}$ **b.** 2 **c.** $5\sqrt{2}$ **d.** 6

Shortest Distance Between Two Straight Lines

27. The shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$ is :

a. $\frac{1}{6}$ **b.** $\frac{1}{\sqrt{6}}$ **c.** $\frac{1}{\sqrt{3}}$ **d.** $\frac{1}{3}$

28. The shortest distance between the lines $\mathbf{r}=(\mathbf{i}+\mathbf{j}-\mathbf{k})+\lambda(3\mathbf{i}-\mathbf{j})$ and $\mathbf{r}=(4\mathbf{i}-\mathbf{k})+\mu(2\mathbf{i}+3\mathbf{k})$ is:

a. 6 **b.** 0

c. 2 **d.** 4

29. If the straight lines $x=1+s, y=3-\lambda s, z=1+\lambda s$ and $x=\frac{t}{2}, y=1+t, z=2-t$ with parameters s and t respectively, are co-planar, then λ equals:

a. 0 **b.**–1

c. –1/2 **d.** –2

Coplanar Lines

30. The ratio in which the plane $x-2y+3z=17$ divides the line joining the point (–2, 4, 7) and (3, –5, 8) is:

a. 10 : 3 **b.** 3 : 1 **c.** 3 : 10 **d.** 10 : 1

31. The xy-plane divides the line joining the points (–1, 3, 4) and (2, –5, 6)

a. Internally in the ratio 2:3 **b.** Internally in the ratio 3:2
c. Externally in the ratio 2:3 **d.** Externally in the ratio 3:2

32. The equation of the plane, which makes with co-ordinate axes a triangle with its centroid (α, β, γ), is:

a. $\alpha x+\beta y+\gamma z=3$ **b.** $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$
c. $\alpha x+\beta y+\gamma z=1$ **d.** $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3$

33. Angle between two planes $x+2y+2z=3$ and $-5x+3y+4z=9$ is:

a. $\cos^{-1}\frac{3\sqrt{2}}{10}$ **b.** $\cos^{-1}\frac{19\sqrt{2}}{30}$
c. $\cos^{-1}\frac{9\sqrt{2}}{20}$ **d.** $\cos^{-1}\frac{3\sqrt{2}}{5}$

34. Distance between two parallel planes $2x + y + 2z = 8$ and $4x+2y+4z+5=0$ is:

a. 9/2 **b.** 5/2 **c.** 7/2 **d.** 3/2

35. A tetrahedron has vertices at $O(0,0,0)$, $A(1,2,1)$, $B(2, 1, 3)$ and $C(-1, 1, 2)$. Then the angle between the faces OAB and ABC will be:

a. $\cos^{-1}\left(\frac{19}{35}\right)$ **b.** $\cos^{-1}\left(\frac{17}{31}\right)$
c. 30° **d.** 90°

36. The distance of the point (2, 1, –1) from the plane $x-2y+4z=9$ is:

a. $\frac{\sqrt{13}}{21}$ **b.** $\frac{13}{21}$ **c.** $\frac{13}{\sqrt{21}}$ **d.** $\sqrt{\frac{13}{21}}$

37. A unit vector perpendicular to plane determined by the points $P(1, -1, 2)$, $Q(2, 0, -1)$ and $R(0, 2, 1)$ is:

a. $\frac{2\mathbf{i}-\mathbf{j}+\mathbf{k}}{\sqrt{6}}$ **b.** $\frac{2\mathbf{i}+\mathbf{j}+k}{\sqrt{6}}$ **c.** $\frac{-2\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{6}}$ **d.** $\frac{2\mathbf{i}+\mathbf{j}-\mathbf{k}}{\sqrt{6}}$

38. If $P = (0, 1, 0)$, $Q = (0, 0, 1)$, then projection of PQ on the plane $x + y + z = 3$ is:

a. $\sqrt{3}$ **b.** 3 **c.** $\sqrt{2}$ **d.** 2

39. The reflection of the point (2, –1, 3) in the plane $3x-2y-z=9$ is:

a. $\left(\frac{26}{7},\frac{15}{7},\frac{17}{7}\right)$ **b.** $\left(\frac{26}{7},\frac{-15}{7},\frac{17}{7}\right)$
c. $\left(\frac{15}{7},\frac{26}{7},\frac{-17}{7}\right)$ **d.** $\left(\frac{26}{7},\frac{17}{7},\frac{-15}{7}\right)$

40. A non-zero vector **a** is parallel to the line of intersection of the plane determined by the vectors **i**, **i** + **j** and the plane determined by the vectors **i** – **j**, **i** + **k**. The angle between **a** and the vector **i** – 2**j** + 2**k** is:

a. $\frac{\pi}{4}$ or $\frac{3\pi}{4}$ **b.** $\frac{2\pi}{4}$ or $\frac{3\pi}{4}$
c. $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ **d.** None of these

41. The d.r.'s of normal to the plane through (1, 0, 0) and (0, 1, 0) which makes an angle $\pi/4$ with plane $x + y = 3$, are:

a. $1,\sqrt{2},1$ **b.** $1,1,\sqrt{2}$ **c.** 1, 1, 2 **d.** $\sqrt{2},1,1$

Plane

42. The angle between the planes $3x-4y+5z=0$ and $2x-y-2z=5$ is:

a. $\frac{\pi}{3}$ **b.** $\frac{\pi}{2}$
c. $\frac{\pi}{6}$ **d.** None of these

43. If a plane cuts off intercepts –6, 3, 4 from the co-ordinate axes, then the length of the perpendicular from the origin to the plane is:

a. $\frac{1}{\sqrt{61}}$ **b.** $\frac{13}{\sqrt{61}}$ **c.** $\frac{12}{\sqrt{29}}$ **d.** $\frac{5}{\sqrt{41}}$

44. The value of k for which the planes $3x-6y-2z=7$ and $2x+y-kz=5$ are perpendicular to each other, is:

a. 0 **b.** 1 **c.** 2 **d.** 3

45. The equation of the plane containing the line of intersection of the planes $2x-y=0$ and $y-3z=0$ and perpendicular to the plane $4x+5y-3z-8=0$ is:

a. $28x-17y+9z=0$ **b.** $28x+17y+9z=0$
c. $28x-17y+9x=0$ **d.** $7x-3y+z=0$

46. A point moves so that its distances from the points (3, 4, –2) and (2, 3, – 3) remains equal. The locus of the point is:

a. A line
b. A plane whose normal is equally inclined to axes
c. A plane which passes through the origin
d. A sphere

47. The equation of a plane which passes through (2, –3, 1) and is normal to the line joining the points (3, 4, –1) and (2, –1, 5) is given by:

a. $x+5y-6z+19=0$ **b.** $x-5y+6z-19=0$
c. $x+5y+6z+19=0$ **d.** $x-5y-6z-19=0$

48. The plane $ax+by+cz=1$ meets the co-ordinate axes in *A*, *B* and *C*. The centroid of the triangle is:

a. $(3a,3b,3c)$ **b.** $\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$

c. $\left(\frac{3}{a},\frac{3}{b},\frac{3}{c}\right)$ **d.** $\left(\frac{1}{3a},\frac{1}{3b},\frac{1}{3c}\right)$

49. If *P* be the point (2, 6, 3), then the equation of the plane through *P* at right angle to *OP*, *O* being the origin, is:

a. $2x+6y+3z=7$ **b.** $2x-6y+3z=7$

c. $2x+6y-3z=49$ **d.** $2x+6y+3z=49$

Projection of a Line on a Plane

50. Value of *k* such that the line $\frac{x-1}{2}=\frac{y-1}{3}=\frac{z-k}{k}$ is perpendicular to normal to the plane $\mathbf{r}(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})=0$ is:

a. –13/4 **b.** –17/4

c. 4 **d.** 5

51. The sine of angle between the straight line $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ and the plane $2x-2y+z=5$ is:

a. $\frac{2\sqrt{3}}{5}$ **b.** $\frac{\sqrt{2}}{10}$ **c.** $\frac{4}{5\sqrt{2}}$ **d.** $\frac{10}{6\sqrt{5}}$

52. The equation of line of intersection of the planes $4x+4y-5z=12$, $8x+12y-13z=32$ can be written as:

a. $\frac{x}{2}=\frac{y-1}{3}=\frac{z-2}{4}$ **b.** $\frac{x}{2}=\frac{y}{3}=\frac{z-2}{4}$

c. $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z}{4}$ **d.** $\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z}{4}$

53. The equation of the plane containing the two lines $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z}{3}$ and $\frac{x}{2}=\frac{y-2}{-1}=\frac{z+1}{-3}$ is:

a. $8x+y-5z-7=0$ **b.** $8x+y+5z-7=0$

c. $8x-y-5z-7=0$ **d.** None of these

54. The plane which passes through the point (3,2,0) and the line $\frac{x-3}{1}=\frac{y-6}{5}=\frac{z-4}{4}$ is:

a. $x-y+z=1$ **b.** $x+y+z=5$

c. $x+2y-z=1$ **d.** $2x-y+z=5$

55. The distance between the line $\mathbf{r}=(\mathbf{i}+\mathbf{j}+2\mathbf{k})+\lambda(2\mathbf{i}+5\mathbf{j}+3\mathbf{k})$ and the plane $\mathbf{r}.(2\mathbf{i}+\mathbf{j}-3\mathbf{k})=5$ is:

a. $\frac{5}{\sqrt{14}}$ **b.** $\frac{6}{\sqrt{14}}$ **c.** $\frac{7}{\sqrt{14}}$ **d.** $\frac{8}{\sqrt{14}}$

Sphere and Angle of Intersection of Two Spheres

56. The intersection of the spheres $x^2+y^2+z^2+7x-2y-z=13$ and $x^2+y^2+z^2-3x+3y+4z=8$ is the same as the intersection of one of the sphere and the plane:

a. $2x-y-z=1$ **b.** $x-2y-z=1$

c. $x-y-2z=1$ **d.** $x-y-z=1$

57. The point at which the line joining the points (2, –3, 1) and (3, –4, –5) intersects the plane $2x+y+z=7$ is:

a. (1, 2, 7) **b.** (1, –2, 7)

c. (–1, 2, 7) **d.** (1, –2, –7)

58. The equation of the plane passing through the lines $\frac{x-4}{1}=\frac{y-3}{1}=\frac{z-2}{2}$ and $\frac{x-3}{1}=\frac{y-2}{-4}=\frac{z}{5}$ is:

a. $11x-y-3z=35$ **b.** $11x+y-3z=35$

c. $11x-y+3z=35$ **d.** None of these

59. The line $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ is parallel to the plane:

a. $3x+4y+5z=7$ **b.** $2x+y-2z=0$

c. $x+y-z=2$ **d.** $2x+3y+4z=0$

60. The equation of the line passing through (1, 2, 3) and parallel to the planes $x-y+2z=5$ and $3x+y+z=6$, is:

a. $\frac{x-1}{-3}=\frac{y-2}{5}=\frac{z-3}{4}$ **b.** $\frac{x-1}{-3}=\frac{y-2}{-5}=\frac{z-1}{4}$

c. $\frac{x-1}{-3}=\frac{y-2}{-5}=\frac{z-1}{-4}$ **d.** None of these

61. The line $\frac{x+3}{3}=\frac{y-2}{-2}=\frac{z+1}{1}$ and the plane $4x+5y+3z-5=0$ intersect at a point:

a. (3, 1, –2) **b.** (3, – 2, 1)

c. (2, –1, 3) **d.** (–1, –2, –3)

62. If a plane passes through the point (1,1,1) and is perpendicular to the line $\frac{x-1}{3}=\frac{y-1}{0}=\frac{z-1}{4}$, then its perpendicular distance from the origin is:

a. $\frac{3}{4}$ **b.** $\frac{4}{3}$ **c.** $\frac{7}{5}$ **d.** 1

63. A plane which passes through the point (3, 2, 0) and the line $\frac{x-3}{1}=\frac{y-6}{5}=\frac{z-4}{4}$ is:

a. $x-y+z=1$ **b.** $x+y+z=5$

c. $x+2y-z=0$ **d.** $2x-y+z=5$

64. The angle between the line $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$ and the plane $3x+2y-3z=4$ is:

a. 45° **b.** 0°

c. $\cos^{-1}\left(\frac{24}{\sqrt{29}\sqrt{22}}\right)$ **d.** 90°

NCERT EXEMPLAR PROBLEMS

More than One Answer

65. If the straight lines $\frac{x-1}{2}=\frac{y+1}{K}=\frac{z}{2}$ and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{k}$ are coplanar, then the plane(s) containing these two lines is/are:

a. $y+2z=-1$ **b.** $y+z=-1$

c. $y-z=-1$ **d.** $y-2z=-1$

66. If $y(x)$ satisfies the differential equation $y'-y\tan x=2x\sec x$ and y(0), then:

a. $y\left(\frac{\pi}{4}\right)=\frac{\pi^2}{8\sqrt{2}}$ **b.** $y'\left(\frac{\pi}{4}\right)=\frac{\pi^2}{18}$

c. $y\left(\frac{\pi}{3}\right)=\frac{\pi^2}{9}$ **d.** $y'\left(\frac{\pi}{3}\right)=\frac{4\pi}{3}+\frac{2\pi^2}{3\sqrt{3}}$

67. A line l passing through the origin is perpendicular to the lines $l_1:(3+t)\hat{i}+(-1+2t)\hat{j}+(4+2t)\hat{k},-\infty<t<\infty$

$l_2:(3+2s)\hat{i}+(3+2s)\hat{j}+(2+s)\hat{k},-\infty<s<\infty$

Then, the coordinate(s) of the point(s) on l_2 at a distance of $\sqrt{17}$ from the point of intersection of l and l_1 is (are)

a. $\left(\frac{7}{3},\frac{7}{3},\frac{5}{3}\right)$ **b.** (–1,–1,0)

c. (1,1,1) **d.** $\left(\frac{7}{9},\frac{7}{9},\frac{8}{9}\right)$

68. Two lines $L_1: x=5,\frac{y}{3-\alpha}=\frac{z}{-2}$ and $L_2: x=\alpha,\frac{y}{-1}=\frac{z}{2-\alpha}$ are coplanar. Then, α can take value(s):

a. 1 **b.** 2 **c.** 3 **d.** 4

69. From a point $P(\lambda,\lambda,\lambda)$, perpendiculars PQ and PR are drawn respectively on the lines $y = x, z = 1$ and $y = -x, z = -1$. If P is such that $\angle QPR$ is a right angle, then the possible value(s) of λ is: (are)

a. $\sqrt{2}$ **b.** 1 **c.** –1 **d.** $-\sqrt{2}$

70. The lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar if

a. $k=0$ **b.** $k=-1$ **c.** $k=-3$ **d.** $k=3$

71. If OABC is a tetrahedron such that $OA^2+BC^2=OB^2+CA^2=OC^2+AB^2$ then:

a. $OA\perp BC$ **b.** $OB\perp CA$

c. $OC\perp AB$ **d.** $AB\perp BC$

72. The equation of a line $4x-4y-z+11=0=x+2y-z-1$ can be put as:

a. $\frac{x}{2}=\frac{y-2}{1}=\frac{z-3}{4}$ **b.** $\frac{x-4}{-2}=\frac{y-4}{2}=\frac{z-11}{2}$

c. $\frac{x-2}{2}=\frac{y}{1}=\frac{z-3}{4}$ **d.** $\frac{x-2}{2}=\frac{y-2}{1}=\frac{z}{4}$

73. If –2, 2, 1are direction ratios of a line, then its direction cosiness are:

a. $-\frac{2}{3},\frac{2}{3},\frac{1}{3}$ **b.** $\frac{2}{3},-\frac{2}{3},\frac{1}{3}$

c. $\frac{2}{3},-\frac{2}{3},-\frac{1}{3}$ **d.** $-\frac{2}{3},\frac{2}{3},-\frac{1}{3}$

74. The equation of a sphere which passes through (1,0,0), (0,1,0) and (0,0,1) and whose centre lies on the curve $4xy=1$is:

a. $x^2+y^2+z^2-x-y-z=0$

b. $x^2+y^2+z^2+x+y+z-2=0$

c. $x^2+y^2+z^2+x+y+z=0$

d. $x^2+y^2+z^2-x-y-z-2=0$

Assertion and Reason

Note: Read the Assertion (A) and Reason (R) carefully to mark the correct option out of the options given below:

a. If both assertion and reason are true and the reason is the correct explanation of the assertion.

b. If both assertion and reason are true but reason is not the correct explanation of the assertion.

c. If assertion is true but reason is false.

d. If the assertion and reason both are false.

e. If assertion is false but reason is true.

75. **Assertion:** The point $A(3,1,6)$ is the mirror image of the point $B(1,3,4)$ in the plane $x-y+z=5$.

Reason: The plane $x-y+z=5$ bisects the line segment joining $A(3,1,6)$ and $B(1,3,4)$.

76. **Assertion:** If the distance of the point $P(1, -2, 1)$ from the plane $x+2y-2z=\alpha$ where $\alpha>0$, is 5, then the foot of the perpendicular from P to the plane is (8/3, 4/3, –7/3)

Reason: A line through $P(1, -2, 1)$and perpendicular to the plane $x+2y-2z=\alpha$ intersects the plane at Q. If $PQ=5$ then $\alpha=10$.

77. Consider the plane $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.
Assertion: The parametric equations of the line of intersection of the given planes are $x = 3 + 14t$, $y = 1 + 2t$, $z15t$; t being the parameter
Reason: the vector $14\hat{i} + 2\hat{j} + 15\hat{k}$ is parallel to the line of intersection of the given planes.

78. **Assertion:** The point $A(1,0,7)$ is the mirror image of the point (1, 6, 3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$
Reason: The line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ bisects the line segment joining $A(1,0,7)$and $B(1,6,3)$

79. $L_1 : \frac{x-1}{1} = \frac{y}{-1} = \frac{z+1}{1}$, $L_2 : \frac{x-2}{1} = \frac{y+1}{2} = \frac{z}{3}$
Assertion: L_1 and L_2 are coplanar and the equation of the plane containing them is $5x + 2y - 3z - 8 = 0$
Reason: L_1 and L_2 intersect at a point.

80. Vertices of a triangle ABC are $A(1,1,0)$, $B(1,0,1)$ and $C(0,1,1)$
Assertion: The radius of the circum circle of the triangle ABC is $\sqrt{2/3}$.
Reason: The centre of the circum circle of the triangle ABC lies on the plane $x + y + z - 2 = 0$

81. Consider the line $L : \frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and the plane $\pi : x + y + z = 0$
Assertion: If P is a point on L at a distance $\sqrt{14}$ from the origin O and N is the foot of the perpendicular from P to the plane π, then $ON = \sqrt{2}$
Reason: If R is a point on L such that the perpendicular distance of R from the plane π is $\sqrt{3}$ then the coordinates of R are (1, 2, 3)

81. $L_1 : \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}$, $L_2 : \frac{x-2}{1} = \frac{y+1}{2} = \frac{z-3}{3}$
Assertion: The unit vector perpendicular to both L_1 and L_2 is $\frac{-\hat{i} - 7\hat{j} + 5\hat{k}}{5\sqrt{3}}$
Reason: The distance of the point (1, 1, 1) from the plane passing through the point (–1, –2, –1) and whose normal is perpendicular to both the lines L_1 and L_2 is $\frac{23}{5\sqrt{3}}$

82. **Assertion:** The distance between the line $r = 2\hat{i} + 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - \hat{j} + 4\hat{k})$ and the plane $r.(\hat{i} + 5\hat{j} + \hat{k}) = 5$ is $\frac{10}{3\sqrt{3}}$
Reason: If a line is parallel to a plane, then the distance between the line and the plane is equal to the length of the perpendicular form any point on the line to the plane.

84. **Assertion:** The direction cosines of the line $6x - 2 = 3y + 1 = 2z - 2$ are same as the direction cosines of the normal to the plane $2x + 3y + z = 14$
Reason: The direction angles of a normal to the plane are $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ and the length of the perpendicular form the origin on the plane is $\sqrt{2}$, equation of the plane is $x + y = 2$

Comprehension Based

Paragraph –I

Consider the lines $L_1 : \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}, L_2 : \frac{x-2}{1} = \frac{y+2}{2} = \frac{z-3}{3}$

85. The unit vector perpendicular to both L_1 and L_2 is:
a. $\frac{-\hat{i} + 7\hat{j} + 7\hat{k}}{\sqrt{99}}$ **b.** $\frac{-\hat{i} - 7\hat{j} + 5\hat{k}}{5\sqrt{3}}$
c. $\frac{-\hat{i} + 7\hat{j} + 5\hat{k}}{5\sqrt{3}}$ **d.** $\frac{7\hat{i} - 7\hat{j} - \hat{k}}{\sqrt{99}}$

86. The shortest distance between L_1 and L_2 is:
a. 0 unit **b.** $17/\sqrt{3}$ unit
c. $41/5\sqrt{3}$ unit **d.** $17/5\sqrt{3}$ unit

87. The distance of the point (1, 1, 1) from the plane passing through the point (–1, –2, –1) and whose normal is perpendicular to both the lines L_1 and L_2 is:
a. $2/\sqrt{75}$ unit **b.** $7/\sqrt{75}$ unit
c. $13/\sqrt{75}$ unit **d.** $23/\sqrt{75}$ unit

Paragraph –II
Let two planes $P_1 : 2x - y + z = 2$ and $P_3 : x + 2y - z = 3$

88. The equation of the plane through the intersection of P_1 and P_2 and the point (3,2,1) is:
a. $3x - y + 2z - 9 = 0$ **b.** $x - 3y + 2z + 1 = 0$
c. $2x - 3y + z - 1 = 0$ **d.** $4x - 3y + 2z - 8 = 0$

89. Equation of the plane which passes through the point (–1,3,2) and is perpendicular to each the planes P_1 and P_2 is:
a. $x + 3y - 5z + 2 = 0$ **b.** $x + 3y + 5z - 18 = 0$
c. $x - 3y - 5z + 20 = 0$ **d.** $x - 3y + 5z = 0$

90. The equation of the acute angle bisector of planes P_1 and P_2 is:

a. $x-3y+2z+1=0$ **b.** $3x+y-5=0$

c. $x+3y-2z+1=0$ **d.** $3x+z+7=0$

91. The equation of the bisector of angle of the planes P_1 and P_2 which not containing origin is:

a. $x - 3y + 2z + 1 = 0$ **b.** $x + 3y = 5$

c. $x + 3y + 2z + 2 = 0$ **d.** $3x + y = 5$

92. The image of plane P_1 in the plane mirror P_2 is:

a. $x + 7y - 4x + 5 = 0$ **b.** $3x + 4y - 5z + 9 = 0$

c. $7x - y + 4z - 9 = 0$ **d.** None of the above

Match the Column

93. Consider the following linear equations $ax + by + cz = 0$, $bx+cy+az=0$, $cx+ay+bz=0$

Column I	Column II
(A) $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	**1.** the equations represent planes meeting only at a single point
(B) $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$	**2.** the equations represent the line $x = y = z$
(C) $a + b + c \neq 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$	**3.** the equations represent identical planes
(D) $a + b + c = 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	**4.** the equations represent the whole of the three dimensional space

a. A→3, B→1, C→4, D→2

b. A→3, B→2, C→1, D→4

c. A→1, B→3, C→2, D→4

d. A→4, B-1, C→3, D→2

94. Consider the lines $L_1:\frac{x-1}{2}=\frac{y}{-1}=\frac{z+3}{1}, L_2:\frac{x-4}{1}=\frac{y+3}{1}=\frac{z+3}{2}$ and the planes $P_1: 7x+y+2z=3, P_2: 3x+5y-6z=4$. Let $ax + by + cz = d$ the equation of the plane passing through the point of intersection of lines L_1 and L_2 and perpendicular to planes P_1 and P_2. Match Column I with Column II and select the correct answer using the code given below the lists:

Column I	Column II
(A) $a =$	**1.** 13
(B) $b =$	**2.** –3
(C) $c =$	**3.** 1
(D) $d =$	**4.** –2

a. A→3, B→2, C→4, D→2

b. A→2, B→4, C→3, D→1

c. A→1, B→3, C→2, D→4

d. A→4, B→1, C→3, D→2

95. Match the statement of Column with those in Column II:

Column I	Column II
(A) If maximum and minimum distances of sphere $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$ from (–0, 0, 10) and λ and μ, then	**1.** $\lambda + \mu = 6$
(B) If maximum and minimum distance of sphere $x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0$ from (0, 3, 4) and λ and μ, then	**2.** $\lambda + \mu = 10$
(C) If maximum and minimum distance of sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ from(–1, 4, –2) are λ and μ, then	**3.** $\lambda + \mu = 14$
	4. $\lambda - \mu = 6$
	5. $\lambda - \mu = 10$

a. A→3,5 B→2,4 C→1,4

b. A→3,4 B→1,4 C→2,4

c. A→3,2 B→2,3 C→1,4

d. A→3,5 B→4,3 C→1,2

Integer

96. A variable plane is at a constant distance p form the origin and meets the axes in A, B and C. If the locus of the centroid of the tetrahedron $OABC$ is $x^{-2} + y^{-2} + z^{-2} = \lambda p^{-2}$ then the value of 160λ must be

97. The lines $\frac{x+4}{3}=\frac{y+6}{5}=\frac{z-1}{-2}$ and $3x - 2y + z + 5 = 0 = 2x + 3y + 4z - k$ are coplanar for k is equal to

98. The shortest distance between the z-axis and the lines $x+y+2z-3=0,\ 2x+3y+4z-4=0$ must be

99. If the volume of tetrahedron formed by planes whose equations are $y + z = 0$, $z + x = 0$, $x + y = 0$ and $x + y + z = 1$ is λ cubic unit then the value of 729λ must be

100. If the angle of intersection of the sphere $x^2+y^2+z^2-2x-4y-6z+10=0$ with the sphere, the extremities of whose diameter are (1, 2, –3) and (5,0,1) is $\cos^{-1}(\lambda)$, then the value of $9999|\lambda|$ must be

ANSWER

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
c	c	a	a	a	b	a	b	a	c
11.	**12.**	**13.**	**14.**	**15.**	**16.**	**17.**	**18.**	**19.**	**20.**
b	c	a	a	a	a	d	d	c	a
21.	**22.**	**23.**	**24.**	**25.**	**26.**	**27.**	**28.**	**29.**	**30.**
a	a	b	b	a	d	b	b	d	c
31.	**32.**	**33.**	**34.**	**35.**	**36.**	**37.**	**38.**	**39.**	**40.**
c	d	a	c	a	c	b	c	b	a
41.	**42.**	**43.**	**44.**	**45.**	**46.**	**47.**	**48.**	**49.**	**50.**
b	b	c	a	a	b	a	d	d	a
51.	**52.**	**53.**	**54.**	**55.**	**56.**	**57.**	**58.**	**59.**	**60.**
b	c	a	a	d	a	b	d	b	a
61.	**62.**	**63.**	**64.**	**65.**	**66.**	**67.**	**68.**	**69.**	**70.**
b	c	a	b	b,c	b,d	b,c	a,d	c	a,c
71.	**72.**	**73.**	**74.**	**75.**	**76.**	**77.**	**78.**	**79.**	**80.**
a,b,c	a,b	a,c	a,b	b	a	d	b	b	b
81.	**82.**	**83.**	**84.**	**85.**	**86.**	**87.**	**88.**	**89.**	**90.**
c	c	a	d	b	d	c	b	c	a
91.	**92.**	**93.**	**94.**	**95.**	**96.**	**97.**	**98.**	**99.**	**100.**
d	c	b	a	a	2560	2	2	486	6666

SOLUTION

Multiple Choice Questions

1. **(c)** Distance $= \sqrt{x^2+z^2} = \sqrt{16+25}$

$= \sqrt{41}$

2. **(c)** Let the point P be $\left(\dfrac{5k+2}{k+1}, \dfrac{k+2}{k+1}, \dfrac{-2k+1}{k+1}\right)$.

$\because$ Given that $\dfrac{5k+2}{k+1} = 4$

$\Rightarrow$ $k = 2$

$\therefore$ z-co-ordinate of $P = \dfrac{-2(2)+1}{2+1} = -1$

3. **(a)** (1, 2, –1) is the centroid of the tetrahedron

$\therefore$ $1 = \dfrac{0+a+1+2}{4}$

$\Rightarrow$ $a = 1,\ 2 = \dfrac{0+2+b+1}{4}$

$\Rightarrow$ $b = 5,\ -1 = \dfrac{0+3+2+c}{4}$

$\Rightarrow$ $c = -9$.

$\therefore$ $(a,b,c) = (1, 5, -9)$.

Its distance from origin $= \sqrt{1+25+81} = \sqrt{107}$

System of co-ordinates, Direction cosines and direction ratios, Projection

4. **(a)** Given that $\beta = \gamma = 60°$

i.e. $m = \cos\beta = \cos 60° = 1/2$, $n = \cos\gamma = \cos 60° = 1/2$

$\because$ $l^2 + m^2 + n^2 = 1$

$\Rightarrow$ $l^2 = 1 - \dfrac{1}{4} - \dfrac{1}{4} = \dfrac{1}{2}$

$\Rightarrow$ $l = \dfrac{1}{\sqrt{2}} \Rightarrow \cos\alpha = \dfrac{1}{\sqrt{2}} \Rightarrow \alpha = 45°$

5. **(a)** Let l, m, n be the d.c.'s of a given line.

Then, as it makes an acute angle with x-axis, therefore $l>0$.

Direction ratios = 4, –4, –2 or 2, –2, –1 and Direction cosines $= \dfrac{2}{3}, \dfrac{-2}{3}, \dfrac{-1}{3}$.

6. **(b)** We know that $l^2 + m^2 + n^2 = 1$

$\Rightarrow$ $\dfrac{1}{c^2} + \dfrac{1}{c^2} + \dfrac{1}{c^2} = 1$

$\Rightarrow$ $\dfrac{3}{c^2} = 1$

$\Rightarrow$ $c = \pm\sqrt{3}$.

7. **(a)** D.r.'s of **r** are 2, –3, 6. Therefore, its d.c.'s are

$l = \dfrac{2}{7}, m = \dfrac{-3}{7}, n = \dfrac{6}{7}$

$\therefore$ $\mathbf{r} = |\mathbf{r}|(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$

$= 21\left[\dfrac{2}{7}\mathbf{i} - \dfrac{3}{7}\mathbf{j} + \dfrac{6}{7}\mathbf{k}\right] = 6\mathbf{i} - 9\mathbf{j} + 18\mathbf{k}$.

8. **(b)** Let AB be the line and its direction cosines be $\cos\alpha$, $\cos\beta$, $\cos\gamma$. Then the projection of line AB on the co-ordinate axes are $AB\cos\alpha$, $AB\cos\beta$, $AB\cos\gamma$.

$\therefore$ $AB\cos\alpha = 2$, $AB\cos\beta = 3$, $AB\cos\gamma = 6$

$\Rightarrow$ $AB^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 2^2 + 3^2 + 6^2 = 49$

$\Rightarrow$ $AB^2(1) = 49 \Rightarrow AB = 7$

9. **(a)** We have, $l + 2m + 3n = 0$ . . .(*i*)

$3lm - 4ln + mn = 0$ . . .(*ii*)

From equation (i), $l = -(2m+3n)$

Putting the value of l in equation (ii)

$\Rightarrow$ $3(-2m-3n)m + mn - 4(-2m-3n)n = 0$

$\Rightarrow$ $-6m^2 - 9mn + mn + 8mn + 12n^2 = 0 \Rightarrow 6m^2 - 12n^2 = 0$

$\Rightarrow$ $m^2 - 2n^2 = 0 \Rightarrow m + \sqrt{2}n = 0$ or $m - \sqrt{2}n = 0$

$l + 2m + 3n = 0$ . . .(*i*)

$0.l + m + \sqrt{2}n = 0$ . . .(*iii*)

$0.l + m - \sqrt{2}n = 0$ . . .(*iv*)

From equation (i) and equation (iii), $\frac{l}{2\sqrt{2}-3}=\frac{m}{-\sqrt{2}}=\frac{n}{1}$

From equation (i) and equation (iv), $\frac{l}{-2\sqrt{2}-3}=\frac{m}{\sqrt{2}}=\frac{n}{1}$

Thus, the direction ratios of two lines are $2\sqrt{2}-3,-\sqrt{2},1$

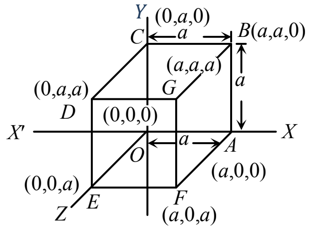
And $-2\sqrt{2}-3,\sqrt{2},1$; $(l_1,m_1,n_1)=(2\sqrt{2}-3,-\sqrt{2},1)$,

$(l_2,m_2,n_2)=(-2\sqrt{2}-3,\sqrt{2},1)$

$l_1l_2+m_1m_2+n_1n_2=0$.

Hence, the angle between them $\pi/2$.

10. (c)



Let side of the cube = a

Then OG, BE and AD, CF will be four diagonals.

d.r.'s of $OG = a, a, a = 1, 1, 1$

d.r.'s of $BE = -a, -a, a = 1, 1, -1$

d.r.'s of $AD = -a, a, a = -1, 1, 1$

d.r.'s of $CF = a, -a, a = 1, -1, 1$

Let d.r.'s of line be l, m, n.

Therefore angle between line and diagonal

$\cos\alpha=\frac{l+m+n}{\sqrt{3}}, \cos\beta=\frac{l+m-n}{\sqrt{3}},$

$\cos\gamma=\frac{-l+m+n}{\sqrt{3}}, \cos\delta=\frac{l-m+n}{\sqrt{3}}$

$\Rightarrow \cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta$

$=\frac{1}{3}[(l+m+n)^2+(l+m-n)^2+(-l+m+n)^2$

$+(l-m+n)^2]=\frac{4}{3}$

$\Rightarrow \sin^2\alpha+\sin^2\beta+\sin^2\gamma+\sin^2\delta=\frac{8}{3}$

11. (b) We have $6x-2=3y+1=2z-2$

$\Rightarrow \frac{6x-(2/6)}{1}=\frac{3y+(1/3)}{1}=\frac{2(z-1)}{1}$

$\Rightarrow \frac{x-(1/3)}{1/6}=\frac{y+(1/3)}{1/3}=\frac{z-1}{1/2}$

$\Rightarrow \frac{x-(1/3)}{1}=\frac{y+(1/3)}{2}=\frac{z-1}{3}$ d.r.'s of line are (1, 2, 3).

Hence, d.c.'s of line are $(1/\sqrt{14},2/\sqrt{14},3/\sqrt{14})$

12. (c) Position vector of A is $\mathbf{a}=3\mathbf{i}+4\mathbf{j}-7\mathbf{k}$ and that of B is $\mathbf{b}=\mathbf{i}-\mathbf{j}+6\mathbf{k}$

We know that equation of line in vector form, $\mathbf{r}=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})$

$\mathbf{r}=(3\mathbf{i}+4\mathbf{j}-7\mathbf{k})+\lambda(-2\mathbf{i}-5\mathbf{j}+13\mathbf{k})$.

13. (a) We have $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$

And $\frac{x-1}{3k}=\frac{y-5}{1}=\frac{z-6}{-5}$

Since lines are $\perp$ to each other.

So, $a_1a_2+b_1b_2+c_1c_2=0$

$(-3)(3k)+(2k)(1)+(2)(-5)=0$

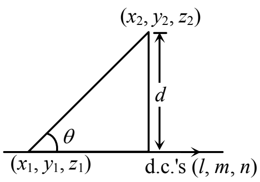
$\Rightarrow -9k+2k-10=0 \Rightarrow -7k=10 \Rightarrow k=-10/7$.

14. (a) Let $\mathbf{r}_1=(x_2-x_1)\mathbf{i}+(y_2-y_1)\mathbf{j}+(z_2-z_1)\mathbf{k}$

$\mathbf{r}_2=l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$

$\therefore \cos\theta=\frac{\mathbf{r}_2.\mathbf{r}_1}{|\mathbf{r}_1||\mathbf{r}_2|}$

Also, $d=|\mathbf{r}_1|\sin\theta$, $d^2=|\mathbf{r}_1|^2\sin^2\theta$



$\Rightarrow d^2=|\mathbf{r}_1|^2(1-\cos^2\theta)$

$\Rightarrow d^2=|\mathbf{r}_1|^2\left(1-\frac{\mathbf{r}_1.\mathbf{r}_2}{|\mathbf{r}_1|^2|\mathbf{r}_2|^2}\right)$

$\Rightarrow d^2=|\mathbf{r}_1|^2-(\mathbf{r}_1.\mathbf{r}_2)^2$, {where $|\mathbf{r}_2|=1$}

$\Rightarrow d=\sqrt{|\mathbf{r}_1|^2-(\mathbf{r}_1.\mathbf{r}_2)^2}$

Therefore, distance of the point (x_1,y_1,z_1) from the line is

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2 -[l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)]^2}$$

15. (a) $\theta=\cos^{-1}\left(\frac{3+0-5}{\sqrt{1+1}\sqrt{9+16+25}}\right)$

$=\cos^{-1}\left(\frac{-2}{\pm 10}\right)=\cos^{-1}\left(\frac{1}{5}\right)$.

16. (a) $\theta=\cos^{-1}\left[\frac{(1)(2)+(2)(-3)+(1)(6)}{\sqrt{1^2+2^2+1^2}\sqrt{2^2+(-3)^2+6^2}}\right]$

$\cos^{-1}\left[\frac{2-6+6}{\sqrt{6}\sqrt{49}}\right]=\cos^{-1}\left[\frac{2}{7\sqrt{6}}\right]$.

17. **(d)** $l+m+n=0, l^2+m^2-n^2=0$

and $l^2+m^2+n^2=1$

Solving above equations, we get $m=\pm\frac{1}{\sqrt{2}}, n=\pm\frac{1}{\sqrt{2}}$

And $l=0$.

$\therefore \quad \theta=\frac{\pi}{3}$ or $\frac{\pi}{2}$.

18. **(d)** $\theta=\cos^{-1}\left(\frac{(2)(1)+(2)(2)+(-1)(2)}{\sqrt{2^2+2^2+1^2}\sqrt{1^2+2^2+2^2}}\right)$

$=\cos^{-1}\frac{4}{9}$.

19. **(c)** $\cos\theta=\frac{1(\sqrt{3}-1)-1(\sqrt{3}+1)+2\times 4}{\sqrt{6}\sqrt{24}}$

$=\frac{6}{12}$

$\Rightarrow \quad \theta=60^\circ$.

20. **(a)** $\theta=\cos^{-1}\frac{(-15-48+65)}{\sqrt{25+144+169}\sqrt{9+16+25}}$

$=\cos^{-1}\left(\frac{2}{(13\sqrt{2})(5\sqrt{2})}\right)=\cos^{-1}\left(\frac{1}{65}\right)$.

21. **(a)** Given lines are, $\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1}=r_1$, (say)

and $\frac{x+3}{-36}=\frac{y-3}{2}=\frac{z-6}{4}=r_2$, (say)

$\therefore \quad x=3r_1+5=-36r_2-3$

$y=-r_1+7=3+2r_2$

and $z=r_1-2=4r_2+6$

On solving, we get $x=21, y=\frac{5}{3}, z=\frac{10}{3}$.

Trick: Check through options

22. **(a)** The given line is $6x-2=3y+1=2z-2$

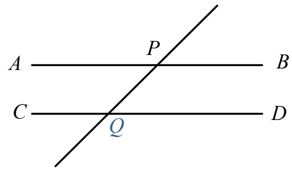
$\Rightarrow \quad \frac{x-1/3}{1}=\frac{y+1/3}{2}=\frac{z-1}{3}$

This show that the given line passes through (1/3, –1/3) and has direction ratio 1, 2, 3.

Position vector $\mathbf{a}=\frac{1}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}+\mathbf{k}$ and is parallel to vector $\mathbf{b}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$.

Hence, $\mathbf{r}=\left(\frac{1}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}+\mathbf{k}\right)+\lambda(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$.

23. **(b)**



Given lines are $\frac{x}{1}=\frac{y+a}{1}=\frac{z}{1}=\lambda$ (say)

$\therefore$ Point is $P(\lambda, \lambda-a, \lambda)$

and $\frac{x+a}{1}=\frac{y}{1/2}=\frac{z}{1/2}$

i.e. $\frac{x+a}{2}=\frac{y}{1}=\frac{z}{1}=\mu$ (say)

$\therefore$ Point $Q\ (2\mu-a, \mu, \mu)$

Since d.r.'s of given lines are 2, 1, 2

and d.r.'s of $PQ=(2\mu-a-\lambda, \mu-\lambda+a, \mu-\lambda)$

According to question, $\frac{2\mu-a-\lambda}{2}=\frac{\mu-\lambda+a}{1}=\frac{\mu-\lambda}{2}$

Then, $\lambda=3a$, $\mu=a$.

Therefore, points of intersection are $P(3a, 2a, 3a)$ and $Q(a, a, a)$.

Alternative Method: Check by option $x=y+a=z$

i.e. $3a=2a+a=3a$

$\Rightarrow \quad a=a=a$

and $x+a=2y=2z$

i.e. $a+a=2a=2a$

$\Rightarrow \quad a=a=a$.

Hence (b) is correct.

24. **(b)** We have, $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}=r_1$ (Let)

$x=2r_1+1,$

$y=3r_1-1,$

$z=4r_1+1$

i.e. point is $(2r_1+1, 3r_1-1, 4r_1+1)$

and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}=r_2$

(Let) *i.e.* point is $(r_2+3, 2r_2+k, r_2)$.

If the lines are intersecting, then they have a common point.

$\Rightarrow \quad 2r_1+1=r_2+3,$

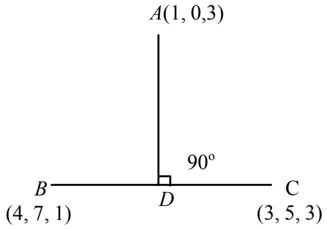
$3r_1-1=2r_2+k,$

$4r_1+1=r_2$

On solving, $r_1=-3/2,\ r_2=-5$

Hence, $k=9/2$.

25. **(a)** Equation of BC, $\dfrac{x-4}{-1}=\dfrac{y-7}{-2}=\dfrac{z-1}{2}$



i.e. $\dfrac{x-4}{1}=\dfrac{y-7}{2}=\dfrac{z-1}{-2}=r$ (say)

Any point on the given line is $D(r+4, 2r+7, -2r+1)$

Then, d.r.'s of $AD=(r+4-1, 2r+7-0, -2r+1-3)$

i.e. d.r.'s of $AD=(r+3, 2r+7, -2r-2)$

and d.r.'s of $BC=(-1, -2, 2)$

Since AD is $\perp$ to given line,

$\therefore$ $(-1)(r+3)+(2r+7)(-2)+(2)(-2r-2)=0$

$\Rightarrow$ $-r-3-4r-14-4r-4=0$

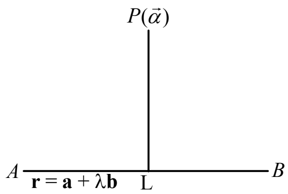
$\Rightarrow$ $-9r-21=0$

$\Rightarrow$ $r=-7/3$

$\therefore$ D is {4 – (7/3), 7– (14/3), (14/3)+1}

i.e. D is (5/3, 7/3, 17/3).

26. **(d)** $\vec{\alpha}=0.\mathbf{i}+0.\mathbf{j}+0.\mathbf{k}$



$\overrightarrow{PL}=(\mathbf{a}-\vec{\alpha})-\left(\dfrac{(\mathbf{a}-\vec{\alpha}).\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$

$\overrightarrow{PL}=(4\mathbf{i}+2\mathbf{j}+4\mathbf{k})-\left[\dfrac{(4\mathbf{i}+2\mathbf{j}+4\mathbf{k}).(3\mathbf{i}+4\mathbf{j}-5\mathbf{k})}{9+16+25}\right]$

$(3\mathbf{i}+4\mathbf{j}-5\mathbf{k})$

$=4\mathbf{i}+2\mathbf{j}+4\mathbf{k}-\left(\dfrac{12+8-20}{50}\right).(3\mathbf{i}+4\mathbf{j}-5\mathbf{k})$

$\overrightarrow{PL}=4\mathbf{i}+2\mathbf{j}+4\mathbf{k}$

The length of PL is magnitude of $\overrightarrow{PL}$

i.e., Length of perpendicular $=|\overrightarrow{PL}|=\sqrt{16+4+16}=6$.

27. **(b)** S.D. $=\dfrac{\begin{vmatrix}2-1 & 4-2 & 5-3\\ 2 & 3 & 4\\ 3 & 4 & 5\end{vmatrix}}{\sqrt{(15-16)^2+(12-10)^2+(8-9)^2}}$.

$=\dfrac{\begin{vmatrix}1 & 2 & 2\\ 2 & 3 & 4\\ 3 & 4 & 5\end{vmatrix}}{\sqrt{1+1+4}}=\dfrac{1}{\sqrt{6}}$

28. **(b)** S.D. $=\left|\dfrac{(\mathbf{b}_1\times\mathbf{b}_2).(\mathbf{a}_2-\mathbf{a}_1)}{|\mathbf{b}_1\times\mathbf{b}_2|}\right|$

$=\left|\dfrac{[(3\mathbf{i}-\mathbf{j})\times(2\mathbf{i}+3\mathbf{k})].(3\mathbf{i}-\mathbf{j})}{|(3\mathbf{i}-\mathbf{j})\times(2\mathbf{i}+3\mathbf{k})|}\right|$

$=\left|\dfrac{(-3\mathbf{i}-9\mathbf{j}+2\mathbf{k}).(3\mathbf{i}-\mathbf{j})}{\sqrt{9+81+4}}\right|=\dfrac{-9+9+0}{\sqrt{94}}$

Hence, S.D. = 0

29. **(d)** We have $\dfrac{x-1}{1}=\dfrac{y+3}{-\lambda}=\dfrac{z-1}{\lambda}=s$

and $\dfrac{2x}{1}=\dfrac{y-1}{1}=\dfrac{z-2}{-1}=t$

i.e. $\dfrac{x-0}{1}=\dfrac{y-1}{2}=\dfrac{z-2}{-2}=\dfrac{t}{2}$

Since, lines are co-planar,

Then, $\begin{vmatrix}x_2-x_1 & y_2-y_1 & z_2-z_1\\ l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\end{vmatrix}=0$

$\Rightarrow$ $\begin{vmatrix}-1 & 4 & 1\\ 1 & -\lambda & \lambda\\ 1 & 2 & -2\end{vmatrix}=0$

On solving, $\lambda=-2$.

30. **(c)** Required ratio $=-\left(\dfrac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}\right)$

$=-\left(\dfrac{-2-8+21-17}{3+10+24-17}\right)=\dfrac{6}{20}=\dfrac{3}{10}$.

31. **(c)** Required ratio $=-\dfrac{z_1}{z_2}=-\left(\dfrac{4}{6}\right)=-\dfrac{2}{3}$

$\therefore$ xy-plane divide externally in the ratio 2 : 3.

32. **(d)** We know that $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$ . . .(*i*)

Centroid $\left(\dfrac{a}{3},\dfrac{b}{3},\dfrac{c}{3}\right)$

i.e. $\alpha=a/3, \beta=b/3, \gamma=c/3$

$\Rightarrow$ $a=3\alpha, b=3\beta, c=3\gamma$

From equation (*i*), $\dfrac{x}{3\alpha}+\dfrac{y}{3\beta}+\dfrac{z}{3\gamma}=1$

$\therefore$ $\dfrac{x}{\alpha}+\dfrac{y}{\beta}+\dfrac{z}{\gamma}=3$.

33. **(a)** We know that, $\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

$$= \frac{1(-5)+2(3)+2(4)}{\sqrt{1+4+4}\sqrt{25+9+16}} = \frac{9}{3.5\sqrt{2}} = \frac{3\sqrt{2}}{10}$$

i.e. $\theta = \cos^{-1}\left(\dfrac{3\sqrt{2}}{10}\right)$.

34. **(c)** We have $2x+y+2z-8=0$ . . .(*i*)

and $4x+2y+4z+5=0$

or $2x+y+2z+5/2=0$ . . .(*ii*)

Distance between the planes $= \dfrac{(5/2)+8}{\sqrt{4+1+4}} = \dfrac{21}{2.3} = \dfrac{7}{2}$.

35. **(a)** Angle between two plane faces is equal to the angle between the normals n_1 and n_2 to the planes. $\mathbf{n_1}$,the normal to the face *OAB* is given by

$$\overrightarrow{OA}\times\overrightarrow{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 5\mathbf{i}-\mathbf{j}-3\mathbf{k} \quad . . .(i)$$

$\mathbf{n_2}$, the normal to the face *ABC*, is given by $\overrightarrow{AB}\times\overrightarrow{AC}$.

$$\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ -2 & -1 & 1 \end{vmatrix} = \mathbf{i}-5\mathbf{j}-3\mathbf{k} \quad . . .(ii)$$

If θ be the angle between $\mathbf{n_1}$ and $\mathbf{n_2}$

Then $\cos\theta = \dfrac{\mathbf{n}_1.\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \dfrac{5.1+5+9}{\sqrt{35}\sqrt{35}}$

$\cos\theta = \dfrac{19}{35} \Rightarrow \theta = \cos^{-1}\left(\dfrac{19}{35}\right)$.

36. **(c)** Distance of the plane from (2, 1, –1)

$$= \left|\frac{2-2(1)+4(-1)-9}{\sqrt{1+4+16}}\right| = \frac{13}{\sqrt{21}}.$$

37. **(b)** We know that, $\dfrac{\overrightarrow{PQ}\times\overrightarrow{PR}}{|\overrightarrow{PQ}\times\overrightarrow{PR}|}$

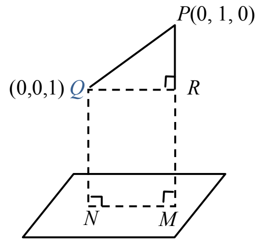
$\overrightarrow{PQ} = \mathbf{i}+\mathbf{j}-3\mathbf{k}$, $\overrightarrow{PR} = -\mathbf{i}+3\mathbf{j}-\mathbf{k}$

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} = 8\mathbf{i}+4\mathbf{j}+4\mathbf{k}$$

And $|\overrightarrow{PQ}\times\overrightarrow{PR}| = 4\sqrt{6}$

Hence, the unit vector is $\dfrac{4(2\mathbf{i}+\mathbf{j}+\mathbf{k})}{4\sqrt{6}}$ *i.e.* $\dfrac{2\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{6}}$.

38. **(c)** Given plane is $x + y + z - 3 = 0$. From point *P* and *Q* draw *PM* and *QN* perpendicular on the given plane and $QR \perp MP$.



$|MP| = \left|\dfrac{0+1+0-3}{\sqrt{1^2+1^2+1^2}}\right| = \dfrac{2}{\sqrt{3}}$ $|NQ| = \dfrac{2}{\sqrt{3}}$

$|PQ| = \sqrt{(0-0)^2+(0-1)^2+(1-0)^2} = \sqrt{2}$

$|RP| = |MP| - |MR| = |MP| - |NQ| = 0$

(*i.e. R* and *P* are the same point)

$\therefore$ $|NM| = |QR| = \sqrt{PQ^2 - RP^2} = \sqrt{(\sqrt{2})^2 - 0} = \sqrt{2}$

39. **(b)** Let *P* be the point (2, –1, 3) and *Q* be its reflection in the given plane.

Then, *PQ* is perpendicular to the given plane

Hence, d.r.'s of *PQ* are 3, –2, 1 and consequently, equations of *PQ* are $\dfrac{x-2}{3} = \dfrac{y+1}{-2} = \dfrac{z-3}{-1}$

Any point on this line is $(3r+2, -2r-1, -r+3)$

Let this point be *Q*. Then midpoint of *PQ*

$$= \left(\frac{3r+2+2}{2}, \frac{-2r-1-1}{2}, \frac{-r+3+3}{2}\right)$$

$$= \left(\frac{3r+4}{2}, -r-1, \frac{-r+6}{2}\right)$$

This point lies in given plane *i.e.*

$$3\left(\frac{3r+4}{2}\right) - 2(-r-1) - \left(\frac{-r+6}{2}\right) = 9$$

$\Rightarrow$ $9r+12+4r+4+r-6=9$

$\Rightarrow$ $14r = 8 \Rightarrow r = \dfrac{4}{7}$

Hence, the required point *Q* is

$$\left(3\left(\frac{4}{7}\right)+2, -2\left(\frac{4}{7}\right)-1, \frac{-4}{7}+3\right). = \left(\frac{26}{7}, \frac{-15}{7}, \frac{17}{7}\right)$$

40. **(a)** Equation of plane containing **i** and **i** + **j** is

$[\mathbf{r}-\mathbf{i}, \mathbf{i}, \mathbf{i}+\mathbf{j}] = 0 \Rightarrow (\mathbf{r}-\mathbf{i}).[\mathbf{i}\times(\mathbf{i}+\mathbf{j})] = 0$

$\Rightarrow$ $[(x-1)\mathbf{i}+y\mathbf{j}+z\mathbf{k}].\mathbf{k} = 0 \Rightarrow z = 0$ . . . (*i*)

Equation of plane containing **i** – **j** and **i** + **k** is

$\Rightarrow$ $[\mathbf{r}-(\mathbf{i}-\mathbf{j}) \quad \mathbf{i}-\mathbf{j} \quad \mathbf{i}+\mathbf{k}] = 0$

$\Rightarrow$ $(\mathbf{r}-\mathbf{i}+\mathbf{j})[(\mathbf{i}-\mathbf{j})\times(\mathbf{i}+\mathbf{k})]=0$

$\Rightarrow$ $x+y-z=0$ ...(*ii*)

Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$. Since **a** is parallel to (*i*) and (*ii*)

$a_3=0, a_1+a_2-a_3=0$

$\Rightarrow$ $a_1=-a_2,\ a_3=0$

Thus a vector in the direction of **a** is $\mathbf{u}=\mathbf{i}-\mathbf{j}$. If θ is the angle between **a** and $\mathbf{i}-2\mathbf{j}+2\mathbf{k}$.

Then $\cos\theta=\pm\frac{1(1)+(-1)(-2)}{\sqrt{1+1}\sqrt{1+4+4}}=\pm\frac{3}{\sqrt{2}.3}$

$\Rightarrow$ $\cos\theta=\pm\frac{1}{\sqrt{2}}\Rightarrow\theta=\pi/4$ or $3\pi/4$

41. **(b)** Let d.r.'s of normal to plane (a, b, c)

$a(x-1)+b(y-0)+c(z-0)=0$ ...(*i*)

It is passes through (0, 1, 0).

$\therefore$ and the plane , is given by.

(a) The line is perpendicular to the plane if and only if.

(b) The line is parallel to the plane if and only if.

(c) The line lies in the plane if and only if and.

42. **(b)** $\theta=\cos^{-1}\left[\frac{6+4-10}{\sqrt{50}\sqrt{9}}\right]=\cos^{-1}(0)=\frac{\pi}{2}$.

43. **(c)** Equation is $\frac{x}{-6}+\frac{y}{3}+\frac{z}{4}=1$ or $-2x+4y+3z=12$

$\therefore$ Length of perpendicular from origin $=\frac{12}{\sqrt{4+16+9}}=\frac{12}{\sqrt{29}}$.

44. **(a)** Planes are perpendicular, if $6-6+2k=0$

$\Rightarrow$ $k=0$.

45. **(a)** Equation of plane containing the line of intersection of planes is, $(2x-y)+\lambda(y-3z)=0$ ... (i)

Also, plane (i) is perpendicular to $4x+5y-3z-8=0$

$\therefore$ $4(2)+5(\lambda-1)-3(-3\lambda)=0$

$\Rightarrow$ $14\lambda=-3\Rightarrow\lambda=-\frac{3}{14}$

Put the value of λ in (i), we get $28x-17y+9z=0$, which is the required plane.

46. **(b)** According to question, $(x-3)^2+(y-4)^2+(z+2)^2$

$=(x-2)^2+(y-3)^2+(z+3)^2$

$\therefore$ The equation reduces to a plane as 2[nd] degree terms cancel out.

The equation is $2x+2y+2z=7$, hence equally inclined to axes.

47. **(a)** Obviously, $(x-2)+5(y+3)-6(z-1)=0$

$\Rightarrow$ $x+5y-6z+19=0$.

48. **(d)** Centroid is $\left(\frac{\frac{1}{a}+0+0}{3},\frac{0+\frac{1}{b}+0}{3},\frac{0+0+\frac{1}{c}}{3}\right)$

i.e., $\left(\frac{1}{3a},\frac{1}{3b},\frac{1}{3c}\right)$.

49. **(d)** Distance of point *P* from origin $OP=\sqrt{4+36+9}=7$

Now d.r's of *OP* = 2–0, 6 – 0, 3 – 0 = 2, 6, 3

$\therefore$ d.c's of *OP* = $\frac{2}{7},\frac{6}{7},\frac{3}{7}$

$\therefore$ Equation of plane in normal form is $lx+my+nz=p$

$\Rightarrow$ $\frac{2}{7}x+\frac{6}{7}y+\frac{3}{7}z=7\Rightarrow 2x+6y+3z=49$.

50. **(a)** We have, $\frac{x-1}{2}=\frac{y-1}{3}=\frac{z-k}{k}$ or vector form of equation of line is $\mathbf{r}=(\mathbf{i}+\mathbf{j}+k\mathbf{k})+\lambda(2\mathbf{i}+3\mathbf{j}+k\mathbf{k})$

i.e. $\mathbf{b}=2\mathbf{i}+3\mathbf{j}+k\mathbf{k}$ and normal to the plane, $\mathbf{n}=2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$.

Given that, $\mathbf{b}.\mathbf{n}=0\Rightarrow(2\mathbf{i}+3\mathbf{j}+k\mathbf{k}).(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})=0$

$\Rightarrow$ $4+9+4k=0\Rightarrow k=-13/4$.

51. **(b)** We know that $\sin\theta=\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$

$\sin\theta=\frac{3(2)+4(-2)+5(1)}{\sqrt{9+16+25}\sqrt{4+4+1}}=\frac{3}{5\sqrt{2}.3}$

Hence, $\sin\theta=\frac{\sqrt{2}}{10}$

52. **(c)** Let equation of line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ ...(*i*)

We have $4x+4y-5z=12$ ...(*ii*)

and $8x+12y-13z=32$ ...(*iii*)

Let $z = 0$. Now putting $z = 0$ in (*ii*) and (*iii*), we get, $4x+4y=12$, $8x+12y=32$,on solving these equations, we get $x=1, y=2$.

Equation of line passing through (1,2,0) is

$\frac{x-1}{l}=\frac{y-2}{m}=\frac{z-0}{n}$

From equation (i) and (ii), $4l+4m-5n=0$

and $8l+12m-13n=0$

$\Rightarrow$ $\frac{l}{8}=\frac{m}{12}=\frac{n}{16}$ *i.e.* $\frac{l}{2}=\frac{m}{3}=\frac{n}{4}$.

Hence, equation of line is $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z}{4}$.

53. **(a)** Any plane through the first line may be written as

$a(x-1)+b(y+1)+c(z)=0$ . . .(*i*)

Where, $2a-b+3c=0$ . . .(*ii*)

It will pass through the second line, if the point (0, 2, –1) on the second line also lies on (*i*)

i.e. if $a(0-1)+b(2+1)+c(-1)=0$,

i.e., $-a+3b-c=0$ . . . (*iii*)

Solving (ii) and (iii), we get $\frac{a}{-8}=\frac{b}{-1}=\frac{c}{5}$

i.e $\frac{a}{8}=\frac{b}{1}=\frac{c}{-5}$

$\therefore$ Required plane is $8(x-1)+1(y+1)-5(z)=0$

$\Rightarrow$ $8x+y-5z-7=0$.

54. **(a)** Any plane through the line $\frac{x-3}{1}=\frac{y-6}{5}=\frac{z-4}{4}$ is

$a(x-3)+b(y-6)+c(z-4)=0$ . . .(*i*)

where, $a+5b+4c=0$ . . .(*ii*)

Plane (i) passes through (3, 2, 0), if

$a(3-3)+b(2-6)+c(0-4)=0$

$-4b-4c=0$ *i.e.* $b+c=0$ . . .(*iii*)

From equation (*ii*) and (*iii*), $a+b=0$.

$\therefore$ $a=-b=c$.

$\therefore$ Required plane is $a(x-3)-a(y-6)+a(z-4)=0$ *i.e.* $x-y+z-3+6-4=0$

i.e. $x-y+z=1$.

Short Trick $\begin{vmatrix} x-3 & y-6 & z-4 \\ 3-3 & 2-6 & 0-4 \\ 1 & 5 & 4 \end{vmatrix} = \begin{vmatrix} x-3 & y-6 & z-4 \\ 0 & -4 & -4 \\ 1 & 5 & 4 \end{vmatrix}$

$\Rightarrow$ $x-y+z=1$.

55. **(d)** The given line is $\mathbf{r}=(\mathbf{i}+\mathbf{j}+2\mathbf{k})+\lambda(2\mathbf{i}+5\mathbf{j}+3\mathbf{k})$

$\mathbf{a}=\mathbf{i}+\mathbf{j}+2\mathbf{k}$, $\mathbf{b}=2\mathbf{i}+5\mathbf{j}+3\mathbf{k}$

Given plane, $\mathbf{r}.(2\mathbf{i}+\mathbf{j}-3\mathbf{k})=5$

$\Rightarrow$ $\mathbf{r}.\mathbf{n}=p$

Since $\mathbf{b}.\mathbf{n}=4+5-9=0$

$\therefore$ The line is parallel to plane. Thus the distance between line and plane is equal to length of perpendicular from a point $\mathbf{a}=\mathbf{i}+\mathbf{j}+2\mathbf{k}$ on line to given plane.

Hence, required distance $=\left|\frac{(\mathbf{i}+\mathbf{j}+2\mathbf{k}).(2\mathbf{i}+\mathbf{j}-3\mathbf{k})-5}{\sqrt{4+1+9}}\right|$

$=\left|\frac{2+1-6-5}{\sqrt{14}}\right|=\frac{8}{\sqrt{14}}$.

56. **(a)** We have the spheres

$x^2+y^2+z^2+7x-2y-z-13=0$

and $x^2+y^2+z^2-3x+3y+4z-8=0$

Required plane is $S_1-S_2=0$

$\therefore$ $(7x+3x)-(2y+3y)-(z+4z)-5=0$

i.e. $10x-5y+(-5z)-5=0$

$\Rightarrow$ $2x-y-z=1$.

57. **(b)** Ratio $-\left[\frac{2\,(2)+(-3)(1)+(1)(1)-7}{2\,(3)+(-4)(1)+(-5)(1)-7}\right]$

$=-\left[\frac{-5}{-10}\right]=-\left(\frac{1}{2}\right)$

$\therefore$ $x=\frac{2(2)-3(1)}{1}=1,\ y=\frac{-3(2)-(-4)}{1}=-2$

and $z=\frac{1(2)-(-5)}{1}=7$.

Therefore, $P(1,-2,7)$.

Trick**:** As (1, – 2, 7) and (– 1, 2, 7) satisfy the equation $2x+y+z=7$, but the point (1, – 2, 7) is collinear with (2, – 3, 1) and (3, – 4, – 5).

Note: If a point dividing the join of two points in some particular ratio, then this point must be collinear with the given points.

58. **(d)** $a(x-4)+b(y-3)+c(z-2)=0$

$\therefore$ $a+b+2c=0$

and $a-4b+5c=0$

$\frac{a}{5+8}=\frac{b}{2-5}=\frac{c}{-4-1}=k$

$\frac{a}{13}=\frac{b}{-3}=\frac{c}{-5}=k$

Therefore, the required equation of plane is $-13x+3y+5z+33=0$.

59. **(b) Trick:** Since line is parallel to plane if $al+bm+cn=0$

$\therefore$ From option b., $3(2)+4(1)+5(-2)=0$.

Clearly, $2x+y-2y=0$ is the required plane.

60. **(a)** $\frac{x-1}{l}=\frac{y-2}{m}=\frac{z-3}{n}$

or $l-m+2n=0$

and $3l+m+n=0$

$\therefore$ $\frac{x-1}{-3}=\frac{y-2}{5}=\frac{z-3}{4}$.

61. **(b)** Line is $\frac{x+3}{3}=\frac{y-2}{-2}=\frac{z+1}{1}=\lambda$ (Let)

$x=3\lambda-3;$

$y=-2\lambda+2;$ $z=\lambda-1$ line intersects plane, therefore,

$4(3\lambda-3)+5(-2\lambda+2)+3(\lambda-1)-5=0$

$\Rightarrow$ $\lambda=2$.

So, $x=3;$

$y=-2;$

$z=1$.

Since the point (3,–2,1) satisfies both the equations.

62. **(c)** According to $\frac{A}{l}=\frac{B}{m}=\frac{C}{n}$, direction ratio of plane are respectively (3, 0, 4).

Equation of plane passing through point (1, 1, 1) is

$\Rightarrow$ $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$

$\Rightarrow$ $3(x-1)+0(y-1)+4(z-1)=0$

$\Rightarrow$ $3x+4z-7=0$

Normal form of plane is, $\frac{3x}{5}+\frac{4z}{5}=\frac{7}{5}$

$\therefore$ Perpendicular distance from $(0,0,0)=\frac{7}{5}$.

63. **(a)** Plane passing through (3, 2, 0) is

$A(x-3)+B(y-2)+c(z-0)=0$ . . . (i)

Plane (i) is passing through the line, $\frac{x-3}{1}=\frac{y-6}{5}=\frac{z-4}{4}$

$\therefore$ $A(3-3)+B(6-2)+C(4-0)=0$

$0.A+4B+4C=0$ . . . (*ii*)

and also $1.A+5B+4C=0$ . . . (*iii*)

Solving (ii) and (iii), we get $x-y+z=1$.

Trick: Required plane is $\begin{vmatrix} x-3 & y-6 & z-4 \\ 3-3 & 2-6 & 0-4 \\ 1 & 5 & 4 \end{vmatrix}=0$

Solving, we get $x-y+z=1$.

64. **(b)** Angle between the plane and line is

$\sin\theta=\frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}$

Here, $aa'+bb'+cc'$

$=2\times3+3\times2-4\times3=0$

$\therefore$ $\sin\theta=0$

$\Rightarrow$ $\theta=0°$.

NCERT Exemplar Problems

More than One Answer

65. **(b, c)** Since, $\frac{x-1}{2}=\frac{y+1}{K}=\frac{z}{2}$ and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{k}$ are coplanar.

$\Rightarrow$ $\begin{vmatrix} 2 & 0 & 0 \\ 2 & K & 2 \\ 5 & 2 & K \end{vmatrix}=0$

$\Rightarrow$ $K^2=4$

$\Rightarrow$ $K=\pm2$

$\therefore$ $\vec{n}_1=\vec{b}_1\times\vec{d}_1=6\hat{j}-6\hat{k},$ for $k=2$

$\therefore$ $\vec{n}_2=\vec{b}_2\times\vec{d}_2=14\hat{j}+14\hat{k},$ for $k=-2$

So, equation of planes are $(\vec{r}-\vec{a})\cdot\vec{n}_1=0$

$\Rightarrow$ $y-z=-1$

and $(\vec{r}-\vec{a})\cdot\vec{n}_2=0$

$\Rightarrow$ $y+z=-1$

66. **(b, d)** $I=\int_{-\pi/2}^{\pi/2}\left(x^2+\log\left(\frac{\pi-x}{\pi+x}\right)\right)\cos x\,dx$

As, $\int_{-a}^{a}f(x)dx=0$, when $f(-x)=-f(x)$

$\therefore$ $I=\int_{-\pi/2}^{\pi/2}x^2\cos x\,dx+0=2\int_0^{\pi/2}(x^2\cos x)\,dx$

$=2\{(x^2\sin x)_0^{\pi/2}-\int_0^{\pi/2}2x\cdot\sin x\,dx\}$

$=2\left[\frac{\pi^2}{4}-2\{(-x\cdot\cos x)_0^{\pi/2}-\int_0^{\pi/2}1\cdot(-\cos x)\,dx\}\right]$

$=2\left[\frac{\pi^2}{4}-2(\sin x)_0^{\pi/2}\right]=2\left[\frac{\pi^2}{4}-2\right]=\left(\frac{\pi^2}{2}-4\right)$

67. **(b, c)** Let $l:\frac{x-0}{a}=\frac{y-0}{b}=\frac{z-0}{c}$

Which is perpendicular to $l_1:(3\hat{i}-\hat{j}+4\hat{k})+t(\hat{i}+2\hat{j}+2\hat{k})$

$l_2:(3\hat{i}+3\hat{j}+2\hat{k})+s(2\hat{i}+2\hat{j}+\hat{k})$

$\therefore$ DR's of l, is $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{vmatrix}=-2\hat{i}+3\hat{j}-2\hat{k}$

$\therefore$ $l:\frac{x}{-2}=\frac{y}{3}=\frac{z}{-2}=k_1,k_2$

Now, $A(-2k_1,3k_1,-2k_1)$ and $B(-2k_2,3k_2,-2k_2)$

Since, A lies on l_1

$\therefore$ $(-2k_1)\hat{i}+(3k_1)\hat{j}-(2k_1)\hat{k}=(3+t)\hat{i}+(-1+2t)\hat{j}+(4+2t)\hat{k}$

$\Rightarrow$ $3+t=-2k_1,-1+2t=3k_1,4+2t=-2k_1$

$\therefore \quad k_1 = -1$

$\Rightarrow \quad A(2,-3,2)$

Let any point on $l_2 (3+2s, 3+2s, 2+5)$

$\Rightarrow \quad \sqrt{(2-3-2s)^2 + (-3-3-2s)^2 + (2-2-s)^2} = \sqrt{17}$

$\Rightarrow \quad 9s^2 + 28s + 37 = 17$

$\Rightarrow \quad 9s^2 + 28s + 20 = 0$

$\Rightarrow \quad 9s^2 + 18s + 10s + 20 = 0$

$\Rightarrow \quad s = -2, \dfrac{-10}{9}$

Hence, (–1, –1, 0) and $\left(\dfrac{7}{9}, \dfrac{7}{9}, \dfrac{8}{9}\right)$ are required points.

68. (a, d) Then, $(x_2 - x_1, y_2 - y_1, z_2 - z_1), (a_1, b_1, c_1)$ and (a_2, b_2, c_2) are coplanar

i.e., $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

Here, $x = 5, \dfrac{y}{3-\alpha} = \dfrac{z}{-2}$

$\Rightarrow \quad \dfrac{x-5}{0} = \dfrac{y-0}{-(\alpha-3)} = \dfrac{z-0}{-2} \qquad \ldots (i)$

and $\quad x = \alpha, \dfrac{y}{-1} = \dfrac{z}{2-\alpha}$

$\Rightarrow \quad \dfrac{x-\alpha}{0} = \dfrac{y-0}{-1} = \dfrac{z-0}{2-\alpha} \qquad \ldots (ii)$

$\Rightarrow \quad \begin{vmatrix} 5-\alpha & 0 & 0 \\ 0 & 3-\alpha & -2 \\ 0 & -1 & 2-\alpha \end{vmatrix} = 0$

$\Rightarrow \quad (5-\alpha)[(3-\alpha)(2-\alpha) - 2] = 0$

$\Rightarrow \quad (5-\alpha)[\alpha^2 - 5\alpha + 4] = 0$

$\Rightarrow \quad (5-\alpha)(\alpha-1)(\alpha-4) = 0$

$\Rightarrow \quad \alpha = 1, 4, 5$

69. (c) Line L_1 given by $y = x$, $z = 1$ can be expressed $L_1 : \dfrac{x}{1} = \dfrac{y}{1} = \dfrac{z-1}{0}$; $\dfrac{x}{1} = \dfrac{y}{1} = \dfrac{z-1}{0} = \alpha$ (say)

$\Rightarrow \quad x = \alpha, y = \alpha, z = 1$

Let the coordinates of Q on L_1 be $(\alpha, \alpha, 1)$

Line L_2 given by $y = -x$, $z = -1$ can be expressed as $L_2 : \dfrac{x}{1} = \dfrac{y}{-1} = \dfrac{z+1}{0}$ $\dfrac{x}{1} = \dfrac{y}{-1} = \dfrac{z+1}{0} = \beta$ (say)

$\Rightarrow \quad x = \beta, y = -\beta, z = -1$

Let the coordinates of R on L_2 be $(\beta, -\beta, -1)$

Direction ratios of PQ are $\lambda - \alpha, \lambda - \alpha, \lambda - 1$.

Now, $PQ \perp L_1$

$\therefore \quad 1(\lambda - \alpha) + 1 \cdot (\lambda - \alpha) + 0 \cdot (\lambda - 1) = 0 \Rightarrow \lambda = \alpha$

$\therefore \quad Q(\lambda, \lambda, 1)$

Direction ratio of PR are $\lambda - \beta, \lambda + \beta, \lambda + 1$.

Now, $PR \perp L_2$

$\therefore \quad 1(\lambda - \beta) + (-1)(\lambda + \beta) + 0(\lambda + 1) = 0$

$\lambda - \beta - \lambda - \beta = 0 \Rightarrow \beta = 0$

$\therefore \quad R(0,0,-1)$

Now, as $\angle QPR = 90^\circ$

(as $a_1a_2 + b_1b_2 + c_1c_2 = 0$, if two lines with DR's $a_1, b_1, c_1; a_2, b_2, c_2$ are perpendicular)

$\therefore \quad (\lambda - \lambda)(\lambda - 0) + (\lambda - \lambda)(\lambda - 0) + (\lambda - 1)(\lambda + 1) = 0$

$\Rightarrow \quad (\lambda - 1)(\lambda + 1) = 0$

$\Rightarrow \quad \lambda = 1$

or $\quad \lambda = -1$

$\lambda = 1$ rejected as P and Q are different points.

$\Rightarrow \quad \lambda = -1$

70. (a, c) The given lines are coplanar, if

$0 = \begin{vmatrix} 2-1 & 3-4 & 4-5 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix}$

Applying $C_2 \to C_2 + C_1$ and $C_3 \to C_3 + C_1$

$= \begin{vmatrix} 1 & \ldots 0 & \ldots 0 \\ 1 & 2 & 1-k \\ k & k+2 & 1+k \end{vmatrix}$

or $\quad 2(1+k) - (k+2)(1-k) = 0$

or $\quad k^2 + 3k = 0$

$\Rightarrow \quad k = 0, 3$

71. (a, b, c) Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}$, then

$\vec{a} \cdot \vec{a} + (\vec{b} - \vec{c}) \cdot (\vec{b} - \vec{c})$

$= \vec{b} \cdot \vec{b} + (\vec{c} - \vec{a}) \cdot (\vec{c} - \vec{a})$

$\Rightarrow \quad 2\vec{b} \cdot \vec{c} = -2\vec{c} \cdot \vec{a}$

$\Rightarrow \quad (\vec{a} - \vec{b}) \cdot \vec{c} = 0$

or $\quad \overrightarrow{AB} \cdot \overrightarrow{OC} = 0$

Hence, $AB \perp OC$

Similarly, $BC \perp OA$ and $CA \perp OB$

72. **(a, b)** The given equation are

$4x-4y-z+11=0$ . . .(*i*)

$x+2y-z+1=0$ . . .(*ii*)

The DR's of normals to the planes (*i*) and (*ii*) are 4,–4,–1 and 1,2,–1 respectively.

Let DR's of line of intersection of plane be *l,m,n*

As the line of intersection of the planes is perpendicular to the normals of the both planes, we get $4l-4m-n=0$ and $l+2m-n=0$

By cross multiplication $\frac{l}{6}=\frac{m}{3}=\frac{n}{12}$

Or $\frac{l}{2}=\frac{m}{1}=\frac{n}{4}$

If x = 0, equation (*i*) and (*ii*) becomes $-4y-z+11=0$, $2y-z-1=0$

Solving, we get $y = 2, z = 3$

$\therefore$ Equation of line is $\frac{x}{2}=\frac{y-2}{1}=\frac{z-3}{4}$

Also $x = 4, y = 4, z = 11$ satisfies equation (*i*) and (*ii*)

Hence, (b) is also the correct option.

73. **(a, c)** The direction cosines of the given line are given by

$\pm\frac{-2}{\sqrt{(-2)^2+2^2+1^2}},\pm\frac{2}{\sqrt{(-2)^2+2^2+1^2}},\pm\frac{1}{\sqrt{(-2)^2+2^2+1^2}}$

$\Rightarrow$ The required direction cosines are $-\frac{2}{3},\frac{2}{2},\frac{1}{3}$

Or $\frac{2}{3},-\frac{2}{3},-\frac{1}{3}$

74. **(a, b)** Let the equation of the sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$

Since, it pass through (1,0,0)(0,1,0) and (0,0,1)

We have, $u=v=w=-\frac{d+1}{2}$

Since the centre $(-u,-v,-w)$ lies on $4xy = 1$

We have, $4\left(\frac{d+1}{2}\right)\left(\frac{d+1}{2}\right)=1$

$\Rightarrow$ $(d+1)^2=1$

$\Rightarrow$ $d+1=\pm 1$

$\Rightarrow$ $d=0$

or $d=-2$

and the equation of the sphere is $x^2+y^2+z^2-x-y-z=0$

or $x^2+y^2+z^2+x+y+z-2=0$

Assertion and Reason

75. **(b)** Direction ratio of *AB* are 1 – 3, 3 – 1, 4 – 6 or 1, –1, 1

$\Rightarrow$ *AB* is normal to the plane $x-y+z=6$. Also the midpoint (2, 2, 5) of *AB* lies on $x-y+z=5$. Thus the plane bisects the line segment *AB* so Reason is true.

As the plane is the perpendicular bisector of the segment *AB*, *A* is the mirror image of *B* in the plane and Assertion is also true but does not follow from Reason alone.

76. **(a)** Equation of the line through *P* perpendicular to the plane is

$\frac{x-1}{1}=\frac{y+2}{2}=\frac{z-1}{-2}=\frac{r}{\sqrt{1+4+4}}=\frac{r}{3}$

Coordinates of a point on this line at a distance *r* form *P* are

$\left(\frac{r}{3}+1,\frac{2r}{3}-2,\frac{-2r}{3}+1\right)$

If this represents the point *Q* then it lies on the plane.

$\frac{r}{3}+1+2\left(\frac{2r}{3}-2\right)-2\left(\frac{-2r}{3}+1\right)=\alpha$

Since *PQ* = 5,*r* = 5

$\Rightarrow$ $\alpha=\frac{5}{3}+1+\frac{4\times5}{3}-4+\frac{4\times5}{3}-2=10$

So the Reason is true. *Q* is now the foot of the perpendicular form P to the plane and its coordinates are (8/3, 4/3, –7/3) and hence Assertion is also true.

77. **(d)** Equation of the line in Assertion can be written as $\frac{x-3}{14}=\frac{y-1}{2}=\frac{z-0}{15}=t$. This is the line of intersection of the planes, then the point (3,1,0) which lies on the line must be on both the planes which is not true and hence the Assertion is false. Direction ratios of the line of intersection of the given planes is (–6)(–2)–(–2)(1), (–2)(2)–(–2)(3), 3(1)–2(–6)

i.e., 14, 2, 15; showing that the vector in Reason is parallel to the line of intersection of the planes and thus Reason is true.

78. **(b)** Midpoint of the segment *AB* is

$\left(\frac{1+1}{2},\frac{0+6}{2},\frac{7+3}{2}\right)=(1, 3, 5)$

Which lies on the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ so Reason is true.

Next, direction ratios of *AB* are 0, 6, –4 and the given line are 1, 2, 3.

Since $0\times1+6\times2+(-4)\times3=0$, the line *AB* is perpendicular to the given line showing than Assertion is also true, but Reason is not a correct explanation for Assertion.

79. **(b)** Since $\begin{vmatrix} 1-2 & 0+1 & -1-0 \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 0$, the lines L_1 and L_2 are coplanar and the equation of the plane containing then is $\begin{vmatrix} x-1 & y & z+1 \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -(5x+2y-3z-8) = 0$

So, Assertion is true.

L_1 and L_2 are not parallel as $\frac{1}{1} \neq \frac{-1}{2} \neq \frac{1}{3}$ and are coplanar so the intersect at a point and the Reason is also true but does not lead to Assertion

80. **(b)** As the triangle ABC is equilateral the circum centre is $\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$, the centroid of the triangle and hence the radius

$= \sqrt{\left(1-\frac{2}{3}\right)^2 + \left(1-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \sqrt{\frac{2}{3}}$

Assertion is the true. Reason is also true as the centre of the circum circle lies on the plane $x + y + z - 2 = 0$ through A, B, C

81. **(c)** Any point on L is $P(r, 2r, 3r)$,

$OP^2 = 14 = r^2 + 4r^2 + 9r^2$

$\Rightarrow$ $r^2 = 1$

$PN^2 = \left(\frac{r+2r+3r}{\sqrt{1+1+1}}\right)^2 = \frac{36}{6} = 12$

$\Rightarrow$ $ON^2 = OP^2 - PN^2 = 2$

$\Rightarrow$ Assertion is true. Let $R(r, 2r, 3r)$,

The $\left|\frac{r+2r+3r}{\sqrt{1+1+1}}\right| = \sqrt{3}$

$\Rightarrow$ $r = \pm\frac{1}{2}$

So, Reason is false.

82. **(c)** L_1 and L_2 are parallel to the vectors $\hat{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ respectively. The vector perpendicular to both L_1 and L_2 is $\hat{a} \times \hat{b}$ and the required unit vector is

$\frac{-\hat{i} - 7\hat{j} + 5\hat{k}}{\sqrt{1+49+25}}$,

So, Assertion is true, In Reason, equation of the plane is $-(x+1) - 7(y+2) + 5(z+1) = 0$ whose distance from $(1, 1, 1)$ is $\frac{13}{5\sqrt{3}}$, so the Reason is false.

83. **(a)** Reason is true, in Assertion, line is parallel to the vector $\hat{i} - \hat{j} + 4\hat{k}$ and the normal to the plane is parallel to $\hat{i} + 5\hat{j} + \hat{k}$. Since $1 - 5 + 4 = 0$ the two vectors are perpendicular so the line is parallel to the plane and using. Reason, required distance is $\left|\frac{2\times1 + 2\times5 + 3\times1 - 5}{\sqrt{1+25+1}}\right| = \frac{10}{3\sqrt{3}}$ and the Assertion is also true.

84. **(d)** Direction ratios of the line is Assertion are 1, 2, 3 and of the plane are 2, 3, 1 so their direction cosines are different and the Assertion is false. In Reason, direction cosines of the normal to the plane are $\cos\frac{\pi}{4}, \cos\frac{\pi}{4}, \cos\frac{\pi}{2}$

i.e., $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$;

So, the equation of the plane is $\left(\frac{1}{\sqrt{2}}\right)x + \left(\frac{1}{\sqrt{2}}\right)y + 0z = \sqrt{2}$

or $x + y = 2$, Reason is true.

Comprehension Based

85. **(b)** The equations of given lines in vector form may be written as $L_1 : \vec{r} = (-\hat{i} - 2\hat{j} - \hat{k}) + \lambda(3\hat{i} + \hat{j} + 2\hat{k})$

and $L_2 : \vec{r} = (2\hat{i} - 2\hat{j} + 3\hat{k}) + \mu(\hat{i} + 2\hat{j} + 3\hat{k})$

Since, the vector perpendicular to both L_1 and L_2.

$\therefore$ $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -\hat{i} - 7\hat{j} + 5\hat{k}$

$\therefore$ Required unit vector

$= \frac{(-\hat{i} - 7\hat{j} + 5\hat{k})}{\sqrt{(-1)^2 + (-7)^2 + (5)^2}} = \frac{1}{5\sqrt{3}}(-\hat{i} - 7\hat{j} + 5\hat{k})$

86. **(d)** The shortest distance between L_1 and L_2 is

$\left|\frac{\{(2-(-1))\hat{i} + (2-2)\hat{j} + (3-(-1))\hat{k}\} \cdot (-\hat{i} - 7\hat{j} + 5\hat{k})}{5\sqrt{3}}\right|$

$= \left|\frac{(3\hat{i} + 4\hat{k}) \cdot (-\hat{i} - 7\hat{j} + 5\hat{k})}{5\sqrt{3}}\right| = \frac{17}{5\sqrt{3}}$ unit

87. **(c)** The equation of the plane passing through the point (–1, –2, –1) and whose normal is perpendicular to both the given lines L_1 and L_2 may be written as

$(x+1) + 7(y+2) - 5(z+1) = 0 \Rightarrow x + 7y - 5z + 10 = 0$

The distance of the point (1, 1, 1) from the plane $= \left|\frac{1+7-5+10}{\sqrt{1+49+25}}\right| = \frac{13}{\sqrt{75}}$ unit.

88. **(b)** The equation of any plane through the intersection of P_1 and P_2 is $P_1 + \lambda P_2 = 0$

$\Rightarrow \quad (2x - y + z - 2) + \lambda(x + 2y - z - 3 = 0) \quad \ldots (i)$

Since, it passes through (3,2,1),

Then $(6 - 2 + 1 - 2) + \lambda(3 + 4 - 1 - 3) = 0$

$\therefore \quad \lambda = -1$

From equation (i), we get $x - 3y + 2z + 1 = 0$

Which is the required plane,

89. **(c)** The equation of any plane through (–1,3,2) is

$a(x+1) + b(y-3), + c(z-2) = 0 \quad \ldots (i)$

If this plane (i) is perpendicular to P_1 then

$2a - b + c = 0 \quad \ldots (ii)$

and if the plane (ii) is perpendicular to P_2, then

$a + 2b - c = 0 \quad \ldots (iii)$

From equation (ii) and (iii), we get $\frac{a}{-1} = \frac{b}{3} = \frac{c}{5}$

Substituting these proportionate values of a, b, c in equation (ii).

We get the required equations as

$= -(x+1) + 3(y-3) + 5(z-2) = 0$

or $\quad x - 3y - 5z + 20 = 0$

90. **(a)** The given planes can be written as $-2x + y - z + 2 = 0$ and $-x - 3y - z + 3 = 0$

Here, $(-2)(-1) + (1)(-2) + (-1)(1) = -1 < 0$

Equation of bisectors $\frac{(-2x + y - z + 2)}{\sqrt{(4+1+1)}} = \pm \frac{(-x - 2y + z + 3)}{\sqrt{1+4+1}}$

$\therefore$ Acute angle bisector is $(-2x + y - z + 2)$

$= (-x - 2y + z + 3)$

$\Rightarrow \quad x - 3y + 2z + 1 = 0$

91. **(d)** Equation of bisector which not containing the origin is

$\frac{(-2x + y - z + 2)}{\sqrt{(4+1+1)}} = -\frac{(-x - 2y + z + 3)}{\sqrt{(1+4+1)}}$

$\Rightarrow \quad 3x + y - 5 = 0$

92. **(c)** The image of plane P_1 in the plane mirror P_2 ,then

$2(2.1 + (-1).2 + 1.(1))(x + 2y - z - 3)$

$= (1 + 4 + 1)(2x - y + z - 2)$

$\Rightarrow \quad -(x + 2y - z - 3)$

$= 3(2x - y + z - 2)$

$\Rightarrow \quad 7x - y + 4z - 9 = 0$

Match the Column

93. **(b)** Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$= -\frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$

(A) If $a + b + c \neq 0$

and $a^2 + b^2 + c^2$

$= ab + bc + ca$

$\Rightarrow \quad \Delta = 0$

and $a = b = c \neq 0$

$\Rightarrow$ the equations represent identical planes.

(B) $a + b + c = 0$

and $a^2 + b^2 + c^2 \neq ab + bc + ca$

$ax + by = (a+b)z$

$bx + cy = (b+c)z$

$\Rightarrow \quad (b^2 - ac)y = (b^2 - ac)z \Rightarrow y = z$

$\Rightarrow \quad ax + by + cy = 0$

$\Rightarrow \quad ax = ay$

$\Rightarrow \quad x = y = z.$

(C) $a + b + c \neq 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$

$\Rightarrow \quad \Delta \neq 0$

$\Rightarrow$ the equation represent planes meeting at only one point.

(D) $a + b + c = 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$

$\Rightarrow \quad a = b = c = 0$

$\Rightarrow$ the equations represent whole of the three dimensional space.

94. **(a)** $L_1 : \frac{x-1}{2} = \frac{y-0}{-1} = \frac{z-(-3)}{1}$

Normal of plane $P : \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 1 & 2 \\ 3 & 5 & -6 \end{vmatrix}$

$= \hat{i}(-16) - \hat{j}(-42 - 6) + \hat{k}(32)$

$= -16\hat{i} + 48\hat{j} + 32\hat{k}$

DR's of normal $\vec{n} = \hat{i} - 3\hat{j} - 2\hat{k}$

Point of intersection of L_1 and L_2.

$\Rightarrow \quad 2K_1 + 1 = K_2 + 4$ and $-k_1 = k_2 - 3$

$\therefore \quad k_1 = 2$ and $k_2 = 1$

$\therefore$ Point of intersection (5, –2, –1)

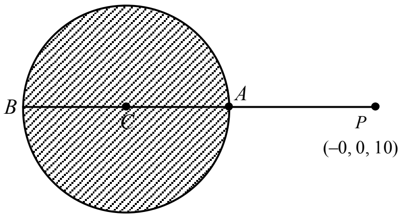
$\therefore$ Equation of plane. $1.(x-5) - 3(y+2) - 2(z+1) = 0$

$\Rightarrow \quad x - 3y - 2z - 13 = 0 \Rightarrow x - 3y - 2z = 13$

$\therefore \quad a = 1, b = -3, c = -2, d = 13$

95. (a)

(A)



Let $f(x, y, z) = x^2 + y^2 + z^2 - 4x + 6y - 8z + 4$

$\therefore$ $f(0, 0, 10) = 0 + 0 + 100 + 0 + 0 - 80 + 4 = 24 > 0$

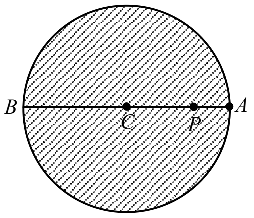
$\therefore$ Point $P(0, 0, 10)$ lies outside the sphere. Centre of the sphere C≡(2, –3, 4) and radius $r = \sqrt{(4+9+16-4)} = 5$

$\therefore$ Maximum distance $\lambda = PB = CB + CP = 5 + 7 = 12$

and minimum distance $\mu = PA = CP - CA = 7 - 5 = 12$

$\therefore$ $\lambda + \mu = 14,\ \lambda - \mu = 10 (3, 5)$

(B)



Let $f(x, y, z) = x^2 + y^2 + z^2 + 2x - 2y - 4z - 19$

$\therefore$ $f(0, 3, 4) = 0 + 9 + 16 + 0 - 6 - 16 - 9$

$\therefore$ Point P(0,3,4) lies inside the sphere. Centre of the sphere C ≡ (–1,1,2)

and radius $r = \sqrt{(1+1+4+19)} = 5$

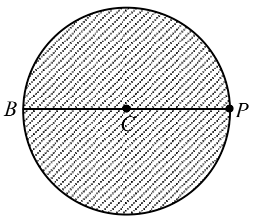
$\therefore$ Maximum distance $\lambda = PB = CB + CP + r + CP = 5 + 3 = 8$

and minimum distance

$\lambda = PB = CA - CP = r - CP = 5 - 3 = 2$

$\therefore$ $\lambda + \mu = 10,\ \lambda - \mu = 6\ (2, 4)$

(C)



Let $f(x, y, z) = x^2 + y^2 + z^2 - 2x - 4y + 2z - 3$

$\therefore$ $f(-1, 4, -2) = 1 + 16 + 4 + 2 - 16 - 4 - 3 = 0$

$\therefore$ Point P(–1,4,–2)lies on the sphere.

Radius of the sphere $r = \sqrt{(1+4+1+3)} = 3$

$\therefore$ Maximum distance $\lambda = PA = 2r = 6$

and minimum distance $\mu = 0$

$\therefore$ $\lambda + \mu = 6,\ \lambda - \mu = 6 (1, 4)$

Integer

96. (2560) Let the equation of the variable plane be

$$\left(\frac{x}{a}\right) + \left(\frac{y}{b}\right) + \left(\frac{z}{c}\right) = 1 \qquad \ldots(i)$$

Given that the plane is at a distance p form (0, 0, 0)

$$\therefore \quad p = \frac{1}{\sqrt{\left\{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2\right\}}}$$

or $$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \qquad \ldots(ii)$$

Also, the plane (i) meets the axes in A, B and C. So the coordinates of O, A, B and C are (0, 0, 0), (a, 0, 0), (0, b, 0) and (0, 0, c) respectively

Let (x,y,z) be the centroid of the tetrahedron $OABC$, then

$$x = \frac{1}{4}(0 + a + 0) = \frac{1}{4}a$$

Similarly $y = \frac{1}{4}b$

and $z = \frac{1}{4}c$

or $a = 4x$,

$b = 4y$,

$c = 4z$

Substituting these values in equation (ii), we get

$$\frac{1}{p^2} = \frac{1}{16x^2} + \frac{1}{16y^2} + \frac{1}{16z^2}$$

or $x^{-2} + y^{-2} + z^{-2}$

$= 16p^{-2}$

$\therefore$ $\lambda = 16$

$\Rightarrow$ 160λ

$= 160 \times 16$

$= 2560$

97. (2) Any point on the first line in symmetrical form is ($3r$ –4, $5r$ – 6, – $2r$ + 1). If the lines are coplanar, this point must lie on both the planes which determine the second line.

$\Rightarrow$ $3(3r - 4) - 2(5r - 6) - 2r + 1 + 5 = 0 \qquad \ldots(i)$

and $2(3r - 4) + 3(5r - 6) + 4(-2r + 1) - k = 0 \qquad \ldots(ii)$

From equation (i)

we get $r = 2$

Now substituting $r = 2$ equation (ii)

then $k = 4$

98. **(2)** The equation of any plane containing the given line is

$(x+y+2z-3)+\lambda(2x+3y+4z-4)=0$

$\Rightarrow \quad (1+2\lambda)x+(1+3\lambda)y+(2+4\lambda)z-(3+4\lambda)=0 \quad \ldots(i)$

If the plane is parallel to z-axis

$\therefore \quad (1+2\lambda)(0)+(1+3\lambda)(0)+(2+4\lambda)(1)=0$

$\Rightarrow \quad \lambda=-\frac{1}{2}$

Put in equation (i), the required plane is

$0+\left(1-\frac{3}{2}\right)y+0-(3-2)=0$

$\Rightarrow \quad y+2=0 \quad \ldots(ii)$

$\therefore$ Shortest distance = distance of any point say (0, 0, 0) on z-axis form plane (ii) $=\frac{2}{\sqrt{(1)^2}}=2$

99. **(486)** The planes are

$y+z=0 \quad \ldots(i)$

$z+x=0 \quad \ldots(ii)$

$x+y=0 \quad \ldots(iii)$

$x+y+z=1 \quad \ldots(iv)$

The point of intersection of the plane (i), (ii) and (iii) is obviously the origin i.e., (0, 0, 0).

Solving equation (ii), (iii) and (iv) we get y = 1, z = 1 and i.e., these planes intersect in (–1, 1, 1). Similarly the other two vertices of the tetrahedron are (1, –1, 1) and (1, 1, –1)

$\therefore$ Required volume $=\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=\frac{1}{6}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

Applying $C_2 \to C_2+C_1$

and $C_3 \to C_3+C_1$, then

$=\frac{1}{6}\begin{vmatrix} -1\ldots & 0 & \ldots 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{vmatrix}=-\frac{1}{6}\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}=-\left(\frac{1}{6}\right)(-4)=\frac{2}{3}$

$\therefore \quad 729\lambda=729\times\frac{2}{3}=486$

100. **(6666)** Let C_1, C_2 be the centres of the spheres and P be their point of intersection. Then the angle between the spheres is the angle between their radi C_1P and C_2P.

$\therefore$ In $\Delta C_1PC_2, C_1P=r_1, \; C_2P=r_2$

and $C_1C_2=d$

$\therefore$ If θ be the required angle, then $\cos\theta=\cos\angle C_1PC_2$

$=\frac{C_1P^2+C_2P^2-C_1C_2^2}{2C_1P\cdot C_2P}=\frac{r_1^2+r_2^2-d^2}{2r_1r_2}$

Now the given spheres are

$x^2+y^2+z^2-2x-4y-6z+10=0 \quad \ldots(i)$

and $(x-1)(x-5)+(y-2)(y-0)+(z+3)(z-1)=0$

or $x^2+y^2+z^2-6x-2y+2z+2=0 \quad \ldots(ii)$

Centre and radius of equation (i) are (1, 2, 3) and 2

Center and radius of equation (ii) are (3, 1, –1) and 3

$\therefore \quad r_1=2, r_2=3, d^2=[(3-1)^2+(1-2)^2+(-1-3)^2]=21$

$\therefore \quad \cos^{-1}\left[\frac{r_1^2+r_2^2-d}{2r_1r_2}\right]=\cos^{-1}\left[\frac{4+9-21}{2\cdot 2\cdot 3}\right]$

$=\cos^{-1}\left(-\frac{2}{3}\right)$

$\therefore \quad \lambda=-\frac{2}{3}$

$\Rightarrow \quad 9999|\lambda|=9999\times\frac{2}{3}$

$=6666$

* * *