Session 6

Trigonometric Ratios of Compound Angles

Trigonometric Ratios of Compound Angles

Algebraic sum of two or more angles is called a compound angle. If A, B, C are any angles then $A+B$, $A-B$, $A+B+C$, $A-B+C$, $A-B-C$, $A+B-C$, etc., are all compound angles.

Till now, we have learnt the values of trigonometric ratios between 0° to 360°. Now, we are going to learn the values of trigonometric ratios of compound angles.

Note

Trigonometric ratios if i.e. sine, cosine, tan, cot, sec and cosec are not distributed over addition and substraction of 2 angles.

i.e. $\sin(A+B) \neq \sin A + \sin B$

Proof : $A = 60°$, $B = 30°$

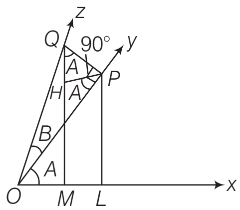
$\sin(90°) \neq \sin 60° + \sin 30°$

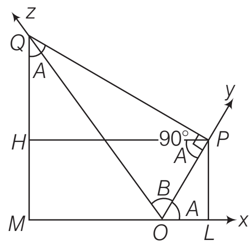
The Addition Formula

(i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(ii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

(iii) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$





Let the revolving line starting from the position OX describe first $\angle XOY = A$ and then proceed further so as to describe $\angle YOZ = B$ in its position OZ.

Then, $\angle XOZ = A + B$

In figure 6.1 $A + B < 90°$ and in figure 6.2 $A + B > 90°$

Let Q be a point on OZ. From Q draw $QM \perp OX$ and $QP \perp OY$. From P draw $PH \perp QM$.

Now, $\angle HPO = \angle POX = A$

$\because \qquad \angle QPO = 90°$

$\therefore \qquad \angle QPH = 90° - A$

$\therefore \qquad \angle HQP = A$

In ΔQOM,

$$\sin(A+B) = \frac{QM}{OQ} = \frac{QH+HM}{OQ} = \frac{QH+PL}{OQ}$$

$$= \frac{QH}{OQ} + \frac{PL}{OQ} = \frac{QH}{QP}\cdot\frac{QP}{OQ} + \frac{PL}{OP}\cdot\frac{OP}{OQ}$$

$$= \frac{PL}{OP}\cdot\frac{OP}{OQ} + \frac{QH}{QP}\cdot\frac{QP}{OQ}$$

$$= \sin POL \cdot \cos POQ + \cos HQP \cdot \sin POQ$$

$$= \sin A \cos B + \cos A \sin B$$

From figure 6.1, $\cos(A+B) = \frac{OM}{Q} = \frac{OL-ML}{OQ} = \frac{OL-PH}{OQ}$

From figure 6.2, $\cos(A+B) = -\frac{OM}{OQ} = -\frac{ML-OL}{OQ}$

$$= \frac{OL-ML}{OQ} = \frac{OL-PH}{OQ}$$

$\therefore$ In both cases $\cos(A+B)$

$$= \frac{OL}{OQ} - \frac{PH}{OQ} = \frac{OL}{OP}\cdot\frac{OP}{OQ} - \frac{PH}{QP}\cdot\frac{QP}{OQ}$$

$$= \cos POL \cdot \cos POQ - \sin PQH \cdot \sin POQ$$

$$= \cos A \cos B - \sin A \sin B$$

In both cases

$$\tan(A+B) = \frac{QM}{OM} = \frac{QH+HM}{OL-ML} = \frac{QH+PL}{OL-PH}$$

$$= \frac{\frac{QH}{OL}+\frac{PL}{OL}}{1-\frac{PH}{OL}} = \frac{\frac{QH}{OL}+\frac{PL}{OL}}{1-\frac{PH}{PL}\cdot\frac{PL}{OL}} \qquad \text{...(i)}$$

From similar ΔQPH and ΔOPL

$$\frac{QH}{OL} = \frac{PH}{PL} = \frac{PQ}{OP} \qquad \text{...(ii)}$$

On putting the value from Eq. (ii) in Eq. (i), we get

$$\tan(A+B) = \frac{\frac{PQ}{OP}+\frac{PL}{OL}}{1-\frac{PQ}{OP}\cdot\frac{PL}{OL}}$$

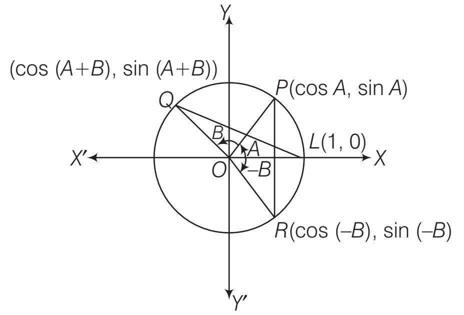
$$= \frac{\tan B + \tan A}{1-\tan B \tan A} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\left[\because \text{ from } \Delta POQ, \frac{PQ}{OP} = \tan B, \text{ from } \Delta POL, \frac{PL}{OL} = \tan A\right]$$

Second Proof of Formulae

1. **$\cos(A+B) = \cos A \cos B - \sin A \sin B$**

Proof Let O be the centre of a unit circle.



Let $\angle LOP = A$ radian, $\angle POQ = B$ radian, $\angle LOR = -B$ radian

(This angle has been measured in clockwise direction)

Now $\angle LOQ = A + B$ and $\angle ROP = A - B$

Since radius of circle is unity

$\therefore \qquad$ arc $LP = A$, arc $PQ = B$, arc $LR = |-B| = B$

[in formulae $\theta = \frac{l}{r}$, θ is always taken a positive]

Also as radius of the circle is 1.

$\therefore \ P \equiv (\cos A, \sin A)$,

$Q \equiv \cos(A+B), \sin(A+B)$,

$R \equiv (\cos(-B), \sin(-B)$ or $R \equiv (\cos B, -\sin B)$

$\therefore \qquad \Delta LOQ \cong \Delta POR$

$\therefore \qquad LQ = PR$

$\Rightarrow \qquad LQ^2 = PR^2$

$\Rightarrow [1-\cos(A+B)]^2 + [0-\sin(A+B)]^2$

$\qquad = [\cos A - \cos(-B)]^2 + [(\sin A - \sin(-B)]^2$

$\Rightarrow 1 + \cos^2(A+B) - 2\cos(A+B) + \sin^2(A+B)$

$\qquad = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$

$\Rightarrow 1 + \cos^2(A+B) + \sin^2(A+B) - 2\cos(A+B)$

$\Rightarrow \qquad = \cos^2 A + \cos^2 B - 2\cos A \cos B + \sin^2 A$

$\qquad + \sin^2 B + 2\sin A \sin B$

$\Rightarrow \qquad 2 - 2\cos(A+B) = (\cos^2 A + \sin^2 A)$

$\qquad + (\cos^2 B + \sin^2 B) - 2(\cos A \cos B - \sin A \sin B)$

$\Rightarrow \ 2 - 2\cos(A+B) = 2 - 2(\cos A \cos B - \sin A \sin B)$

$\Rightarrow \qquad \cos(A+B) = \cos A \cos B - \sin A \sin B \qquad$ ...(i)

2. **Putting $-B$ in place of B in (1), we get**

$\cos(A-B) = \cos A \cos(-B) - \sin A \sin(-B)$

$\qquad = \cos A \cos B + \sin A \sin B \qquad$ ...(ii)

3. $\sin(A+B)=\cos\left[\frac{\pi}{2}-(A+B)\right]$

$$=\cos\left[\left(\frac{\pi}{2}-A\right)-B\right]$$

$$=\cos\left(\frac{\pi}{2}-A\right)\cos B+\sin\left(\frac{\pi}{2}-A\right)\sin B$$

$$=\sin A\cos B+\cos A\sin B \quad \text{...(iii)}$$

4. $\tan(A+B)=\dfrac{\sin(A+B)}{\cos(A+B)}$

$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

$$=\frac{\tan A+\tan B}{1-\tan A\tan B} \quad \text{...(iv)}$$

[dividing numerator and denominator by $\cos A\cos B$]

5. Putting $-B$ in place of B in (3), we get

$$\sin(A-B)=\sin A\cos B-\cos A\sin B \quad \text{...(v)}$$

6. Putting $-B$ in place of B in (4), we get

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B} \quad \text{...(vi)}$$

7. $\cot(A+B)=\dfrac{\cos(A+B)}{\sin(A+B)}$

$$=\frac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}$$

$$=\frac{\cot A\cot B-1}{\cot B+\cot A} \quad \text{...(vii)}$$

[dividing numerator and denominator by $\sin A\sin B$]

8. Putting $-B$ in place of B in (7), we get

$$\cot(A-B)=\frac{-\cot A\cot B-1}{-\cot B+\cot A}$$

$$=\frac{\cot A\cot B+1}{\cot B-\cot A} \quad \text{...(viii)}$$

Third Proof by Complex Number Method

The result of the sine, cosine and tangent of compound angle can also be derived using the concept of complex numbers as discussed.

$$\cos(A\pm B)+i\sin(A\pm B)=e^{i(A\pm B)}$$

$$=e^{iA}\cdot e^{i(\pm B)}=(\cos A+i\sin A)(\cos(\pm B)+i\sin(\pm B))$$

$$=(\cos A\cos B\pm i\cos A\sin B+i\sin A\cos B\mp\sin A\sin B)$$

$$=(\cos A\cos B\mp\sin A\sin B)+i(\sin A\cos A\pm\cos A\sin B)$$

Comparing real and imaginary parts of the left and right hand side, we get,

$$\cos(A\pm B)=(\cos A\cos B\mp\sin A\sin B)$$

$$\sin(A\pm B)=(\sin A\cos B\pm\cos A\sin B)$$

TWO VERY IMPORTANT IDENTITIES

(a) $\sin(A+B)\cdot\sin(A-B)$

$$=\sin^2 A-\sin^2 B=\cos^2 B-\cos^2 A$$

(b) $\cos(A+B)\cdot\cos(A-B)=\cos^2 A-\sin^2 B$

Proof : (a) $\sin(A+B)\cdot\sin(A-B)$

$$=(\sin A\cos B+\cos A\sin B)(\sin A\cos B-\cos A\sin B)$$

$$=\sin^2 A\cos^2 B-\cos^2 A\sin^2 B$$

$$=\sin^2 A(1-\sin^2 B)-\sin^2 B(1-\sin^2 A)$$

$$=\sin^2 A-\sin^2 B$$

(b) $\cos(A+B)\cdot\cos(A-B)$

$$=(\cos A\cos B-\sin A\sin B)(\cos A\cos B+\sin A\sin B)$$

$$=\cos^2 A\cos^2 B-\sin^2 A\sin^2 B$$

$$=\cos^2 A(1-\sin^2 B)-(1-\cos^2 A)\sin^2 B$$

$$=\cos^2 A-\sin^2 B$$

Example 44. ***Find the value of*** $\tan 105°$.

Sol. $\tan 105°=\tan(60°+45°)=\dfrac{\tan 60°+\tan 45°}{1-\tan 60°\tan 45°}$

$$=\frac{\sqrt{3}+1}{1-\sqrt{3}\cdot 1}=\frac{(\sqrt{3}+1)^2}{1-3}=-(2+\sqrt{3})$$

$\therefore \quad \tan 105°=-(2+\sqrt{3})$

Example 45. Prove that $\tan 70°=\tan 20°+2\tan 50°$.

Sol. $\tan 70°=\tan(20°+50°)=\dfrac{\tan 20°+\tan 50°}{1-\tan 20°\tan 50°}$

or $\quad \tan 70°-\tan 20°\cdot\tan 50°\cdot\tan 70°=\tan 20°+\tan 50°$

or $\quad \tan 70°=\tan 70°\tan 50°\tan 50°+\tan 20°+\tan 50°$

$$=\cot 20°\tan 50°\tan 20°+\tan 20°+\tan 50°$$

[$\because \tan 70°=\tan(90°-20°)=\cot 20°$]

$$=2\tan 50°+\tan 20°$$

Example 46. If $A+B=45°$, then show that $(1+\tan A)(1+\tan B)=2$.

Sol. $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$; $1=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

[as $A+B=45°$, $\tan(A+B)=1$]

$\therefore \quad \tan A+\tan B+\tan A\tan B=1$

or $\quad 1+\tan A+\tan B+\tan A\tan B=1+1$

[$\because$ adding '1' on both sides]

$\Rightarrow \quad (1+\tan A)+\tan B(1+\tan A)=2$

$\Rightarrow \quad (1+\tan A)(1+\tan B)=2$

Example 47. Find the value of $\frac{\tan 495^\circ}{\cot 855^\circ}$

Sol. $\tan 495^\circ = \tan(2.180^\circ + 135^\circ) = \tan 135^\circ = -1$

$\cot 855^\circ = \cot(4.180^\circ + 135^\circ)$

$= \cot 135^\circ = -1 \quad [\because \cot(4.180^\circ + \theta) = \cot\theta]$

$\therefore \frac{\tan 495^\circ}{\cot 855^\circ} = \frac{-1}{-1} = 1$

Example 48. Evaluate $\sin\left\{n\pi + (-1)^n \frac{\pi}{4}\right\}$; where n is an integer.

Sol. $\because \quad \sin(\pi + \theta) = -\sin\theta$

$\therefore \sin(n\pi + \theta) = (-1)^n \sin\theta \Rightarrow \sin\left\{n\pi + (-1)^n \frac{\pi}{4}\right\}$

$= (-1)^n \sin\left\{(-1)^n \frac{\pi}{4}\right\}$

$= (-1)^n (-1)^n \sin\frac{\pi}{4} \quad [\because \sin(-\theta) = -\sin\theta]$

$\therefore \sin\{(-1)^n\theta\} = (-1)^n \sin\theta$

$= (-1)^{2n} \sin\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Example 49. Prove that $\cos 18^\circ - \sin 18^\circ = \sqrt{2}\sin 27^\circ$

Sol. R.H.S. $= \sqrt{2}\sin 27^\circ = \sqrt{2}\sin(45^\circ - 18^\circ)$

$= \sqrt{2}(\sin 45^\circ \cos 18^\circ - \cos 45^\circ \sin 18^\circ)$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos 18^\circ - \frac{1}{\sqrt{2}}\sin 18^\circ\right)$

$= \cos 18^\circ - \sin 18^\circ$

= L.H.S.

Example 50. Show that $\cot\left(\frac{\pi}{4} + x\right) \cdot \cot\left(\frac{\pi}{4} - x\right) = 1$

Sol. L.H.S. $= \frac{\cos\left(\frac{\pi}{4} + x\right)\cos\left(\frac{\pi}{4} - x\right)}{\sin\left(\frac{\pi}{4} + x\right)\sin\left(\frac{\pi}{4} - x\right)}$

$= \frac{\cos^2\frac{\pi}{4} - \sin^2 x}{\sin^2\frac{\pi}{4} - \sin^2 x} = \frac{\frac{1}{2} - \sin^2 x}{\frac{1}{2} - \sin^2 x} = 1$

Example 51. If $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$, Prove that $1 + \cot\alpha\tan\beta = 0$

Sol. Given, $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$

$\Rightarrow \quad \cos\alpha\cos\beta - \sin\alpha\sin\beta = 1$

$\Rightarrow \quad \cos(\alpha + \beta) = 1 \quad$ ...(i)

Now, $1 + \cos\alpha\tan\beta = 1 + \frac{\cos\alpha}{\sin\alpha} \cdot \frac{\sin\beta}{\cos\beta}$

$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta} = \frac{\sin(\alpha + \beta)}{\sin\alpha\cos\beta}$

$= \frac{0}{\sin\alpha\cos\beta} = 0$

$[\because \sin^2(\alpha + \beta) = 1 - \cos^2(\alpha + \beta) = 1 - 1 = 0]$

Example 52. Prove that $\frac{\sin(B - C)}{\cos B\cos C} + \frac{\sin(C - A)}{\cos C\cos A} + \frac{\sin(A - B)}{\cos A\cos B} = 0$

Sol. First term of L.H.S.

$= \frac{\sin(B - C)}{\cos B\cos C} = \frac{\sin B\cos C - \cos B\sin C}{\cos B\cos C}$

$= \frac{\sin B\cos C}{\cos B\cos C} - \frac{\cos B\sin C}{\cos B\cos C}$

$= \tan B - \tan C$

Similarly, second term of L.H.S. = $\tan C - \tan A$ and 3rd term of L.H.S. = $\tan A - \tan B$

Now L.H.S. $= (\tan B - \tan C) + (\tan C - \tan A) + (\tan A - \tan B) = 0$

Example 53. Show that $\tan 75^\circ + \cot 75^\circ = 4$.

Sol. $\tan 75^\circ = \tan(45^\circ + 30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$

$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \quad$ ...(i)

and $\cot 75^\circ = \frac{1}{\tan 75^\circ} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \quad$ ...(ii)

Now, L.H.S. $= \tan 75^\circ + \cot 75^\circ$

$= \frac{\sqrt{3} + 1}{\sqrt{3} - 1} + \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \quad$ [from Eqs. (i) and (ii)]

$= \frac{(\sqrt{3} + 1)^2 + (\sqrt{3} - 1)^2}{(\sqrt{3} - 1)(\sqrt{3} + 1)}$

$= \frac{(4 + 2\sqrt{3}) + (4 - 2\sqrt{3})}{3 - 1} = \frac{8}{2} = 4 =$ R.H.S.

Example 54. If $\tan\beta = \frac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}$. Prove that $\tan(\alpha - \beta) = (1 - n)\tan\alpha$.

Sol. $\tan\beta = \frac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}$

$= \frac{\frac{n\sin\alpha\cos\alpha}{\cos^2\alpha}}{\frac{1}{\cos^2\alpha} - \frac{n\sin^2\alpha}{\cos^2\alpha}}$

[dividing numerator and denominator by $\cos\alpha$]

$$= \frac{n \tan\alpha}{\sec^2\alpha - n\tan^2\alpha}$$

$$= \frac{n \tan\alpha}{1 + \tan^2\alpha - n\tan^2\alpha}$$

$$= \frac{n \tan\alpha}{1 + (1-n)\tan^2\alpha} \quad \text{...(i)}$$

Now, L.H.S. $= \tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$

$$= \frac{\tan\alpha - \dfrac{n\tan\alpha}{1+(1-n)\tan^2\alpha}}{1 + \tan\alpha \dfrac{n\tan\alpha}{1+(1-n)\tan^2\alpha}} \quad \text{[from Eqs. (i)]}$$

$$= \frac{\tan\alpha + (1-n)\tan^3\alpha - n\tan\alpha}{1 + (1-n)\tan^2\alpha + n\tan^2\alpha}$$

$$= \frac{(1-n)\tan\alpha + (1-n)\tan^3\alpha}{1 + \tan^2\alpha}$$

$$= \frac{(1-n)\tan\alpha\,(1+\tan^2\alpha)}{1+\tan^2\alpha}$$

$$= (1-n)\tan\alpha$$

Example 55. Show that $\cos^2\theta + \cos^2(\alpha+\theta) - 2\cos\alpha\cos\theta\cos(\alpha+\theta)$ in independent of θ.

Sol. $\cos^2\theta + \cos^2(\alpha+\theta) - 2\cos\alpha\cos\theta\cos(\alpha+\theta)$

$$= \cos^2\theta + \cos(\alpha+\theta)[\cos(\alpha+\theta) - 2\cos\alpha\cos\theta]$$

$$= \cos^2\theta + \cos(\alpha+\theta)[\cos\alpha\cos\theta - \sin\alpha\sin\theta - 2\cos\alpha\cos\theta]$$

$$= \cos^2\theta - \cos(\alpha+\theta)[\cos\alpha\cos\theta + \sin\alpha\sin\theta]$$

$$= \cos^2\theta - \cos(\alpha+\theta)\cos(\alpha-\theta)$$

$$= \cos^2\theta - [\cos^2\alpha - \sin^2\theta]$$

$$= \cos^2\theta + \sin^2\theta - \cos^2\alpha$$

$= 1 - \cos^2\alpha$, which is independent of θ.

Example 56. If $3\tan\theta\tan\phi = 1$, then prove that $2\cos(\theta+\phi) = \cos(\theta-\phi)$.

Sol. Given, $3\tan\theta\tan\phi = 1$ or $\cot\theta\cot\phi = 3$

or $\dfrac{\cos\theta\cos\phi}{\sin\theta\sin\phi} = \dfrac{3}{1}$

By componendo and dividendo, we get

$$\frac{\cos\theta\cos\phi + \sin\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi} = \frac{3+1}{3-1}$$

or $\dfrac{\cos(\theta-\phi)}{\cos(\theta+\phi)} = 2$

or $2\cos(\theta+\phi) = \cos(\theta-\phi)$

Example 57. Let A, B, C be the three angles such that $A+B+C=\pi$. If $\tan A \cdot \tan B = 2$, then find the value of $\dfrac{\cos A\cos B}{\cos C}$.

Sol. Given, $\tan A \cdot \tan B = 2$

Let $y = \dfrac{\cos A\cos B}{\cos C} = -\dfrac{\cos A \cdot \cos B}{\cos(A+B)}$

$[\because \cos C = \cos(\pi - (A+B)) = -\cos(A+B)]$

$$= \frac{\cos A \cdot \cos B}{\sin A\sin B - \cos A\cos B}$$

$$= \frac{1}{\tan A\tan B - 1} = \frac{1}{2-1} = 1$$

Example 58. Prove that $\dfrac{\cos 10^\circ + \sin 10^\circ}{\cos 10^\circ - \sin 10^\circ} = \tan 55^\circ$.

Sol. $\dfrac{\cos 10^\circ + \sin 10^\circ}{\cos 10^\circ - \sin 10^\circ} = \dfrac{1+\tan 10^\circ}{1-\tan 10^\circ} = \dfrac{\tan 45^\circ + \tan 10^\circ}{1 - \tan 45^\circ\tan 10^\circ}$

$= \tan(45^\circ + 10^\circ) = \tan 55^\circ$ (dividing by $\cos 10^\circ$)

Example 59. If $\sin(A-B) = \dfrac{1}{\sqrt{10}}$, $\cos(A+B) = \dfrac{2}{\sqrt{29}}$, find the value of $\tan 2A$ where A and B lie between 0 and $\dfrac{\pi}{4}$.

Sol. $\tan 2A = \tan[(A+B)+(A-B)]$

$$= \frac{\tan(A+B) + \tan(A-B)}{1 - \tan(A+B)\tan(A-B)} \quad \text{...(i)}$$

Given that, $0 < A < \dfrac{\pi}{4}$ and $0 < B < \dfrac{\pi}{4}$. Therefore,

$$0 < A + B < \frac{\pi}{2}$$

Also, $-\dfrac{\pi}{4} < A - B < \dfrac{\pi}{4}$ and $\sin(A-B) = \dfrac{1}{\sqrt{10}}$

$\therefore$ $0 < A - B < \dfrac{\pi}{4}$

Now, $\sin(A-B) = \dfrac{1}{\sqrt{10}}$

$\Rightarrow$ $\tan(A-B) = \dfrac{1}{3}$ ...(ii)

$\cos(A+B) = \dfrac{2}{\sqrt{29}}$

$\Rightarrow$ $\tan(A+B) = \dfrac{5}{2}$ ...(iii)

From Eqs. (i), (ii) and (iii), we get

$$\tan 2A = \frac{\dfrac{5}{2} + \dfrac{1}{3}}{1 - \dfrac{5}{2}\times\dfrac{1}{3}} = \frac{17}{6}\times\frac{16}{1} = 17$$

Example 60. Prove that $(1+\tan 1^\circ)(1+\tan 2^\circ)\ldots(1+\tan 45^\circ)=2^{23}$.

Sol. $(1+\tan x^\circ)(1+\tan(45^\circ - x^\circ))$

$$= (1+\tan x^\circ)\left(1+\frac{1-\tan x^\circ}{1+\tan x^\circ}\right) = 2$$

$\therefore (1+\tan 1^\circ)(1+\tan 44^\circ)$

$$= (1+\tan 2^\circ)(1+\tan 43^\circ)$$
$$= (1+\tan 3^\circ)(1+\tan 42^\circ)$$
$$\ldots$$
$$\ldots$$
$$= (1+\tan 22^\circ)(1+\tan 23^\circ)$$
$$= 2$$

$(1+\tan 1^\circ)(1+\tan 2^\circ)\ldots(1+\tan 45^\circ)=2^{23}$

(as $1+\tan 45^\circ = 2$)

Example 61. If $\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)=-\frac{3}{2}$, Prove that $\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0$

Sol. Given, $\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)=-\frac{3}{2}$

or $3+2\cos(\beta-\gamma)+2\cos(\gamma-\alpha)+2\cos(\alpha-\beta)=0$

or $3+2(\cos\beta\cos\gamma+\sin\beta\sin\gamma)$
$+2(\cos\gamma\cos\alpha+\sin\gamma\sin\alpha)$
$+2(\cos\alpha\cos\beta+\sin\alpha\sin\beta)=0$

or $(\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)+(\cos^2\gamma+\sin^2\gamma)$
$+2(\cos\beta\cos\gamma+\sin\beta\sin\gamma)+2(\cos\gamma\cos\alpha+\sin\gamma\sin\alpha)$
$+2(\cos\alpha\cos\beta+\sin\alpha\sin\beta)=0$

or $(\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta+2\cos\beta\cos\gamma$
$+2\cos\gamma\cos\gamma)+(\sin^2\alpha+\sin^2\beta+\sin^2\gamma$
$+2\sin\alpha\sin\beta+2\sin\beta\sin\gamma+2\sin\gamma\sin\alpha)=0$

or $(\cos\alpha+\cos\beta+\cos\gamma)^2+(\sin\alpha+\sin\beta+\sin\gamma)^2=0$

which is possible only when

$\cos\alpha+\cos\beta+\cos\gamma=0$ and $\sin\alpha+\sin\beta+\sin\gamma=0$

Example 62. Find the value of $\frac{\cos 25^\circ+\cot 55^\circ}{\tan 25^\circ+\tan 55^\circ}+\frac{\cot 55^\circ+\cot 100^\circ}{\tan 55^\circ+\tan 100^\circ}+\frac{\cot 100^\circ+\cot 25^\circ}{\tan 100^\circ+\tan 25^\circ}$.

Sol. $E=\frac{\cot 25^\circ+\cot 55^\circ}{\tan 25^\circ+\tan 55^\circ}+\frac{\cot 55^\circ+\cot 100^\circ}{\tan 55^\circ+\tan 100^\circ}+\frac{\cot 100^\circ+\cot 25^\circ}{\tan 100^\circ+\tan 25^\circ}$

$$=\frac{1}{\tan 55^\circ\tan 100^\circ}+\frac{1}{\tan 55^\circ\tan 100^\circ}+\frac{1}{\tan 100^\circ\tan 25^\circ}$$

$$=\frac{\tan 25^\circ+\tan 55^\circ+\tan 100^\circ}{\tan 25^\circ\cdot\tan 55^\circ\cdot\tan 100^\circ}$$

Since, $25^\circ+55^\circ+100^\circ=180^\circ$

$\tan 25^\circ+\tan 55^\circ+\tan 100^\circ=\tan 25^\circ\tan 55^\circ\tan 100^\circ$

$\Rightarrow \quad E=1$

Example 63. Prove that $\sum_{k=1}^{100}\sin(kx)\cos(101-k)x=50\sin(101x)$

Sol. Let $S=\sum_{k=1}^{100}\sin(kx)\cos(101-k)x$

$\Rightarrow \quad S=\sin x\cos 100x+\sin 2x\cos 99x$
$+\ldots+\sin 100x\cos x$ ...(i)

$S=\cos x\sin 100x+\cos 2x\sin 99x+\ldots+$
$\sin x\cos 100x$ ...(ii)

(on writing in reverse order)

On adding Eqs. (i) and (ii), we get

$2S=(\sin x\cos 100x+\cos x\sin 100x)$
$+(\sin 2x\cos 99x+\cos 2x\sin 99x)$
$\ldots$
$\ldots$
$+(\sin 100x\cos x+\sin x\cos 100x)$

$=\sin 101x+\sin 101x+\ldots+\sin 101x$ (100 times)

Hence, $S=50\sin(101x)$

Example 64. If $A=\frac{\pi}{5}$, then find the value of $\sum_{r=1}^{8}\tan(rA)\cdot\tan((r+1)A)$.

Sol. $\tan((r+1)A-(rA))=\frac{\tan(r+1)A-\tan(rA)}{1+\tan(r+1)A\cdot\tan(rA)}$

$$\Rightarrow \quad S=\sum_{r=1}^{8}\tan(rA)\cdot\tan(r+1)A$$

$$=\sum_{r=1}^{8}(-1)+\frac{1}{\tan A}\sum_{r=1}^{8}(\tan(r+1)A-\tan(rA))$$

$$=-8+\frac{1}{\tan A}\cdot(\tan 9A-\tan A)$$

Now, $\tan 9A=\tan\frac{9\pi}{5}$

$$=\tan\left(2\pi-\frac{\pi}{5}\right)$$

$$=-\tan\frac{\pi}{5}$$

$$\Rightarrow \quad S=-8+\frac{1}{\tan A}(-2\tan A)$$

$$=-8-2=-10$$

Example 65. Prove that
$\sin\theta\cdot\sec 3\theta+\sin 3\theta\cdot\sec 3^2\theta+\sin 3^2\,\theta\cdot\sec 3^3\,\theta+\dots$ upto n terms $=\frac{1}{2}[\tan 3^n\theta-\tan\theta]$

Sol. $\sin\theta\cdot\sec 3\theta+\sin 3\theta\cdot\sec 3^2\,\theta+\sin 3^2\theta\cdot\sec 3^3\,\theta+\dots$ upto n terms

$$=\sum_{r=1}^{n}\sin 3^{r-1}\,\theta\cdot\sec 3^r\,\theta$$
$$=\sum_{r=1}^{n}\frac{2\cos 3^{r-1}\,\theta\sin 3^{r-1}\,\theta}{2\cos 3^{r-1}\,\theta\cdot\cos 3^r\,\theta}$$
$$=\frac{1}{2}\sum_{r=1}^{n}\frac{\sin(2\cdot 3^{r-1}\,\theta)}{\cos 3^{r-1}\,\theta\cdot\cos 3^r\,\theta}$$
$$=\frac{1}{2}\sum_{r=1}^{n}\frac{\sin(3^r\theta-3^{r-1}\,\theta)}{\cos 3^{r-1}\,\theta\cdot\cos 3^r\theta}$$
$$=\frac{1}{2}\sum_{r=1}^{n}\frac{\sin 3^r\,\theta\cdot\cos 3^{r-1}\,\theta-\cos 3^r\,\theta\cdot\sin 3^{r-1}\,\theta}{\cos 3^{r-1}\,\theta\cdot\cos 3^r\,\theta}$$
$$=\frac{1}{2}\sum_{r=1}^{n}(\tan 3^r\,\theta-\tan 3^{r-1}\,\theta)$$
$$=\frac{1}{2}[\tan 3^n\theta-\tan\theta]$$

Example 66. In a triangle ABC, if $\sin A\sin(B-C)=\sin C\sin(A-B)$, then prove that $\cot A$, $\cot B$, $\cot C$ are in AP.

Sol. $\sin A\sin(B-C)=\sin C\sin(A-B)$

$$\Rightarrow\quad\frac{\sin(B-C)}{\sin C\sin B}=\frac{\sin(A-B)}{\sin A\sin B}$$
$$\Rightarrow\frac{\sin B\cos C-\sin C\cos B}{\sin C\sin B}=\frac{\sin A\cos B-\sin B\cos A}{\sin A\sin B}$$
$$\Rightarrow\quad\cot C-\cot B=\cot B-\cot A$$
$$\Rightarrow\quad 2\cot B=\cot A+\cot C$$

$\therefore$ $\cot A$, $\cot B$, $\cot C$ are in A.P.

Example 67. If $0<\beta<\alpha<\pi/4$, $\cos(\alpha+\beta)=3/5$ and $\cos(\alpha-\beta)=4/5$, then evaluate $\sin 2\alpha$.

Sol. We know, $\sin(2\alpha)=\sin\{(\alpha+\beta)+(\alpha-\beta)\}$

$$=\sin(\alpha+\beta)\cdot\cos(\alpha-\beta)+\sin(\alpha-\beta)\cdot\cos(\alpha+\beta).$$
$$=\frac{4}{5}\cdot\frac{4}{5}+\frac{3}{5}\cdot\frac{3}{5}$$

[using $\cos(\alpha+\beta)=3/5$, $\cos(\alpha-\beta)=4/5$
$\Rightarrow\sin(\alpha+\beta)=4/5$, $\sin(\alpha-\beta)=3/5$]

$$=\frac{16+9}{25}=1\ \Rightarrow\ \sin 2\alpha=1$$

Example 68. If $\cos\alpha=\frac{1}{2}\left(x+\frac{1}{x}\right)$, $\cos\beta=\frac{1}{2}\left(y+\frac{1}{y}\right)$, then evaluate $\cos(\alpha-\beta)$.

Sol. $\cos\alpha=\frac{1}{2}\left(x+\frac{1}{x}\right)$

$$\Rightarrow x^2-2x\cos\alpha+1=0\ \Rightarrow\ x=\frac{2\cos\alpha\pm\sqrt{4\cos^2\alpha-4}}{2}$$
$$\Rightarrow\quad x=\frac{2\cos\alpha\pm 2i\sin\alpha}{2}\qquad\{\text{as }\sqrt{-1}=i\}$$
$$\therefore\quad x=\cos\alpha\pm i\sin\alpha$$

Similarly, $y=\cos\beta\pm i\sin\beta$

$$\therefore\quad\frac{x}{y}=\frac{\cos\alpha\pm i\sin\alpha}{\cos\beta\pm i\sin\beta}=\cos(\alpha-\beta)\pm i\sin(\alpha-\beta)\qquad\text{...(i)}$$

and $$\frac{y}{x}=\frac{\cos\beta\pm i\sin\beta}{\cos\alpha\pm i\sin\alpha}=\cos(\alpha-\beta)\mp i\sin(\alpha-\beta)\qquad\text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$\frac{x}{y}+\frac{y}{x}=2\cos(\alpha-\beta)$$

i.e. $$\cos(\alpha-\beta)=\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right).$$

Example 69. If $2\sin\alpha\cos\beta\sin\gamma=\sin\beta\sin(\alpha+\gamma)$. Then, show $\tan\alpha$, $\tan\beta$ and $\tan\gamma$ are in harmonic progression.

Sol. We have, $2\sin\alpha\cos\beta\sin\gamma=\sin\beta\sin(\alpha+\gamma)$

or $2\sin\alpha\cos\beta\sin\gamma=\sin\beta\{\sin\alpha\cos\gamma+\cos\alpha\sin\gamma\}$

$\Rightarrow$ $2\sin\alpha\cos\beta\sin\gamma=\sin\alpha\sin\beta\cos\gamma+\cos\alpha\sin\beta\sin\gamma$

On dividing both sides by $\sin\alpha\sin\beta\sin\gamma$, we get

$$2\cot\beta=\cot\alpha+\cot\gamma\quad\text{or}\quad\frac{2}{\tan\beta}=\frac{1}{\tan\alpha}+\frac{1}{\tan\gamma}$$

i.e. $\frac{1}{\tan\alpha},\frac{1}{\tan\beta},\frac{1}{\tan\gamma}$ are in AP

or $\tan\alpha,\tan\beta,\tan\gamma$ are in HP.

Exercise for Session 6

1. If α lies in II quadrant, β lies in III quadrant and $\tan(\alpha+\beta) > 0$, then $(\alpha+\beta)$ lies in .... quadrants.
2. If $3\tan A \tan B = 1$, then prove that $\dfrac{\cos(A-B)}{\cos(A+B)} = 2$.
3. If $\tan\alpha = \dfrac{m}{m+1}$ and $\tan\beta = \dfrac{1}{2m+1}$, the find the value of $\alpha+\beta$.
4. If $\cos(\alpha+\beta) = \dfrac{4}{5}$, $\sin(\alpha-\beta) = \dfrac{5}{13}$ and $\alpha, \beta \in \left(0, \dfrac{\pi}{4}\right)$, then find the value of $\tan 2\alpha$.
5. If $\alpha+\beta = \dfrac{\pi}{2}$ and $\beta+\gamma = \alpha$, then find the value of $\tan\alpha$.
6. If $\cos(\theta-\alpha) = a$ and $\cos(\theta-\beta) = b$ then the value of $\sin^2(\alpha-\beta) + 2ab\cos(\alpha-\beta)$.
7. If $2\cos A = x + \dfrac{1}{x}$, $2\cos\beta = y + \dfrac{1}{y}$ then show that $2\cos(A-B) = \dfrac{x}{y} + \dfrac{y}{x}$.
8. If $y = (1+\tan A)(1-\tan B)$, where $A-B = \dfrac{\pi}{4}$, then find the value of $(y+1)^{y+1}$.

Answer

Exercise for Session 6

1. I and IV **3.** $\frac{\pi}{4}$ **4.** $\frac{65}{33}$ **5.** $\tan\beta + 2\tan\gamma$

6. $a^2 + b^2$ **8.** 27