CHAPTER TWENTY TWO

Vectors

A vector is a quantity having both magnitude and direction. For instance displacement, velocity, force and acceleration are vector quantities. Any portion of a straight line, where the two end-points are distinguished as *initial* and *terminal*, is called a *directed line segment*. The directed line segment with initial point A and terminal point B is denoted by the symbol **AB** or $\overrightarrow{AB}$ and the length of the vector **AB** by |**AB**|. Graphically, a vector is represented by a directed line segment. We denote it by a letter with an arrow over it, as in $\vec{a}$, or in bold type, as in **a**. A scalar is a quantity having magnitude only but no direction, such as mass, length, time, temperature and any real number. A vector whose initial and terminal points are the same is called a zero vector, **0**.

Two vectors are equal if they have the same magnitude; they lie on the same line or on parallel lines and they have the same direction.

Let **a** be any vector and α a scalar then $\alpha\mathbf{a}$ is a vector whose length is equal to $|\alpha|\,|\mathbf{a}|$, **a** is a vector on the same line or on a line parallel to the line on which **a** lies. Moreover, direction of $\alpha\mathbf{a}$ is same as that of **a** if $\alpha > 0$ and is opposite to that of **a** if $\alpha < 0$.

If **a** = **AB** and **b** = **BC** then **a** + **b** is the vector **AC**.

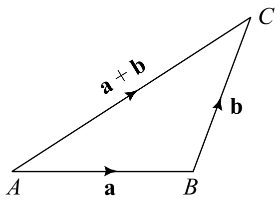

Fig. 22.1

The following are some elementary properties of addition and scalar multiplication:

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (addition is commutative) (Fig. 22.2)
2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ (addition is associative) (Fig. 22.3)
3. $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$
4. $(\alpha + \beta)\,\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$

   $0\mathbf{a} = 0$, $1\mathbf{a} =$ and $(-1)\,\mathbf{a} = -\,\mathbf{a}$

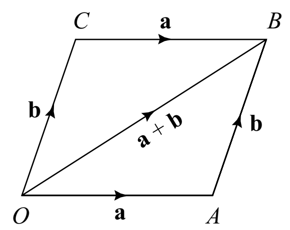

Fig. 22.2

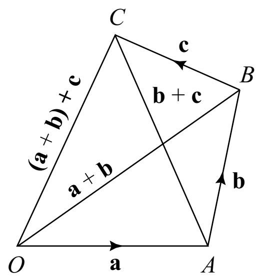

Fig. 22.3

The position vector **r** of any point P with respect to the origin of reference O is the vector **OP**. Two vectors **a** and **b** are said to be collinear if they are supported on the same or parallel lines. For such vectors, $\mathbf{b} = x\mathbf{a}$ for some scalar x. A set of vectors is said to be *coplanar* if they lie in the same plane, or the planes in which the different vectors lie are all parallel to the same plane. Three vectors **a**, **b** and **c** are coplanar if and only if $\mathbf{c} = x\mathbf{a} + y\mathbf{b}$ for some scalars x and y.

LINEAR COMBINATIONS

The vector $\mathbf{r} = \alpha_1\,\mathbf{a}_1 + \alpha_2\,\mathbf{a}_2 + \cdots + \alpha_n\,\mathbf{a}_n$, where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are scalars, is called a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$. The following results are useful in determining coplanar and collinear vectors:

1. If **a** and **b** are non-collinear vectors, then

   $x\mathbf{a} + y\mathbf{b} = x'\mathbf{a} + y'\mathbf{b} \quad \Leftrightarrow \quad x = x',\ y = y'$
2. *Fundamental theorem in plane*. If **a** and **b** are non-collinear vectors, then any vector **r**, coplanar with **a** and **b**, can be expressed uniquely as a linear combination of **a** and **b**. That is, there exist unique x and $y \in \mathbf{R}$ such that $\mathbf{r} = x\mathbf{a} + y\mathbf{b}$.
3. If **a**, **b** and **c** are *non-coplanar* vectors, then

   $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = x'\mathbf{a} + y'\mathbf{b} + z'\mathbf{c}$

   $\Leftrightarrow \quad x = x',\ y = y',\ z = z'$
4. *Fundamental theorem in space*. If **a**, **b** and **c** are non-coplanar vectors in space, then any vector **r**

can be uniquely expressed as a linear combination of **a**, **b** and **c**. That is, there exist unique $x, y, z \in \mathbf{R}$ such that $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$. If **i**, **j** and **k** are three unit vectors along x-axis, y-axis and z-axis respectively, then any vector **r** can be represented uniquely as $\mathbf{r} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, where a_1, a_2 and a_3 are the *coordinates of* ***r***.

5. *Section formula*. The position vector of a point P which divides the line joining the points A and B with position vectors **a** and **b** respectively in the ratio $m : n$, is

$$\frac{n\mathbf{a} + m\mathbf{b}}{m + n} \quad (m \neq -n)$$

The position vector of mid-point M of AB, is (1/2) $(\mathbf{a} + \mathbf{b})$. The point A with position vector **a** is written $A(\mathbf{a})$. If $A(\mathbf{a})$, $B(\mathbf{b})$ and $C(\mathbf{c})$ are the vertices of a triangle ABC, then the centroid of this triangle is (1/3) $(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

6. *Test of collinearity*. Three points $A(\mathbf{a})$, $B(\mathbf{b})$, and $C(\mathbf{c})$ are collinear if and only if there exist scalars x, y and z, not all zero, such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0}$, where $x + y + z = 0$.

7. *Test of coplanarity*. Four points $A(\mathbf{a})$, $B(\mathbf{b})$, $C(\mathbf{c})$ and $D(\mathbf{d})$ are coplanar if and only if there exist scalars x, y, z and w, not all zero, such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + w\mathbf{d} = \mathbf{0}$, $x + y + z + w = 0$.

SCALAR OR DOT PRODUCT

The scalar product of two vectors **a** and **b** is given by $|\mathbf{a}|\,|\mathbf{b}| \cos\theta$, where θ $(0 \le \theta \le \pi)$ is the angle between the vectors **a** and **b**. It is denoted by $\mathbf{a} \cdot \mathbf{b}$.

Properties of the scalar product

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 = \mathbf{a}^2$.
2. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.
3. Two non-zero vectors **a** and **b** make an acute angle if $\mathbf{a} \cdot \mathbf{b} > 0$, an obtuse angle if $\mathbf{a} \cdot \mathbf{b} < 0$ and are inclined at a right angle if $\mathbf{a} \cdot \mathbf{b} = 0$.
4. $\mathbf{a} \cdot \mathbf{b} = (\text{projection of } \mathbf{a} \text{ on } \mathbf{b})|\mathbf{b}|$.
5. $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a}^2 - \mathbf{b}^2$; $(\mathbf{a} + \mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{a} \cdot \mathbf{b}$,
   $(\mathbf{a} - \mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a} \cdot \mathbf{b}$.
6. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$, then

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3, \; |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

and
$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

7. Vector component of **u** orthogonal to a vector **a** is $\mathbf{u} - \text{proj}_{\mathbf{a}}\,\mathbf{u}$ where $\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}.\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}$ and $\text{proj}_{\mathbf{a}}\,\mathbf{u}$ is vector component of **u** along **a.**

VECTOR OR CROSS PRODUCT

The vector product of two vectors **a** and **b**, denoted $\mathbf{a} \times \mathbf{b}$, is the vector **c** with $|\mathbf{c}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$, where θ is the angle between **a** and **b**, with $0 \le \theta \le \pi$. **c** is supported by the line perpendicular to **a** and **b**, and the direction of **c** is such that **a**, **b** and **c** form a right-handed system.

Properties of the vector product

1. $\mathbf{a} \times \mathbf{b} = -\,\mathbf{b} \times \mathbf{a}$.
2. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.
4. $(\mathbf{a} \times \mathbf{b})^2 = \mathbf{a}^2\mathbf{b}^2 - (\mathbf{a} \cdot \mathbf{b})^2$.
5. Two non-zero vectors **a** and **b** are collinear if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.
6. If **i**, **j** and **k** are unit vectors along positive x-axis, y-axis and z-axis then $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$, $\mathbf{k}\times\mathbf{i} = \mathbf{j}$.
7. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

8. The area of the parallelogram whose adjacent sides are represented by the vectors $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$, is $|\mathbf{a} \times \mathbf{b}|$, and the area of the triangle OAB is (1/2) $|\mathbf{a} \times \mathbf{b}|$. The vector area of the above parallelogram is $\mathbf{a} \times \mathbf{b}$.
9. A unit vector perpendicular to the plane of **a** and **b** is

$$\frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$$

and a vector of magnitude λ perpendicular to the plane of **a** and **b** is

$$\pm\frac{\lambda(\mathbf{a} \times \mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}$$

SCALAR TRIPLE PRODUCT

The scalar triple product of the three vectors **a**, **b** and **c** is denoted by $[\mathbf{a}\;\mathbf{b}\;\mathbf{c}]$, and is defined as $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$.

Properties of the scalar triple product

1. $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.
2. $[\mathbf{a}\;\mathbf{b}\;\mathbf{c}] = [\mathbf{b}\;\mathbf{c}\;\mathbf{a}] = [\mathbf{c}\;\mathbf{a}\;\mathbf{b}] = -\,[\mathbf{b}\;\mathbf{a}\;\mathbf{c}] = -\,[\mathbf{c}\;\mathbf{b}\;\mathbf{a}] = -\,[\mathbf{a}\;\mathbf{c}\;\mathbf{b}]$.
3. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = [\mathbf{a}\;\mathbf{b}\;\mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

In particular $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = 0$ and $\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}) = 0$.

4. The volume of the parallelopiped whose adjacent sides are represented by the vectors **a**, **b** and **c**, is $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$.
5. The volume of a tetrahedron *ABCD* is equal to $\frac{1}{6} |\mathbf{AB} \times \mathbf{AC} \cdot \mathbf{AD}|$.
6. Any three vectors **a**, **b** and **c** are coplanar if and only if $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$.
7. $[\mathbf{a} + \mathbf{b}\ \mathbf{c}\ \mathbf{d}] = [\mathbf{a}\ \mathbf{c}\ \mathbf{d}] + [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]$.
8. Three vectors **a**, **b** and **c** form a right-handed or left-handed system, according as $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] > $ or < 0.
9. $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$

10 $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\ [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \begin{vmatrix} \mathbf{a}\cdot\mathbf{u} & \mathbf{b}\cdot\mathbf{u} & \mathbf{c}\cdot\mathbf{u} \\ \mathbf{a}\cdot\mathbf{v} & \mathbf{b}\cdot\mathbf{v} & \mathbf{c}\cdot\mathbf{v} \\ \mathbf{a}\cdot\mathbf{w} & \mathbf{b}\cdot\mathbf{w} & \mathbf{c}\cdot\mathbf{w} \end{vmatrix}$

11. Four points with position vectors **a**, **b**, **c** and **d** will be coplanar if
$$[\mathbf{d}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{d}\ \mathbf{c}\ \mathbf{a}] + [\mathbf{d}\ \mathbf{a}\ \mathbf{b}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$
12. $[\mathbf{a} + \mathbf{b}\quad \mathbf{b} + \mathbf{c}\quad \mathbf{c} + \mathbf{a}] = 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$
13. $[\mathbf{b} \times \mathbf{c}\quad \mathbf{c} \times \mathbf{a}\quad \mathbf{a} \times \mathbf{b}] = 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

VECTOR TRIPLE PRODUCT

The vector triple product of three vectors **a**, **b** and **c** is the vector $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$,

Since $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is coplanar with **b** and **c** and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is coplanar with **a** and **b**, we have

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$

Clearly, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ in general. In fact, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ if and only if the vectors **a** and **c** are collinear.

If **a**,**b**,**c** and **d** lie in the same plane then $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{0}$

GEOMETRICAL AND PHYSICAL APPLICATIONS

Bisector of an angle If **a** and **b** are unit vectors along the sides of an angle, then $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are respectively the vectors along the internal and external bisectors of the angle. The bisectors of the angles between the lines $\mathbf{r} = x\mathbf{a}$ and $\mathbf{r} = y\mathbf{b}$ are given by
$$\mathbf{r} = \lambda \left(\frac{\mathbf{a}}{|\mathbf{a}|} \pm \frac{\mathbf{b}}{|\mathbf{b}|} \right) \qquad (\lambda \in \mathbf{R})$$
Equation of a line passing through a point with position vector **a** and parallel to a vector **b** is $\mathbf{r} = \mathbf{a} + t\mathbf{b}$

Reciprocal systems of vectors Let **a**, **b** and **c** be a system of three non-coplanar vectors. Then the system $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$, which satisfies
$$\mathbf{a} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{b}' = \mathbf{c} \cdot \mathbf{c}' = 1$$
and $$\mathbf{a} \cdot \mathbf{b}' = \mathbf{a} \cdot \mathbf{c}' = \mathbf{b} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{c}' = \mathbf{c} \cdot \mathbf{a}' = \mathbf{c} \cdot \mathbf{b}' = 0$$
is called the reciprocal system to the vectors **a**, **b** and **c**.

Equation of a plane. The equation of a plane passing through the point with position vector **a** and parallel to the plane containing **b** and **c**, is
$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c} \text{ or } [\mathbf{r} - \mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0.$$
λ and μ being parameters. The equation of a plane through three points whose position vectors are **a**, **b** and **c** is
$$\mathbf{r} = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$
or $$\mathbf{r} \cdot (\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}) = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$
λ and μ being parameters.

Equation of a plane which is at a distance d from the origin having a unit normal **n** is $\mathbf{r} \cdot \mathbf{n} = d$. Equation of a plane passing through a point with position vector **a** having a unit normal **n** is $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$.

Work done. If a force **F** acts at a point *A* and displaces it to the point *B*, then the work done by the force **F** is $\mathbf{F} \cdot \mathbf{AB}$. The moment of a force **F** applied at *B* about the point *A* is the vector $\mathbf{AB} \times \mathbf{F}$.

SOLVED EXAMPLES

Concept Based
Straight Objective Type Questions

Example 1: If in a triangle *OAC*, *B* is the mid point of *AC* and $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$ then

(a) $\mathbf{OC} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ (b) $\mathbf{OC} = 2\mathbf{b} - 2\mathbf{a}$

(c) $\mathbf{OC} = 2\mathbf{b} - \mathbf{a}$ (d) $\mathbf{OC} = 3\mathbf{b} - 2\mathbf{a}$

Ans. (c)

Solution: Let *O* be the origin of reference. The position vector of *A* is **a** and that *B* is **b**. If position vector of *C* is **c**, then
$$\mathbf{b} = \frac{\mathbf{a} + \mathbf{c}}{2} \quad \Rightarrow \quad \mathbf{c} = 2\mathbf{b} - \mathbf{a}$$

Example 2: The angle between the vectors $\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $-\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is

(a) 45° (b) 60°

(c) 90° (d) 135°

Ans. (c)

Solution: $(\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot (-\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = -1 - 1 + 2 = 0$

So the angle is 90°.

Example 3: A unit vector **c** perpendicular to **a** = **i** – **j** and coplanar with **a** and **b** = **i** + **k** is

(a) $\frac{1}{\sqrt{6}}(\mathbf{i}+\mathbf{j}+2\mathbf{k})$ (b) $\frac{1}{\sqrt{3}}(\mathbf{i}-\mathbf{j}+\mathbf{k})$

(c) $\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}-\mathbf{k})$ (d) $\frac{1}{\sqrt{6}}(\mathbf{i}-\mathbf{j}+2\mathbf{k})$

Ans. (a)

Solution: $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$

$= \lambda(\mathbf{i} - \mathbf{j}) + \mu(\mathbf{i} + \mathbf{k})$

$= (\lambda + \mu)\mathbf{i} - \lambda\mathbf{j} + \mu\mathbf{k}$

since **c** is a unit vector so $(\lambda + \mu)^2 + \lambda^2 + \mu^2 = 1$ (i)

Also $\mathbf{a}.\mathbf{c}. = 0$, but $\mathbf{a}.\mathbf{c} = \lambda\,|a|^2 + \mu\,\mathbf{a}.\mathbf{b}$

$\Rightarrow$ $0 = \lambda.\,2 + \mu\,.1$

$\Rightarrow$ $\mu = -2\lambda$

(i) $\Rightarrow$ $(1-2)^2\,\lambda^2 + \lambda^2 + 4\lambda^2 = 1$

$6\lambda^2 = 1$

$\lambda^2 = \frac{1}{6}$

$\Rightarrow \mathbf{c} = \pm\frac{1}{\sqrt{6}}\,(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

Example 4: If **a** and **b** are two non-parallel vectors satisfying |**a**| = |**b**|, then the vector (**a** + **b**) × (**a** × **b**) is parallel to

(a) **a** (b) **a** – **b**
(c) **a** + **b** (d) **b**

Ans. (b)

Solution: $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} \times \mathbf{b})$

$= \mathbf{a} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{b} \times (\mathbf{a} \times \mathbf{b})$

$= (\mathbf{a}\cdot\mathbf{b})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{b} + (\mathbf{b}\cdot\mathbf{b})\mathbf{a} - (\mathbf{b}\cdot\mathbf{a})\mathbf{b}$

$= (\mathbf{a}\cdot\mathbf{b})\,(\mathbf{a} - \mathbf{b}) + |\mathbf{a}|^2\,(\mathbf{a} - \mathbf{b})$

$= (\mathbf{a}\cdot\mathbf{b} + |\mathbf{a}|^2)\,(\mathbf{a} - \mathbf{b})$

So (**a** + **b**) × (**a** × **b**) is parallel to **a** – **b**.

Example 5: If **a** = **i** + **j** + **k** and **b** = **i** – **j** + 2**k** then the projection of **a** on **b** is given by

(a) $\frac{1}{2}(\mathbf{i}-\mathbf{j}+2\mathbf{k})$ (b) $\frac{1}{3}(\mathbf{i}+\mathbf{j}+\mathbf{k})$

(c) $\frac{1}{3}(\mathbf{i}-\mathbf{j}-\mathbf{k})$ (d) $\frac{1}{3}(\mathbf{i}-\mathbf{j}+2\mathbf{k})$

Ans. (d)

Solution: $\text{Proj}_{\mathbf{b}}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\,\mathbf{b}$

$= \frac{2}{6}(\mathbf{i}-\mathbf{j}+2\mathbf{k})$

$= \frac{1}{3}(\mathbf{i}-\mathbf{j}+2\mathbf{k})$

Example 6: If **a**, **b**, **c** are unit vectors such that **a** – **b** + **c** = **0** then **c** · **a** is equal to

(a) $\frac{3}{2}$ (b) $-\frac{1}{2}$

(c) $\frac{1}{3}$ (d) $-\frac{1}{3}$

Ans. (b)

Solution: Taking dot product with **a**, **b**, **c** in the relation **a** – **b** + **c** = 0, we have

$|\mathbf{a}|^2 - \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c} = 0$

$\mathbf{a}\cdot\mathbf{b} - |\mathbf{b}|^2 + \mathbf{b}\cdot\mathbf{c} = 0$

$\mathbf{a}\cdot\mathbf{c} - \mathbf{b}\cdot\mathbf{c} + |\mathbf{c}|^2 = 0$

Adding, we get $2\,\mathbf{a}\cdot\mathbf{c} = -(|\mathbf{a}|^2 - |\mathbf{b}|^2 + |\mathbf{c}|^2)$

$= -1$

$\mathbf{a}\cdot\mathbf{c} = -\frac{1}{2}$

Example 7: The non-zero vectors **a**, **b** and **c** are related as **b** = 5**a** and **c** = –2**b**. The angle between **a** and **c** is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$

(c) π (d) $\frac{\pi}{3}$

Ans. (c)

Solution: cosine of angle between **a** and **c** is

$$= \frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}||\mathbf{c}|} = \frac{\frac{1}{5}\mathbf{b}\cdot -2\mathbf{b}}{\frac{1}{5}|\mathbf{b}|\;2|\mathbf{b}|}$$

$= -1$

Hence the angle is π.

Example 8: A vector **b** collinear with $\mathbf{a} = 2\sqrt{2}\,\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ of length 10 is given by

(a) $3(2\sqrt{2}\mathbf{i}-\mathbf{j}+4\mathbf{k})$ (b) $2(2\sqrt{2}\mathbf{i}+\mathbf{j}-4\mathbf{k})$

(c) $2(2\sqrt{2}\mathbf{i}+\mathbf{j}+4\mathbf{k})$ (d) $2(2\sqrt{2}\mathbf{i}-\mathbf{j}+4\mathbf{k})$

Ans. (d)

Solution: $\mathbf{b} = \lambda\mathbf{a}$ and $10 = |\mathbf{b}| = |\lambda||\mathbf{a}|$

but $|\mathbf{a}|^2 = 8 + 1 + 16 = 25 \;\Rightarrow\; |\mathbf{a}| = 5$

Thus $|\lambda| = 2.$

So $\mathbf{b} = \pm 2(2\sqrt{2}\mathbf{i}-\mathbf{j}+4\mathbf{k})$

Example 9: The vector $\mathbf{p} = (\mathbf{a}\cdot\mathbf{c})\;\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\;\mathbf{c}$ is perpendicular to

(a) **c** (b) **b**
(c) **a** (d) **c** + **b**

Ans. (a)

Solution: **p** = **a** × (**b** × **c**) which is perpendicular to **a**.

Example 10: The angle between **a** + 2**b** and **a** – 3**b** if |**a**| = 1, |**b**| = 2 and angle between **a** and **b** is 60° is

(a) an acute angle (b) $\cos^{-1}\dfrac{-24}{\sqrt{21}\sqrt{31}}$

(c) $\cos^{-1}\dfrac{24}{\sqrt{21}\sqrt{31}}$ (d) $\cos^{-1}-\dfrac{1}{3}$

Ans. (b)

Solution: $|\mathbf{a}+2\mathbf{b}|^2 = |\mathbf{a}|^2 + 4|\mathbf{b}|^2 + 4\mathbf{a}\cdot\mathbf{b}$

$$= 1 + 16 + 4\,|\mathbf{a}||\mathbf{b}|\cos 60°$$

$$= 1 + 16 + 4.1.2.\frac{1}{2} = 21$$

$$|\mathbf{a}-3\mathbf{b}|^2 = |\mathbf{a}|^2 + 9|\mathbf{b}|^2 - 6\mathbf{a}\cdot\mathbf{b}$$

$$= 1 + 36 - 6 = 31$$

$$(\mathbf{a}+2\mathbf{b})\cdot(\mathbf{a}-3\mathbf{b}) = |\mathbf{a}|^2 - 6|\mathbf{b}|^2 - \mathbf{a}\cdot\mathbf{b}$$

$$= 1 - 24 - 1.2.\frac{1}{2} = -24$$

The angle between **a** + 2**b** and **a** – 3**b**

$$= \cos^{-1}\frac{(\mathbf{a}+2\mathbf{b})\cdot(\mathbf{a}-3\mathbf{b})}{|\mathbf{a}+2\mathbf{b}|\,|\mathbf{a}-3\mathbf{b}|}$$

$$= \cos^{-1}\frac{-24}{\sqrt{21}\sqrt{31}}.$$

LEVEL 1

Straight Objective Type Questions

Example 11: Let L_1: $\mathbf{r} = (\mathbf{i}+5\mathbf{j}+5\mathbf{k}) + t(4\mathbf{i}-4\mathbf{j}+5\mathbf{k})$ and L_2: $\mathbf{r} = (2\mathbf{i}+4\mathbf{j}+5\mathbf{k}) + t\,(8\mathbf{i}-3\mathbf{j}+\mathbf{k})$ be two lines then

(a) L_1 is parallel to L_2
(b) L_1 is perpendicular to L_2
(c) L_1 is not parallel to L_2
(d) none of these

Ans. (c)

Solution: The line L_1 is parallel to the vector $4\mathbf{i}-4\mathbf{j}+5\mathbf{k}$ and the line L_2 is parallel to the vector $8\mathbf{i}-3\mathbf{j}+\mathbf{k}$. These vectors are not parallel since neither is a scalar multiple of the other. Also $(4\mathbf{i}-4\mathbf{j}+5\mathbf{k}).(8\mathbf{i}-3\mathbf{j}+\mathbf{k})$ = 32 + 12 + 5 = 49 ≠ 0. So L_1 is not perpendicular to L_2.

Example 12: The angle between a diagonal of a cube and one of its edges is

(a) $\cos^{-1}(1/\sqrt{3})$ (b) $\pi/4$

(c) $\pi/6$ (d) $\pi/3$

Ans. (a)

Solution: Let **a** = a_1 **i**, **b** = a_1 **j** and **c** = a_1**k**. Then the vector **d** = $a_1(\mathbf{i}+\mathbf{j}+\mathbf{k})$ is a diagonal of the cube. The angle θ between **d** and **a** is given by

$$\cos\theta = \frac{\mathbf{a}\,.\,\mathbf{d}}{|\mathbf{a}|\,|\mathbf{d}|} = \frac{a_1^2}{a_1\left(\sqrt{3a_1^2}\right)} = \frac{1}{\sqrt{3}}.$$

Example 13: Let $\mathbf{u} = 2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $\mathbf{a} = 4\mathbf{i}-\mathbf{j}+2\mathbf{k}$. The vector component of **u** orthogonal to **a** is

(a) $-\frac{1}{7}(6\mathbf{i}+2\mathbf{j}-11\mathbf{k})$ (b) $\frac{1}{7}(-6\mathbf{i}+2\mathbf{j}-11\mathbf{k})$

(c) $-\frac{1}{7}(6\mathbf{i}-2\mathbf{j}+11\mathbf{k})$ (d) $-\frac{1}{7}(-6\mathbf{i}+2\mathbf{j}+11\mathbf{k})$

Ans. (a)

Solution: $\text{Proj}_{\mathbf{a}}\,\mathbf{u} = \dfrac{\mathbf{u.a}}{|\mathbf{a}|^2}\mathbf{a} = \dfrac{5}{7}(4\mathbf{i}-\mathbf{j}+2\mathbf{k})$ and vector component of **u** orthogonal to **a** is

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\,\mathbf{u} = (2\mathbf{i}-\mathbf{j}+3\mathbf{k}) - \frac{5}{7}(4\mathbf{i}-\mathbf{j}+2\mathbf{k})$$

$$= -\frac{6}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{11}{7}\mathbf{k}.$$

Example 14: Volume of the tetrahedron with vertices $P(-1, 2, 0)$, $Q\,(2, 1, -3)$, $R\,(1, 0, 1)$ and $S\,(3, -2, 3)$ is

(a) 1/3 (b) 2/3

(c) 1/4 (d) 3/4

Ans. (b)

Solution: $\mathbf{PQ} = 3\mathbf{i}-\mathbf{j}-3\mathbf{k}$, $\mathbf{PR} = 2\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $\mathbf{PS} = 4\mathbf{i}-4\mathbf{j}+3\mathbf{k}$ so the required volume

$$= \frac{1}{6}|\mathbf{PQ}\cdot(\mathbf{PR}\times\mathbf{PS})|$$

$$= \frac{1}{6}\left|\begin{vmatrix} 3 & -1 & -3 \\ 2 & -2 & 1 \\ 4 & -4 & 3 \end{vmatrix}\right| = \frac{|-4|}{6} = \frac{2}{3}.$$

Example 15: The distance between a point P whose position vector is $5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and the line $\mathbf{r} = (3\mathbf{i} + 7\mathbf{j} + \mathbf{k}) + t(\mathbf{j} + \mathbf{k})$ is

(a) 3 (b) 4
(c) 5 (d) 6

Ans. (d)

Solution: Let $\mathbf{u} = \mathbf{j} + \mathbf{k}$ and the point Q whose position vector is $3\mathbf{i}+7\mathbf{j}+\mathbf{k}$ is on the line, so

$$\mathbf{v} = \mathbf{QP} = (5-3)\mathbf{i} + (1-7)\mathbf{j} + (3-1)\mathbf{k} = 2\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$$

$$d = |\mathbf{v}|\sin\theta = \frac{|\mathbf{u}\times\mathbf{v}|}{|\mathbf{u}|} = \frac{|8\mathbf{i}+2\mathbf{j}-2\mathbf{k}|}{|\mathbf{j}+\mathbf{k}|} = \frac{\sqrt{64+4+4}}{\sqrt{2}} = 6$$

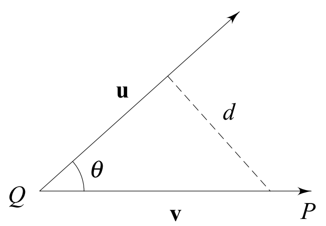

Fig. 22.4

Example 16: Let **a**, **b**, **c** be the three vectors such that $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}+\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}+\mathbf{b}) = 0$ and $|\mathbf{a}| = 1$, $|\mathbf{b}| = 4$, $|\mathbf{c}| = 8$, then $|\mathbf{a}+\mathbf{b}+\mathbf{c}| =$

(a) 13 (b) 81
(c) 9 (d) 5

Ans. (c)

Solution: $|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2 = (\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c})$

Adding $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}+\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}+\mathbf{b}) = 0$,

we get $2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{a}\cdot\mathbf{c}) = 0$

$\Rightarrow |\mathbf{a}+\mathbf{b}+\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c})$

$= 1 + 16 + 64 = 81$. Hence $|\mathbf{a}+\mathbf{b}+\mathbf{c}| = 9$.

Example 17: If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + \mathbf{j}$, then t such that $\mathbf{a} + t\mathbf{b}$ is at right angle to **c** will be equal to

(a) 5 (b) 4
(c) 6 (d) 2

Ans. (a)

Solution: Since $\mathbf{a} + t\mathbf{b}$ is at right angle to **c** so $(\mathbf{a}+t\mathbf{b})\cdot\mathbf{c} = 0$. But $\mathbf{a}\cdot\mathbf{c} = 5$ and $\mathbf{b}\cdot\mathbf{c} = -1$ so $5 - t = 0 \Rightarrow t = 5$.

Example 18: If $|\mathbf{a}| = 2$, $|\mathbf{b}| = 5$ and $|\mathbf{a}\times\mathbf{b}| = 8$ then $|\mathbf{a}\cdot\mathbf{b}|$ is equal to

(a) 4 (b) 6
(c) 5 (d) none of these

Ans. (b)

Solution: Since $\sin\theta = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} = \frac{8}{10} = \frac{4}{5}$ so $\cos\theta = \pm 3/5$

Thus $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta = 10\cdot(\pm 3/5) = \pm 6$.

Example 19: If $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ is equal to

(a) 0 (b) 1
(c) – 1 (d) $|\mathbf{a}|\,|\mathbf{b}|\,|\mathbf{c}|$

Ans. (d)

Solution: Since $\mathbf{a}\cdot\mathbf{b} = \mathbf{c}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{c} = 0$ so **a**, **b** and **c** are mutually perpendicular. Thus **a** is collinear with $\mathbf{b}\times\mathbf{c}$. Hence $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = |\mathbf{a}|\,|\mathbf{b}\times\mathbf{c}| = |\mathbf{a}||\mathbf{b}||\mathbf{c}|$.

Example 20: Let **a**, **b** and **c** be three non-coplanar vectors, and let **p**, **q** and **r** be the vectors defined by the relations

$$\mathbf{p} = \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\quad \mathbf{q} = \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} \text{ and } \mathbf{r} = \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

Then the value of the expression $(\mathbf{a}+\mathbf{b})\cdot\mathbf{p} + (\mathbf{b}+\mathbf{c})\cdot\mathbf{q} + (\mathbf{c}+\mathbf{a})\cdot\mathbf{r}$ is equal to

(a) 0 (b) 1
(c) 2 (d) 3

Ans. (d)

Solution: $\mathbf{a}\cdot\mathbf{p} = \frac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = 1 = \mathbf{b}\cdot\mathbf{q} = \mathbf{c}\cdot\mathbf{r}$

$$\mathbf{b}\cdot\mathbf{p} = \frac{\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{0}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = 0 = \mathbf{c}\cdot\mathbf{q} = \mathbf{a}\cdot\mathbf{r}$$

Therefore, the given expression is equal to $1 + 0 + 1 + 0 + 1 + 0 = 3$.

Example 21: The volume of the parallelopiped whose sides are given by $\mathbf{OA} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{OB} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{OC} = 3\mathbf{i} - \mathbf{k}$ is

(a) 4/13 (b) 4
(c) 2/7 (d) none of these

Ans. (b)

Solution: The volume of the parallelopiped

$$= |[\mathbf{OA}\ \mathbf{OB}\ \mathbf{OC}]| = \left|\begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix}\right| = 4.$$

Example 22: The points with position vectors $60\mathbf{i} + 3\mathbf{j}$, $40\mathbf{i} - 8\mathbf{j}$, $a\mathbf{i} - 52\mathbf{j}$ are collinear if

(a) $a = -40$ (b) $a = 40$
(c) $a = 20$ (d) none of these

Ans. (a)

Solution: Denoting **a**, **b**, **c** by the given vectors respectively. These vectors will be collinear if there is some constant K such that $\mathbf{c} - \mathbf{a} = K(\mathbf{b} - \mathbf{a})$

$\Rightarrow a - 60 = -20K$ and $-55 = -11K$

$\Rightarrow a = -100 + 60 = -40$.

Example 23: If $|\mathbf{a}| = 2$, $|\mathbf{b}| = 3$ $|\mathbf{c}| = 4$ and $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ then the value of $\mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b}$ is equal to

(a) 19/2 (b) –19/2
(c) 29/2 (d) –29/2

Ans. (d)

◎ **Solution:** $0 = |\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c})$

$\Rightarrow \quad 29 + 2\,(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = 0$

$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = -29/2.$

◉ **Example 24:** If A, B, C, D are four points in a space and $|\mathbf{AB} \times \mathbf{CD} + \mathbf{BC} \times \mathbf{AD} + \mathbf{CA} \times \mathbf{BD}| = \lambda$ (area of the triangle ABC). Then the value of λ is

(a) 1 (b) 2
(c) 3 (d) 4

Ans. (d)

◎ **Solution:** Let D be the origin of reference and $\mathbf{DA} = \mathbf{a}$, $\mathbf{DB} = \mathbf{b}$, $\mathbf{DC} = \mathbf{c}$

$$|\mathbf{AB}\times\mathbf{CD}+\mathbf{BC}\times\mathbf{AD}+\mathbf{CA}\times\mathbf{BD}|$$
$$= |(\mathbf{b}-\mathbf{a})\times(-\mathbf{c})+(\mathbf{c}-\mathbf{b})\times(-\mathbf{a})+(\mathbf{a}-\mathbf{c})\times(-\mathbf{b})|$$
$$= |\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}-\mathbf{b}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}-\mathbf{c}\times\mathbf{b}|$$
$$= 2\,|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|$$
$$= 2\,(2 \text{ area of } \Delta ABC) = 4 \text{ area of } \Delta ABC.$$

Hence $\lambda = 4$.

◉ **Example 25:** Given $\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. A unit vector perpendicular to both $\mathbf{a} + \mathbf{b}$ and $\mathbf{b} + \mathbf{c}$ is

(a) $\dfrac{2\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{6}}$ (b) $\mathbf{j}$

(c) $\mathbf{k}$ (d) $\dfrac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$

Ans. (c)

◎ **Solution:** $(\mathbf{a} + \mathbf{b}) \times (\mathbf{b} + \mathbf{c}) = 3\mathbf{j}\times(-2\mathbf{i}+4\mathbf{j}) = 6\mathbf{k}$

Hence the required unit vector is $\mathbf{k}$.

◉ **Example 26:** If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are unit coplanar vectors, then the scalar triple product $[2\mathbf{a} - \mathbf{b}\ 2\mathbf{b} - \mathbf{c}\ 2\mathbf{c} - \mathbf{a}] =$

(a) 0 (b) 1
(c) $-\sqrt{3}$ (d) $\sqrt{3}$

Ans. (a)

◎ **Solution:** As $\mathbf{a}\,\mathbf{b}, \mathbf{c}$ are coplanar vectors, $2\mathbf{a} - \mathbf{b}$, $2\mathbf{b} - \mathbf{c}$ and $2\mathbf{c} - \mathbf{a}$ are also coplanar vectors. Thus $[2\mathbf{a} - \mathbf{b}\ 2\mathbf{b} - \mathbf{c}\ 2\mathbf{c} - \mathbf{a}] = 0$.

◉ **Example 27:** If the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ form the sides BC, CA and AB respectively, of a triangle ABC, then

(a) $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}=0$

(b) $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$

(c) $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}$

(d) $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}=\mathbf{0}$

Ans. (b)

◎ **Solution:** $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{BC}+\mathbf{CA}+\mathbf{AB}=\mathbf{BA}+\mathbf{AB}$

So $\quad \mathbf{a}\times(\mathbf{a}+\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{0}=\mathbf{0}$

$\Rightarrow \quad \mathbf{a}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}=\mathbf{0} \quad\Rightarrow\quad \mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{a}$

Similarly, $\quad \mathbf{b}\times(\mathbf{a}+\mathbf{b}+\mathbf{c})=\mathbf{0}$

$\Rightarrow \quad \mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{c}=\mathbf{0} \quad\Rightarrow\quad \mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}$. Thus

$\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{a}=\mathbf{b}\times\mathbf{c}$.

◉ **Example 28:** If $\mathbf{a}$ and $\mathbf{b}$ are two unit vectors such that $\mathbf{a} + 2\mathbf{b}$ and $5\mathbf{a} - 4\mathbf{b}$ are perpendicular to each other then the angle between $\mathbf{a}$ and $\mathbf{b}$ is

(a) 45º (b) 60º
(c) $\cos^{-1}(1/\sqrt{3})$ (d) $\cos^{-1}(2/7)$

Ans. (b)

◎ **Solution:** According to the given conditions $|\mathbf{a}| = |\mathbf{b}| = 1$ and $(\mathbf{a} + 2\mathbf{b})\cdot(5\mathbf{a} - 4\mathbf{b}) = 0$ so

$$5|\mathbf{a}|^2 - 8|\mathbf{b}|^2 + 6\mathbf{a}\cdot\mathbf{b} = 0$$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = 3/6 = 1/2$

$\Rightarrow \quad |\mathbf{a}|\,|\mathbf{b}| \cos\theta = 1/2$

$\Rightarrow \quad \cos\theta = 1/2 \quad\Rightarrow\quad \theta = 60º.$

◉ **Example 29:** Let $\mathbf{v} = 2\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{w} = \mathbf{i}+3\mathbf{k}$. If $\mathbf{u}$ is a unit vector, then the maximum value of the scalar triple product $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$ is

(a) -1 (b) $\sqrt{10}+\sqrt{16}$
(c) $\sqrt{59}$ (d) $\sqrt{60}$

Ans. (c)

◎ **Solution:** $\mathbf{v}\times\mathbf{w} = 3\mathbf{i}-7\mathbf{j}-\mathbf{k}$

Now $\quad [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \mathbf{u}\cdot(3\mathbf{i}-7\mathbf{j}-\mathbf{k})$

$= |\mathbf{u}|\ |3\mathbf{i}-7\mathbf{j}-\mathbf{k}|\ \cos\theta$ (where θ is the angle between $\mathbf{u}$ and $\mathbf{v}\times\mathbf{w}$)

$= \sqrt{59}\ \cos\theta$

Thus $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$ is maximum if $\cos\theta = 1$ i.e. $\theta = 0$ or take $\mathbf{u} = \dfrac{1}{\sqrt{59}}(3\mathbf{i}-7\mathbf{j}-\mathbf{k})$. Hence the maximum value is $\sqrt{59}$.

◉ **Example 30:** A vector $\mathbf{c}$ perpendicular to the vectors $2\mathbf{i}+3\mathbf{j}-\mathbf{k}$ and $\mathbf{i}-2\mathbf{j}+3\mathbf{k}$ satisfying $\mathbf{c}.(2\mathbf{i}-\mathbf{j}+\mathbf{k})=-6$ is

(a) $-2\mathbf{i}+\mathbf{j}-\mathbf{k}$ (b) $2\mathbf{i}-\mathbf{j}-\dfrac{4}{3}\mathbf{k}$

(c) $-3\mathbf{i}+3\mathbf{j}+3\mathbf{k}$ (d) $3\mathbf{i}-3\mathbf{j}+3\mathbf{k}$

Ans. (c)

◎ **Solution:** Vector $\mathbf{c}$ perpendicular to $\mathbf{a}=2\mathbf{i}+3\mathbf{j}-\mathbf{k}$ and $\mathbf{b} = \mathbf{i}-2\mathbf{j}+3\mathbf{k}$ is of the form $\alpha\,(\mathbf{a}\times\mathbf{b}) = \alpha\,(7\mathbf{i} - 7\mathbf{j} - 7\mathbf{k})$.

Since $\mathbf{c}\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k}) = -6$ so $\alpha[14+7-7] = -6 \Rightarrow \alpha = -3/7$.

Hence $\mathbf{c} = -3\mathbf{i}+3\mathbf{j}+3\mathbf{k}$.

Example 31: If **a**, **b**, **c** be three units vectors such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (1/2)\,\mathbf{b}$; **b** and **c** being non-parallel then

(a) the angle between **a** and **c** is $\pi/3$
(b) the angle between **a** and **c** is $\pi/2$
(c) the angle between **a** and **b** is $\pi/3$
(d) the angle between **a** and **b** is $\pi/6$

Ans. (a)

Solution: $(1/2)\ \mathbf{b} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$$\Rightarrow \qquad \left(\mathbf{a}\cdot\mathbf{c} - \frac{1}{2}\right)\mathbf{b} = (\mathbf{a}\cdot\mathbf{b})\ \mathbf{c}.$$

The last relation is possible only if $\mathbf{a}\cdot\mathbf{c} = 1/2$ and $\mathbf{a}\cdot\mathbf{b} = 0$ as **b** and **c** are non-parallel. Hence the angle between **a** and **c** is $\pi/3$ and between **a** and **b** it is $\pi/2$.

Example 32: A vector **a** of magnitude 50 is collinear with the vector $6\mathbf{i} - 8\mathbf{j} - (15/2)\mathbf{k}$ making on obtuse angle with the z-axis is

(a) $24\mathbf{i} - 32\mathbf{j} - 30\mathbf{k}$ (b) $-24\mathbf{i} + 32\mathbf{j} + 30\mathbf{k}$
(c) $24\mathbf{i} + 32\mathbf{j} - 30\mathbf{k}$ (d) none of these

Ans. (a)

Solution: A unit vector along $\mathbf{b} = 6\mathbf{i} - 8\mathbf{j} - \frac{15}{2}\mathbf{k}$, is $\pm\frac{2}{25}\left(6\mathbf{i} - 8\mathbf{j} - \frac{15}{2}\mathbf{k}\right)$, so a vector of length 50 along **b** is $\pm 4\left(6\mathbf{i} - 8\mathbf{j} - \frac{15}{2}\mathbf{k}\right)$. Since **a** makes obtuse angle with z-axis so we must have $\mathbf{a}\cdot\mathbf{k} < 0$. Thus $\mathbf{a} = 24\mathbf{i} - 32\mathbf{j} - 30\mathbf{k}$.

Example 33: Let there be two points A, B on the curve $y = x^2$ in the plane OXY satisfying $\mathbf{OA}\cdot\mathbf{i} = 1$ and $\mathbf{OB}\cdot\mathbf{i} = -2$ then the length of the vector $2\mathbf{OA} - 3\mathbf{OB}$ is

(a) $\sqrt{14}$ (b) $2\sqrt{51}$
(c) $3\sqrt{41}$ (d) none of these

Ans. (d)

Solution: Let $\mathbf{OA} = x_1\,\mathbf{i} + y_1\,\mathbf{j}$ and $\mathbf{OB} = x_2\,\mathbf{i} + y_2\,\mathbf{j}$. Since $1 = \mathbf{OA}.\,\mathbf{i} = x_1$ and $-2 = \mathbf{OB}.\,\mathbf{i} = x_2$. Moreover, $y_1 = x_1^2 = 1$ and $y_2 = x_2^2 = 4$, so
$\mathbf{OA} = \mathbf{i} + \mathbf{j}$ and $\mathbf{OB} = -2\,\mathbf{i} + 4\,\mathbf{j}$. Hence $|2\mathbf{OA} - 3\mathbf{OB}| = |8\mathbf{i} - 10\mathbf{j}| = \sqrt{164} = 2\sqrt{41}$.

Example 34: If A, B, C, D are four points in space satisfying $\mathbf{AB}\cdot\mathbf{CD} = K\left[|\mathbf{AD}|^2 + |\mathbf{BC}|^2 - |\mathbf{AC}|^2 - |\mathbf{BD}|^2\right]$ then the value of K is

(a) 2 (b) 1/3
(c) 1/2 (d) 1

Ans. (c)

Solution: Let A be the origin of reference and the position vector of B, C, D be **b**, **c**, **d**, w.r.t. A. So $\mathbf{AB} = \mathbf{b}$, $\mathbf{CD} = \mathbf{d} - \mathbf{c}$, $\mathbf{AD} = \mathbf{d}$, $\mathbf{BC} = \mathbf{c} - \mathbf{b}$, $\mathbf{AC} = \mathbf{c}$ and $\mathbf{BD} = \mathbf{d} - \mathbf{b}$. The L.H.S. is equal to $\mathbf{b}\cdot(\mathbf{d} - \mathbf{c})$ .The R.H.S. is

$$K\left[|\mathbf{d}|^2 + |\mathbf{c} - \mathbf{b}|^2 - |\mathbf{c}|^2 - |\mathbf{d} - \mathbf{b}|^2\right]$$
$$= K\left[\mathbf{d}\cdot\mathbf{d} + \mathbf{c}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{b} - 2\mathbf{c}\cdot\mathbf{b} - \mathbf{c}\cdot\mathbf{c} - \mathbf{d}\cdot\mathbf{d} - \mathbf{b}\cdot\mathbf{b} + 2\mathbf{d}\cdot\mathbf{b}\right]$$
$= 2K\,[\mathbf{b}\cdot(\mathbf{d} - \mathbf{c})]$. Hence $K = 1/2$.

Example 35: The distance of the point B with position vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ from the line passing through the point A with position vector $4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and parallel to the vector $2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ is

(a) $\sqrt{10}$ (b) $\sqrt{5}$
(c) $\sqrt{6}$ (d) none of these

Ans. (a)

Solution: $\mathbf{AB} = -3\mathbf{i} + \mathbf{k}$. Since $\mathbf{AB}\cdot(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}) = -6 + 6 = 0$. Hence **AB** is perpendicular to the given line. Thus required distance is equal to $|\mathbf{AB}| = \sqrt{9+1} = \sqrt{10}$.

Example 36: If **a**, **b** and **c** are unit vectors then $|\mathbf{a} - \mathbf{b}|^2 + |\mathbf{b} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{a}|^2$ does not exceed.

(a) 4 (b) 9
(c) 8 (d) 6

Ans. (b)

Solution: Since $|\mathbf{a} + \mathbf{b} + \mathbf{c}| \geq 0$

$$\Rightarrow \qquad 0 \leq |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b})$$
$$= 3 + 2\,(\boldsymbol{b}.\boldsymbol{c} + \boldsymbol{c}.\boldsymbol{a} + \boldsymbol{a}.\boldsymbol{b})$$
$$\Rightarrow \qquad \boldsymbol{b}.\boldsymbol{c} + \boldsymbol{c}.\boldsymbol{a} + \boldsymbol{a}.\boldsymbol{b} \geq -3/2$$
$$|\mathbf{a} - \mathbf{b}|^2 + |\mathbf{b} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{a}|^2$$
$$= 2\left(|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 - \mathbf{b}.\mathbf{c} - \mathbf{c}.\mathbf{a} - \mathbf{a}.\mathbf{b}\right)$$
$$\leq 2\,(1 + 1 + 1) + 3 = 9$$

The maximum value is attained e.g. when

$$\boldsymbol{a} = \mathbf{i},\ \boldsymbol{b} = \frac{1}{2}\left(-\mathbf{i} + \sqrt{3}\mathbf{j}\right) \text{ and } \mathbf{c} = \frac{1}{2}\left(-\mathbf{i} - \sqrt{3}\mathbf{j}\right).$$

Example 37: The value of a so that the volume of the parallelepiped formed by $\boldsymbol{i} + a\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{j} + a\boldsymbol{k}$ and $a\,\boldsymbol{i} + \boldsymbol{k}$ becomes minimum is

(a) -3 (b) 3
(c) $1/\sqrt{3}$ (d) $\sqrt{3}$

Ans. (c)

Solution: Volume of the parallelepiped is

$$V(a) = \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = a^3 - a + 1$$

$$V'(a) = 3a^2 - 1 = 0 \text{ if } a = a \pm \frac{1}{\sqrt{3}}$$

$$V''(a) = 6a > 0 \text{ if } a = \frac{1}{\sqrt{3}}$$

Thus $V(a)$ is minimum when $a = \dfrac{1}{\sqrt{3}}$.

Example 38: If $\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}$, $\mathbf{a}\cdot\mathbf{b} = 1$ and $\mathbf{a}\times\mathbf{b} = -\mathbf{j} + \mathbf{k}$, then $\mathbf{k}$ is equal to

(a) $\mathbf{i} + \mathbf{j} - \mathbf{k}$ (b) $-2\mathbf{j} + \mathbf{k}$
(c) $\mathbf{i}$ (d) $-2\mathbf{i} + \mathbf{k}$

Ans (c)

Solution: let $\mathbf{b} = \alpha\,\mathbf{i} + \beta\,\mathbf{j} + \gamma\mathbf{k}$ then $\mathbf{a}\cdot\mathbf{b} = 1$

$\Rightarrow \quad \alpha - \beta - \gamma = 1. \quad (1)$

And $\mathbf{a}\times\mathbf{b} = \mathbf{j} + \mathbf{k} \quad \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ \alpha & \beta & \gamma \end{vmatrix} = \mathbf{j} + \mathbf{k}$

$\Rightarrow \quad (\beta - \gamma)\,\mathbf{i} - (\gamma + \alpha)\mathbf{j} + (\beta + \gamma)\,\mathbf{k} = -\mathbf{j} + \mathbf{k}$

$\Rightarrow \quad \beta - \gamma = 0,\ \gamma + \alpha = 1,\ \beta + \alpha = 1$

$\Rightarrow \quad \beta = \gamma,\ \alpha = 1 - \gamma$

Putting there values in (1)

$1 - \gamma - \gamma - \gamma = 1 \Rightarrow \gamma = 0$ so $\alpha = 1;\ \beta = 0$

Thus $\quad \mathbf{b} = \mathbf{i}.$

Example 39: The unit vector which is orthogonal to the vector $5\,\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ and is coplanar with the vectors $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$ is

(a) $\frac{1}{\sqrt{41}}(2\mathbf{i} - 6\mathbf{j} + \mathbf{k})$ (b) $\frac{1}{\sqrt{29}}(2\mathbf{i} - 5\mathbf{j})$

(c) $\frac{1}{\sqrt{10}}(3\mathbf{j} - \mathbf{k})$ (d) $\frac{1}{\sqrt{69}}(2\mathbf{i} - 8\mathbf{j} + \mathbf{k})$

Ans (c)

Solution: A vector coplanar with $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$ is of the form

$$\mathbf{a} = \alpha\,(2\mathbf{i} + \mathbf{j} + \mathbf{k}) + \beta\,(\mathbf{i} - \mathbf{j} + \mathbf{k})$$
$$= (2\alpha + \beta)\,\mathbf{i} + (\alpha - \beta)\,\mathbf{j} + (\alpha + \beta)\mathbf{k}$$

This vector will be orthogonal to $5\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ if

$$5(2\alpha + \beta) + 2(\alpha - \beta) + 6(\alpha + \beta) = 0$$

$\Rightarrow \quad 18\alpha + 9\beta = 0 \quad \Rightarrow \quad \beta = -2\alpha$

So $\mathbf{a}$ is of the form $\alpha\,(3\mathbf{j} - \mathbf{k})$

Thus, a required unit vector is

$$\frac{1}{\sqrt{10}}(3\mathbf{j} - \mathbf{k}).$$

Example 40: Let $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} - \mathbf{k}$. A vector in the plane of $\mathbf{a}$ and $\mathbf{b}$ whose projection on $\mathbf{c}$ is $1/\sqrt{3}$ is

(a) $4\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ (b) $3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$
(c) $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ (d) $4\mathbf{i} + \mathbf{j} - 4\mathbf{k}$

Ans. (c)

Solution: A vector in the plane of $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{d} = \alpha\,\mathbf{a} + \beta\,\mathbf{b}$$
$$= (\alpha + \beta)\,\mathbf{i} + (2\alpha - \beta)\mathbf{j} + (\alpha + \beta)\mathbf{k}$$

Length of projection of $\mathbf{d}$ on $\mathbf{c}$ is

$$\frac{\mathbf{d}\cdot\mathbf{c}}{|\mathbf{c}|} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \frac{(\alpha + \beta) + (2d - \beta) - (d + \beta)}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}$$

This is satisfied when $\alpha = 1$, $\beta = 1$.

Example 41: If $|\mathbf{a}| = 1$, $|\mathbf{b}| = 2$ and $|\mathbf{a} - 2\mathbf{b}| = 4$ then $|\mathbf{a} + 3\mathbf{b}|$ is equal to

(a) 8 (b) $\sqrt{\frac{51}{2}}$

(c) $\frac{\sqrt{19}}{2}$ (d) $\sqrt{\frac{77}{2}}$

Ans (d)

Solution: $16 = |\mathbf{a} - 2\mathbf{b}|^2 = |\mathbf{a}|^2 + 4\,|\mathbf{b}|^2 - 4\,\mathbf{a}\cdot\mathbf{b}$

$$= 1 + 16 - 4\,\mathbf{a}\cdot\mathbf{b}$$

$$4\,\mathbf{a}\cdot\mathbf{b} = 1 \quad \Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = \frac{1}{4}$$

$$|\mathbf{a} + 3\mathbf{b}|^2 = |\mathbf{a}|^2 + 9\,|\mathbf{b}|^2 + 6\,\mathbf{a}\cdot\mathbf{b}$$
$$= 1 + 36 + \frac{6}{4} = \frac{77}{2}.$$

Example 42: If $|\mathbf{a}|^2 = 8$ and $\mathbf{a}\times(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = \mathbf{0}$ then the value of $\mathbf{a}\cdot(-\mathbf{i} + \mathbf{j} + 4\mathbf{k})$ is

(a) $\frac{4}{\sqrt{3}}$ (b) $\frac{16}{\sqrt{3}}$

(c) $\frac{8}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{3}}$

Ans (b)

Solution: Since $\mathbf{a}\times(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = \mathbf{0}$ so $\mathbf{a}$ is parallel to $\mathbf{i} + \mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{a} = \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

but $\quad 8 = |\mathbf{a}|^2 = \lambda^2\cdot 6 \Rightarrow \lambda^2 = \frac{4}{3}.$

$$\mathbf{a}.(-\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})\cdot(-\mathbf{i} + \mathbf{j} + 4\mathbf{k})$$
$$= \lambda\,(-1 + 1 + 8) = 8\lambda = \frac{8.2}{\sqrt{3}} = \frac{16}{\sqrt{3}}$$

Example 43: If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are unit vectors, then the maximum value of $|\mathbf{a} + 2\mathbf{b}|^2 + |\mathbf{b} + 3\mathbf{c}|^2 + |\mathbf{c} + 4\mathbf{a}|^2$ is

(a) 28 (b) 21
(c) 48 (d) 58

Ans (b)

Solution: $|\mathbf{a} + 2\mathbf{b}|^2 + |\mathbf{b} + 3\mathbf{c}|^2 + |\mathbf{c} + 4\mathbf{a}|^2$

$$= |\mathbf{a}|^2 + 4|\mathbf{b}|^2 + 2\times\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 + 9\,|\mathbf{c}|^2 + 6\,\mathbf{b}\cdot\mathbf{c}$$
$$+ |\mathbf{c}|^2 + 16\,|\mathbf{a}|^2 + 8\,\mathbf{a}\cdot\mathbf{c}$$
$$\le 1 + 4 + 2 + 1 + 9 + 6 + 1 + 16 + 8 = 48$$

Example 44: Let $\mathbf{a} = 2\,\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j}$. If $\mathbf{c}$ is a vector such that $\mathbf{a}\cdot\mathbf{c} = |\mathbf{c}|$ and $|\mathbf{c} - \mathbf{a}| = 2\sqrt{2}$ and angle between $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}$ is 30°, then $|(\mathbf{a}\times\mathbf{b})\times\mathbf{c}|$ equals:

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$

(c) 2 (d) $\frac{\sqrt{3}}{2}$

Ans. (a)

Solution: $|\mathbf{a}|^2 = 4 + 1 + 4 = 9$, $|\mathbf{b}|^2 = 1 + 1 = 2$.

Also, $|\mathbf{c} - \mathbf{a}|^2 = 8 \Rightarrow |\mathbf{c}|^2 + |\mathbf{a}|^2 - 2\,\mathbf{c}.\mathbf{a} = 8$

$\Rightarrow$ $|\mathbf{c}|^2 + 9 - 2|\mathbf{c}| = 8$

$\Rightarrow$ $|\mathbf{c}|^2 - 2|\mathbf{c}| + 1 = 0 \Rightarrow (|\mathbf{c}| - 1)^2 = 0 \Rightarrow |\mathbf{c}| = 1.$

Also, $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$\Rightarrow$ $|\mathbf{a} \times \mathbf{b}| = 3$

Also, $|(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| = |\mathbf{a} \times \mathbf{b}|\,|\mathbf{c}| \sin 30° = (3)\,(1)\,(1/2) = 3/2.$

Example 45: The non-zero vectors **a**, **b** and **c** are related by **a** = 8**b** and **c**= – 7**b**. Then the angle between **a** and **c** is

(a) 0 (b) $\pi/4$

(c) $\pi/2$ (d) π

Ans. (d)

Solution: $\mathbf{a} = 8\mathbf{b} = -\frac{8}{7}\mathbf{c}$ so **a** and **c** are parallel and **a** and **c** have opposite direction so angle between **a** and **c** is π.

Example 46: If **u**, **v**, **w** are non-coplanar vectors and p, q are real numbers, then the equality $[3\mathbf{u}\ p\mathbf{v}\ p\mathbf{w}] - [p\mathbf{v}\ \mathbf{w}\ q\mathbf{u}] - [2\mathbf{w}\ q\mathbf{v}\ q\mathbf{u}] = 0$ holds for

(a) more than two but not all values of (p, q)

(b) all values of (p, q)

(c) exactly one values of (p, q)

(d) exactly two values of (p, q)

Ans (c)

Solution: $0 = [3\mathbf{u}\ p\mathbf{v}\ p\mathbf{w}] - [p\mathbf{v}\ \mathbf{w}\ q\mathbf{u}] - [2\mathbf{w}\ q\mathbf{v}\ q\mathbf{u}]$

$= (3p^2 - pq + 2q^2)\,[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$

As **u**, **v**, **w** are non – coplanar so $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] \neq 0$, so

$3p^2 - pq + 2q^2 = 0$

$\Rightarrow$ $2\left(q - \frac{1}{4}p\right)^2 + \frac{23}{8}p^2 = 0$

$\Rightarrow$ $p = 0, q = \frac{1}{4}p = 0$

Thus there is exactly one value of (p, q).

Example 47: If the vectors $\mathbf{a} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \lambda\mathbf{i} + \mathbf{j} + \mu\mathbf{k}$ are mutually orthogonal then $(\lambda, \mu) =$

(a) (–2, 3) (b) (3, –2)

(c) (–3, 2) (d) (2, –3)

Ans. (c)

Solution: As **a**, **b** and **c** are mutually orthogonal,

$\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$

$\Rightarrow$ $\lambda - 1 + 2\mu = 0$ and $2\lambda + 4 + \mu = 0$

$\Rightarrow$ $\lambda + 2\mu = 1$ and $2\lambda + \mu = -4$

$\Rightarrow$ $\lambda = -3, \mu = 2.$

Example 48: Let $\mathbf{a} = \mathbf{j} - \mathbf{k}$ and $\mathbf{c} = \mathbf{i} - \mathbf{j} - \mathbf{k}$. Then the vector **b** satisfying $(\mathbf{a} \times \mathbf{b}) + \mathbf{c} = \mathbf{0}$ and $\mathbf{a} \cdot \mathbf{b} = 3$ is

(a) $\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ (b) $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

(c) $-\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ (d) $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

Ans (c)

Solution: Let $\mathbf{b} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$.

We have

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ \alpha & \beta & \gamma \end{vmatrix} = (\beta + \gamma)\mathbf{i} - \alpha\mathbf{j} - \alpha\mathbf{k}$$

As $(\mathbf{a} \times \mathbf{b}) + \mathbf{c} = \mathbf{0}$, we get

$(\beta + \gamma + 1)\mathbf{i} - (\alpha + 1)\mathbf{j} - (\alpha + 1)\,\mathbf{k} = \mathbf{0}$

$\Rightarrow$ $\beta + \gamma + 1 = 0, \alpha = -1$

Also, as $\mathbf{a} \cdot \mathbf{b} = 3$, we get $\beta - \gamma = 3$

Thus $\alpha = -1$, $b = 1$, $\gamma = -2$

Hence, $\mathbf{b} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Example 49: If $\mathbf{a} = \frac{1}{\sqrt{10}}(3\mathbf{i} + k)$ and $\mathbf{b} = \frac{1}{7}(2\mathbf{i} + 3\mathbf{j} - 6k)$, then the value of $(2\mathbf{a} - \mathbf{b}).\,[(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})]$ is:

(a) 3 (b) –5

(c) –3 (d) 5

Ans. (b)

Solution: we have $\mathbf{a}\,.\,\mathbf{a} = 1$, $\mathbf{b}.\,\mathbf{b} = 1$ and

$\mathbf{a}.\,\mathbf{b} = \frac{1}{7\sqrt{10}} = ((3)\,(2) + (1)\,(-6)) = 0 = \mathbf{b}\,.\,\mathbf{a}$

Now, $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})$

$= (\mathbf{a}\,.\,(\mathbf{a} + 2\mathbf{b}))\,\mathbf{b} - (\mathbf{b}\,.\,(\mathbf{a} + 2\mathbf{b}))\,\mathbf{a}$

$= (\mathbf{a}\,.\,\mathbf{a})\,\mathbf{b} - 2\,(\mathbf{b}\,.\,\mathbf{b})\mathbf{a}\ (\mathbf{a}\,.\,\mathbf{b} = 0)$

$= \mathbf{b} - 2\mathbf{a}.$

Thus $(2\mathbf{a} - \mathbf{b})\,.\,[(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})]$

$= (2\mathbf{a} - \mathbf{b})\,.\,(\mathbf{b} - 2\mathbf{a}) = -\,|2\mathbf{a} - \mathbf{b}|^2$

$= -\left[4|\mathbf{a}|^2 + |\mathbf{b}|^2 - 4\mathbf{a}.\mathbf{b}\right] = -5.$

Example 50: The vectors **a** and **b** are not perpendicular and **c** and **d** are the vectors satisfying $\mathbf{b} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$ and $\mathbf{a}\,.\,\mathbf{d} = 0$. Then the vector **d** is equal to :

(a) $\mathbf{c} - \left(\frac{\mathbf{a}.\mathbf{c}}{\mathbf{a}.\mathbf{b}}\right)\mathbf{b}$ (b) $\mathbf{b} - \left(\frac{\mathbf{b}.\mathbf{c}}{\mathbf{a}.\mathbf{b}}\right)\mathbf{c}$

(c) $\mathbf{c} + \left(\frac{\mathbf{a}.\mathbf{c}}{\mathbf{a}.\mathbf{b}}\right)\mathbf{b}$ (d) $\mathbf{b} + \left(\frac{\mathbf{b}.\mathbf{c}}{\mathbf{a}.\mathbf{b}}\right)\mathbf{c}$

And. (a)

◎ **Solution:** $\mathbf{b} \times \mathbf{c} = \mathbf{b} \times \mathbf{d} \Rightarrow \mathbf{b} \times (\mathbf{c} - \mathbf{d}) = \mathbf{0}$

$\Rightarrow \quad \mathbf{c} - \mathbf{d} \parallel \mathbf{b}$

$\Rightarrow \quad \mathbf{c} - \mathbf{d} = \alpha\mathbf{b}$ for some $\alpha \in \mathbf{R}$

$\Rightarrow \quad \mathbf{d} = \mathbf{c} - \alpha\,\mathbf{b}$

Also, $\mathbf{a} \cdot \mathbf{d} = \mathbf{a} . \mathbf{c} - \alpha\,\mathbf{a} . \mathbf{b}$

$\Rightarrow \quad 0 = \mathbf{a} \cdot \mathbf{c} - \alpha\mathbf{a} . \mathbf{b}$

$\Rightarrow \quad \alpha = \dfrac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}$

Thus, $\quad \mathbf{d} = \mathbf{c} - \dfrac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}\,\mathbf{b}$.

◉ **Example 51:** If vectors p**i** + q**j** + **k**, **i** + q**j** + **k** and **i** + **j** + r**k** ($p \neq q \neq r \neq 1$) are coplanar, then the value of $p\,q\,r - (p + q + r)$ is

(a) 2 (b) 0
(c) –1 (d) –2

Ans. (d)

◎ **Solution:** Since the given vectors are coplanar so

$$\begin{vmatrix} p & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & r \end{vmatrix} = 0$$

$$\Rightarrow \quad p\begin{vmatrix} q & 1 \\ 1 & r \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & r \end{vmatrix} + \begin{vmatrix} 1 & q \\ 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad p(qr-1) - (r-1) + (1-q) = 0$$

$$\Rightarrow \quad pqr - p - q - r = -2$$

◉ **Example 52:** Let **a**, **b**, **c** be three non-zero vectors which are pairwise non-collinear. If **a** + 3**b** is collinear with **c** and **b** + 2**c** is collinear with **a**, then **a** + 3**b** + 6**c** is

(a) **a** (b) **b**
(c) **0** (d) **a** + **c**

Ans. (c)

◎ **Solution:** We are given that

$$\mathbf{a} + 3\mathbf{b} = \alpha\mathbf{c}$$

and $\quad \mathbf{b} + 2\mathbf{c} = \beta\mathbf{a}$

for some $\alpha, \beta \in \mathbf{R}$.

We have

$$\mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = \alpha\mathbf{c} + 6\mathbf{c} = (6 + \alpha)\mathbf{c}$$

and $\quad \mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = \mathbf{a} + 3(\mathbf{b} + 2\mathbf{c})$

$$= \mathbf{a} + 3\beta\mathbf{a} = (1 + 3\beta)\mathbf{b}$$

Thus, $\quad (6 + \alpha)\mathbf{c} = (1 + 3\beta)\mathbf{a}$

If $6 + \alpha \neq 0$, we get $\mathbf{c} = \dfrac{1+3\beta}{6+\alpha}\,\mathbf{a}$, so **c** is collinear with **a**.

Thus $6 + \alpha = 0$, therefore, **a** + 3**b** + 6**c** = 0.

◉ **Example 53:** Let **a** = **i** + **j** + **k**, **b** = **i** – **j** + **k** and **c** = **i** – **j** – **k** be three vectors. A vector **v** in the plane of **a** and **b**, whose projection on **c** is $1/\sqrt{3}$, , is given by

(a) **i** – 3**j** + 3**k** (b) –3**i** – 3**j** – **k**
(c) 3**i** – **j** + **k** (d) **i** + 3**j** – 3**k**

Ans. (c)

◎ **Solution:** $\mathbf{v} = \alpha\,\mathbf{a} + \beta\mathbf{b}$, where $\alpha, \beta \in \mathbf{R}$

$\Rightarrow \quad \mathbf{v} = (\alpha + \beta)\mathbf{i} + (\alpha - \beta)\mathbf{j} + (\alpha + \beta)\,\mathbf{k}$

The projection of **v** on **c** is $1/\sqrt{3}$, so

$$\frac{|\mathbf{c}\cdot\mathbf{v}|}{|\mathbf{c}|} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \frac{|(\alpha+\beta)-(\alpha+\beta)-(\alpha+\beta)|}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad |\alpha-\beta| = 1 \Rightarrow \alpha - \beta = \pm 1.$$

The only possible answer is (c).

◉ **Example 54:** Suppose that **a** and **b** are two unit vectors. If the vectors **c** = **a** + 2**b** and **d** = 5**a** – 4**b** are perpendicular to each other, then the angle between **a** and **b** is

(a) $\pi/2$ (b) $\pi/3$
(c) $\pi/4$ (d) $\pi/6$

Ans. (b)

◎ **Solution:** $\mathbf{c} . \mathbf{d} = 0 \Rightarrow (\mathbf{a} + 2\mathbf{b}) . (5\mathbf{a} - 4\mathbf{b}) = 0$

$\Rightarrow \quad 5\,|\mathbf{a}|^2 + 10\mathbf{b} . \mathbf{a} - 4\,\mathbf{a} . \mathbf{b} - 8\,|\mathbf{b}|^2 = 0$

$\Rightarrow \quad 5 + 6\,\mathbf{a} . \mathbf{b} - 8 = 0$

$\Rightarrow \quad \mathbf{a} . \mathbf{b} = 1/2$

$\Rightarrow \quad |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta = 1/2$

$\Rightarrow \quad \cos\theta = 1/2 \Rightarrow \theta = \pi/3.$

◉ **Example 55:** Let *ABCD* be a parallelogram such that *AB* = **q**, *AD* = **p** and *BAD* be an acute angle. If **r** is the vector that coincides with the altitude directed from the vertex *B* to the side *AD*, then **r** is given by :

(a) $\mathbf{r} = -\mathbf{q} + \dfrac{(\mathbf{p}\cdot\mathbf{q})}{\mathbf{p}\cdot\mathbf{p}}\,\mathbf{p}$

(b) $\mathbf{r} = \mathbf{q} - \dfrac{(\mathbf{p}\cdot\mathbf{q})}{\mathbf{p}\cdot\mathbf{p}}\,\mathbf{p}$

(c) $\mathbf{r} = -3\mathbf{q} + \dfrac{3(\mathbf{p}\cdot\mathbf{q})}{\mathbf{p}\cdot\mathbf{p}}\,\mathbf{p}$

(d) $\mathbf{r} = 3\mathbf{q} - \dfrac{3(\mathbf{p}\cdot\mathbf{q})}{\mathbf{p}\cdot\mathbf{p}}\,\mathbf{p}$

Ans. (a)

◎ **Solution:** Let *M* be the foot of perpendicular from B to AD.

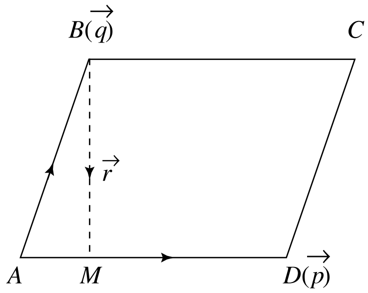

Fig. 22.5

Suppose $\mathbf{AM} = \alpha\mathbf{p}$

Also, $\mathbf{AM} = \mathbf{AB} + \mathbf{BM} = \mathbf{q} + \mathbf{r}$

As $\mathbf{q} + \mathbf{r} = \alpha\mathbf{p}$, we get

$$(\mathbf{q} + \mathbf{r}).\mathbf{p} = \alpha\, \mathbf{p}.\mathbf{p}$$

$$\Rightarrow \quad \mathbf{q}.\mathbf{p} + \mathbf{r}.\mathbf{p} = \alpha\ \mathbf{p}.\mathbf{p} \Rightarrow \alpha = \frac{\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}\cdot\mathbf{p}} \quad [\because\ \mathbf{r}.\mathbf{p} = 0]$$

Thus, $$\mathbf{r} = -\,\mathbf{q} + \frac{\mathbf{q}\cdot\mathbf{p}}{\mathbf{p}\cdot\mathbf{p}}\ \mathbf{p}.$$

Assertion-Reason Type Questions

Example 56: **Statement-1:** If vectors **a** and **c** are non-collinear then the lines

$$\mathbf{r} = 6\mathbf{a} - \mathbf{c} + \lambda(2\mathbf{c} - \mathbf{a})$$
$$\mathbf{r} = \mathbf{a} - \mathbf{c} + \mu(\mathbf{a} + 3\mathbf{c}) \text{ are coplanar}$$

Statement-2: There exist λ and μ such that the two values of **r** become same

Ans. (a)

Solution: If lines have a common point then there exists λ and μ such that

$$6 - \lambda = 1 + \mu \text{ and } -1 + 2\lambda = -1 + 3\mu$$

$\Rightarrow \quad \lambda = 3, \mu = 2.$

Example 57: Given that **a**, **b**, **c** are the position vectors of the vertices of a ΔABC

Statement-1: The area of ΔABC is $\frac{1}{2}[\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}]$

Statement-2: Cross product is distributive over addition of vectors

Ans. (b)

Solution: Required area = $\frac{1}{2}(\mathbf{AB}\times\mathbf{AC})$

$$= \frac{1}{2}((\mathbf{b} - \mathbf{a})\times(\mathbf{c} - \mathbf{a}))$$

$$= \frac{1}{2}[\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}].$$

Example 58: Let A, B, C be three points with position vectors $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $2\mathbf{i} + 3\mathbf{k}$, $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

Statement-1: The angle between **AB** and **AC** is acute

Statement-2: If θ is the angle between **AB** and **AC** then

$$\cos\theta = \frac{17}{\sqrt{21}\sqrt{22}}$$

Ans. (c)

Solution: $\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$

$\mathbf{AC} = 2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$

$$\cos\theta = \frac{\mathbf{AB}.\mathbf{AC}}{|\mathbf{AB}|\,|\mathbf{AC}|} = \frac{20}{\sqrt{21}\sqrt{22}}$$

Example 59: **Statement-1:** $((\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c})).\ \mathbf{d} = \mathbf{b}.\mathbf{d}\ [\mathbf{abc}]$

Statement-2: $(\mathbf{a}\times\mathbf{b}).\mathbf{c} = \mathbf{a}.(\mathbf{b}\times\mathbf{c})$

Ans. (a)

Solution: $((\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c}))\ .\ \mathbf{d}$

$= (\mathbf{a}\times\mathbf{b})\ .\ ((\mathbf{a}\times\mathbf{c})\times\mathbf{d})$

$= (\mathbf{a}\times\mathbf{b})\ .\ [(\mathbf{a}\ .\ \mathbf{d})\ \mathbf{c} - (\mathbf{c}\ .\ \mathbf{d})\ \mathbf{a}]$

$= (\mathbf{a}\ .\ \mathbf{d})\ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ (the last scalar product is zero)

Example 60: If **a**, **b**, **c** be three unit vectors such that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \frac{1}{2}\mathbf{b}$, **b**, **c** being non-parallel.

Statement-1: The angle between **a** and **b** is $\pi/2$.

Statement-2: The angle between **a** and **c** is $\frac{\pi}{3}$.

Ans. (b)

Solution: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \frac{1}{2}\ \mathbf{b}$

$\Rightarrow (\mathbf{a}\ .\ \mathbf{c})\ \mathbf{b} - (\mathbf{a}\ .\ \mathbf{b})\ \mathbf{c} = \frac{1}{2}\ \mathbf{b}$

$\Rightarrow (\mathbf{a}\ .\ \mathbf{c} - 1/2)\ \mathbf{b} - (\mathbf{a}\ .\ \mathbf{b})\ \mathbf{c} = 0$

$\Rightarrow \mathbf{a}\ .\ \mathbf{c} = 1/2$ and $\mathbf{a}\ .\ \mathbf{b} = 0$

Cosine of angle between **a** and **c** = 1/2 $\Rightarrow$ angle between **a** and **c** is $\pi/3$ and angle between **a** and **b** is $\pi/2$.

Example 61: Let **u**, **v**, **w** be three non-coplanar vectors, then $(\mathbf{v} - \mathbf{w}).\ [(\mathbf{w} - \mathbf{u})\times(\mathbf{u} - \mathbf{v})] = 0$

Statement-1: $\mathbf{v} - \mathbf{w} = \lambda(\mathbf{w} - \mathbf{u}) + \mu(\mathbf{u} - \mathbf{v})$ for some $\lambda, \mu \in \mathbf{R}$.

Statement-2: The vectors $\mathbf{v} - \mathbf{w}$, $\mathbf{w} - \mathbf{u}$, $\mathbf{u} - \mathbf{v}$ are coplanar.

Ans. (a)

◎ **Solution:** The given condition implies **v** – **w**, **w** – **u**, and **u** – **v** are coplanar. So one vector is linear combination of other two.

◉ **Example 62:** Let **r** be a vector such that $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$.

Statement-1: **r** is linear combination of **a** and **b**.

Statement-2: $\mathbf{r} = \mathbf{a}$

Ans. (c)

◎ **Solution:** $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b} \quad \Rightarrow \quad (\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$

$\Rightarrow$ $\mathbf{r} - \mathbf{a}$ is parallel to **b** so

$\mathbf{r} - \mathbf{a} = \lambda\, \mathbf{b}$, for some $\lambda \in \mathbf{R}$.

$\Rightarrow \quad \mathbf{r} = \mathbf{a} + \lambda\, \mathbf{b}$.

◉ **Example 63:** If **a**, **b**, **c** are non-coplanar vectors then

Statement-1: $\mathbf{b} \times \mathbf{c}$, $\mathbf{c} \times \mathbf{a}$, $\mathbf{a} \times \mathbf{b}$ are non-coplanar.

Statement-2: $[\mathbf{b} \times \mathbf{c} \quad \mathbf{c} \times \mathbf{a} \quad \mathbf{a} \times \mathbf{b}] = 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

Ans. (c)

◎ **Solution:** $[\mathbf{b} \times \mathbf{c} \quad \mathbf{c} \times \mathbf{a} \quad \mathbf{a} \times \mathbf{b}]$

$= ((\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})) \cdot (\mathbf{a} \times \mathbf{b})$

$= ((\mathbf{b}.(\mathbf{c} \times \mathbf{a}))\,\mathbf{c} - (\mathbf{c}.(\mathbf{c} \times \mathbf{a})\mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b})$

$= [\mathbf{a}.\ \mathbf{b}.\ \mathbf{c}]\ \mathbf{c}.(\mathbf{a} \times \mathbf{b})\ (\mathbf{c}.(\mathbf{c} \times \mathbf{a}) = 0)$

$= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

Since R.H.S. is non-zero so is L.H.S. Hence $\mathbf{b} \times \mathbf{c}$, $\mathbf{c} \times \mathbf{a}$, $\mathbf{a} \times \mathbf{b}$ are non-coplanar.

LEVEL 2

Straight Objective Type Questions

◉ **Example 64:** The vector **c**, directed along the internal bisector of the angle between the vectors $\mathbf{a} = 7\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, with $|\mathbf{c}| = 5\sqrt{6}$ is

(a) $\pm (5/3)\ (\mathbf{i} - 7\mathbf{j} + 2\mathbf{k})$

(b) $(5/3)\ (5\mathbf{i} + 5\mathbf{j} + 2\mathbf{k})$

(c) $\pm\ (5/3)\ (\mathbf{i} + 7\mathbf{j} + 2\mathbf{k})$

(d) $(5/3)\ (-5\mathbf{i} + 5\mathbf{j} + 2\mathbf{k})$

Ans. (a)

◎ **Solution:** The required vector **c** is given by

$$\lambda\left(\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}\right)$$

Now $\quad \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{1}{9}(7\mathbf{i} - 4\mathbf{j} - 4\mathbf{k})$ and $\dfrac{\mathbf{b}}{|\mathbf{b}|} = \dfrac{1}{3}(-2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

$$\Rightarrow \quad \mathbf{c} = \lambda\left(\frac{1}{9}\mathbf{i} - \frac{7}{9}\mathbf{j} + \frac{2}{9}\mathbf{k}\right) \quad \Rightarrow \quad |\mathbf{c}|^2 = \lambda^2 \times \frac{54}{81}$$

$\Rightarrow \quad \lambda^2 = 225 \Rightarrow \lambda = \pm 15. \Rightarrow \mathbf{c} = \pm (5/3)\ (\mathbf{i} - 7\mathbf{j} + 2\mathbf{k})$

◉ **Example 65:** Let **a**, **b** and **c** be three non-zero vectors, no two of which are collinear. If the vectors $\mathbf{a} + 2\mathbf{b}$ is collinear with **c**, and $\mathbf{b} + 3\mathbf{c}$ is collinear with **a**, then $\mathbf{a} + 2\mathbf{b} + 6\mathbf{c}$ is equal to, (λ being some non-zero scalar)

(a) $\lambda\,\mathbf{a}$ (b) $\lambda\,\mathbf{b}$

(c) $\lambda\,\mathbf{c}$ (d) $\mathbf{0}$

Ans. (d)

◎ **Solution:** Let $\mathbf{a} + 2\boldsymbol{b} = x\mathbf{c}$ and $\mathbf{b} + 3\mathbf{c} = y\mathbf{a}$. Then $\mathbf{a} + 2\mathbf{b} + 6\mathbf{c} = (x + 6)\mathbf{c}$ and also, $\mathbf{a} + 2\mathbf{b} + 6\mathbf{c} = (1 + 2y)\mathbf{a}$. So $(x + 6)\mathbf{c} = (1 + 2y)\mathbf{a}$. Since **a** and **c** are non-zero and non-collinear, we have $x + 6 = 0$ and $1 + 2y = 0$, i.e., $x = -6$ and $y = -1/2$. In either case, we have $\mathbf{a} + 2\mathbf{b} + 6\mathbf{c} = \mathbf{0}$.

◉ **Example 66:** The value of k for which the points $A(1, 0, 3)$, $B(-1, 3, 4)$, $C(1, 2, 1)$ and $D(k, 2, 5)$ are coplanar, are

(a) 1 (b) 2

(c) 0 (d) –1

Ans. (d)

◎ **Solution:** Let $\mathbf{a} = (1, 0, 3)$, $\mathbf{b} = (-1, 3, 4)$, $\mathbf{c} = (1, 2, 1)$ and $\mathbf{d} = (k, 2, 5)$. Since A, B, C and D are coplanar, we have

$$[\mathbf{d}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{d}\ \mathbf{c}\ \mathbf{a}] + [\mathbf{d}\ \mathbf{a}\ \mathbf{b}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

$$\Rightarrow \quad \begin{vmatrix} k & 2 & 5 \\ -1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} + \begin{vmatrix} k & 2 & 5 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix} + \begin{vmatrix} k & 2 & 5 \\ 1 & 0 & 3 \\ -1 & 3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 3 \\ -1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}$$

$\Rightarrow \quad -5k - 15 + 6k - 14 - 9k + 1 = -20$

$\Rightarrow \quad -8k - 28 = -20 \quad \Rightarrow \quad k = -1$

Alternate Solution

A, B, C, D are coplanar $\Leftrightarrow$

$[\mathbf{AB}\ \mathbf{AC}\ \mathbf{AD}] = 0$

$$\begin{vmatrix} -2 & 3 & 1 \\ 0 & 2 & -2 \\ k-1 & 2 & 2 \end{vmatrix} = 0 \Leftrightarrow k = -1$$

◉ **Example 67:** Let a, b, c be distinct non-negative numbers. If the vectors $a\mathbf{i} + a\mathbf{j} + c\mathbf{k}$, $\mathbf{i} + \mathbf{k}$ and $c\mathbf{i} + c\mathbf{j} + b\mathbf{k}$ lie in a plane, then c is

(a) the arithmetic mean of a and b

(b) the geometric mean of a and b
(c) the harmonic mean of a and b
(d) equal to zero

Ans. (b)

Solution: The three vectors are coplanar if and only if

$$\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0 \text{ or } \begin{vmatrix} 0 & a & c \\ 1 & 0 & 1 \\ 0 & c & b \end{vmatrix} = 0. \ (C_1 \to C_1 - C_2)$$

$$\Leftrightarrow \quad -1\,(ab - c^2) = 0 \quad \Leftrightarrow \quad ab = c^2.$$

Example 68: Let **p**, **q**, **r** be three mutually perpendicular vectors of the same magnitude. If a vector **x** satisfies the equation

$$\mathbf{p} \times ((\mathbf{x} - \mathbf{q}) \times \mathbf{p}) + \mathbf{q} \times ((\mathbf{x} - \mathbf{r}) \times \mathbf{q}) + \mathbf{r} \times ((\mathbf{x} - \mathbf{p}) \times \mathbf{r}) = \mathbf{0}$$

then **x** is given by

(a) $(1/2)(\mathbf{p} + \mathbf{q} - 2\mathbf{r})$ (b) $(1/2)(\mathbf{p} + \mathbf{q} + \mathbf{r})$
(c) $(1/3)(\mathbf{p} + \mathbf{q} + \mathbf{r})$ (d) $(1/3)(2\mathbf{p} + \mathbf{q} - \mathbf{r})$

Ans. (b)

Solution: Let $|\mathbf{p}| = |\mathbf{q}| = |\mathbf{r}| = K$. Let **a, b, c** be unit vectors along **p**, **q**, **r** respectively. Clearly **p**, **q**, **r** are mutually perpendicular vectors, so any vector **x** can be written as $a_1\,\mathbf{a} + a_2\,\mathbf{b} + a_3\,\mathbf{c}$.

$$\mathbf{p} \times ((\mathbf{x} - \mathbf{q}) \times \mathbf{p}) = (\mathbf{p}\cdot\mathbf{p})\,(\mathbf{x} - \mathbf{q}) - (\mathbf{p}\cdot(\mathbf{x} - \mathbf{q}))\,\mathbf{p}$$
$$= K^2\,(\mathbf{x} - \mathbf{q}) - (\mathbf{p}\cdot\mathbf{x})\,\mathbf{p}\ (\mathbf{p}\cdot\mathbf{q} = 0)$$
$$= K^2\,(\mathbf{x} - \mathbf{q}) - K\,\mathbf{a}\cdot(a_1\,\mathbf{a} + a_2\,\mathbf{b} + a_3\,\mathbf{c})\,K\,\mathbf{a}$$
$$= K^2\,(\mathbf{x} - \mathbf{q} - a_1\,\mathbf{a})$$

Similarly $\mathbf{q} \times ((\mathbf{x} - \mathbf{r}) \times \mathbf{q}) = K^2(\mathbf{x} - \mathbf{r} - a_2\mathbf{b})$

and $\mathbf{r} \times ((\mathbf{x} - \mathbf{p}) \times \mathbf{r}) = K^2(\mathbf{x} - \mathbf{p} - a_3\mathbf{c})$

According to the given condition

$$K^2\,(\mathbf{x} - \mathbf{q} - a_1\,\mathbf{a} + \mathbf{x} - \mathbf{r} - a_2\,\mathbf{b} + \mathbf{x} - \mathbf{p} - a_3\,\mathbf{c}) = 0$$
$$\Rightarrow \quad \{3\mathbf{x} - (\mathbf{p} + \mathbf{q} + \mathbf{r}) - (a_1\,\mathbf{a} + a_2\,\mathbf{b} + a_3\,\mathbf{c})\} = 0$$
$$\Rightarrow \quad [2\mathbf{x} - (\mathbf{p} + \mathbf{q} + \mathbf{r})] = \mathbf{0}$$
$$\Rightarrow \quad x = (1/2)\,(\mathbf{p} + \mathbf{q} + \mathbf{r}).$$

Example 69: If **a**, **b**, **c** and **d** are unit vectors, then $|\mathbf{a} - \mathbf{b}|^2 + |\mathbf{b} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{d}|^2 + |\mathbf{d} - \mathbf{a}|^2 + |\mathbf{c} - \mathbf{a}|^2 + |\mathbf{b} - \mathbf{d}|^2$ does not exceed

(a) 4 (b) 12
(c) 8 (d) 16

Ans. (b)

Solution: We have

$$0 \le |\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + |\mathbf{d}|^2 + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}$$
$$+ \mathbf{a}\cdot\mathbf{d} + \mathbf{c}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{d}$$
$$= 4 + 2\,\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{c}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{d}$$

So $\quad \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{d} + \mathbf{c}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{d} \ge -2.$

Now

$$|\mathbf{a}-\mathbf{b}|^2 + |\mathbf{b}-\mathbf{c}|^2 + |\mathbf{c}\cdot\mathbf{d}|^2 + |\mathbf{d}-\mathbf{a}|^2 + |\mathbf{c}-\mathbf{a}|^2 + |\mathbf{b}-\mathbf{d}|^2$$
$$= 2(|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + |\mathbf{d}|^2 - (\mathbf{a}\cdot\mathbf{b} + \mathbf{c}\cdot\mathbf{d} + \mathbf{d}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{c})$$
$$\le 2\,(4 + 2) = 12.$$

Example 70: Let $\mathbf{a} = \mathbf{i} - \mathbf{k}$, $\mathbf{b} = x\mathbf{i} + \mathbf{j} + (1 - x)\mathbf{k}$ and $\mathbf{c} = y\mathbf{i} + x\mathbf{j} + (1 + x - y)\mathbf{k}$. Then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ depends on

(a) only x
(b) only y
(c) neither x nor y
(d) both x and y

Ans. (c)

Solution: $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ y & x & 1+x \end{vmatrix} \text{ (using } C_3 \to C_3 + C_1)$$

$= (1 + x) - x = 1$, which depends neither on x nor on y.

Example 71: Let $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j}$. If **c** is a vector such that $\mathbf{a}\cdot\mathbf{c} = |\mathbf{c}|, |\mathbf{c} - \mathbf{a}| = 2\sqrt{2}$ and the angle between $\mathbf{a} \times \mathbf{b}$ and **c** is 30º, then $|(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| =$

(a) 2/3 (b) 3/2
(c) 2 (d) 3

Ans. (b)

Solution: $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$\Rightarrow \quad |\mathbf{a} \times \mathbf{b}| = \sqrt{4+4+1} = 3$. Also $|\mathbf{c} - \mathbf{a}|^2 = 8$

so $\quad |\mathbf{c}|^2 + |\mathbf{a}|^2 - 2\mathbf{a}.\mathbf{c} = 8 \quad \Rightarrow \quad |\mathbf{c}|^2 + |\mathbf{a}|^2 - 2|\mathbf{c}| = 8$

$\Rightarrow \quad |\mathbf{c}|^2 + 9 - 2|\mathbf{c}| = 8 \Rightarrow |\mathbf{c}|^2 - 2|\mathbf{c}| + 1 = 0$

$\Rightarrow \quad (|\mathbf{c}| - 1)^2 \quad \Rightarrow \quad |\mathbf{c}| = 1$

Now $\quad |(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| = |\mathbf{a} \times \mathbf{b}|\,|\mathbf{c}|\ \sin 30º = 3.1.\dfrac{1}{2} = \dfrac{3}{2}.$

Example 72: A tangent is drawn to the curve $y = 8/x^2$ in xy-plane at the point $A\,(x_0, y_0)$, where $x_0 = 2$, and the tangent cuts the x-axis at a point B. Then $\mathbf{AB}\cdot\mathbf{OB}$ is equal to

(a) 2 (b) 1
(c) 0 (d) 3

Ans. (d)

Solution: Since $x_0 = 2$ so $y_0 = 8/4 = 2$

Equation of tangent is $Y - 2 = \dfrac{8(-2)}{2^3}(X - 2)$ i.e. $Y = -2X + 6$.

This tangent cuts x-axis at (3, 0)

Hence $B = (3, 0)$. Therefore $\mathbf{AB} = (3-2)\mathbf{i} + (0-2)\mathbf{j}$ and $OB = 3\mathbf{i}$, thus $\mathbf{AB}\cdot\mathbf{OB} = 3$.

Example 73: Let P, Q, R be points with position vectors $\mathbf{r}_1 = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $\mathbf{r}_2 = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{r}_3 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ relative to an origin 0. The distance of P from the plane OQR is (magnitude)

(a) 2 (b) 3

(c) 1 (d) $11/\sqrt{3}$

Ans. (b)

Solution: Equation of the plane OQR is $\mathbf{r} = \lambda\mathbf{r}_2 + \mu\mathbf{r}_3$, i.e. $\mathbf{r}\cdot(\mathbf{r}_2\times\mathbf{r}_3) = 0$

So the distance of P from the plane OQR is $\left|\dfrac{\mathbf{r}_1\cdot(\mathbf{r}_2\times\mathbf{r}_3)}{|\mathbf{r}_2\times\mathbf{r}_3|}\right|$.

Since $\mathbf{r}_2\times\mathbf{r}_3 = -10\,\mathbf{i} + 10\,\mathbf{j} - 5\mathbf{k}$ so $|\mathbf{r}_2\times\mathbf{r}_3| = 15$.

Required distance $= \dfrac{1}{15}\left|\begin{vmatrix} 3 & -2 & -1 \\ 1 & 3 & 4 \\ 2 & 1 & -2 \end{vmatrix}\right| = 3$.

Example 74: For unit vectors $\mathbf{b}$ and $\mathbf{c}$ and any non zero vector $\mathbf{a}$, the value of $\{\{(\mathbf{a}+\mathbf{b})\times(\mathbf{a}+\mathbf{c})\}\times(\mathbf{b}\times\mathbf{c})\}\cdot(\mathbf{b}+\mathbf{c})$ is

(a) $|\mathbf{a}|^2$ (b) $2\,|\mathbf{a}|^2$

(c) $3\,|\mathbf{a}|^2$ (d) none of these

Ans. (d)

Solution: The given expression

$$= \{\{\mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{a} + \mathbf{b}\times\mathbf{c}\}\times(\mathbf{b}\times\mathbf{c})\}\,.\,(\mathbf{b}+\mathbf{c})$$

$$= \{(\mathbf{a}\times\mathbf{c})\times(\mathbf{b}\times\mathbf{c}) + (\mathbf{b}\times\mathbf{a})\times(\mathbf{b}\times\mathbf{c})\}\,.\,(\mathbf{b}+\mathbf{c})$$

$$= [(\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}))\mathbf{c} - (\mathbf{c}\cdot(\mathbf{b}\times\mathbf{c}))\mathbf{a} + (\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c}))\,\mathbf{a}$$

$$- (\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\mathbf{b})]\;.\;(\mathbf{b}+\mathbf{c})$$

$$= [\,(\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}))\,(\mathbf{c}-\mathbf{b})\cdot(\mathbf{b}+\mathbf{c})\,]$$

$$= (\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}))\left[|\,\mathbf{c}\,|^2 - |\,\mathbf{b}\,|^2\right] = 0$$

$[\because\ |\mathbf{b}| = |\mathbf{c}| = 1]$.

Example 75: Three non-coplanar vector $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are drawn from a common initial point. The angle between the plane passing through the terminal points of these vectors and the vector $\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}$ is

(a) $\pi/4$ (b) $\pi/2$

(c) $\pi/3$ (d) none of these

Ans. (b)

Solution: Let the terminal points be A, B, C and the common initial point be the origin of reference so that $\mathbf{AB} = \mathbf{b} - \mathbf{a}$ and $\mathbf{AC} = \mathbf{c} - \mathbf{a}$. The vector $\mathbf{AB}\times\mathbf{AC}$ is perpendicular to the plane ABC. $\mathbf{AB}\times\mathbf{AC} = (\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a}) = \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a} + \mathbf{a}\times\mathbf{b}$. Hence the angle between the plane and the given vector is $\pi/2$.

Example 76: A unit tangent vector at $t = 2$ on the curve $x = t^2 + 2$, $y = 4t^3 - 5$, $z = 2t^2 - 6t$ is

(a) $\dfrac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$ (b) $\dfrac{1}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})$

(c) $\dfrac{1}{\sqrt{6}}(2\mathbf{i}+\mathbf{j}+\mathbf{k})$ (d) none of these

Ans. (d)

Solution: The position vector of any point at t is

$$\mathbf{r} = (2+t^2)\,\mathbf{i} + (4t^3-5)\,\mathbf{j} + (2t^2-6t)\,\mathbf{k}$$

$$\therefore \quad \frac{d\mathbf{r}}{dt} = 2t\,\mathbf{i} + 12t^2\,\mathbf{j} + (4t-6)\mathbf{k},\quad \left.\frac{d\mathbf{r}}{dt}\right|_{t=2} = 4\mathbf{i} + 48\mathbf{j} + 2\mathbf{k}$$

$$\left|\frac{d\mathbf{r}}{dt}\right|_{t=2} = \sqrt{16+2304+4} = \sqrt{2320}.$$

Hence the required unit tangent vector at $t = 2$ is $(1/\sqrt{580})\ (2\mathbf{i} + 24\mathbf{j} + \mathbf{k})$.

Example 77: A particle moves along a curve so that its coordinates at time t are $x = t$, $y = \dfrac{1}{2}t^2$, $z = \dfrac{1}{3}t^3$. The acceleration at $t = 1$ is

(a) $\mathbf{j} + 2\mathbf{k}$ (b) $\mathbf{j} + \mathbf{k}$

(c) $2\mathbf{j} + \mathbf{k}$ (d) none of these

Ans. (a)

Solution: Let $\mathbf{r} = t\mathbf{i} + \dfrac{1}{2}t^2\,\mathbf{j} + \dfrac{1}{3}t^3\,\mathbf{k}$ then the velocity $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \mathbf{i} + t\,\mathbf{j} + t^2\,\mathbf{k}$ and the acceleration $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \mathbf{j} + 2t\,\mathbf{k}$. Hence $\mathbf{a}\,|_{(t=1)} = \mathbf{j} + 2\,\mathbf{k}$.

Example 78: Consider the parallelopipped with sides $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ then the angle between $\mathbf{a}$ and the plane containing the face determined by $\mathbf{b}$ and $\mathbf{c}$ is

(a) $\sin^{-1}(1/3)$ (b) $\cos^{-1}(9/14)$

(c) $\sin^{-1}(9/14)$ (d) $\sin^{-1}(2/3)$

Ans. (c)

Solution: $\mathbf{b}\times\mathbf{c} = -3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. If θ is the angle between $\mathbf{a}$ and the plane containing $\mathbf{b}$ and $\mathbf{c}$

then. $\cos(90° - \theta) = \left|\dfrac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}{|\mathbf{a}|\,|\mathbf{b}\times\mathbf{c}|}\right|$

$$= \frac{1}{\sqrt{14}}\frac{1}{\sqrt{14}}\,|(-9-2+2)| = \frac{9}{14}.$$

Hence $\theta = \sin^{-1}(9/14)$.

Example 79: A unit vector $\mathbf{n}$ perpendicular to the plane determined by the points A (0, –2, 1), B (1, – 1, –2) and C (–1, 1, 0)

(a) $\dfrac{1}{3}(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$ (b) $1/4\sqrt{6}\,(8\mathbf{i}+4\mathbf{j}+4\mathbf{k})$

(c) $\dfrac{1}{\sqrt{3}}(\mathbf{i}-\mathbf{j}+\mathbf{k})$ (d) $\dfrac{1}{\sqrt{14}}(3\mathbf{i}+\mathbf{j}+2\mathbf{k})$

Ans. (b)

◎ **Solution:** $\mathbf{AB} = \mathbf{i} + \mathbf{j} - 3\,\mathbf{k}$ and $\mathbf{AC} = -\,\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{AB} \times \mathbf{AC} = 8\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$. Hence $\mathbf{n} = (1/4\sqrt{6})\,(8\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$.

◉ **Example 80:** If the vectors $\mathbf{AB} = -3\mathbf{i} + 4\mathbf{k}$ and $\mathbf{AC} = 5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ are the sides of a triangle ABC. Then the length of the median through A is

(a) $\sqrt{14}$ (b) $\sqrt{18}$

(c) $\sqrt{29}$ (d) none of these

Ans. (b)

◎ **Solution:** Let A be the origin of reference so that the position vectors of B and C are $-3\mathbf{i} + 4\mathbf{j}$ and $-\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ respectively. The position vector of mid point of BC is $\mathbf{i} - \mathbf{j} + 4\mathbf{k}$. Thus the length of the median is $\sqrt{1+1+16} = \sqrt{18}$.

◉ **Example 81:** If $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ and $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$ and $|\mathbf{c}| = 7$ then the angle between $\mathbf{a}$ and $\mathbf{b}$ is

(a) $\pi/6$ (b) $2\pi/3$

(c) $\pi/3$ (d) $5\pi/3$

Ans. (c)

◎ **Solution:** $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0} \Rightarrow \mathbf{a} + \mathbf{b} = -\mathbf{c} \Rightarrow (\mathbf{a} + \mathbf{b})\cdot(\mathbf{a} + \mathbf{b}) = |\mathbf{c}|^2$.

Thus $|\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\,|\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{c}|^2$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Therefore, $\cos\theta = \dfrac{49-9-25}{2\cdot3\cdot5} = \dfrac{1}{2} \Rightarrow \theta = \pi/3$.

◉ **Example 82:** The vector $((\mathbf{i}-\mathbf{j})\times(\mathbf{j}-\mathbf{k}))\times(\mathbf{i}+5\mathbf{k})$ is equal to

(a) $5\mathbf{i} - 4\mathbf{j} - \mathbf{k}$ (b) $3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$

(c) $4\mathbf{i} - 5\mathbf{j} - \mathbf{k}$ (d) $5\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

Ans. (a)

◎ **Solution:** The given expression

$$((\mathbf{i}-\mathbf{j})\cdot(\mathbf{i}+5\mathbf{k}))\,(\mathbf{j}-\mathbf{k}) - ((\mathbf{j}-\mathbf{k})\cdot(\mathbf{i}+5\mathbf{k}))\,(\mathbf{i}-\mathbf{j})$$

$$= 1\,(\mathbf{j}-\mathbf{k}) + 5(\mathbf{i}-\mathbf{j}) = 5\mathbf{i} - 4\mathbf{j} - \mathbf{k}.$$

◉ **Example 83:** The position vector of a point P is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $x, y, z \in \mathbf{N}$ and $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. If $\mathbf{r}\cdot\mathbf{u} = 10$, then the number of possible positions of P is

(a) 72 (b) 36

(c) 60 (d) 108

Ans. (b)

◎ **Solution:** $\mathbf{r}\cdot\mathbf{u} = 10 \Rightarrow x + y + z = 10$. The number of positive integral solution of this equation is ${}^9C_2 = 36$ (see Chapter 6).

◉ **Example 84:** If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors and θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ then $\sin(\theta/2)$ is equal to

(a) $\dfrac{1}{2}|\mathbf{a}-\mathbf{b}|$ (b) 1

(c) $\dfrac{1}{2}|\mathbf{a}+\mathbf{b}|$ (d) 0

Ans. (a)

◎ **Solution:** $\left|\dfrac{\mathbf{a}-\mathbf{b}}{2}\right|^2 = \dfrac{1}{4}\left[|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}.\mathbf{b}\right]$

$$= \frac{1}{2}[1 - |\mathbf{a}||\mathbf{b}|\cos\theta] = \frac{1}{2}[1-\cos\theta] = \sin^2\frac{\theta}{2}$$

Hence $\sin\left(\dfrac{\theta}{2}\right) = \left(\dfrac{1}{2}\right)|\mathbf{a}-\mathbf{b}|$.

◉ **Example 85:** The vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} + m\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + (m+1)\,\mathbf{k}$ and $\mathbf{c} = \mathbf{i} - \mathbf{j} + m\mathbf{k}$ are coplanar if m is equal to

(a) 1 (b) 4

(c) 3 (d) none of these

Ans. (d)

◎ **Solution:** $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar if

$$\begin{vmatrix} 1 & 1 & m \\ 1 & 1 & m+1 \\ 1 & -1 & m \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 0 & m \\ 1 & 0 & m+1 \\ 1 & -2 & m \end{vmatrix} = 0$$

$\Rightarrow\quad 2 = 0.$

So there no value of m for which the vectors are coplanar.

◉ **Example 86:** If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar unit vectors such that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \dfrac{\mathbf{b}+\mathbf{c}}{\sqrt{2}}$, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is

(a) $3\pi/4$ (b) $\pi/4$

(c) $\pi/2$ (d) π

Ans. (a)

◎ **Solution:** The given equality implies

$$(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c} = \frac{\mathbf{b}+\mathbf{c}}{\sqrt{2}}$$

$$\Rightarrow \left(\mathbf{a}\cdot\mathbf{c} - \frac{1}{\sqrt{2}}\right)\mathbf{b} = \left(\frac{1}{\sqrt{2}} + \mathbf{a}\cdot\mathbf{b}\right)\mathbf{c}$$

$\Rightarrow$ $\mathbf{b}$ and $\mathbf{c}$ are collinear thus $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar if $\mathbf{a}\cdot\mathbf{c} \neq \dfrac{1}{\sqrt{2}}$ and $\mathbf{a}\cdot\mathbf{b} \neq -\dfrac{1}{\sqrt{2}}$. Hence $\mathbf{a}\cdot\mathbf{b} = -1/\sqrt{2}$

$\Rightarrow\quad \cos\theta = -1/\sqrt{2}$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

So $\theta = 3\pi/4$.

EXERCISE

Concept Based Straight Objective Type Questions

1. If M is the mid point of AB and O is any point, then
   (a) $\mathbf{OM} = \mathbf{OA} + \mathbf{MA}$
   (b) $\mathbf{OM} = \mathbf{OA} - \mathbf{MA}$
   (c) $\mathbf{OM} = \frac{1}{2}(\mathbf{OA} - \mathbf{OB})$
   (d) $\mathbf{OM} = \frac{1}{2}(\mathbf{OB} + \mathbf{OA})$

2. The angle between $3\mathbf{i} + 4\mathbf{j}$ and $2\mathbf{j} - 5\mathbf{k}$ is
   (a) $\frac{\pi}{2}$ (b) $\cos^{-1}\frac{8}{5\sqrt{29}}$
   (c) $\frac{\pi}{6}$ (d) $\cos^{-1}\frac{1}{3}$

3. A unit vector $\mathbf{c}$ perpendicular to $\mathbf{a}$ and coplanar with $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j}$ is given by
   (a) $\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{k})$ (b) $\frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j})$
   (c) $\frac{1}{\sqrt{2}}(\mathbf{j}+\mathbf{k})$ (d) $\frac{1}{\sqrt{2}}(-\mathbf{j}+\mathbf{k})$

4. A vector $\mathbf{b}$, which is collinear with vector $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and satisfies $\mathbf{a}\cdot\mathbf{b} = 2$ is given by
   (a) $\frac{1}{2}(2\mathbf{i}+\mathbf{j}-\mathbf{k})$ (b) $\frac{1}{3}(2\mathbf{i}+\mathbf{j}-\mathbf{k})$
   (c) $\frac{1}{4}(2\mathbf{i}+\mathbf{j}-\mathbf{k})$ (d) $\frac{1}{2}(-2\mathbf{i}-\mathbf{j}+\mathbf{k})$

5. If $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ then the magnitude of projection of $\mathbf{u} \times \mathbf{v}$ on $\mathbf{w}$ is given by
   (a) $\sqrt{\frac{1}{2}}$ (b) $\sqrt{\frac{1}{3}}$
   (c) $\sqrt{\frac{3}{4}}$ (d) $\sqrt{\frac{3}{2}}$

6. If $\mathbf{a}$ and $\mathbf{b}$ are non-collinear vectors, then the value of λ for which
   $\mathbf{u} = (\lambda + 2)\,\mathbf{a} + \mathbf{b}$ and
   $\mathbf{v} = (1 + 4\lambda)\,\mathbf{a} - 2\mathbf{b}$ are collinear is
   (a) $\frac{1}{2}$ (b) $\frac{3}{2}$
   (c) $\frac{3}{4}$ (d) $\frac{1}{3}$

7. The area of the triangle formed by A (1, 0, 0), B (0, 1, 0), C (1, 1, 1) is
   (a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{4}$
   (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{4}$

8. A unit vector perpendicular to $3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$ is
   (a) $\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$
   (b) $\frac{1}{\sqrt{14}}(\mathbf{i}-2\mathbf{j}+3\mathbf{k})$
   (c) $\frac{1}{\sqrt{74}}(4\mathbf{i}+3\mathbf{j}-7\mathbf{k})$
   (d) $\frac{1}{\sqrt{74}}(4\mathbf{i}-3\mathbf{j}-7\mathbf{k})$

9. The value of scalar triple product $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{j} + \mathbf{k}$ is
   (a) 12 (b) 10
   (c) 14 (d) 16

10. The vector $[(\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (2\mathbf{i} - 3\mathbf{j} - \mathbf{k})] \times [(-3\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (2\mathbf{j} + \mathbf{k})]$ is given by
   (a) $3\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$
   (b) $-5(3\mathbf{i} - 5\mathbf{j} - 3\mathbf{k})$
   (c) $5(3\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})$
   (d) $(15\mathbf{i} - 25\mathbf{j} + 15\mathbf{k})$

LEVEL 1

Straight Objective Type Questions

11. Let |**a**| = 3 and |**b**| = 4. The value of μ for which the vectors **a** + μ**b** and **a** – μ**b** will be perpendicular is
    (a) $3/4$ (b) $2/3$
    (c) $-5/2$ (d) $-2/3$
12. The value of α for which the vectors 2**i** – **j** + **k**, **i** + 2**j** + α**k** and 3**i** – 4**j** + 5**k** are coplanar is
    (a) 3 (b) –3
    (c) 2 (d) none of these
13. The area of a parallelogram having diagonals **a** = 3**i**+ **j** – 2**k** and **b** = **i** – 3**j** + 4**k** is
    (a) $5\sqrt{3}$ (b) $2\sqrt{3}$
    (c) $4\sqrt{3}$ (d) none of these
14. If **r** satisfies the equation **r** × (**i** + 2**j** + **k**) = **i** – **k**, then for any scalar t, **r** is equal to
    (a) $\mathbf{i} + t(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$
    (b) $\mathbf{j} + t(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$
    (c) $\mathbf{k} + t(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$
    (d) $\mathbf{i} - \mathbf{k} + t(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$
15. The vectors **a**, **b** and **c** are equal in length and taken pairwise, make equal angles. If **a** = **i** + **j**, **b** = **j** + **k** and **c** make an obtuse angle with the base vector **i**, then **c** is equal to
    (a) $\mathbf{i} + \mathbf{k}$ (b) $-\mathbf{i} + 4\mathbf{j} - \mathbf{k}$
    (c) $-\frac{1}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$ (d) $\frac{1}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$
16. The vectors **AB** = 3**i** – 2**j** + 2**k** and **BC** = – **i** – 2**k** are the adjacent sides of a parallelogram. An angle between its diagonals is
    (a) $\pi/4$ (b) $\pi/3$
    (c) $\pi/2$ (d) $2\pi/3$
17. Let the unit vectors **a** and **b** be perpendicular and the unit vector **c** be inclined at an angle θ to both **a** and **b**. If **c** = α**a** + β**b** + γ(**a** × **b**) then
    (a) $\alpha = 2\beta$ (b) $\gamma^2 = 1 + 2\alpha^2$
    (c) $\gamma^2 = \cos 2\theta$ (d) $\beta^2 = \frac{1 + \cos 2\theta}{2}$
18. If the unit vectors **a** and **b** are inclined at an angle 2θ and |**a** – **b**| < 1, then if $0 \le \theta \le \pi$, θ lies in the interval
    (a) $\left[0, \frac{\pi}{6}\right)$ (b) $\left(\frac{5\pi}{6}, 2\pi\right]$
    (c) $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ (d) $\left[\frac{\pi}{2}, \frac{5\pi}{6}\right]$
19. For non-coplanar vectors **a**, **b** and **c**, |(**a** × **b**)·**c**| = |**a**| |**b**| |**c**| holds if and only if
    (a) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$
    (b) $\mathbf{a}\cdot\mathbf{b} = 0 = \mathbf{b}\cdot\mathbf{c}$
    (c) $\mathbf{a}\cdot\mathbf{b} = 0 = \mathbf{c}\cdot\mathbf{a}$
    (d) $\mathbf{b}\cdot\mathbf{c} = 0 = \mathbf{c}\cdot\mathbf{a}$
20. The volume of the tetrahedron whose vertices are the points with position vectors **i** – 6**j** + 10**k**, –**i** – 3**j** + 7**k**, 5**i** – **j** + λ**k** and 7**i** – 4**j** + 7**k** is 11 cubic units if the value of λ is
    (a) –1 (b) 1
    (c) –7 (d) 5
21. The vectors (x, x + 1, x + 2), (x + 3, x + 4, x + 5) and (x + 6, x + 7, x+ 8) are coplanar for
    (a) only finite number of values of x
    (b) $x < 0$
    (c) Only $x = z$
    (d) none of these
22. A vector of length $\sqrt{7}$ which is perpendicular to $2\mathbf{j} - \mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and makes an obtuse angle with y-axis is
    (a) $(1/\sqrt{5})\left(4\mathbf{i} - \mathbf{j} + \sqrt{18}\mathbf{k}\right)$
    (b) $(1/\sqrt{3})(4\mathbf{i} - \mathbf{j} - 2\mathbf{k})$
    (c) $(1/\sqrt{3})(-4\mathbf{i} + \mathbf{j} + 2\mathbf{k})$
    (d) $(1/\sqrt{3})(-4\mathbf{i} - \mathbf{j} + 2\mathbf{k})$
23. Let |**a**| = |**b**| = 2 and **p** = **a** + **b**, **q** = **a** – **b**. If | **p** × **q** | = 2(k – (**a** . **b**)^2)$^{1/2}$ then
    (a) $k = 16$ (b) $k = 8$
    (c) $k = 4$ (d) $k = 1$
24. If **r** · **a** = 0, **r** · **b** = 0 and **r** · **c** = 0 for some non-zero vector **r**, then the value of [**a** **b** **c**] is
    (a) 0 (b) $1/2$
    (c) 1 (d) 2

25. The position vectors of three consecutive vertices of a parallelogram are $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and $7\mathbf{i} + 9\mathbf{j} + 11\mathbf{k}$. The position vector of the fourth vertex is

(a) $6(\mathbf{i} + \mathbf{j} + \mathbf{k})$ (b) $7(\mathbf{i} + \mathbf{j} + \mathbf{k})$
(c) $2\mathbf{j} - 4\mathbf{k}$ (d) $6\mathbf{i} + 8\mathbf{j} + 10\mathbf{k}$.

26. The volume of the parallelopiped whose sides are given by $\mathbf{OA} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{OB} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{OC} = 3\mathbf{i} - \mathbf{k}$ is

(a) $4/13$ (b) 4
(c) $2/7$ (d) none of these.

27. The value of $|\mathbf{a} \times \mathbf{i}|^2 + |\mathbf{a} \times \mathbf{j}|^2 + |\mathbf{a} \times \mathbf{k}|^2$ is

(a) $\mathbf{a}^2$ (b) $2\mathbf{a}^2$
(c) $3\mathbf{a}^2$ (d) none of these.

28. If **a**, **b** and **c** are any three vectors, then $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ if and only if

(a) **b** and **c** are collinear
(b) **a** and **c** are collinear
(c) **a** and **b** are collinear
(d) none of these.

29. The value of $\mathbf{i} \times (\mathbf{a} \times \mathbf{i}) + \mathbf{j} \times (\mathbf{a} \times \mathbf{j}) + \mathbf{k} \times (\mathbf{a} \times \mathbf{k})$ is

(a) **a** (b) 2**a**
(c) **0** (d) 3**a**

30. The value of $[\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}]$ is

(a) $2\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ (b) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$
(c) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$ (d) 0

31. Given vectors $\mathbf{a} = (3, -1, 5)$ and $\mathbf{b} = (1, 2, -3)$. A vector **c** which is perpendicular to z-axis and satisfying $\mathbf{c} \cdot \mathbf{a} = 9$ and $\mathbf{c} \cdot \mathbf{b} = -4$ is

(a) $(2, -2, 0)$ (b) $(4, -2, 0)$
(c) $(2, -3, 0)$ (d) $(1, 2, 4)$

32. Area of the parallelogram on the vectors $\mathbf{a} + 3\mathbf{b}$ and $3\mathbf{a} + \mathbf{b}$ if $|\mathbf{a}| = |\mathbf{b}| = 1$ and the angle between **a** and **b** is $\pi/6$ is

(a) 2 (b) 4
(c) 8 (d) 16

33. If $\mathbf{a} = x\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{c} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ are coplanar then the value of x is

(a) 1 (b) -2
(c) -1 (d) none of these

34. If $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 3$ then

(a) $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = -3$
(b) $\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = -3$
(c) $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) = 3$
(d) $(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b} = 3$

35. Let $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and **b** be another vector such that $\mathbf{a} \cdot \mathbf{b} = 14$ and $\mathbf{a} \times \mathbf{b} = 3\mathbf{i} + \mathbf{j} - 8\mathbf{k}$ then the vector **b** is equal to

(a) $5\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ (b) $5\mathbf{i} - \mathbf{j} - 2\mathbf{k}$
(c) $5\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ (d) $3\mathbf{i} + \mathbf{j} + 4\mathbf{k}$

36. ABCDEF is a regular hexagon with centre at the origin such that $\mathbf{AD} + \mathbf{EB} + \mathbf{FC} = \lambda\,\mathbf{ED}$. Then λ equals

(a) 2 (b) 4
(c) 6 (d) 3.

37. A non-zero vector **a** is parallel to the line of intersection of the plane determined by the vectors $\mathbf{i}$, $\mathbf{i} + \mathbf{j}$ and the plane determined by the vectors $\mathbf{i} - \mathbf{j}$, $\mathbf{i} + \mathbf{k}$. The angle between **a** and $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ is:

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{3}$
(c) $\dfrac{\pi}{6}$ (d) $\dfrac{\pi}{4}$

38. Let P, Q, R and S be the points on the plane with position vectors $-2\mathbf{i} - \mathbf{j}$, $4\mathbf{i}$, $3\mathbf{i} + 3\mathbf{j}$ and $-3\mathbf{i} + 2\mathbf{j}$ respectively. The quadrilateral $PQRS$ must be a

(a) parallelogram, which is neither rhombus nor a rectangle
(b) square
(c) rectangle but not a square
(d) rhombus, but not a square

39. If **a**, **b**, **c** and **d** are unit vectors such that $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = 1$ and $\mathbf{a} \cdot \mathbf{c} = 1/2$. Then

(a) **a** , **b** , **c** are non-coplanar
(b) **b**, **c**, **d** are non-coplanar
(c) **b**, **d** are non-parallel
(d) **a**, **d** are parallel and **b**, **c** are parallel

40. The edges of a paralllelopiped are of unit length and parallel to non-coplanar unit length and are parallel to non-coplanar unit vectors **a**, **b**, **c** such that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 1/2$. Then the volume of the parallelopiped is

(a) $1/\sqrt{2}$ (b) $1/2\sqrt{2}$
(c) $\sqrt{3}/2$ (d) $1/\sqrt{3}$

41. The set of all $a \in \mathbf{R}$ for which, the vector $a\mathbf{i} + 2a\mathbf{j} - 3a\mathbf{k}$, $(2a + 1)\,\mathbf{i} + (2a + 3)\,\mathbf{j} + (a + 1)\mathbf{k}$, $(3a + 5)\mathbf{i} + (a + 5)\,\mathbf{j} + (a + 2)\mathbf{k}$ are coplanar is

(a) $\{0\}$ (b) $(0, \infty)$
(c) $(-\infty, 1)$ (d) $(1, \infty)$

42. If ($\mathbf{a} \times \mathbf{b}$) $\times$ ($\mathbf{c} \times \mathbf{d}$) = $p\mathbf{a} + q\mathbf{b}$, where **a**, **b** are non-collinear and **c**, **d** are also non-collinear, then
(a) $p = [\mathbf{c}\ \mathbf{b}\ \mathbf{d}]$ (b) $p = [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]$
(c) $p = [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]$ (d) $p = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

43. The unit vectors **a** and **b** are perpendicular. Suppose that a unit vector **c** which is equal to $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma(\mathbf{a} \times \mathbf{b})$ and is equally inclined to **a** and **b**. Then
(a) $\alpha = \beta$ (b) $\alpha = 2\beta$
(c) $\alpha = \dfrac{\beta}{2}$ (d) $\beta^2 - \dfrac{1+\alpha}{2}$

Assertion-Reason Type Questions

44. If $\mathbf{a} \cdot \mathbf{c} = 3/2$, $\mathbf{b} \cdot \mathbf{d} = 2$, $\mathbf{a} \cdot \mathbf{d} = 3$ and $\mathbf{b} \cdot \mathbf{c} = 1/2$
Statement-1: $\mathbf{a} \times \mathbf{b}$, **c**, **d** are non-coplanar
Statement-2: $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})$

45. Suppose that **a**, **b**, **c** are non-coplanar such $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 8$
Statement-1: $\mathbf{a} \times \mathbf{b}$, $\mathbf{b} \times \mathbf{c}$, $\mathbf{c} \times \mathbf{a}$ are coplanar
Statement-2: $[\mathbf{a} \times \mathbf{b}\ \ \mathbf{b} \times \mathbf{c}\ \ \mathbf{c} \times \mathbf{a}] = 64$

46. Let $|\mathbf{a} + \mathbf{b}| = 2\,|\mathbf{a} - \mathbf{b}| = 1$
Statement-1: The angle between **a** and **b** is acute
Statement-2: $4\mathbf{a} \cdot \mathbf{b} = |\mathbf{a} + \mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2$

47. **Statement-1:** Unit vectors perpendicular to $\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ are $\pm\dfrac{1}{\sqrt{4}}(\mathbf{i} - 2\mathbf{j} - 3\mathbf{k})$
Statement-2: Unit vectors perpendicular to **a** and **b** are $\pm\dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$.

48. If **a**, **b**, **c** are non-coplanar vectors then
Statement-1: $\mathbf{a} + \mathbf{b}$, $\mathbf{b} + \mathbf{c}$, $\mathbf{c} + \mathbf{a}$ are non-coplanar
Statement-2: $[\mathbf{a} + \mathbf{b}\ \ \mathbf{b} + \mathbf{c}\ \ \mathbf{c} + \mathbf{a}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

LEVEL 2

Straight Objective Type Questions

49. A line makes angles α, β, γ and δ with diagonals of a cube. The value of $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ is
(a) 1 (b) 1/3
(c) 8/3 (d) 4/3

50. If ($\mathbf{a} \times \mathbf{b}$) $\times$ ($\mathbf{c} \times \mathbf{d}$) = $[\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\,\mathbf{c} + k\,\mathbf{d}$ then the value of k is
(a) $[\mathbf{b}\ \mathbf{a}\ \mathbf{c}]$ (b) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$
(c) $[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]$ (d) $[\mathbf{c}\ \mathbf{b}\ \mathbf{d}]$

51. The one of the value of x for which the angle between $\mathbf{c} = x\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{d} = \mathbf{i} + x\mathbf{j} + \mathbf{k}$ is $\pi/3$ is
(a) $1 + \sqrt{2}$ (b) $2 + \sqrt{2}$
(c) $3 + \sqrt{2}$ (d) none of these

52. The line $x = -2$, $y = 4 + 2t$, $z = -3 + t$ intersect
(a) the xy-plane
(b) the xz-plane in $(-2, 0, -4)$
(c) the yz-plane
(d) none of these

53. Let $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. The vector component of **u** orthogonal to **a** is
(a) $(1/7)\,(20\mathbf{i} - 5\mathbf{j} + 10\mathbf{k})$
(b) $(1/7)\,(4\mathbf{i} + 24\mathbf{j} + 4\mathbf{k})$
(c) $(1/7)\,(11\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})$
(d) $(-1/7)\,(6\mathbf{i} + 2\mathbf{j} - 11\mathbf{k})$

54. If **a, b, c, d** lie in the same plane then ($\mathbf{a} \times \mathbf{b}$) $\times$ ($\mathbf{c} \times \mathbf{d}$) is equal to
(a) $\mathbf{c} + \mathbf{d}$ (b) $\mathbf{0}$
(c) $[\mathbf{a}, \mathbf{b}, \mathbf{c}]\,\mathbf{a} + 2\mathbf{b}$ (d) $[\mathbf{b}, \mathbf{c}, \mathbf{d}]\,\mathbf{c} + \mathbf{d}$

55. If $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) + k\,(\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$ then the value of k is
(a) 1 (b) 0
(c) -2 (d) -1

56. The distance between (5, 1, 3) and the line $x = 3$, $y = 7 + t$, $z = 1 + t$ is
(a) 4 (b) 2
(c) 6 (d) 8

57. The distance between the lines $x = 1 - 4t$, $y = 2 + t$, $z = 3 + 2t$ and $x = 1 + s$, $y = 4 - 2s$, $z = -1 + s$ is

(a) 8 (b) $16/\sqrt{90}$

(c) $8/\sqrt{5}$ (d) $16/\sqrt{110}$

58. A triangle ABC is defined by the coordinates of vertices $A(1, -2, 2)$, B (1, 4, 0) and C (– 4, 1, 1). The vector **BM**, where M is the foot of the altitude drawn from B to AC is

(a) $-\frac{20}{3}\mathbf{i}-10\mathbf{j}+\frac{10}{3}\mathbf{k}$

(b) $-\frac{10}{7}\mathbf{i}-\frac{30}{7}\mathbf{j}+\frac{10}{7}\mathbf{k}$

(c) $\frac{20}{7}\mathbf{i}+5\mathbf{j}-\frac{10}{7}\mathbf{k}$

(d) $-\frac{20}{7}\mathbf{i}-\frac{30}{7}\mathbf{j}+\frac{10}{7}\mathbf{k}$

59. If **a, b, c,** are non-coplanar vectors such that $(2h + k)\,\mathbf{a} + (3 - 4h + l)\mathbf{b} + (1 + h + k)\,\mathbf{c} = h\mathbf{a} + k\mathbf{b} + l\mathbf{c}$ then

(a) $h = 1, k = -4/3, l = 4/3$

(b) $h = 4/3, k = -4/3, l = 1$

(c) $h = 1/3, k = -1/3, l = 2/3$

(d) none of these.

60. The angle between two diagonals of a cube is

(a) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (b) $\cos^{-1}\left(\frac{2}{\sqrt{3}}\right)$

(c) $\cos^{-1}\left(\frac{1}{3}\right)$ (d) $\cos^{-1}\left(\frac{2}{3}\right)$

61. The point of intersection of the lines $\mathbf{r} \times \mathbf{a} = \mathbf{b} \times \mathbf{a}$, $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$ is

(a) **a** (b) **b – a**

(c) **a – b** (d) **a + b**

62. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$; $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$; $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$; $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$ and

$$k\,(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} -\mathbf{a} & -\mathbf{b} & \mathbf{c} & \mathbf{d} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

(formal expression) then

(a) $k = 1$ (b) $k = 2$

(c) $k = 4$ (d) none of these

63. The value of $(\mathbf{b} \times \mathbf{c})\cdot(\mathbf{a} \times \mathbf{d}) + (\mathbf{c} \times \mathbf{a})\cdot(\mathbf{b} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{b})\cdot(\mathbf{c} \times \mathbf{d})$ is

(a) **[a, b, c] – [b, c, d]**

(b) **[a, b, c] + [b, c, d]**

(c) 0

(d) none of these

64. The lines $\mathbf{r} = \mathbf{b} - 2\mathbf{c} + \lambda(\mathbf{a} + \mathbf{b})$ and $\mathbf{r} = 2\mathbf{b} - \mathbf{c} + \mu(\mathbf{b} + \mathbf{c})$ intersect at the point.

(a) **b – 2c** (b) **b + 2c**

(c) **b + c** (d) **c – b**

65. If $\mathbf{a} = \mathbf{i}+2\mathbf{j}-3\mathbf{k}, \mathbf{b} = 2\mathbf{i}+\mathbf{j}-\mathbf{k}$ then the vector **v** satisfying $\mathbf{a} \times \mathbf{v} = \mathbf{a} \times \mathbf{b}$ and $\mathbf{a}\cdot\mathbf{v} = 0$ is $\mathbf{b} + t\,\mathbf{a}$, t being a scalar for

(a) all values of t

(b) for no value of t

(c) finite number of values of t

(d) $t = -1/4$

66. The value of

$$|\mathbf{a}\times(\mathbf{i}\times\mathbf{j})|^2 + |\mathbf{a}\times(\mathbf{j}\times\mathbf{k})|^2 + |\mathbf{a}\times(\mathbf{k}\times\mathbf{i})|^2 \text{ is}$$

(a) $|\mathbf{a}|^2$ (b) $2|\mathbf{a}|^2$

(c) $3|\mathbf{a}|^2$ (d) none of these

67. The locus of a point equidistant from two points whose position vectors are **a** and **b** is

(a) $(\mathbf{r}-(\mathbf{a}+\mathbf{b}))\cdot\mathbf{b} = 0$

(b) $\left(\mathbf{r}-\frac{1}{2}(\mathbf{a}+\mathbf{b})\right)\cdot\mathbf{a} = 0$

(c) $\left(\mathbf{r}-\frac{1}{2}(\mathbf{a}+\mathbf{b})\right)\cdot(\mathbf{a}-\mathbf{b}) = 0$

(d) $\left(\mathbf{r}-\frac{1}{2}(\mathbf{a}+\mathbf{b})\right)\cdot(\mathbf{a}+\mathbf{b}) = 0$

68. A vector $\mathbf{a} = (x, y, z)$ of length $2\sqrt{3}$ which makes equal angles with the vectors $\mathbf{b} = (y, -2z, 3x)$ and $\mathbf{c} = (2z, 3x, -y)$ and is perpendicular to $\mathbf{d} = (1, -1, 2)$ and makes an obtuse angle with y-axis is

(a) (–2, 2, 2) (b) $\left(1, 1, \sqrt{10}\right)$

(c) (2, –2, –2) (d) none of these

69. If $\mathbf{a} \times \mathbf{b} = \mathbf{c}$ and $\mathbf{b} \times \mathbf{c} = \mathbf{a}$, then

(a) **a**, **b**, **c** are orthogonal in pairs but $|\mathbf{a}| \neq |\mathbf{c}|$

(b) **a**, **b**, **c** are orthogonal in pairs but $|\mathbf{b}| \neq 1$

(c) **a**, **b**, **c** are not orthogonal to each other in pairs

(d) **a**, **b**, **c** are orthogonal in pairs and $|\mathbf{a}| = |\mathbf{c}|$, $|\mathbf{b}| = 1$

70. The acute angle between the lines $x = -2 + 2t$, $y = 3 - 4t$, $z = -4 + t$ and $x = -2 - t$, $y = 3 + 2t$, $z = -4 + 3t$ is

(a) $\sin^{-1}\frac{1}{\sqrt{3}}$ (b) $\cos^{-1}\frac{1}{\sqrt{6}}$

(c) $\cos^{-1}\frac{1}{\sqrt{5}}$ (d) $\cos^{-1} 2/3$

71. Let $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$, $b = b_1\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$ and $c = c_1\text{i} + c_2\mathbf{j} + c_3\mathbf{k}$ be three non zero vectors such that $\boldsymbol{c}$ is unit vector perpendicular to both the vectors **a** and **b**. If the angle between **a** and **b** is $\pi/6$, then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$ is equal to
   (a) 0
   (b) 1
   (c) $\left(\frac{1}{4}\right)\left(\sum_{i=1}^{3} a_i^2\right)\left(\sum_{i=1}^{3} b_i^2\right)$
   (d) none of these

72. A vector **a** has components $2p$ and 1 with respect to a rectangular cartesian system. This system is rotated through a certain angle about the origin in the counterclockwise sense. If, with respect to the new system, **a** has components $p + 1$ and 1, then
   (a) $p = 0$
   (b) $p = 1$ or $p = 1/3$
   (c) $p = 1$ or $p = 1/3$
   (d) $p = 1$ or $p = -1$

73. $\mathbf{a}\cdot((\mathbf{b}\times\mathbf{c})\times(\mathbf{a}+(\mathbf{b}\times\mathbf{c}))$ is equal to
   (a) 0 (b) $2\,[\mathbf{a\,b\,c}]$
   (c) $[\mathbf{a\,b\,c}]$ (d) none of these

74. If $\mathbf{X}\cdot\mathbf{A} = \mathbf{X}\cdot\mathbf{B} = \mathbf{X}\cdot\mathbf{C}$ for some non-zero vector **X**, then $[\mathbf{A\,B\,C}]$ is equal to
   (a) $|\mathbf{A}||\mathbf{B}|\mathbf{C}|$ (b) 0
   (c) $2\,|\mathbf{A}||\mathbf{B}|\mathbf{C}|$ (d) none of these

75. Given the vectors $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. A vector **c** which is perpendicular to the z-axis and satisfies $\mathbf{c}\cdot\mathbf{a} = 9$ and $\mathbf{c}\cdot\mathbf{b} = -4$ is
   (a) $2\mathbf{i} - 3\mathbf{j}$ (b) $-2\mathbf{i} + 3\mathbf{j}$
   (c) $-4\mathbf{i} - 4\mathbf{j}$ (d) $\mathbf{i} - \mathbf{j} + \mathbf{k}$

76. A unit vector in XZ plane making angles $\pi/4$ and $\pi/3$ respectively with $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = \mathbf{j} - \mathbf{k}$ is
   (a) $\frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{k})$ (b) $\frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{k})$
   (c) $-\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k})$ (d) none of these

77. The vector $\mathbf{a} + \mathbf{b}$ bisects the angle between **a** and **b** if
   (a) $|\mathbf{a}| = 2|\mathbf{b}|$ (b) $|\mathbf{a}| + |\mathbf{b}|^2 = |\mathbf{a} + \mathbf{b}|^2$
   (c) $|\mathbf{a}| = |\mathbf{b}|$ (d) $|\mathbf{a}| - |\mathbf{b}| = |\mathbf{a} - \mathbf{b}|$

78. The vectors $\mathbf{AB} = -3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{AC} = 5\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$ are the sides of the triangle ABC. The length of the median **AM** is
   (a) $\sqrt{5}$ (b) $\sqrt{14}$
   (c) $\sqrt{17}$ (d) $\sqrt{18}$

79. $ABCD$ is a quadrilateral such that $\mathbf{AB} = \mathbf{b}$, $\mathbf{AD} = \mathbf{d}$, $\mathbf{AC} = m\mathbf{b} + p\mathbf{d}$ where $m, p > 0$. The area of the quadrilateral $ABCD$ is
   (a) $\frac{1}{2}\,(m + p)\,|\mathbf{b} \times \mathbf{d}|$ (b) $(m + p)\,|\mathbf{b} \times \mathbf{d}|$
   (c) $2(m + p)\,|\mathbf{b} \times \mathbf{d}|$ (d) $\frac{1}{2}\,|\,m - p|\,|\mathbf{b} \times \mathbf{d}|$

80. If $\mathbf{u} = \mathbf{a} + \mathbf{b}$ and $\mathbf{v} = \mathbf{a} - \mathbf{b}$ and $|\mathbf{a}| = |\mathbf{b}| = k$ then $|\mathbf{u} \times \mathbf{v}|$ is equal to
   (a) $2(k^2 - (\mathbf{a}\,.\,\mathbf{b})^2)$ (b) $2(k^4 - (\mathbf{a}\,.\,\mathbf{b})^2)^{1/2}$
   (c) $(k^4 + (\mathbf{a}\,.\,\mathbf{b})^2)^{1/2}$ (d) $(k^4 + (\mathbf{a}\,.\,\mathbf{b})^2)^{1/2}$

Previous Years' AIEEE/JEE Main Questions

1. If $|\mathbf{a}| = 4$, $|\mathbf{b}| = 2$ and the angle between **a** and **b** is $\frac{\pi}{6}$ then $(\mathbf{a} \times \mathbf{b})^2$ is equal to
   (a) 48 (b) 16
   (c) 9 (d) none of these **[2002]**

2. If **a**, **b**, **c** are vectors such that $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 4$ then $[\mathbf{a}\times\mathbf{b}\quad \mathbf{b}\times\mathbf{c}\quad \mathbf{c}\times\mathbf{a}] =$
   (a) 16 (b) 64
   (c) 4 (d) 8 **[2002]**

3. If **a**, **b**, **c** are vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ and $|\mathbf{a}| = 7$, $|\mathbf{b}| = 5$, $|\mathbf{c}| = 3$ then the angle between vector **b** and **c** is
   (a) 60° (b) 30°
   (c) 45° (d) 90° **[2002]**

4. If $|\mathbf{a}| = 5$, $|\mathbf{b}| = 4$, $|\mathbf{c}| = 3$ then the value of $(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a})$ given that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$
   (a) 25 (b) 50
   (c) –25 (d) –50 **[2002]**

5. $\mathbf{a} = 3\mathbf{i} - 5\mathbf{j}$ and $\mathbf{b} = 6\mathbf{i} + 3\mathbf{j}$ are two vectors and $\mathbf{c}$ is a vector such that $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ then $|\mathbf{a}|:|\mathbf{b}|:|\mathbf{c}|$
   (a) $\sqrt{34} : \sqrt{45} : \sqrt{39}$ (b) $\sqrt{34} : \sqrt{45} : 39$
   (c) 34 : 39 : 45 (d) 39 : 35 : 34 **[2002]**

6. Let $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$ and $\boldsymbol{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. If $\mathbf{n}$ is a unit vector such that $\mathbf{u}.\mathbf{n} = 0$ then $|\mathbf{w}.\mathbf{n}|$ is equal to
   (a) 1 (b) 2
   (c) 3 (d) 0 **[2003]**

7. A particle acted by constant forces $4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ is displaced from the point $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ to the point $5\mathbf{i} + 4\mathbf{j} + \mathbf{k}$. The total work done by the forces is
   (a) 30 units (b) 40 units
   (c) 50 units (d) 20 units **[2003, 04]**

8. The vector $\mathbf{AB} = 3\mathbf{i} + 4\mathbf{k}$ and $\mathbf{AC} = 5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ are the sides of a triangle ABC. The length of the median through A is
   (a) $\sqrt{72}$ (b) $\sqrt{33}$
   (c) $\sqrt{288}$ (d) $\sqrt{18}$ **[2003]**

9. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar vectors and λ is a real number, then the vectors $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}$, $\lambda\mathbf{b} + 4\mathbf{c}$ and $(2\lambda - 1)\mathbf{c}$ are non-coplanar for
   (a) all except two values of λ
   (b) all except one value of λ
   (c) for all values of λ
   (d) no value of λ **[2004]**

10. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be such that $|\mathbf{u}| = 1$, $|\mathbf{v}| = 2$, $|\mathbf{w}| = 3$. If the projection $\mathbf{v}$ along $\mathbf{u}$ is equal to that of $\mathbf{w}$ along $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{w}$ are perpendicular to each other than $|\mathbf{u} - \mathbf{v} + \mathbf{w}|$ equals
    (a) $\sqrt{14}$ (b) $\sqrt{7}$
    (c) 2 (d) 14 **[2004]**

11. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be non-zero vectors such that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$. If θ is acute angle between the vectors $\mathbf{b}$ and $\mathbf{c}$, then $\sin\theta$ equals
    (a) 2/3 (b) $\sqrt{2}/3$
    (c) 1/3 (d) $2\sqrt{2}/3$ **[2004]**

12. If C is the mid point of AB and P is any point outside AB, then
    (a) $\mathbf{PA} + \mathbf{PB} + 2\mathbf{PC} = \mathbf{0}$
    (b) $\mathbf{PA} + \mathbf{PB} + \mathbf{PC} = \mathbf{0}$
    (c) $\mathbf{PA} + \mathbf{PB} = 2\mathbf{PC}$
    (d) $\mathbf{PA} + \mathbf{PB} = \mathbf{PC}$ **[2005]**

13. For any vector $\mathbf{a}$, the value of $(\mathbf{a} \times \mathbf{i})^2 + (\mathbf{a} \times \mathbf{j})^2 + (\mathbf{a} \times \mathbf{k})^2$ is equal to
    (a) $2\mathbf{a}^2$ (b) $4\mathbf{a}^2$
    (c) $3\mathbf{a}^2$ (d) $\mathbf{a}^2$ **[2005]**

14. Let a, b and c be distinct non-negative numbers. If the vectors $a\mathbf{i} + a\mathbf{j} + c\mathbf{k}$, $\mathbf{i} + \mathbf{k}$ and $c\mathbf{i} + c\mathbf{j} + b\mathbf{k}$ lie in a plane, then $\mathbf{c}$ is
    (a) equal to zero
    (b) the harmonic mean of a and b
    (c) the geometric mean of a and b
    (d) the arithmetic mean of a and b. **[2005]**

15. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar vectors and λ is a real number then
    $$[\lambda(\mathbf{a} + \mathbf{b})\ \lambda^2\mathbf{b}\ \lambda\mathbf{c}] = [\ \mathbf{a}\ \mathbf{b} + \mathbf{c}\ \mathbf{b}] \text{ for}$$
    (a) exactly three values of λ
    (b) exactly two values of λ
    (c) exactly one value of λ
    (d) no value of λ **[2005]**

16. Let $\mathbf{a} = \mathbf{i} - \mathbf{k}$, $\mathbf{b} = x\mathbf{i} + \mathbf{j} + (1 - x)\mathbf{k}$, $\mathbf{c} = y\mathbf{i} + x\mathbf{j} + (1 + x - y)\mathbf{k}$ then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ depends on
    (a) both x and y (b) neither on x nor on y
    (c) only y (d) only x **[2005]**

17. If $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are any three vectors such that $\mathbf{a}, \mathbf{b} \neq 0$, $\mathbf{b} \,.\, \mathbf{c} \neq \mathbf{0}$, then $\mathbf{a}$ and $\mathbf{c}$ are
    (a) parallel
    (b) inclined at an angle of $\pi/3$ between then
    (c) inclined at an angle of $\pi/6$ between them
    (d) perpendicular **[2006]**

18. The value of a, for which the points A, B, C with position vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ and $a\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ respectively are vertices of a right-angled triangle with $C = \pi/2$ are
    (a) 2 and – 1 (b) 2 and 1
    (c) – 2 and – 1 (d) – 2 and 1 **[2007]**

19. Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = \mathrm{x}\mathbf{i} + (x - 2)\mathbf{j} - \mathbf{k}$. If the vector $\mathbf{c}$ lies in the plane of $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{x}$ equals
    (a) 0 (b) 1
    (c) – 4 (d) – 2 **[2007]**

20. If $\mathbf{u}$ and $\mathbf{v}$ are unit vectors and θ is the angle between them, then $2\mathbf{u} \times 3\mathbf{v}$ is a unit vector for
    (a) exactly two values of θ
    (b) more than two value of θ
    (c) no value of θ
    (d) Exactly one value of θ **[2007]**

21. The vector $\mathbf{a} = \alpha\mathbf{i} + 2\mathbf{j} + \beta\mathbf{k}$ lies in the plane of the vectors $\mathbf{b} = \mathbf{i} + \mathbf{j}$ and $\mathbf{c} = \mathbf{j} + \mathbf{k}$ and bisects the angle between **b** and **c**. Then which one of the following gives possible values of α and β
    (a) $\alpha = 2, \beta = 2$ (b) $\alpha = 1, \beta = 2$
    (c) $\alpha = 2, \beta = 1$ (d) $\alpha = 1, \beta = 1$ **[2008]**
22. The non-zero vectors **a**, **b** and **c** are related by $\mathbf{a} = 8\mathbf{b}$ and $\mathbf{c} = -7\mathbf{b}$. Then the angle between **a** and **c** is
    (a) 0 (b) $\pi/4$
    (c) $\pi/2$ (d) π **[2008]**
23. If **u**, **v**, **w** are non-coplanar vectors and p, q are real numbers, then the equality
    $[3\mathbf{u}\ p\mathbf{v}\ p\mathbf{w}] - [p\mathbf{v}\ \mathbf{w}\ q\mathbf{u}] - [2\mathbf{w}\ q\mathbf{v}\ q\mathbf{u}]$
    holds for
    (a) more than two but not all values of (p, q)
    (b) all value of (p, q)
    (c) exactly one value of (p, q)
    (d) exactly two values of (p, q) **[2009]**
24. If the vectors $\mathbf{a} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \lambda\mathbf{i} + \mathbf{j} + \mu\mathbf{k}$ are mutually orthogonal, then $(\lambda, \mu) =$
    (a) $(-2, 3)$ (b) $(3, -2)$
    (c) $(-3, 2)$ (d) $(2, -3)$ **[2010]**
25. Let $\mathbf{a} = \mathbf{j} - \mathbf{k}$ and $\mathbf{c} = \mathbf{i} - \mathbf{j} - \mathbf{k}$. Then the vector **b** satisfying $\mathbf{a} \times \mathbf{b} + \mathbf{c} = \mathbf{0}$ and $\mathbf{a}\ .\ \mathbf{b} = 3$ is
    (a) $\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ (b) $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
    (c) $-\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ (d) $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ **[2010]**
26. If $\mathbf{a} = \frac{1}{\sqrt{10}}(3\mathbf{i} + \mathbf{k})$ and $\mathbf{b} = \frac{1}{7}(2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$, then the value of $(2\mathbf{a} - \mathbf{b})\ .\ [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})]$ is
    (a) 3 (b) –5
    (c) –3 (d) 5 **[2011]**
27. The vectors **a** and **b** are not perpendicular and **c** and **d** are vectors satisfying $\mathbf{b} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$ and $\mathbf{a}\ .\ \mathbf{d} = 0$. Then the vector **d** is equal to
    (a) $\mathbf{c} - \left(\frac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}\right)\mathbf{b}$ (b) $\mathbf{b} - \left(\frac{\mathbf{b}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}\right)\mathbf{c}$
    (c) $\mathbf{c} + \left(\frac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}\right)\mathbf{b}$ (d) $\mathbf{b} + \left(\frac{\mathbf{b}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}\right)\mathbf{c}$ **[2011]**
28. If the vectors $p\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} + q\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + r\mathbf{k}$ $(p \neq q \neq r \neq 1)$ are coplanar, then the value of $pqr - (p + q + r)$ is
    (a) 2 (b) 0
    (c) –1 (d) – 2 **[2011]**
29. Let **a**, **b**, **c** be three non-zero vectors which are pairwise non-collinear. If $\mathbf{a} + 3\mathbf{b}$ is collinear with **c** and $\mathbf{b} + 2\mathbf{c}$ is collinear with **a**, then $\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}$ is
    (a) **a** (b) **c**
    (c) **0** (d) $\mathbf{a} + \mathbf{c}$ **[2011]**
30. Let **a** and **b** be two unit vectors. If the vectors $\mathbf{c} = \mathbf{a} + 2\mathbf{b}$ and $\mathbf{d} = 5\mathbf{a} - 4\mathbf{b}$ are perpendicular to each other, then the angle between **a** and **b** is
    (a) π/2 (b) π/3
    (c) π/4 (d) π/6 **[2012]**
31. Let ABCD be a parallelogram such that $\mathbf{AB} = \mathbf{q}$, $\mathbf{AD} = \mathbf{p}$ and ∠BAD be an acute. If **r** is the vector that coincides with the attitude directed from the vertex B to the side AD, then **r** is given by
    (a) $\mathbf{r} = -\mathbf{q} + \left(\frac{\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}\cdot\mathbf{p}}\right)\mathbf{p}$
    (b) $\mathbf{r} = \mathbf{q} - \left(\frac{\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}\cdot\mathbf{p}}\right)\mathbf{p}$
    (c) $\mathbf{r} = -3\mathbf{q} + 3\left(\frac{\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}\cdot\mathbf{p}}\right)\mathbf{p}$
    (d) $\mathbf{r} = 3\mathbf{q} - 3\left(\frac{\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}\cdot\mathbf{p}}\right)\mathbf{p}$ **[2012]**
32. If the vectors $\mathbf{AB} = 3\mathbf{i} + 4\mathbf{k}$ and $\mathbf{AC} = 5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ are the sides of a triangle ABC, then the length of the median through A is
    (a) $\sqrt{72}$ (b) $\sqrt{33}$
    (c) $\sqrt{45}$ (d) $\sqrt{18}$ **[2013]**
33. If **a** and **b** are non-collinear vectors, then the value of α for which the vectors $\mathbf{u} = (\alpha - 2)\ \mathbf{a} + \mathbf{b}$ and $\mathbf{v} = (2 + 3\alpha)\ \mathbf{a} - 3\mathbf{b}$ are collinear is:
    (a) $\frac{3}{2}$ (b) $\frac{2}{3}$
    (c) $-\frac{3}{2}$ (d) $-\frac{2}{3}$ **[2013, online]**
34. Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ be three vectors. A vector of the type $\mathbf{b} + \lambda\mathbf{c}$ for some scalar λ, whose projection on **a** is of magnitude $\sqrt{\frac{2}{3}}$ is
    (a) $2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ (b) $2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$
    (c) $2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ (d) $2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ **[2013, online]**
35. If $[\mathbf{a}\times\mathbf{b}\quad \mathbf{b}\times\mathbf{c}\quad \mathbf{c}\times\mathbf{a}] = \lambda\ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$ then λ is equal to
    (a) 2 (b) 3
    (c) 0 (d) 1 **[2014]**

36. If $|\mathbf{a}| = 2$, $|\mathbf{b}| = 3$ and $|2\mathbf{a} - \mathbf{b}| = 5$, then $|2\mathbf{a} + \mathbf{b}|$ equals
(a) 17 (b) 7
(c) 5 (d) 1 **[2014, online]**

37. If $|\mathbf{c}|^2 = 60$ and $\mathbf{c} \times (\mathbf{i} + \mathbf{j} + 5\mathbf{k}) = \mathbf{0}$, then a value of $\mathbf{c}\cdot(-7\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ is:
(a) $4\sqrt{2}$ (b) 12
(c) 24 (d) $12\sqrt{2}$ **[2014, online]**

38. If $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are three unit vectors in three dimensional space, then the minimum value of $|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{y} + \mathbf{z}|^2 + |\mathbf{z} + \mathbf{x}|^2$ is
(a) $\frac{3}{2}$ (b) 3
(c) $3\sqrt{3}$ (d) 6 **[2014, online]**

39. If $\mathbf{x} = 3\mathbf{i} - 6\mathbf{j} - \mathbf{k}$, $\mathbf{y} = \mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and $\mathbf{z} = 3\mathbf{i} - 4\mathbf{j} - 12\mathbf{k}$, then the magnitude of the projection of $\mathbf{x} \times \mathbf{y}$ on $\mathbf{z}$ is
(a) 12 (b) 15
(c) 14 (d) 13 **[2014, online]**

40. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non zero vectors such that no two of them are collinear and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \frac{1}{3}|\mathbf{b}|\,|\mathbf{c}|\,\mathbf{a}$. If θ is the angle between vectors $\mathbf{b}$ and $\mathbf{c}$, then a value of $\sin\theta$ is:
(a) $\frac{2\sqrt{2}}{3}$ (b) $\frac{-\sqrt{2}}{3}$
(c) $\frac{2}{3}$ (d) $\frac{-2\sqrt{3}}{3}$ **[2015]**

41. In a parallelogram *ABCD*, $|\mathbf{AB}| = a$, $|\mathbf{AD}| = b$ and $|\mathbf{AC}| = c$, then $\mathbf{DB}.\ \mathbf{AB}$ has the value: (choices modified from original)
(a) $\frac{1}{2}\left(3a^2 + b^2 - c^2\right)$ (b) $\frac{1}{4}\left(a^2 + b^2 - c^2\right)$
(c) $\frac{1}{3}\left(b^2 + c^2 - a^2\right)$ (d) $\frac{1}{2}\left(a^2 + b^2 + c^2\right)$
[2015, online]

42. Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors such that $|\mathbf{a} + \mathbf{b}| = \sqrt{3}$. If
$\mathbf{c} = \mathbf{a} + 2\mathbf{b} + 3(\mathbf{a} \times \mathbf{b})$, then $2|\mathbf{c}|$ is equal to:
(a) $\sqrt{55}$ (b) $\sqrt{51}$
(c) $\sqrt{43}$ (d) $\sqrt{37}$ **[2015, online]**

43. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three unit vectors such that
$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \frac{\sqrt{3}}{2}(\mathbf{b} \times \mathbf{c})$.
If $\mathbf{b}$ is not parallel to $\mathbf{c}$ then the angle between $\mathbf{a}$ and $\mathbf{b}$ is:
(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{2}$
(c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$ **[2016]**

44. In a triangle ABC, right angled at the vertex A, if the position vectors of A, B and C are respectively $3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$, $-\boldsymbol{i} + 3\boldsymbol{j} + p\boldsymbol{k}$ and $5\boldsymbol{i} + q\boldsymbol{j} - 4\boldsymbol{k}$, then the point (p, q) lies on a line:
(a) making an obtuse angle with the positive direction of x-axis.
(b) parallel to x-axis.
(c) parallel to y-axis.
(d) making an acute angle with the positive direction of x-axis. **[2016, online]**

45. Let ABC be a triangle whose circumcentre is at P. If the positive vectors of A, B, C and P are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ respectively, then the position vector of the orthocentre of this triangle, is:
(a) $-\frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ (b) $\mathbf{a} + \mathbf{b} + \mathbf{c}$
(c) $\frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ (d) $\mathbf{0}$ **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors such that
$\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$
If $|\mathbf{u}| = 3$, $|\mathbf{v}| = 4$ and $|\mathbf{w}| = 5$, then
$\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{u}$ is
(a) –25 (b) 0
(c) 25 (d) 47 **[2006]**

2. If $\mathbf{a}$ and $\mathbf{b}$ are two non-parallel vectors having equal magnitude, then the vector $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} \times \mathbf{b})$ is parallel to
(a) $\mathbf{b}$ (b) $\mathbf{a} - \mathbf{b}$
(c) $\mathbf{a} + \mathbf{b}$ (d) $\mathbf{a}$ **[2007]**

3. If numbers **a**, **b**, **c** are distinct and non-negative. If three vectors $a\mathbf{i} + a\mathbf{j} + c\mathbf{k}$, $c\mathbf{i} + c\mathbf{j} + b\mathbf{k}$ and $\mathbf{i} + \mathbf{k}$ are coplanar, then c is
(a) geometric mean of a, b
(b) harmonic mean of a, b
(c) equal to zero
(d) arithmetic mean of a, b **[2007]**

4. Let **x**, **y** and **z** be unit vectors such that
$|\mathbf{x} - \mathbf{y}|^2 + |\mathbf{y} - \mathbf{z}|^2 + |\mathbf{z} - \mathbf{x}|^2 = 9$
Then $|\mathbf{x} + \mathbf{y} - \mathbf{z}|^2 - 4\mathbf{x}.\mathbf{y} =$
(a) 1 (b) 4
(c) 6 (d) 8 **[2008]**

5. If **a**, **b** and **c** are three unit vectors satisfying $2\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{c} = \mathbf{0}$ then the acute angle between **a** and **b** is
(a) $\frac{\pi}{5}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$ **[2009]**

6. If $\mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{c} = \mathbf{j} + 2\mathbf{k}$ and **a** is a unit vector, then the maximum value of the scalar triple product [**a b c**] is
(a) $\sqrt{30}$ (b) $\sqrt{29}$
(c) $\sqrt{26}$ (d) $\sqrt{60}$ **[2009]**

7. **Statement-1:** If **a** and **b** are two vectors such that $|\mathbf{a}| = 2$, $|\mathbf{b}| = 3$, $|2\mathbf{a} - \mathbf{b}| = 5$ then $|2\mathbf{a} + \mathbf{b}| = 5$
Statement-2: For any two vectors **c** and **d**, $|\mathbf{c} - \mathbf{d}| = |\mathbf{c} + \mathbf{d}|$ **[2010]**

8. If **a**, **b** and **c** are non-zero vectors such that $\mathbf{a} \times \mathbf{b} = \mathbf{c}$, $\mathbf{b} \times \mathbf{c} = \mathbf{a}$ and $\mathbf{c} \times \mathbf{a} = \mathbf{b}$ then
(a) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$
(b) $\mathbf{a} = \mathbf{b} = \mathbf{c}$
(c) $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$
(d) $|\mathbf{a}| + |\mathbf{b}| - |\mathbf{c}| = 0$ **[2010]**

9. Let $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = 2\mathbf{b} + 10\mathbf{a}$ and $\mathbf{OC} = \mathbf{b}$ where O is the origin. If p is the area of the quadrilateral $OABC$ and q is the area of the parallelogram with OA and OC as adjacent sides then p is equal to
(a) q^6 (b) $6q$
(c) $q/6$ (d) $6 - q$ **[2011]**

10. If **a** and **b** are two vectors such that $2\mathbf{a} + \mathbf{b} = \mathbf{e}_1$ and $\mathbf{a} + 2\mathbf{b} = \mathbf{e}_2$, where $\mathbf{e}_1 = (1, 1, 1)$ and $\mathbf{e}_2 = (1, 1, -1)$, then the angle between **a** and **b** is
(a) $\cos^{-1}\left(\frac{7}{9}\right)$ (b) $\cos^{-1}\left(\frac{7}{11}\right)$
(c) $\cos^{-1}\left(-\frac{7}{11}\right)$ (d) $\cos^{-1}\left(-\frac{7}{9}\right)$ **[2012]**

11. If **u**, **v**, **w** are unit vectors satisfying $2\mathbf{u} + 2\mathbf{v} + 3\mathbf{w} = \mathbf{0}$, then $|\mathbf{u} - \mathbf{v}|$ equals
(a) $\frac{7}{4}$ (b) $\frac{\sqrt{5}}{2}$
(c) $\frac{\sqrt{7}}{2}$ (d) $\frac{5}{4}$ **[2012]**

12. Let $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 3\mathbf{k}$. If **u** is a unit vector, then the maximum value of the scalar triple product [**u v w**] is
(a) $\sqrt{6}$ (b) $\sqrt{10} + \sqrt{6}$
(c) $\sqrt{59}$ (d) $\sqrt{60}$ **[2013]**

13. Unit vectors **a**, **b**, **c** are coplanar. A unit vector **d** is perpendicular to them. If
$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \frac{1}{6}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$
and the angle between **a** and **b** is 30°, then **c** is/are
(a) $\pm\frac{1}{3}(-\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$ (b) $\frac{1}{3}(2\mathbf{i} + \mathbf{j} - \mathbf{k})$
(c) $\pm\frac{1}{3}(-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ (d) $\frac{1}{3}(-2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ **[2014]**

14. Let $\mathbf{x} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$ and $\mathbf{y} = \boldsymbol{i} + \boldsymbol{j}$. If **z** is a vector such that $\mathbf{x}.\mathbf{z} = |\mathbf{z}|$, $|\mathbf{z} - \mathbf{x}| = 2\sqrt{2}$ and the angle between $\mathbf{x} \times \mathbf{y}$ and **z** is 30°, then the magnitude of the vector $(\mathbf{x} \times \mathbf{y}) \times \mathbf{z}$ is:
(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{3}{2}$
(c) $\frac{1}{2}$ (d) $\frac{3\sqrt{3}}{2}$ **[2015]**

15. From a point A with position vector $p(\mathbf{i} + \mathbf{j} + \mathbf{k})$, AB and AC are drawn perpendicular to the lines $\mathbf{r} = \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = -\mathbf{k} + \mu(\mathbf{i} - \mathbf{j})$ respectively. A value of p is equal to
(a) –1 (b) $\sqrt{2}$
(c) 2 (d) –2 **[2016]**

Answers

Concept Based

1. (d)	**2.** (b)	**3.** (d)	**4.** (b)
5. (d)	**6.** (b)	**7.** (c)	**8.** (d)
9. (a)	**10.** (b)		

Level 1

11. (a)	**12.** (b)	**13.** (a)	**14.** (b)
15. (c)	**16.** (a)	**17.** (d)	**18.** (a)
19. (a)	**20.** (b)	**21.** (b)	**22.** (b)

23. (a)	**24.** (a)	**25.** (b)	**26.** (b)
27. (b)	**28.** (b)	**29.** (b)	**30.** (c)
31. (c)	**32.** (b)	**33.** (d)	**34.** (b)
35. (a)	**36.** (b)	**37.** (d)	**38.** (a)
39. (c)	**40.** (a)	**41.** (a)	**42.** (a)
43. (a)	**44.** (c)	**45.** (d)	**46.** (b)
47. (a)	**48.** (c)		

Level 2

49. (d)	**50.** (a)	**51.** (d)	**52.** (a)
53. (d)	**54.** (b)	**55.** (d)	**56.** (c)
57. (d)	**58.** (d)	**59.** (b)	**60.** (c)
61. (d)	**62.** (b)	**63.** (c)	**64.** (a)
65. (c)	**66.** (b)	**67.** (c)	**68.** (c)
69. (d)	**70.** (b)	**71.** (c)	**72.** (c)
73. (a)	**74.** (b)	**75.** (a)	**76.** (b)
77. (c)	**78.** (d)	**79.** (a)	**80.** (b)

Previous Years' AIEEE/JEE Main Questions

1. (b)	**2.** (a)	**3.** (a)	**4.** (c)
5. (b)	**6.** (c)	**7.** (b)	**8.** (b)
9. (a)	**10.** (a)	**11.** (d)	**12.** (c)
13. (a)	**14.** (c)	**15.** (d)	**16.** (b)
17. (a)	**18.** (b)	**19.** (d)	**20.** (d)
21. (d)	**22.** (d)	**23.** (c)	**24.** (c)
25. (c)	**26.** (b)	**27.** (a)	**28.** (d)
29. (c)	**30.** (b)	**31.** (a)	**32.** (b)
33. (b)	**34.** (b)	**35.** (d)	**36.** (c)
37. (d)	**38.** (b)	**39.** (c)	**40.** (a)
41. (a)	**42.** (a)	**43.** (d)	**44.** (d)
45. (c)			

Previous Years' B-Architecture Entrance Examination Questions

1. (a)	**2.** (c)	**3.** (a)	**4.** (c)
5. (d)	**6.** (a)	**7.** (c)	**8.** (c)
9. (b)	**10.** (c)	**11.** (c)	**12.** (c)
13. (c)	**14.** (b)	**15.** (a) (b) (c) (d)	

Hints and Solutions

Concept Based

1. Let O be the origin of reference. Then the position vector of A, B, M are **OA**, **OB** and **OM** respectively

so $\mathbf{OM} = \frac{1}{2}(\mathbf{OA}+\mathbf{OB})$

2. $(3\mathbf{i} + 4\mathbf{j})\cdot(2\mathbf{j} - 5\mathbf{k}) = 8$, $|3\mathbf{i} + 4\mathbf{j}| = 5$,

$|2\mathbf{j} - 5\mathbf{k}| = \sqrt{29}$. Hence the required angle is

$$\cos^{-1}\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \cos^{-1}\frac{8}{5\sqrt{29}}.$$

3. $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b} = (\lambda + \mu)\mathbf{i} + (\lambda + 2\mu)\mathbf{j} + \lambda\mathbf{k}$

$$0 = \mathbf{a}.\mathbf{c} = \lambda\,|\mathbf{a}|^2 + \mu\,\mathbf{a}.\mathbf{b}$$
$$= 3\lambda + 3\mu \quad \Rightarrow \quad \lambda + \mu = 0$$

Also $(\lambda + \mu)^2 + (\lambda + 2\mu)^2 + \lambda^2 = 1$

$$\Rightarrow \quad 2\lambda^2 = 1 \quad \Rightarrow \quad \lambda = \pm\frac{1}{\sqrt{2}}$$

Hence $\mathbf{c} = -\lambda\mathbf{j} + \lambda\mathbf{k} = \pm\frac{1}{\sqrt{2}}(-\mathbf{j}+\mathbf{k})$

4. $\mathbf{b} = \lambda\mathbf{a} = \lambda(2\mathbf{i} + \mathbf{j} - \mathbf{k})$. so

$\lambda\,(2\mathbf{i} + \mathbf{j} - \mathbf{k})\cdot(2\mathbf{i} + \mathbf{j} - \mathbf{k}) = 2$

$$\Rightarrow \quad 6\lambda = 2 \quad \Rightarrow \quad \lambda = \frac{1}{3}$$

Hence $\mathbf{b} = \frac{1}{3}(2\mathbf{i}+\mathbf{j}-\mathbf{k})$

5. $$\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & -1\\ 2 & 1 & 1\end{vmatrix} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$$

so $$\text{Proj}_{\mathbf{w}}\,\mathbf{u}\times\mathbf{v} = \frac{(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}$$
$$= \frac{2-3-2}{6}(\mathbf{i}+\mathbf{j}+2\mathbf{k})$$
$$= -\frac{1}{2}(\mathbf{i}+\mathbf{j}+2\mathbf{k})$$

$$|\text{Proj}_{\mathbf{w}}\,\mathbf{u}\times\mathbf{v}|^2 = \frac{1}{4}\cdot 6 = \frac{3}{2}$$

6. $\mathbf{u} = t\mathbf{v} \Rightarrow (\lambda + 2)\mathbf{a} + \mathbf{b} = t(1 + 4\lambda)\mathbf{a} - 2t\,\mathbf{b}$

$\Rightarrow \quad (\lambda + 2 - t - 4t\,\lambda)\mathbf{a} = (-2t - 1)\mathbf{b}$

Since **a** and **b** are non-collinear, so

$t = -\frac{1}{2}$ and $\lambda + 2 - t - 4 + \lambda = 0$

$$\Rightarrow \quad \lambda + 2 - \frac{1}{2} - 2\lambda = 0$$

$$\Rightarrow \quad -\lambda + \frac{3}{2} = 0 \quad \Rightarrow \quad \lambda = \frac{3}{2}.$$

7. $\mathbf{AB} = -\mathbf{i} + \mathbf{j}$, $\mathbf{BC} = \mathbf{i} + \mathbf{k}$

Area of triangle $ABC = \frac{1}{2}|\mathbf{AB}\times\mathbf{BC}|$

$$= \frac{1}{2}\left|\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 1 & 0\\ 1 & 0 & 1\end{vmatrix}\right| = \frac{1}{2}|\mathbf{i}+\mathbf{j}-\mathbf{k}| = \frac{\sqrt{3}}{2}$$

8. $(3\mathbf{i} + 4\mathbf{j}) \times (\mathbf{i} - \mathbf{j} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 0 \\ 1 & -1 & 1 \end{vmatrix}$

$= 4\mathbf{i} - 3\mathbf{j} - 7\mathbf{k}$

The length of this vector $\sqrt{16+9+49} = \sqrt{74}$.

A unit vector perpendicular to given vectors is

$\frac{1}{\sqrt{74}}(4\mathbf{i} - 3\mathbf{j} - 7\mathbf{k})$

9. $\begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \times 2 - 2 \times (-2) + 3 \times 2$

$= 2 + 4 + 6 = 12$

10. $\mathbf{u} = (\mathrm{i} - \mathrm{j} + \mathrm{k}) \times (2\mathbf{i} - 3\mathbf{j} - \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & -3 & -1 \end{vmatrix}$

$= 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

$\mathbf{v} = (-3\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (2\mathbf{j} + \mathbf{k})$

$= -\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$

$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & -1 \\ -1 & 3 & -6 \end{vmatrix} = -15\mathbf{i} + 25\mathbf{j} + 15\mathbf{k}$

Level 1

11. $0 = (\mathbf{a} + \mu\mathbf{b}) \,.\, (\mathbf{a} - \mu\mathbf{b}) = |\,\mathbf{a}\,|^2 - \mu^2\,|\,\mathbf{b}\,|^2$

$\Rightarrow 9 - 16\mu^2 = 0 \quad \Rightarrow \quad \mu = \pm 3/4.$

12. $0 = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & \alpha \\ 3 & -4 & 5 \end{vmatrix} \quad \Rightarrow \quad 15 + 5\alpha = 0$

$\Rightarrow \alpha = -3.$

13. If $\mathbf{p}$ and $\mathbf{q}$ are adjacent sides of the parallelogram, their sum gives one of the diagonals and their difference gives the other that is

$\mathbf{p} + \mathbf{q} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{p} - \mathbf{q} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$

$\Rightarrow \mathbf{p} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{q} = \mathbf{i} + 2\mathbf{k} - 3\mathbf{k}$

Required area

$= |\,\mathbf{p} \times \mathbf{q}\,| = |\,\mathbf{i}(3-2) - \mathbf{j}(-6-1) + \mathbf{k}(4+1)\,|$

$= \sqrt{1^2 + 7^2 + 5^2} = 5\sqrt{3}.$

14. Put $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The given equation implies $(y - 2z)\mathbf{i} + (z - x)\mathbf{j} + (2x - y)\mathbf{k} = \mathbf{i} - \mathbf{k}$

That is $y - 2z = 1$, $z - x = 0$ and $2x - y = -1$, from which we get $x = z = t$ (say), so that $y = 1 + 2t$. Substituting these, we get

$\mathbf{r} = t\mathbf{i} + (1 + 2t)\mathbf{j} + t\mathbf{k} = \mathbf{j} + t(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

15. $|\,\mathbf{a}\,| = \sqrt{2} = |\,\mathbf{b}\,| = |\,\mathbf{c}\,|$. Let $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, since $\mathbf{c}$ makes an obtuse angle with $\mathbf{i}$, we must have $\mathbf{c} \,.\, \mathbf{i} = c_1 < 0$. It is also given that the angles between the vectors are equal i.e.,

$\cos^{-1}\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \cos^{-1}\frac{1}{2} = \cos^{-1}\frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}|\,|\mathbf{c}|} = \cos^{-1}\frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}|\,|\mathbf{c}|}$

But $\mathbf{a}.\,\mathbf{b} = 1$, $\mathbf{a} \,.\, \mathbf{c} = c_1 + c_2$ and $\mathbf{b} \,.\, \mathbf{c} = c_2 + c_3$

This gives $c_1 + c_2 = 1$ and $c_2 + c_3 = 1 \Rightarrow c_3 = c_1$ and $c_2 = 1 - c_1$. Putting in $c_1^2 + c_2^2 + c_3^2 = 2$, we get $c_1 = 1, -1/3$. Since $c_1 < 0$ so $c_1 = -1/3 = c_3$ and $c_2 = 4/3$. Hence

$\mathbf{c} = -\frac{1}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$

16. The two diagonals are

$\mathbf{AB} - \mathbf{BC} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{AB} + \mathbf{BC} = 2\mathbf{i} - 2\mathbf{j}$

These vectors have magnitudes 6 and $2\sqrt{2}$ respectively, and their dot product is 12. Therefore, the angle between them is

$\cos^{-1}\frac{12}{(6)(2\sqrt{2})} = \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}.$

17. $|\,\mathbf{a}\,| = |\,\mathbf{b}\,| = 1 = |\,\mathbf{c}\,|$. It is also given that the angle θ between $\mathbf{a}$ and $\mathbf{c}$ equals that between $\mathbf{b}$ and $\mathbf{c}$

$\frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}|\,|\mathbf{c}|} = \mathbf{a}\,.\,\mathbf{c} = \cos\theta = \frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}|\,|\mathbf{c}|} = \mathbf{b}\,.\,\mathbf{c}$

Since $\mathbf{a} \,.\, \mathbf{b} = 0$, we get from the given value of $\mathbf{c}$,

$\mathbf{a} \,.\, \mathbf{c} = \alpha\,\mathbf{a} \,.\, \mathrm{a} + \beta\,\mathbf{a} \,.\, \mathbf{b} + \mathrm{v}\mathbf{a} \,.\, (\mathbf{a} \times \mathbf{b}) = \alpha$

i.e., $\alpha = \cos\theta$, and similarly $\mathbf{b}.\mathbf{c} = \cos\theta = \beta$

$1 = \mathbf{c} \,.\, \mathbf{c} = 2\alpha^2 + \gamma^2\,|\,\mathbf{a} \times \mathbf{b}\,|^2 = 2\alpha^2 + \gamma^2[\,|\,\mathbf{a}\,|^2\,|\,\mathbf{b}\,|^2 - (\mathbf{a}.\mathbf{b})^2] = 2\alpha^2 + \gamma^2$

$\Rightarrow \gamma^2 = 1 - 2\alpha^2 = 1 - 2\cos^2\theta = -\cos 2\theta$

$\Rightarrow \alpha^2 = \beta^2 = \frac{1-\gamma^2}{2} = \frac{1+\cos 2\theta}{2}.$

18. $|\,\mathbf{a} - \mathbf{b}\,|^2 = \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a}.\mathbf{b} = 4\sin^2\theta$

$\Rightarrow \quad |\,\mathbf{a} - \mathbf{b}\,| = 2\,|\sin\theta\,|.$

$|\,\mathbf{a} - \mathbf{b}| < 1 \Rightarrow |\sin\theta\,| < 1/2$

$\Rightarrow \theta \in [0, \pi/6]$ or $\left(\frac{5\pi}{6}, \pi\right)$.

19. If θ is angle between $\mathbf{a}$ and $\mathbf{b}$, and ϕ the angle between $\mathbf{c}$ and $\mathbf{a} \times \mathbf{b}$. Then

$|\,(\mathbf{a} \times \mathbf{b}) \,.\, \mathbf{c}\,| = |\,\mathbf{a}\,|\,|\,\mathbf{b}\,|\,|\,\mathbf{c}\,|\sin\theta\cos\phi$

So we must have $\sin\theta\cos\phi = 1 \Rightarrow \sin\theta = 1$, $\cos\phi = 1 \Rightarrow \theta = \pi/2$, $\phi = 0 \Rightarrow \mathbf{a}$ and $\mathbf{b}$ are perpendicular so $\mathbf{a} \,.\, \mathbf{b} = 0$. $\phi = 0 \Rightarrow \mathbf{c}$ must be perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, so that $\mathbf{b} \,.\, \mathbf{c} = 0 = \mathbf{a} \,.\, \mathbf{c}$.

20. Let A, B, C and D be the points with the given position vectors. Then $\mathbf{AB} = -2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$, $\mathbf{AC} = 4\mathbf{i} + 5\mathbf{j} + (\lambda - 10)\mathbf{k}$ and $\mathbf{AD} = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

The volume of the tetrahedron is

$$\frac{1}{6}\ |(\mathbf{AB}\times\mathbf{AC})\ .\ \mathbf{AD}|$$

$$= \frac{1}{6}\left|\begin{vmatrix} -2 & 3 & -3 \\ 4 & 5 & \lambda-10 \\ 6 & 2 & -3 \end{vmatrix}\right|$$

$$= \frac{1}{6}\ |-88+22\lambda| = 11 \text{ (given)}$$

$\Rightarrow \quad \lambda = 1$ or 7

21. $\begin{vmatrix} x & x+1 & x+2 \\ x+3 & x+4 & x+5 \\ x+6 & x+7 & x+8 \end{vmatrix} = \begin{vmatrix} x & x+1 & x+2 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} = 0$

for all x so for $x < 0$.

22. Required vector = $K(2\mathbf{j} - \mathbf{k}) \times (-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$ = $K(-4\mathbf{i} + \mathbf{j} + 2\mathbf{k})$. The length of this vector is $\sqrt{21}\ |K|$ so $|K| = 1/\sqrt{3}$. Thus $K = \pm\ 1/\sqrt{3}$. But it will make obtuse angle with y-axis if $K = -1/\sqrt{3}$. Thus the required vector is $1/\sqrt{3}\,(4\mathbf{i} - \mathbf{j} - 2\mathbf{k})$

23. $|\mathbf{p}\times\mathbf{q}| = |(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})| = 2\ |\mathbf{a}\times\mathbf{b}|$
$4\ |\mathbf{a}|^2\ |\mathbf{b}|^2 \cos^2\theta = 4\ (K - |\mathbf{a}|^2\ |\mathbf{b}|^2\cos^2\theta)$
$\Rightarrow |\mathbf{a}|^2\ |\mathbf{b}|^2 = K \Rightarrow K = 16.$

24. The given conditions mean that **r** is perpendicular to all three vectors **a**, **b** and **c**. This is possible only if they are coplanar which means$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$

25. Let A, B and C be the three given vertices and D the fourth vertex, with the position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Since ABCD is a parallelogram, the diagonals AC and BD bisect each other so

$$\frac{(1+7)\mathbf{i}+(1+9)\mathbf{j}+(1+11)\mathbf{k}}{2}$$

$$= \frac{(1+x)\mathbf{i}+(3+y)\mathbf{j}+(5+z)\mathbf{k}}{2}$$

$\Rightarrow x = 7,\ y = 7$ and $z = 7$, so that D is the point $7\ (\mathbf{i} + \mathbf{j} + \mathbf{k})$.

26. Required volume = $(\mathbf{OA}\times\mathbf{OB})\ .\ \mathbf{OC}$

$$= \begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix} = 4.$$

27. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ then $|\mathbf{a}\times\mathbf{i}|^2 = a_2^2 + a_3^2$
Therefore $|\mathbf{a}\times\mathbf{i}|^2 + |\mathbf{a}\times\mathbf{j}|^2 + |\mathbf{a}\times\mathbf{k}|^2$
$= 2(a_1^2 + a_2^2 + a_3^2) = 2\mathbf{a}^2.$

28. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\ .\ \mathbf{c})\mathbf{b} - (\mathbf{a}\ .\ \mathbf{b})\mathbf{c}$ and $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\ .\ \mathbf{c})\mathbf{b} - (\mathbf{b}\ .\ \mathbf{c})\mathbf{a}$, so the given equality implies $(\mathbf{a}\ .\ \mathbf{b})\ \mathbf{c} = (\mathbf{b}\ .\ \mathbf{c})\ \mathbf{a}$ which means that **a** and **c** are collinear.

29. The given expression = $(\mathbf{i}\ .\ \mathbf{i})\mathbf{a} - (\mathbf{i}\ .\ \mathbf{a})\mathbf{i} + (\mathbf{j}\ .\ \mathbf{j})\mathbf{a} - (\mathbf{j}\ .\ \mathbf{a})\mathbf{j} + (\mathbf{k}\ .\ \mathbf{k})\mathbf{a} - (\mathbf{k}\ .\ \mathbf{a})\mathbf{k}$
$= 3\mathbf{a} - [(\mathbf{i}\ .\ \mathbf{a})\mathbf{i} + (\mathbf{j}\ .\ \mathbf{a})\mathbf{j} + (\mathbf{k}\ .\ \mathbf{a})\mathbf{k}]$
$= 3\mathbf{a} - \mathbf{a} = 2\mathbf{a}$

30. $[\ \mathbf{a}\times\mathbf{b}\quad \mathbf{b}\times\mathbf{c}\quad \mathbf{c}\times\mathbf{a}]$
$= ((\mathbf{a}\times\mathbf{b})\times(\mathbf{b}\times\mathbf{c})).\ (\mathbf{c}\times\mathbf{a})$
$= ((\mathbf{a}\times\mathbf{b}\ .\ \mathbf{c})\mathbf{b} - ((\mathbf{a}\times\mathbf{b})\ .\ \mathbf{b})\ \mathbf{c}).\ (\mathbf{c}\times\mathbf{a})$
$= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\ (\mathbf{b}\ .\ \mathbf{c}\times\mathbf{a}) = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2.$

31. A vector **c** perpendicular to z-axis is of the form $c_1\ \mathbf{i} + c_2\ \mathbf{j}$. According to given conditions $3c_1 - c_2 = 9$ and $c_1 + 2c_2 = -4$
$\Rightarrow c_2 = 2$ and $c_2 = -3$. Thus required point is (2, – 3, 0).

32. Required area = $|(\mathbf{a} + 3\mathbf{b})\times(3\mathbf{a}+\mathbf{b})|$
$= |9(\mathbf{b}\times\mathbf{a}) + \mathbf{a}\times\mathbf{b}| = 8\ |\mathbf{a}\times\mathbf{b}| =$
$8\ |\mathbf{a}|\ |\mathbf{b}|\sin\pi/6 = 4.$

33. $\begin{vmatrix} x & 5 & 7 \\ 1 & 1 & -1 \\ 1 & 2 & 2 \end{vmatrix} = 0 \Rightarrow x = 2.$

34. $\mathbf{a}\ .\ (\mathbf{b}\times\mathbf{c}) = -\ \mathbf{a}\ .\ (\mathbf{c}\times\mathbf{b})$ so $\mathbf{a}\ .\ (\mathbf{c}\times\mathbf{b}) = -3.$

35. If $\mathbf{b} = b_1\ \mathbf{i} + b_2\ \mathbf{j} + b_3\ \mathbf{k}$, we have the following relations $2b_1 + 2b_2 + b_3 = 14$ and $\mathbf{a}\times\mathbf{b} = (2b_3 - b_2)\mathbf{i} + (b_1 - 2b_3)\mathbf{d} + 2(b_2 - b_1)\mathbf{k}$. So $2b_3 - b_2 = 3$, $1 = b_1 - 2b_3$, $b_2 - b_1 = -4$. Solving $b_1 = 5$, $b_2 = 1$, $b_3 = 2$.

36. Let **AB** = **a**, **BC** = **b**,
then **FC** = 2**a**,
AD = 2**b**.
Also, **DC** = **AC** – **AD**
= **a** + **b** – 2**b**
= **a** – **b**
$\Rightarrow$ **EB** = 2 **DC** = 2(**a** – **b**)
Now, **AD** + **EB**+ **FC**
= 2**b** + 2(**a** – **b**) + 2**a** = 4**a** = 4 **AB** = 4 **ED**
$\therefore \quad \lambda = 4$

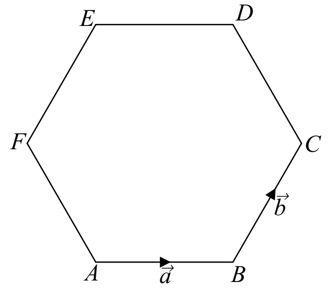

Fig. 22.6

37. Let N_1 be a vector normal to plane determined by vectors **i**, **i** + **k** and N_2 be a vector normal to the plane determined by vectors **i** – **j**, **i** + **k** we have
$\mathbf{N}_1 = \mathbf{i}\times(\mathbf{i}+\mathbf{j}) = \mathbf{k}$

and $\mathbf{N}_2 = (\mathbf{i} - \mathbf{j}) \times (\mathbf{i} + \mathbf{k})$
$= -\mathbf{i} - \mathbf{j} + \mathbf{k}$

Note that **a** is parallel to $\mathbf{N}_1 \times \mathbf{N}_2 = \mathbf{i} - \mathbf{j}$

If θ is the angle between **a** and $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, then

$$\cos\theta = \pm \frac{1+2}{\sqrt{2}\sqrt{1+4+4}} = \pm \frac{1}{\sqrt{2}}$$

Thus, θ may be taken as $\pi/4$.

38. Let M be the mid point of PR, then position vector of M is

$$\frac{1}{2}(-2\mathbf{i} - \mathbf{j} + 3\mathbf{i} + 3\mathbf{j}) = \frac{1}{2}\mathbf{i} + \mathbf{j}$$

Let N be the mid point of QS, then position vector of N is

$$\frac{1}{2}(4\mathbf{i} - 3\mathbf{i} + 2\mathbf{j}) = \frac{1}{2}\mathbf{i} + \mathbf{j}$$

Thus, $PQRS$ is a parallelogram.

Next $\quad PQ = |\mathbf{PQ}| = |4\mathbf{i} - (-2\mathbf{i} - \mathbf{j})|$

$= |6\mathbf{i} + \mathbf{j}| = \sqrt{37}$

and $\quad QR = |\mathbf{QR}| = |3\mathbf{i} + 3\mathbf{j} - 4\mathbf{i}| = |-\mathbf{i} + 3\mathbf{j}|$

$= \sqrt{1+9} = \sqrt{10}$

Thus, $PQRS$ is not a rhombus

Also, $\quad PR = |\mathbf{PR}| = |3\mathbf{i} + 3\mathbf{j} - (-2\mathbf{i} - \mathbf{j})|$

$= |5\mathbf{i} + 4\mathbf{j}| = \sqrt{25+16} = 41$

and $\quad OS = |\mathbf{OS}| = |-3\mathbf{i} + 2\mathbf{j} - (-4\mathbf{i})|$

$= |\mathbf{i} + 2\mathbf{j}| = \sqrt{1+4} = \sqrt{5}$.

$\therefore$ $PQRS$ is not a rectangle.

Thus, $PQRS$ is a parallelogram which is neither a rhombus nor a rectangle.

39. Let θ_1 be the angle between **a** and **b** and θ_2 be the angle between **c** and **d**.

Now,

$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta_1\, \mathbf{n}_1 = \sin\theta_1\, \mathbf{n}_1$

and $\quad \mathbf{c} \times \mathbf{b} = |\mathbf{c}|\,|\mathbf{d}| \sin\theta_2\, \mathbf{n}_2 = \sin\theta_2\, \mathbf{n}_2$

where $\mathbf{n}_1$ is a unit vector parallelogram to the plane of **a** and **b**; and $\mathbf{n}_2$ is a unit vector perpendicular to the plane of **c** and **d**.

As $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = 1$, we get

$(\sin\theta_1)(\sin\theta_2)\, \mathbf{n}_1 \cdot \mathbf{n}_2 = 1$

$(\sin\theta_1)(\sin\theta_2)\, |\mathbf{n}_1|\,|\mathbf{n}_2| \cos\phi = 1$

where ϕ is angle between $\mathbf{n}_1$ and $\mathbf{n}_2$.

$\Rightarrow$ $(\sin\theta_1)(\sin\theta_2) = \cos\phi = 1$

$\Rightarrow$ $\theta_1 = \pi/2$, $\theta_2 = \pi/2$ and $\phi = 0$

As $\phi = 0$ we get $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel.

Therefore, **a**, **b**, **c** and **d** are coplanar.

Also $\mathbf{a}.\ \mathbf{c} = 1/2 \Rightarrow |\mathbf{a}|\,|\mathbf{c}| \cos\theta = 1/2$,

So the angle between **a** and **c** is $\pi/3$

$\Rightarrow$ the angle between **b** and **d** is $\pi/3$

$\therefore$ **b** and **d** are nonparallel.

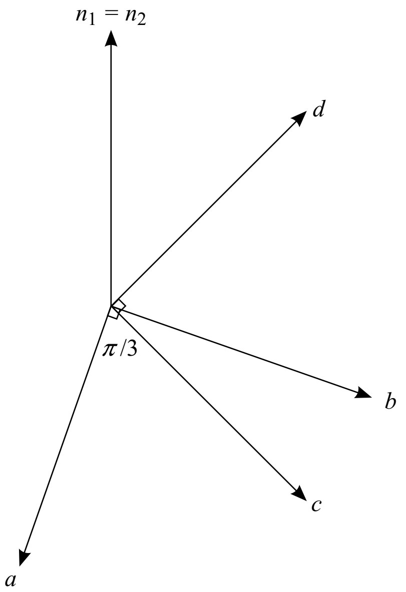

Fig. 22.7

40. If V is volume of the parallelopiped, then

$$V^2 = [\mathbf{abc}]^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1/2 & 1/2 \\ 1 & 1 & 1/2 \\ 1 & 1/2 & 1 \end{vmatrix}$$

$[C_1 \to C_1 + C_2 + C_3]$

$$= 2\begin{vmatrix} 1 & 1/2 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{vmatrix} = \frac{1}{2}$$

$\Rightarrow V = 1/\sqrt{2}$

41. $$\begin{vmatrix} a & 2a & -3a \\ 2a+1 & 2a+3 & a+1 \\ 3a+5 & a+5 & a+2 \end{vmatrix}$$

$$= a\begin{vmatrix} 1 & 2 & -3 \\ 2a+1 & 2a+3 & 4+1 \\ 3a+5 & a+5 & a+2 \end{vmatrix}$$

$$= a\begin{vmatrix} 1 & 0 & 0 \\ 2a+1 & 1-2a & 7a+4 \\ 3a+5 & -5a-5 & 10a+17 \end{vmatrix}$$

$[C_2 \to C_2 - 2C_1,\ C_3 \to C_3 + 3C_1]$

$= a\ (15a^2 + 31a + 37) = 0$ if and only if $a = 0$ as $15a^2 + 31a + 37 > 0$ for any a.

42. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$
$= (\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d}))\mathbf{b} - (\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d}))\ \mathbf{a}$
$= p\mathbf{a} + q\mathbf{b}$
So $p = -(\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})) = \mathbf{b} \cdot (\mathbf{d} \times \mathbf{c}) = [\mathbf{b}\ \mathbf{d}\ \mathbf{c}]$
$= [\mathbf{c}\ \mathbf{b}\ \mathbf{d}]$.

43. $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$. Also
$$\cos\theta = \mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c}$$
$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma(\mathbf{a} \times \mathbf{b}) \text{ so}$$
$$\mathbf{a} \cdot \mathbf{c} = \alpha + \beta(\mathbf{a} \cdot \mathbf{b}) = \alpha$$
$$\mathbf{b} \cdot \mathbf{c} = \alpha(\mathbf{a} \cdot \mathbf{b}) + \beta = \beta$$
Thus $\alpha = \mathbf{a} \cdot \mathbf{c} = \cos\theta = \mathbf{b} \cdot \mathbf{c} = \beta$.

44. $(\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d}) = ((\mathbf{a} \times \mathbf{b}) \times \mathbf{c})\ .\ \mathbf{d}$
$= ((\mathbf{a}\ .\ \mathbf{c})\ \mathbf{b} - (\mathbf{b}\ .\ \mathbf{c})\ \mathbf{a})\ .\ \mathbf{d}$
$= (\mathbf{a}\ .\ \mathbf{c})\ (\mathbf{b}\ .\ \mathbf{d}) - (\mathbf{b}\ .\ \mathbf{c})\ (\mathbf{a}\ .\ \mathbf{d})$

45. $[\mathbf{a} \times \mathbf{b},\ \mathbf{b} \times \mathbf{c},\ \mathbf{c} \times \mathbf{a}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

46. $|\ \mathbf{a} + \mathbf{b}\ |^2 - |\ \mathbf{a} - \mathbf{b}\ |^2 = |\ \mathbf{a}\ |^2 + |\ \mathbf{b}\ |^2 + 2\mathbf{a}\ .\ \mathbf{b}$
$- (|\ \mathbf{a}\ |^2 + |\ \mathbf{b}\ |^2 - 2\mathbf{a}\ .\ \mathbf{b})$
$= 4\mathbf{a}\ .\ \mathbf{b}$

47. $(\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} - \mathbf{k})$
$= -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

48. $[\mathbf{a} + \mathbf{b}\ \ \mathbf{b} + \mathbf{c}\ \ \mathbf{c} + \mathbf{a}] = 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

Level 2

49. Suppose that l, m, n are direction cosine of a line and the cube be a unit cube. The direction ratios of the diagonals will 1,1,1; 1, –1; 1, 1,1; –1,1,1. Hence
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$
$$= \frac{(l+m+n)^2}{1\cdot 3} + \frac{(l-m+n)^2}{1\cdot 3} + \frac{(-l+m+n)^2}{1\cdot 3} + \frac{(l+m-n)^2}{1\cdot 3}$$
$$= \frac{1}{3}[4(l^2 + m^2 + n^2) + 2(lm + mn + ln - lm + ln - mn + lm - ln - mn - lm + mn - ln)]$$
$$= \frac{4}{3}.$$

50. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = ((\mathbf{a} \times \mathbf{b}).\ \mathbf{d})\ \mathbf{c} - ((\mathbf{a} \times \mathbf{b})\cdot\mathbf{c})\ \mathbf{d}$
$= [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{d}$
so, $k = -[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{a}\ \mathbf{c}]$.

51. $\frac{1}{2} = \cos\frac{\pi}{3} = \frac{\mathbf{c}\cdot\mathbf{d}}{|\mathbf{c}||\mathbf{d}|} = \frac{x + x + 1}{\sqrt{x^2+2}\sqrt{x^2+2}} = \frac{2x+1}{x^2+2}$
$\Rightarrow x^2 + 2 = 4x + 2 \Rightarrow x = 0, 4$.

52. For $t = 3$, the line intersect xy plane at (–2, 10, 0).

53. The vector component of u orthogonal to $\mathbf{a}$ is
$$\mathbf{u} - \frac{\mathbf{u}.\mathbf{a}}{|\mathbf{a}|^2}\ \mathbf{a}$$
$$= (2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) - \frac{8+1+6}{\left(\sqrt{16+1+4}\right)^2}\ (4\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$
$$= \frac{1}{7}[(14\mathbf{i} - 7\mathbf{i} + 21\mathbf{k}) - 20\mathbf{i} + 5\mathbf{j} - 10\mathbf{k}]$$
$$= \frac{1}{7}(-6\mathbf{i} - 2\mathbf{j} + 11\mathbf{k})$$
$$= -\frac{1}{7}(6\mathbf{i} - 2\mathbf{j} - 11\mathbf{k}).$$

54. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [a\ b\ d]c - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{d}$
a, **b**, **c**, **d** lie in the same plane so $[\mathbf{a}\ \mathbf{b}\ \mathbf{d}] = 0$ and $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \mathbf{O}$ so $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = 0$.

55. $(\mathbf{a} \times \mathbf{b}).\ (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$
$= (\mathbf{a}\cdot\mathbf{c})\ (\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})\ (\mathbf{b}\cdot\mathbf{c})$
so $k = -1$.

56. Any point on the given line is of the form $B(3, 7 + t, 1 + t)$. The given line is parallel to the vector $\mathbf{j} + \mathbf{k}$. The direction ratio of AB will be $3 - 5, 6 + t, -2 + t$ where A is (5, 1, 3). If AB is perpendicular to the given line then
$$-2.\ 0 + (6 + t).\ 1 + (-2 + t)\ 1 = 0$$
$$\Rightarrow t = -2$$
$$|\mathbf{AB}| = \sqrt{(3-5)^2 + (5-1)^2 + (-1-3)^2}$$
$$= \sqrt{4+16+16} = 6.$$

57. The vector equation of two lines are
$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + t\ (-4\ \mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and
$\mathbf{r} = \mathbf{i} + 4\mathbf{j} - \mathbf{k} + \mathbf{s}\ (\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ so distance between them is (Ch. 18, P 6, formula 13)
$$\frac{\left|\begin{vmatrix} 1 & 4 & -1 \\ -4 & 1 & 2 \\ 1 & -2 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 3 \\ -4 & 1 & 2 \\ 1 & -2 & 1 \end{vmatrix}\right|}{\left|\begin{vmatrix} i & j & k \\ -4 & 1 & 2 \\ 1 & -2 & 1 \end{vmatrix}\right|}$$
$$= \frac{|22 - 38|}{|5\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}|}$$
$$= \frac{16}{\sqrt{110}}.$$

58. The vector AC is given by
$\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + t\ (-5\mathbf{i} + 3\mathbf{j} - \mathbf{k})$

Any point M on **AC** is of the form

$(1 - 5t, -2 + 3t, 2 - t)$

d.r. of **BM** are $-5t, -6 + 3t, 2 - t$. Since **BM** is an altitude so

$-5(-5t) + 3(-6 + 3t) - 1(2 - t) = 0$

$\Rightarrow t = \dfrac{4}{7}$

The point M is given by $\left(\dfrac{-13}{7}, \dfrac{-2}{7}, \dfrac{10}{7}\right)$

BM is given by $\dfrac{-20}{7}\mathbf{i} \dfrac{-30}{7}\mathbf{j} + \dfrac{10}{7}\mathbf{k}$.

59. $h = 2h + k,\ 3 - 4h + l = k,\ 1 + h + k = l$

$\Rightarrow k = -h$ so $3 - 3h + l = 0,\ l = 1$

$\Rightarrow h = 4/3,\ k = -4/3,\ l = 1.$

60. For a unit cube d.r. of diagonal are 1, 1, 1 and 1, – 1, 1 so the angle between two diagonals will be $\cos^{-1}\dfrac{1-1+1}{\sqrt{3}\sqrt{3}} = \cos^{-1}\dfrac{1}{3}$.

61. Since $(\mathbf{a} + \mathbf{b}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a}$ and $(\mathbf{a} + \mathbf{b}) \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$ so the intersection is $\mathbf{a} + \mathbf{b}$.

62. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\ \mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{d}$

$$= \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}\mathbf{c} - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\mathbf{d}$$

Also

$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\mathbf{b} - [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\mathbf{a}$

$$= \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}\mathbf{b} - \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}\mathbf{a}$$

Adding, we have $k = 2$.

63. $(\mathbf{b} \times \mathbf{c}).(\mathbf{a} \times \mathbf{d}) + (\mathbf{c} \times \mathbf{a}).(\mathbf{b} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d})$.

$$= \begin{vmatrix} \mathbf{b}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{a} \\ \mathbf{b}\cdot\mathbf{d} & \mathbf{c}\cdot\mathbf{d} \end{vmatrix} + \begin{vmatrix} \mathbf{c}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{c}\cdot\mathbf{d} & \mathbf{a}\cdot\mathbf{d} \end{vmatrix} + \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$$

$= (\mathbf{b}\cdot\mathbf{a})(\mathbf{c}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{d})(\mathbf{c}\cdot\mathbf{a}) + (\mathbf{c}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{d}) - (\mathbf{c}\cdot\mathbf{d})(\mathbf{a}\cdot\mathbf{b}) + (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$

$= 0.$

64. Equating the two equations we get $\lambda = 0$ and $\mu = -1$ so the point of intersection is $\mathbf{b} - 2\mathbf{c}$.

65. $\mathbf{a} \times \mathbf{v} = \mathbf{a} \times \mathbf{b} \Rightarrow \mathbf{a} \times (\mathbf{v} - \mathbf{b}) = 0$

$\Rightarrow \mathbf{v} - \mathbf{b} = t\,\mathbf{a} \Rightarrow \mathbf{v} = \mathbf{b} + t\,\mathbf{a}$

$\mathbf{a}\cdot\mathbf{v} = 0 \Rightarrow \mathbf{a}\cdot\mathbf{b} + t\,\mathbf{a}\cdot\mathbf{a} = 0$

$\Rightarrow t = -\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2} = -\dfrac{2+2+3}{\sqrt{1+4+9}\sqrt{4+1+1}}$

$= -\dfrac{7}{\sqrt{14}\sqrt{7}} = -\dfrac{1}{\sqrt{2}}$.

66. For $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$, we have

$\mathbf{a} \times (\mathbf{i} \times \mathbf{j}) = (\mathbf{a}\cdot\mathbf{i})\mathbf{j} - (\mathbf{a}\cdot\mathbf{j})\mathbf{i}$

$|\mathbf{a} \times (\mathbf{i} \times \mathbf{j})|^2 = (\mathbf{a}\cdot\mathbf{i})^2 + (\mathbf{a}\cdot\mathbf{j})^2 = a_1^2 + a_2^2$

$|\mathbf{a} \times (\mathbf{j} \times \mathbf{k})|^2 = a_2^2 + a_3^2$

$|\mathbf{a} \times (\mathbf{k} \times \mathbf{i})|^2 = a_3^2 + a_1^2$

so the required value $= 2(a_1^2 + a_2^2 + a_3^2) = 2|\mathbf{a}|^2$.

67. The mid point is given by $\dfrac{1}{2}(\mathbf{a} + \mathbf{b})$. The required locus passes through $\dfrac{1}{2}(\mathbf{a} + \mathbf{b})$ so required locus is of the form

$r = \dfrac{1}{2}(\mathbf{a} + \mathbf{b}) + t\,\mathbf{c}$ (i)

where $\mathbf{c}$ is perpendicular to $\mathbf{b} - \mathbf{a}$. Thus

$\mathbf{c}\,.\,(\mathbf{b} - \mathbf{a}) = 0$

Taking dot product in (i) by $\mathbf{a} - \mathbf{b}$, we get $(\mathbf{r} - \dfrac{1}{2}(\mathbf{a} + \mathbf{b})).(\mathbf{a} - \mathbf{b}) = 0$.

68. Since $\mathbf{a}$ is perpendicular to $\mathbf{d}$, so

$x - y + 2z = 0$ (i)

Moreover, $|\mathbf{b}| = |\mathbf{c}|$ so $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c}$ as $\mathbf{a}$ makes equal angles with $\mathbf{b}$ and $\mathbf{c}$.

Thus $xy - 2yz + 3xz = 2xz + 3xy - yz$

$\Rightarrow xz - 2xy - yz = 0$ (ii)

Also $x^2 + y^2 + z^2 = 12$ (iii)

and $y < 0$.

Substituting the value of y from (i) in (ii) we get

$x^2 + 2xz + z^2 = 0$

So $x = -z$ and $y = z$

Again substituting these values in (iii) we get $z^2 = 4$ i.e. $z = \pm 2$ but $y < 0$ and $y = z$

so $z = -2 = y$ and $x = 2$.

69. $\mathbf{a} \times \mathbf{b} = \mathbf{c} \Rightarrow \mathbf{c}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$.
$\mathbf{b} \times \mathbf{c} = \mathbf{a} \Rightarrow \mathbf{a}$ is perpendicular to $\mathbf{b}$ and $\mathbf{c}$

Thus $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are orthogonal in pairs. Since

$|\mathbf{c}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|$ and similarly $|\mathbf{a}| = |\mathbf{b}|\,|\mathbf{c}|$ so $|\mathbf{c}| = |\mathbf{c}|\,|\mathbf{b}|^2$

Hence $|\mathbf{b}| = 1$ and $|\mathbf{c}| = |\mathbf{a}|$.

70. Given lines has vector form as

$\mathbf{r} = -2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + t\,(2\mathbf{i} - 4\mathbf{i} + \mathbf{k})$

$\mathbf{r} = -2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + t\,(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

$$\cos\theta = \frac{-2-8+3}{\sqrt{4+16+1}\sqrt{1+4+9}} = \frac{-7}{\sqrt{21}\sqrt{14}} = -\frac{1}{\sqrt{6}}$$

Acute angle $= \pi - \cos^{-1}\left(-\frac{1}{\sqrt{6}}\right) = \cos^{-1}\frac{1}{\sqrt{6}}$.

71. According to the given conditions,

$c_1^2 + c_3^2 + c_3^2 = 1$, $\mathbf{a}\cdot\mathbf{c} = 0$, $\mathbf{b}\cdot\mathbf{c} = 0$ and

$$\frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{\sum a_i^2}\sqrt{\sum b_i^2}}$$

Thus $a_1c_1 + a_2c_2 + a_3c_3 = 0 = b_1c_1 + b_2c_2 + b_3c_3$

and $\frac{\sqrt{3}}{2} = \sqrt{\sum a_i^2}\sqrt{\sum b_i^2} = \sum a_i\, b_i$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \begin{vmatrix} \sum a_i^2 & \sum a_ib_i & \sum a_ic_i \\ \sum a_ib_i & \sum b_i^2 & \sum b_ic_i \\ \sum a_ic_i & \sum b_ic_i & \sum c_i^2 \end{vmatrix}$$

$$= \begin{vmatrix} \sum a_i^2 & \sum a_ib_i & 0 \\ \sum a_ib_i & \sum b_i^2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= \left(\sum a_i^2\right)\left(\sum b_i^2\right) - \left(\sum a_ib_i\right)^2$

$= \left(\sum a_i^2\right)\left(\sum b_i^2\right) - \frac{3}{4}\left(\sum a_i^2\right)\left(\sum a_i^2\right)$

$= \frac{1}{4}\left(\sum a_i^2\right)\left(\sum b_i^2\right)$.

72. Since the rotation of axes does not affect the distance between the origin and the point, we have

$4p^2 + 1 = (p + 1)^2 + 1$

$\Rightarrow p + 1 = \pm 2p \Rightarrow p = 1$ or $-1/3$.

73. $\mathbf{a}.\ ((\mathbf{b} \times \mathbf{c}) \times (\mathbf{a} + (\mathbf{b} \times \mathbf{c})) = \mathbf{a}.\ ((\mathbf{b} \times \mathbf{c}) \times \mathbf{a})$

$= \mathbf{a}.\ ((\mathbf{a}\cdot\mathbf{b})\mathbf{c} - (\mathbf{a}\cdot\mathbf{c})\mathbf{b}) = 0$.

74. $\mathbf{X}\cdot\mathbf{A} = 0$, $\mathbf{X}\cdot\mathbf{B} = \mathbf{X}\cdot\mathbf{C} = 0 \Rightarrow \mathbf{A}, \mathbf{B}, \mathbf{C}$ are perpendicular to $\mathbf{X}$. Thus $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are coplanar $\Rightarrow [\mathbf{A}, \mathbf{B}, \mathbf{C}] = 0$.

75. Let $\mathbf{c} = c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$. $\mathbf{c}\cdot\mathbf{k} = 0 \Rightarrow c_3 = 0$

$$\left.\begin{array}{l} \mathbf{c}\cdot\mathbf{a} = 9 \Rightarrow 3c_1 - 3c_2 = 9 \\ \mathbf{c}\cdot\mathbf{b} = -4 \Rightarrow c_1 + 2c_2 = -4 \end{array}\right] \Rightarrow c_1 = 2, c_2 = -3$$

Thus $\mathbf{c} = 2\mathbf{i} - 3\mathbf{j}$.

76. A unit vector in XZ plane is of the form $\mathbf{c} = c_1\,\mathbf{i} + c_3\,\mathbf{k}$ where $c_1^2 + c_3^2 = 1$

Also $\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} = \frac{2c_1 - c_3}{\sqrt{9}} \Rightarrow \frac{3}{\sqrt{2}} = 2c_1 - c_3$

$\frac{1}{2} = \cos\frac{\pi}{3} = \frac{-c_3}{\sqrt{2}} \Rightarrow c_3 = -\frac{1}{\sqrt{2}}$

so $c_1 = \frac{1}{\sqrt{2}}$. Hence $\mathbf{c} = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{k})$.

77. According to the given condition

$$\frac{\mathbf{a}.(\mathbf{a}+\mathbf{b})}{|\mathbf{a}|\,|\mathbf{a}+\mathbf{b}|} = \frac{\mathbf{b}.(\mathbf{a}+\mathbf{b})}{|\mathbf{b}|\,|\mathbf{a}+\mathbf{b}|}$$

$$\Rightarrow \frac{|\mathbf{a}|^2 + \mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2}{|\mathbf{b}|}$$

$$\Rightarrow |\mathbf{a}| + \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} + |\mathbf{b}|$$

$$\Rightarrow |\mathbf{a}| - |\mathbf{b}| = \mathbf{a}\cdot\mathbf{b}\left(\frac{|\mathbf{a}| - |\mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|}\right)$$

If $|\mathbf{a}| \neq |\mathbf{b}|$ then $|\mathbf{a}|\,|\mathbf{b}| = \mathbf{a}\cdot\mathbf{b}$

$\Rightarrow \theta = 0$ not possible

so $|\mathbf{a}| = |\mathbf{b}|$.

78. Treating A as the origin of reference. The position vector M is given by

$$\frac{5-3}{2}\mathbf{i} + \frac{-2+4}{2}\mathbf{j} + \frac{8}{2}\mathbf{k} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}.$$

The $|\mathbf{AM}| = \sqrt{18}$.

79. Area of the triangle $\mathbf{ABC} = \frac{1}{2}\,|\mathbf{b} \times \mathbf{AC}|$

$= \frac{1}{2}\,|\mathbf{b} \times (m\mathbf{b} + p\mathbf{d})|$

$= \frac{1}{2}\,|\mathrm{p}\ (\mathbf{b} \times \mathbf{d})|$

Similarly area of the triangle ACD

$= \frac{1}{2}\,\mathbf{d} \times (m\mathbf{b} + p\mathbf{d}) = \frac{1}{2}\,|m|\,|\mathbf{b} \times \mathbf{d}|$

Area of the quadrilateral $ABCD$

$= \frac{1}{2}\,(|p| + |m|)\,|\mathbf{b} \times \mathbf{d}|$.

80. $|\mathbf{u} \times \mathbf{v}|^2 = |(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} - \mathbf{b})|^2 = |\mathbf{b} \times \mathbf{a} - \mathbf{a} \times \mathbf{b}|^2$

$= 4|\mathbf{a} \times \mathbf{b}|^2 = 4|\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2 \theta$

$= 4k^4 (1 - \cos^2\theta) = 4k^4\left(1 - \frac{(\mathbf{ab})^2}{|\mathbf{a}|^2 |\mathbf{b}|^2}\right)$

$= 4k^4 \left(1 - \frac{(\mathbf{a} \cdot \mathbf{b})^2}{k^4}\right)$

$|\mathbf{u} \times \mathbf{v}| = 2(k^4 - (\mathbf{a}\cdot\mathbf{b})^2)^{1/2}$.

Previous Years' AIEEE/JEE Main Questions

1. $(\mathbf{a} \times \mathbf{b})^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2 (\pi/6)$

   $= (4^2)(2^2)(1/2)^2 = 16$

2. $[\mathbf{a} \times \mathbf{b}\ \ \mathbf{b} \times \mathbf{c}\ \ \mathbf{c} \times \mathbf{a}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 = 4^2 = 16$

3. $|\mathbf{b} + \mathbf{c}|^2 = |-\mathbf{a}|^2$

   $\Rightarrow |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2\mathbf{b}.\mathbf{c} = |\mathbf{a}|^2$

   $\Rightarrow 25 + 9 + 2|\mathbf{b}|\ |\mathbf{c}| \cos \theta = 49$

   $\Rightarrow 2(5)(3) \cos \theta = 15 \Rightarrow \cos \theta = 1/2$

   $\Rightarrow \theta = 60°$

4. $0 = |\mathbf{a} + \mathbf{b} + \mathbf{c}|^2$

   $= |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a}.\mathbf{b} + \mathbf{b}.\mathbf{c} + \mathbf{c}.\mathbf{a})$

   $\Rightarrow \mathbf{a}.\mathbf{b} + \mathbf{b}.\mathbf{c} + \mathbf{c}.\mathbf{a} = -\frac{1}{2}\left(5^2 + 4^2 + 3^2\right) = -25$

5. $\mathbf{c} = \mathbf{a} \times \mathbf{b} = 39\boldsymbol{k}$

   $|\mathbf{a}| = \sqrt{34}$, $|\mathbf{b}| = \sqrt{45}$, $|\mathbf{c}| = 39$

   $\therefore |\mathbf{a}| : |\mathbf{b}| : |\mathbf{c}| = \sqrt{34} : \sqrt{45} : 39$

6. $\mathbf{u} \times \mathbf{v} = -2\boldsymbol{k} \Rightarrow \boldsymbol{n} = \boldsymbol{k}$.

   Thus, $\mathbf{w}.\ \boldsymbol{n} = 3 \Rightarrow |\mathbf{w}.\ \boldsymbol{n}| = 3$

7. $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = 7\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$

   $\mathbf{d} = 5\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k} - (\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})$

   $= 4\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$

   Thus, work done $= \mathbf{w} = \mathbf{F}.\mathbf{d} = 28 + 4 + 8 = 40$

8. Let M be mid point of BC,

   $\mathbf{AM} = \frac{1}{2}(\mathbf{AB} + \mathbf{AC})$

   $= 4\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$

   $\Rightarrow \text{AM} = |\mathbf{AM}| = \sqrt{33}$

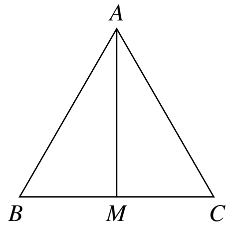

Fig. 22.8

9. As $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0$

   Now $[\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}\ \ \lambda\mathbf{b} + 4\mathbf{c}\ \ (2\lambda - 1)\mathbf{c}]$

   $= (2\lambda - 1)\ [\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}\ \ \lambda\mathbf{b} + 4\mathbf{c}\ \ \mathbf{c}]$

   $= (2\lambda - 1)\ [\mathbf{a} + 2\mathbf{b}\ \ \lambda\mathbf{b}\ \ \mathbf{c}]$

   $= (2\lambda - 1)\lambda\ [\mathbf{a} + 2\mathbf{b}\ \ \mathbf{b}\ \ \mathbf{c}]$

   $= (2\lambda - 1)\lambda\ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0$

   if $\lambda \neq 0, 1/2$

10. We are given

    $\frac{\mathbf{v}.\mathbf{u}}{|\mathbf{u}|} = \frac{\mathbf{w}.\mathbf{u}}{|\mathbf{u}|}$

    $\Rightarrow \mathbf{v}.\ \mathbf{u} = \mathbf{w}.\ \mathbf{u}$

    Also, $\mathbf{v}.\ \mathbf{w} = 0$

    Now, $|\mathbf{u} - \mathbf{v} + \mathbf{w}|^2$

    $= |\mathbf{u}|^2 + |\mathbf{v}|^2 + |\mathbf{w}|^2 - 2\mathbf{u}.\mathbf{v} + 2\mathbf{u}.\mathbf{w} - 2\mathbf{v}.\mathbf{w}$

    $= 1 + 4 + 9 = 14$

    $\Rightarrow |\mathbf{u} - \mathbf{v} + \mathbf{w}| = \sqrt{14}$

11. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \frac{1}{3}|\mathbf{b}|\ |\mathbf{c}|\ \mathbf{a}$

    $\Rightarrow (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{b}.\mathbf{c})\mathbf{a} = \frac{1}{3}|\mathbf{b}|\ |\mathbf{c}|\ \mathbf{a}$

    A possible solution is $\mathbf{a}.\mathbf{c} = 0$, $\mathbf{b}.\mathbf{c} = -\frac{1}{3}|\mathbf{b}|\ |\mathbf{c}|$

    $\Rightarrow |\mathbf{b}|\ |\mathbf{c}| \cos \theta = -\frac{1}{3}|\mathbf{b}|\ |\mathbf{c}| \Rightarrow \cos \theta = -1/3$

    $\therefore \sin \theta = 2\sqrt{2}/3$

12. Take P as origin

    $\therefore \mathbf{PC} = \frac{1}{2}(\mathbf{PA} + \mathbf{PB})$

    $\Rightarrow \mathbf{PA} + \mathbf{PB} = 2\mathbf{PC}$

13. Let $\mathbf{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$

    $\Rightarrow \mathbf{a} \times \boldsymbol{i} = -a_2\boldsymbol{k} + a_3\boldsymbol{j}$

    $\Rightarrow |\mathbf{a} \times \boldsymbol{i}|^2 = a_2^2 + a_3^2$

    Similarly, $|\mathbf{a} \times \mathbf{j}|^2 = a_1^2 + a_3^2$

    and $|\mathbf{a} \times \boldsymbol{k}|^2 = a_1^2 + a_2^2$

    Thus, $|\mathbf{a} \times \boldsymbol{i}|^2 + |\mathbf{a} \times \mathbf{j}|^2 + |\mathbf{a} \times \boldsymbol{k}|^2$

    $= 2(a_1^2 + a_2^2 + a_3^2) = 2|\mathbf{a}|^2$

14. $\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$

    $\Rightarrow \begin{vmatrix} 0 & a & c \\ 1 & 0 & 1 \\ 0 & c & b \end{vmatrix} = 0 \Rightarrow -ab + c^2 = 0$

    $\Rightarrow a, b, c$ are in G.P.

15. $[\lambda(\mathbf{a}+\mathbf{b})\ \lambda^2\mathbf{b}\ \lambda\mathbf{c}] = [\mathbf{a}\ \mathbf{b}+\mathbf{c}\ \mathbf{b}]$

$\Rightarrow (\lambda)(\lambda^2)(\lambda)\ [\mathbf{a}+\mathbf{b}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{a}\ \mathbf{c}\ \mathbf{b}]$

$\Rightarrow \lambda^4\ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = -[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

As $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0$, $\lambda^4 = -1$.

Not possible for any real value of λ.

16. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ y & x & 1+x \end{vmatrix}$ $[C_3 \to C_3 + C_1]$

$= 1$

$\Rightarrow [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ is independent of x, y.

17. $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

$\Rightarrow (\mathbf{a}.\mathbf{c})\ \mathbf{b} - (\mathbf{b}.\mathbf{c})\ \mathbf{a} = (\mathbf{a}.\mathbf{c})\ \mathbf{b} - (\mathbf{a}.\mathbf{b})\ \mathbf{c}$

$\Rightarrow \mathbf{a} = \dfrac{\mathbf{a}.\mathbf{b}}{\mathbf{b}.\mathbf{c}}\mathbf{c}$

$\therefore$ **a** is parallel to **c**

18. $\mathbf{CA} = (a-2)\boldsymbol{i} - 2\boldsymbol{j} + 0\boldsymbol{k}$

$\mathbf{CB} = (a-1)\boldsymbol{i} + 0\boldsymbol{j} + 6\boldsymbol{k}$

As $\angle C = \pi/2$, $\mathbf{CA}.\mathbf{CB} = 0$

$\Rightarrow (a-2)(a-1) = 0 \Rightarrow a = 1, 2$

19. As **c** are lies in the plane of **a** and **b**, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$

$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ x & x-2 & -1 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ -1 & -3 & 2 \\ x+1 & x-1 & -1 \end{vmatrix} = 0$

[using $C_1 \to C_1 - C_3$, $C_2 \to C_2 - C_3$]

$\Rightarrow -(x-1) + 3(x+1) = 0 \Rightarrow x = -2$

20. $|2\mathbf{u}\times 3\mathbf{v}| = 1 \Rightarrow |\mathbf{u}\times\mathbf{v}| = \dfrac{1}{6}$

$\Rightarrow |\mathbf{u}|\ |\mathbf{v}| \sin\theta = \dfrac{1}{6} \Rightarrow \sin\theta = \dfrac{1}{6}$

$\Rightarrow \theta = \sin^{-1}(1/6),\ \pi - \sin^{-1}(1/6)$

21. Unit vector along angle bisector of **b** and **c** is

$\mathbf{d} = \dfrac{1}{2}\left(\dfrac{b}{|b|} + \dfrac{c}{|c|}\right)$

$= \dfrac{1}{2\sqrt{2}}\ (\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$

Now, $\mathbf{a} = \lambda\mathbf{d}$

$\Rightarrow a\boldsymbol{i} + 2\boldsymbol{j} + \beta\boldsymbol{k} = \dfrac{\lambda}{2\sqrt{2}}(\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$

Equating coefficients of $\boldsymbol{j}$, we get

$\lambda = 2\sqrt{2}$

Thus, $\alpha = 1$, $\beta = 1$

22. $\mathbf{a} = 8\mathbf{b} = -\dfrac{8}{7}(7\mathbf{b}) = -\dfrac{8}{7}\mathbf{c}$

$\Rightarrow$ angle between **a** and **c** is π.

23. $[3\mathbf{u}\ p\mathbf{v}\ p\mathbf{w}] - [p\mathbf{v}\ \mathbf{w}\ q\mathbf{u}]$

$-[2\mathbf{w}\ q\mathbf{v}\ q\mathbf{u}] = 0$

$\Rightarrow 3p^2\ [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] - pq[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] + 2q^2\ [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = 0$

$\Rightarrow (3p^2 - pq + 2q^2)\ [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = 0$

As **u**, **v**, **w** are non-coplanar,

$3p^2 - pq + 2q^2 = 0$

$\Rightarrow p^2 - \dfrac{1}{3}pq + \dfrac{2}{3}q^2 = 0$

$\Rightarrow \left(p - \dfrac{1}{6}q\right)^2 + \dfrac{23}{36}q^2 = 0$

$\Rightarrow p = \dfrac{1}{6}q,\ q = 0.$

$\therefore p = 0,\ q = 0$

That is, there is exactly one value of (p, q).

24. $\mathbf{a}.\mathbf{c} = 0$, $\mathbf{b}.\mathbf{c} = 0$

$\Rightarrow \lambda - 1 + 2\mu = 0,\ 2\lambda + 4 + \mu = 0$

$\Rightarrow \lambda = -3,\ \mu = 2$

25. Let $\mathbf{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$

$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 1 & -1 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$= (b_2 + b_3)\boldsymbol{i} - b_1\boldsymbol{j} - b_1\boldsymbol{k}$

Now, $\mathbf{a}\times\mathbf{b} + \mathbf{c} = \mathbf{0}$

$\Rightarrow (b_2 + b_3 + 1)\boldsymbol{i} + (-b_1 - 1)\boldsymbol{j} + (-b_1 - 1)\boldsymbol{k} = \mathbf{0}$

$\Rightarrow b_2 + b_3 + 1 = 0,\ b_1 = -1$

Also, $\mathbf{a}.\mathbf{b} = 3$

$\Rightarrow b_2 - b_3 = 3$

Thus, $\mathbf{b}_2 = 1$, $\mathbf{b}_3 = -2$

Hence, $\mathbf{b} = -\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

26. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})$

$= (\mathbf{a}.(\mathbf{a} + 2\mathbf{b}))\ \mathbf{b} - (\mathbf{b}.(\mathbf{a} + 2\mathbf{b}))\ \mathbf{a}$

$= (|\mathbf{a}|^2 + 2\mathbf{a}.\mathbf{b})\ \mathbf{b} - (\mathbf{b}.\mathbf{a} + 2|\mathbf{b}|^2)\ \mathbf{a}$

But $\mathbf{a}.\mathbf{b} = 0$, $|\mathbf{a}| = 1$, $|\mathbf{b}| = 1$

Thus, $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b}) = \mathbf{b} - 2\mathbf{a}$

$\Rightarrow (2\mathbf{a} - \mathbf{b}).\ [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})]$

$= -\ |2\mathbf{a} - \mathbf{b}|^2 = -(4|\mathbf{a}|^2 + |\mathbf{b}|^2) = -\ 5.$

27. $\mathbf{b} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$

$\Rightarrow \mathbf{b} \times (\mathbf{c} - \mathbf{d}) = \mathbf{0}$

$\Rightarrow \mathbf{c} - \mathbf{d} = \alpha\mathbf{b}$ for same $\alpha \in \mathbf{R}$.

$\Rightarrow \mathbf{d} = \mathbf{c} - \alpha\mathbf{b}$

As $\mathbf{a}.\mathbf{d} = 0$, we get $\mathbf{a}.\mathbf{c} - \alpha\ \mathbf{a}.\mathbf{b} = 0$

$\Rightarrow \alpha = \dfrac{\mathbf{a}.\mathbf{c}}{\mathbf{a}.\mathbf{b}}$

Thus, $\mathbf{d} = \mathbf{c} - \left(\dfrac{\mathbf{a}.\mathbf{c}}{\mathbf{a}.\mathbf{b}}\right)\mathbf{b}$

28. As $p\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{i} + q\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{i} + \boldsymbol{j} + r\boldsymbol{k}$ are coplanar, so

$$\begin{vmatrix} p & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & r \end{vmatrix} = 0$$

$\Rightarrow pqr + 2 - p - q - r = 0$

$\Rightarrow pqr - p - q - r = -2$

29. $\mathbf{a} + 3\mathbf{b} = \alpha\mathbf{c}$ and $\mathbf{b} + 2\mathbf{c} = \beta\mathbf{a}$

Now, $\mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = (\alpha + 6)\mathbf{c}$

and $\mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = (1 + 3\beta)\mathbf{a}$

$\therefore\ (\alpha + 6)\ \mathbf{c} = (1 + 3\beta)\ \mathbf{a}$

As $\mathbf{a}$ and $\mathbf{c}$ are non-collinear,

We get $\alpha + 6 = 0$, $1 + 3\beta = 0$.

$\therefore\ \mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = \mathbf{0}$

30. As $\mathbf{c}.\mathbf{d} = 0$

$\Rightarrow (\mathbf{a} + 2\mathbf{b}).(5\mathbf{a} - 4\mathbf{b}) = 0$

$\Rightarrow 5|\mathbf{a}|^2 + 10\mathbf{b}.\mathbf{a} - 4\mathbf{a}.\mathbf{b} - 8|\mathbf{b}|^2 = 0$

$\Rightarrow 5 + 6|\mathbf{a}|\ |\mathbf{b}| \cos\theta - 8 = 0$

$\Rightarrow \cos\theta = 1/2 \Rightarrow \theta = \pi/3$

31. Let $\mathbf{AE} = \alpha\ \mathbf{p}$,

for same $\alpha \in \mathbf{R}$.

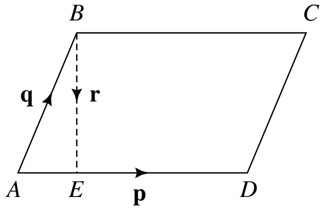

Fig. 22.9

$\mathbf{BE} = \mathbf{AE} - \mathbf{AB}$

$\Rightarrow \mathbf{r} = \alpha\ \mathbf{p} - \mathbf{q}$

As, $\mathbf{BE}.\mathbf{AD} = 0$

$\Rightarrow 0 = \mathbf{r}.\mathbf{p} = \alpha\mathbf{p}.\mathbf{q} - \mathbf{p}.\mathbf{q}$

$\Rightarrow \alpha = \dfrac{\mathbf{p}.\mathbf{q}}{\mathbf{p}.\mathbf{p}}$

Thus, $\mathbf{r} = -\mathbf{q} + \dfrac{\mathbf{p}.\mathbf{q}}{\mathbf{p}.\mathbf{p}}\mathbf{p}$

32. $\mathbf{AD} = \dfrac{1}{2}\ (\mathbf{AB} + \mathbf{AC})$

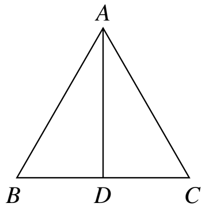

Fig. 22.10

$= 4\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$

$\Rightarrow |\mathbf{AD}| = \sqrt{33}$

33. As $\mathbf{u}$ and $\mathbf{v}$ are collinear,

$\mathbf{u} = k\mathbf{v}$ for some $k \in \mathbf{R}$.

$\Rightarrow \alpha - 2 = k(2 + 3\alpha)$, $1 = -\ 3k$

$\Rightarrow \alpha - 2 = -\dfrac{1}{3}(2 + 3\alpha) \Rightarrow \alpha = 2/3$

34. Projection of $\mathbf{b} + \lambda\mathbf{c}$ on $\mathbf{a}$

$= \dfrac{(\mathbf{b} + \lambda\mathbf{c}).\mathbf{a}}{|\mathbf{a}|}$

$\Rightarrow \pm\sqrt{\dfrac{2}{3}} = \dfrac{\mathbf{b}.\mathbf{a} + \lambda\mathbf{c}.\mathbf{a}}{\sqrt{6}}$

$\Rightarrow \pm 2 = -\ 1 + \lambda(-1) \Rightarrow \lambda = -3, 1$

For $\lambda = 1$, $-\ \mathbf{b} + \lambda\mathbf{c} = 2\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k}$

35. $\lambda = 1$ [see Theory]

36. $|2\mathbf{a} - \mathbf{b}|^2 + |2\mathbf{a} + \mathbf{b}|^2$

$= 2(4|\mathbf{a}|^2 + |\mathbf{b}|^2)$

$\Rightarrow 25 + |2\mathbf{a} + \mathbf{b}|^2 = 2(4(2^2) + 3^2)$

$\Rightarrow |2\mathbf{a} + \mathbf{b}| = 5$

37. As $\mathbf{c} \times (\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}) = \mathbf{0}$, is parallel to $\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}$. Let

$\mathbf{c} = \alpha(\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k})$ for same $\alpha \in \mathbf{R}$.

$\Rightarrow 60 = |\mathbf{c}|^2 = \alpha^2(1 + 4 + 25) \Rightarrow \alpha^2 = 2$

Now, $\mathbf{c}.\ (-7\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})$

$= \alpha(-7 + 4 + 15)$

$= 12\alpha = 12\sqrt{2}$

38. $|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{y} + \mathbf{z}|^2 + |\mathbf{z} + \mathbf{x}|^2 - |\mathbf{x} + \mathbf{y} + \mathbf{z}|^2$

$= |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}.\ \mathbf{y} + |\mathbf{y}|^2 + |\mathbf{z}|^2 + 2\mathbf{y}.\mathbf{z} + |\mathbf{z}|^2 + |\mathbf{x}|^2 + 2\mathbf{z}.\mathbf{x} - (|\mathbf{x}|^2 + |\mathbf{y}|^2 + |\mathbf{z}|^2 + 2\mathbf{x}.\mathbf{y} + 2\mathbf{x}.\mathbf{z} + 2\mathbf{y}.\mathbf{z})$

$= |\mathbf{x}|^2 + |\mathbf{y}|^2 + |\mathbf{z}|^2 = 3$

Thus, $u = |\mathbf{x} + \mathbf{y}|^2 + |\mathbf{y} + \mathbf{z}|^2 + |\mathbf{z} + \mathbf{x}|^2$

$= 3 + |\mathbf{x} + \mathbf{y} + \mathbf{z}|^2 \geq 3.$

$\therefore$ minimum value u is 3 and its attained when $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}$.

39. $\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & -6 & -1 \\ 1 & 4 & -3 \end{vmatrix}$

$= 22\boldsymbol{i} + 8\boldsymbol{j} + 18\boldsymbol{k}$

Projection of $\mathbf{x} \times \mathbf{y}$ on $\mathbf{z}$

$= \dfrac{|(\mathbf{x} \times \mathbf{y}).\mathbf{z}|}{|\mathbf{z}|} = \dfrac{|66 - 32 - 216|}{\sqrt{9 + 16 + 144}} = \dfrac{182}{13} = 14$

40. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \dfrac{1}{3}|\mathbf{b}|\ |\mathbf{c}|\ \mathbf{a}$

$\Rightarrow (\mathbf{a}.\mathbf{c})\ \mathbf{b} - (\mathbf{b}.\mathbf{c})\ \mathbf{a} = \dfrac{1}{3}\ |\mathbf{b}|\ |\mathbf{c}|\ \mathbf{a}$

$\Rightarrow |\mathbf{a}|\ |\mathbf{c}| \cos \varphi \mathbf{b} = |\mathbf{b}|\ |\mathbf{c}| \left(\cos\theta + \dfrac{1}{3}\right)\mathbf{a}$

As $\mathbf{a}$ and $\mathbf{b}$ are non-collinear,

$\cos \theta + \dfrac{1}{3} = 0$ and $\cos \varphi = 0$

$\Rightarrow \sin^2 \theta = 1 - 1/9 = 8/9$

$\Rightarrow \sin \theta = \pm 2\sqrt{2}/3$.

Thus, a value of $\sin \theta$ is $2\sqrt{2}/3$

41. $\mathbf{AC} = \mathbf{AB} + \mathbf{BC}$

$\Rightarrow \mathbf{c} = \mathbf{a} + \mathbf{b}$

and $\mathbf{DB} = \mathbf{AB} - \mathbf{AD} = \mathbf{a} - \mathbf{b}$.

Now, $\mathbf{DB}.\mathbf{AB} = (\mathbf{a} - \mathbf{b}).\mathbf{a} = \mathbf{a}^2 - \mathbf{b}.\mathbf{a}$

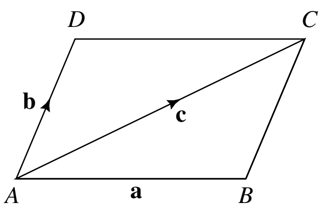

Fig. 22.11

Also

$\mathbf{c}^2 = |\mathbf{c}|^2 = |\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}.\mathbf{b}$

$\Rightarrow \mathbf{a}.\mathbf{b} = \dfrac{1}{2}\left(c^2 - a^2 - b^2\right)$

Thus,

$\mathbf{DB}.\mathbf{AB} = a^2 - \dfrac{1}{2}\left(c^2 - a^2 - b^2\right)$

$= \dfrac{1}{2}\left(3a^2 - b^2 - c^2\right)$

42. $\sqrt{3} = |\mathbf{a} + \mathbf{b}|$

$\Rightarrow 3 = |\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}.\mathbf{b}$

$\Rightarrow \mathbf{a}.\mathbf{b} = 1/2$

Also, $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2\ |\mathbf{b}|^2 - (\mathbf{a}.\mathbf{b})^2$

$= 1 - 1/4 = 3/4$

Now,

$|\mathbf{c}|^2 = |\mathbf{a}|^2 + 9|\mathbf{b}|^2 + 9|\mathbf{a} \times \mathbf{b}|^2 + 4\mathbf{a}.\mathbf{b} + 6\mathbf{a}.(\mathbf{a} \times \mathbf{b}) + 12\ \mathbf{b}.(\mathbf{a} \times \mathbf{b})$

$= 1 + 4 + 9(3/4) + 4(1/2) + 0 + 0$

$\Rightarrow 4|\mathbf{c}|^2 = 55$

$\Rightarrow 2|\mathbf{c}| = \sqrt{55}$.

43. $(\mathbf{a}.\mathbf{c})\ \mathbf{b} - (\mathbf{a}.\mathbf{b})\ \mathbf{c} = \dfrac{\sqrt{3}}{2}\ (\mathbf{b} + \mathbf{c})$

As $\mathbf{b}$ and $\mathbf{c}$ are not parallel

$\mathbf{a}.\mathbf{c} = \dfrac{\sqrt{3}}{2}$, $\mathbf{a}.\mathbf{b} = -\dfrac{\sqrt{3}}{2}$

Now, $\mathbf{a}.\mathbf{b} = -\sqrt{3}/2 \Rightarrow |\mathbf{a}|\ |\mathbf{b}| \cos \theta = -\sqrt{3}/2$

$\Rightarrow \cos \theta = -\sqrt{3}/2 \Rightarrow \theta = 5\pi/6$

44. $\mathbf{AB} = -4\boldsymbol{i} + 2\boldsymbol{j} + (p + 1)\boldsymbol{k}$

and $\mathbf{AC} = 2\boldsymbol{i} + (q - 1)\boldsymbol{j} - 3\boldsymbol{k}$

As $\angle CAB = \pi/2$, we get

$(-4)\ (2) + 2(q - 1) + (p + 1)\ (-3) = 0$

$\Rightarrow -3p + 2q - 13 = 0$

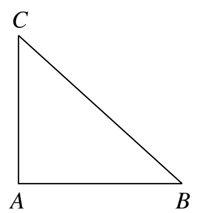

Fig. 22.12

Thus, (p, q) lies on $3x - 2y + 13 = 0$ which makes an acute angle with the x-axis.

45. Let $\mathbf{p}$ be the position vectors of the circumcentre and orthocentre be $\mathbf{p}$ and $\mathbf{h}$.

As G divides HP in the ratio 2:1,

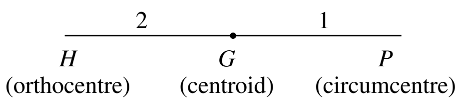

Fig. 22.13

$\dfrac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \dfrac{1}{3}(2\mathbf{p} + \mathbf{h})$

$\Rightarrow \mathbf{h} = \dfrac{1}{2}\ (\mathbf{a} + \mathbf{b} + \mathbf{c})$

Previous Years' B-Architecture Entrance Examination Questions

1. $0 = |\mathbf{u} + \mathbf{v} + \mathbf{w}|^2$

   $= |\mathbf{u}|^2 + |\mathbf{v}|^2 + |\mathbf{w}|^2 + 2\mathbf{u}.\mathbf{v} + 2\mathbf{v}.\mathbf{w} + 2\mathbf{w}.\mathbf{u}$

   $= 9 + 16 + 25 + 2(\mathbf{u}.\mathbf{v} + \mathbf{v}.\mathbf{w} + \mathbf{w}.\mathbf{u})$

   $\Rightarrow \mathbf{u}.\mathbf{v} + \mathbf{v}.\mathbf{w} + \mathbf{w}.\mathbf{u} = -25$

2. Let $\mathbf{c} = (\mathbf{a} - \mathbf{b}) \times (\mathbf{a} \times \mathbf{b})$

   $= ((\mathbf{a} - \mathbf{b}).\mathbf{b})\mathbf{a} - ((\mathbf{a} - \mathbf{b}).\mathbf{a})\, \mathbf{b}$

   $\Rightarrow (\mathbf{a}.\mathbf{b} - |\mathbf{b}|^2)\mathbf{a} - (|\mathbf{a}|^2 - \mathbf{b}.\mathbf{a})\mathbf{b}$

   $= (\mathbf{a}.\mathbf{b} - |\mathbf{a}|^2)\,(\mathbf{a} - \mathbf{b})$ $[\because |\mathbf{a}| = |\mathbf{b}|]$

   $\Rightarrow \mathbf{c}$ is parallel to $\mathbf{a} - \mathbf{b}$

3. See solution to Question 14 in Previous Years' AIEEE/JEE Main Questions.

4. $|\mathbf{x} - \mathbf{y}|^2 + |\mathbf{y} - \mathbf{z}|^2 + |\mathbf{z} - \mathbf{x}|^2 = 9$

   $\Rightarrow |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2\mathbf{x}.\mathbf{y} + |\mathbf{y}|^2 + |\mathbf{z}|^2 - 2\mathbf{y}.\mathbf{z} + |\mathbf{z}|^2 + |\mathbf{x}|^2 - 2\mathbf{z}.\mathbf{x} = 9$

   $\Rightarrow 2\mathbf{x}.\mathbf{y} + 2\mathbf{y}.\mathbf{z} + 2\mathbf{z}.\mathbf{x} = -3$ $[|\mathbf{x}| = |\mathbf{y}| = |\mathbf{z}| = 1]$

   Now, $|\mathbf{x} + \mathbf{y} - \mathbf{z}|^2 - 4\mathbf{x}.\mathbf{y}$

   $= |\mathbf{x}|^2 + |\mathbf{y}|^2 + |\mathbf{z}|^2 + 2\mathbf{x}.\mathbf{y} - 2\mathbf{x}.\mathbf{z} - 2\mathbf{y}.\mathbf{z} - 4\mathbf{x}.\mathbf{y}$

   $= 3 - (-3) = 6$

5. $2\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{c} = \mathbf{0}$

   $\Rightarrow 2(\mathbf{a}.\mathbf{b})\mathbf{a} - 2(\mathbf{a}.\mathbf{a})\mathbf{b} + \mathbf{c} = 0$

   $\Rightarrow \mathbf{c} = 2\mathbf{b} - 2(\mathbf{a}.\mathbf{b})\mathbf{a}$

   $[\because |\mathbf{a}| = 1]$

   $\Rightarrow 1 = |\mathbf{c}|^2 = 4|\mathbf{b}|^2 + 4(\mathbf{a}.\mathbf{b})^2\,|\mathbf{a}|^2 - 8(\mathbf{a}.\mathbf{b})\,(\mathbf{b}.\mathbf{a})$

   $\Rightarrow 1 = 4 - 4(\mathbf{a}.\mathbf{b})^2 \Rightarrow (\mathbf{a}.\mathbf{b})^2 = 3/4$

   $\Rightarrow |\mathbf{a}|^2\,|\mathbf{b}|\cos^2\theta = 3/4 \Rightarrow \cos\theta = \pm\sqrt{3}/2$

   $\therefore$ *a* value of θ is $\pi/6$

6. $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & 3 \\ 0 & 1 & 2 \end{vmatrix}$

   $= -5\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$

   Now, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \mathbf{a}.(\mathbf{b} \times \mathbf{c})$

   $= |\mathbf{a}|\,|\mathbf{b} \times \mathbf{c}|\cos\theta$

   where θ = angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$

   $\Rightarrow [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \le (1)\,(29)$

   Max $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 29$ when $\cos\theta = 1$.

7. Statement-2 is false as

   $|\boldsymbol{i} - \boldsymbol{j}| = \sqrt{2} = |\boldsymbol{i} + \boldsymbol{j}|$

   Also, $|2\mathbf{a} - \mathbf{b}|^2 + |2\mathbf{a} + \mathbf{b}|^2$

   $= 2[|2\mathbf{a}|^2 + |\mathbf{b}|^2] = 2[4(4) + 9] = 50$

   $\Rightarrow |2\mathbf{a} + \mathbf{b}|^2 = 50 - 5^2 = 25$

   $\Rightarrow |2\mathbf{a} + \mathbf{b}| = 5$

   $\therefore$ Statement-1 is true.

8. $\mathbf{c} = \mathbf{a} \times \mathbf{b}$

   $\Rightarrow |\mathbf{c}|^2 = \mathbf{c}.\mathbf{c} = (\mathbf{a} \times \mathbf{b}).\mathbf{c} = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

   Similarly, $|\mathbf{a}|^2 = |\mathbf{b}|^2 = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

   Thus, $|\mathbf{a}|^2 = |\mathbf{b}|^2 = |\mathbf{c}|^2$

   $\Rightarrow |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$

9. p = Area of quadrilateral

   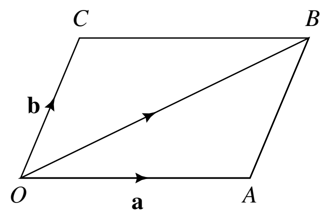

   Fig. 22.14

   = area (Δ *OAB*) + area (Δ *OBC*)

   $= \frac{1}{2}|\mathbf{a} \times (2\mathbf{b} + 10\mathbf{a})| + \frac{1}{2}|(2\mathbf{b} + 10\mathbf{a}) \times \mathbf{b}|$

   $= |\mathbf{a} \times \mathbf{b}| + 5|\mathbf{a} \times \mathbf{b}| = 6|\mathbf{a} \times \mathbf{b}| = 6q$

10. Solving two equations, we get

    $\mathbf{a} = \frac{2}{3}\mathbf{e}_1 - \frac{1}{3}\mathbf{e}_2,\ \mathbf{b} = -\frac{1}{3}\mathbf{e}_1 + \frac{2}{3}\mathbf{e}_2$

    $\Rightarrow \mathbf{a} = \frac{1}{3}(1, 1, 3)$

    $\mathbf{b} = \frac{1}{3}(1, 1, -3)$

    $\therefore \mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$

    $\Rightarrow \frac{1}{9}\ (1 + 1 - 9) = \frac{1}{9}\ (11)\cos\theta$

    $\Rightarrow \theta = \cos^{-1}(-7/11)$

11. $2\mathbf{u} + 2\mathbf{v} = -3\mathbf{w}$

    $\Rightarrow 9 = 9|\mathbf{w}|^2 = 4|\mathbf{u}|^2 + 4|\mathbf{v}|^2 + 8\mathbf{u}.\mathbf{v}$

    $\Rightarrow \mathbf{u}.\mathbf{v} = 1/8$

    Now, $|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\mathbf{u}.\mathbf{v}$

    $= 1 + 1 - 1/4 = 7/4$

    $\Rightarrow |\mathbf{u} - \mathbf{v}| = \sqrt{7}/2$

12. $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & -1 \\ 1 & 0 & 3 \end{vmatrix}$

$= 3\boldsymbol{i} - 7\boldsymbol{j} - \boldsymbol{k}$

$[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \mathbf{u}.(\mathbf{v} \times \mathbf{w})$

$= |\mathbf{u}|\ |\mathbf{v} \times \mathbf{w}| \cos\theta$

where θ = angle between $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$

$\Rightarrow [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = (1)\ \sqrt{59}\cos\theta \le \sqrt{59}$

$\therefore$ Max $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \sqrt{59}$

for $\cos\theta = 1$

13. $\frac{1}{6}\ (\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}) = (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$

$= \{(\mathbf{a} \times \mathbf{b}).\mathbf{d}\}\mathbf{c} - \{(\mathbf{a} \times \mathbf{b}).\mathbf{c}\}\mathbf{d}$

$= [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{d}$

As $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$

We have $[\mathbf{a}\ \mathbf{b}\ \mathbf{d}] = (\mathbf{a} \times \mathbf{b}).\mathbf{d}$

$= (\pm |\mathbf{a}|\ |\mathbf{b}| \sin\theta\ \mathbf{d}).\mathbf{d}$

$= \pm\frac{1}{2}$

Thus, $\pm\frac{1}{2}\mathbf{d} = \frac{1}{6}\ (\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$

$\Rightarrow \mathbf{d} = \pm\frac{1}{3}(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$

$= \pm\frac{1}{3}\ (-\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k})$

14. $|\mathbf{z} - \mathbf{x}|^2 = 8 \Rightarrow |\mathbf{z}|^2 - 2\mathbf{z}.\mathbf{x} + |\mathbf{x}|^2 = 8$

$\Rightarrow |\mathbf{z}|^2 - 2|\mathbf{z}| + 9 - 8 = 0 \Rightarrow |\mathbf{z}| = 1.$

Also,

$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix}$

$= 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

$\Rightarrow |\mathbf{x} \times \mathbf{y}| = 3$

Now,

$|(\mathbf{x} \times \mathbf{y}) \times \mathbf{z}|^2 = |\mathbf{x} \times \mathbf{y}|^2\ |\mathbf{z}|^2 \sin^2(30°)$

$= 9(1)\left(\frac{1}{4}\right)$

$\Rightarrow |(\mathbf{x} \times \mathbf{y}) \times \mathbf{z}| = \frac{3}{2}$

15. $\mathbf{AB} = \boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j}) - p(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$and $\mathbf{AC} = -\boldsymbol{k} + \mu(\boldsymbol{i} - \boldsymbol{j}) - p(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

As AB is perpendicular to $\mathbf{r} = \boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j})$

$\boldsymbol{k} \cdot (\boldsymbol{i} + \boldsymbol{j}) + \lambda(\boldsymbol{i} + \boldsymbol{j}).(\boldsymbol{i} + \boldsymbol{j}) - p(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}).(\boldsymbol{i} + \boldsymbol{k}) = 0$

$\Rightarrow 0 + 2\lambda - p(2) \Rightarrow \lambda = p$

Similarly $\mu(1 + 1) - p(1 - 1) \Rightarrow \mu = 0$

Thus for any given value of p, $B(p, p, 1)$ lying on the line

$\mathbf{r} = \boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j})$ (1)

is such that AB is perpendicular to (1), and $C(0, 0, -1)$ lying on the line

$\mathbf{r} = -\boldsymbol{k} + \mu(\boldsymbol{i} - \boldsymbol{j})$ (2)

is such that AC is perpendicular to (2).

$\therefore$ all the four answers are correct.