INDEFINITE INTEGRATION

If f & F are function of x such that F' (x) = f(x) then the function F is called a **PRIMITIVE OR ANTIDERIVATIVE OR INTEGRAL** of f(x) w.r.t. x and is written symbolically as

$\int f(x)\,dx = F(x) + c \Leftrightarrow \frac{d}{dx}\{F(x) + c\} = f(x)$, where c is called the **constant of integration.**

Note : If $\int f(x)dx = F(x) + c$, then $\int f(ax+b)dx = \frac{F(ax+b)}{a} + c, a \neq 0$

1. STANDARD RESULTS :

(i) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c;\ n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\ell n\left| ax+b \right| + c$

(iii) $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$

(iv) $\int a^{px+q} dx = \frac{1}{p}\frac{a^{px+q}}{\ell n\, a}(a > 0) + c$

(v) $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$

(vi) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$

(vii) $\int \tan(ax+b)dx = \frac{1}{a}\ell n\,|\sec(ax+b)| + c$

(viii) $\int \cot(ax+b)dx = \frac{1}{a}\ell n\,|\sin(ax+b)| + c$

(ix) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + c$

(x) $\int \text{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + c$

(xi) $\int \text{cosec}\,(ax+b).\cot(ax+b)dx = -\frac{1}{a}\text{cosec}\,(ax+b)+c$

(xii) $\int \sec\,(ax+b).\tan(ax+b)dx = \frac{1}{a}.\sec(ax+b)+c$

(xiii) $\int \sec x\,dx = \ell n\left|\sec x+\tan x\right|+c$

OR $\int \sec x dx = \ell n \tan\left|\frac{\pi}{4}+\frac{x}{2}\right|+c$

(xiv) $\int \text{cosec}\,x\,dx = \ell n\left|\text{cosec}\,x-\cot x\right|+c$

OR $\int \text{cosec}\,x\,dx = \ell n\left|\tan\frac{x}{2}\right|+c$ OR $-\ell n(\text{cosec}\,x+\cot x)+c$

(xv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+c$

(xvi) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$

(xvii) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}+c$

(xviii) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ell n\left[x+\sqrt{x^2+a^2}\right]+c$

(xix) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ell n\left[x+\sqrt{x^2-a^2}\right]+c$

(xx) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ell n\left|\frac{a+x}{a-x}\right|+c$

(xxi) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ell n\left|\frac{x-a}{x+a}\right|+c$

(xxii) $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$

(xxiii) $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ell n\left(x+\sqrt{x^2+a^2}\right)+c$

(xxiv) $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ell n\left(x+\sqrt{x^2-a^2}\right)+c$

(xxv) $\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

(xxvi) $\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

2. TECHNIQUES OF INTEGRATION :

(a) Substitution or change of independent variable :

Integral I = $\int f(x)\,dx$ is changed to $\int f(\phi(t))\phi'(t)\,dt$, by a suitable substitution x = ϕ(t) provided the later integral is easier to integrate.

Some standard substitution :

(1) $\int [f(x)]^n f'(x)dx$ OR $\int \frac{f'(x)}{[f(x)]^n}dx$ put f(x) = t & proceed.

(2) $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\;dx$

Express ax^2 + bx + c in the form of perfect square & then apply the standard results.

(3) $\int \frac{px+q}{ax^2+bx+c}dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Express px + q = A (differential coefficient of quadratic term of denominator) + B.

(4) $\int e^x[f(x)+f'(x)]dx = e^x.f(x) + c$

(5) $\int [f(x) + xf'(x)]dx = x\,f(x) + c$

(6) $\int \frac{dx}{x(x^n+1)}$ $n \in N$, take x^n common & put $1 + x^{-n} = t$.

(7) $\int \frac{dx}{x^2\left(x^n+1\right)^{(n-1)/n}}$ $n \in N$, take x^n common & put $1 + x^{-n} = t^n$

(8) $\int \frac{dx}{x^n(1+x^n)^{1/n}}$, take x^n common and put $1 + x^{-n} = t$.

(9) $\int \frac{dx}{a + b\sin^2 x}$ OR $\int \frac{dx}{a + b\cos^2 x}$

OR $\int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}$

Multiply N^r & D^r by $\sec^2 x$ & put $\tan x = t$.

(10) $\int \frac{dx}{a + b\sin x}$ OR $\int \frac{dx}{a + b\cos x}$ OR $\int \frac{dx}{a + b\sin x + c\cos x}$

Convert sines & cosines into their respective tangents of half the angles, put $\tan \frac{x}{2} = t$

(11) $\int \frac{a.\cos x + b.\sin x + c}{p.\cos x + q.\sin x + r} dx.$

Express Numerator $(N^r) \equiv \ell(D^r) + m\frac{d}{dx}(D^r) + n$ & proceed.

(12) $\int \frac{x^2 + 1}{x^4 + Kx^2 + 1} dx$ OR $\int \frac{x^2 - 1}{x^4 + Kx^2 + 1} dx$,

where K is any constant.

Divide Nr & Dr by x^2, then put $x - \frac{1}{x} = t$ OR $x + \frac{1}{x} = t$ respectively & proceed

(13) $\int \frac{dx}{(ax + b)\sqrt{px + q}}$ & $\int \frac{dx}{(ax^2 + bx + c)\sqrt{px + q}}$; put $px + q = t^2$

(14) $\int \frac{dx}{(ax + b)\sqrt{px^2 + qx + r}}$, put $ax + b = \frac{1}{t}$;

$\int \frac{dx}{(ax^2 + bx + c)\sqrt{px^2 + qx + r}}$, put $x = \frac{1}{t}$

(15) $\int \sqrt{\frac{x - \alpha}{\beta - x}}\, dx$ OR $\int \sqrt{(x - \alpha)(\beta - x)}$; put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\int \sqrt{\frac{x - \alpha}{x - \beta}}\, dx$ OR $\int \sqrt{(x - \alpha)(x - \beta)}$; put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

$\int \frac{dx}{\sqrt{(x - \alpha)(x - \beta)}}$; put $x - \alpha = t^2$ or $x - \beta = t^2$.

(16) To integrate $\int \sin^m x \cos^n x \, dx$.

(i) If m is odd positive integer put cos x = t.

(ii) If n is odd positive integer put sin x = t

(iii) If m + n is negative even integer then put tan x = t.

(iv) If m and n both even positive integer then use

$$\sin^2 x = \frac{1-\cos 2x}{2}, \cos^2 x = \frac{1+\cos 2x}{2}$$

(b) Integration by parts : $\int u.v \, dx = u\int v \, dx - \int\left[\frac{du}{dx}.\int v \, dx\right] dx$

where u & v are differentiable functions.

Note : While using integration by parts, choose u & v such that

(i) $\int vdx$ & (ii) $\int\left[\frac{du}{dx}.\int v \, dx\right] dx$ is simple to integrate.

This is generally obtained, by keeping the order of u & v as per the order of the letters in **ILATE**, where; I-Inverse function, L-Logarithmic function, A-Algebraic function, T-Trigonometric function & E-Exponential function.

(c) Partial fraction : Rational function is defined as the ratio of two polynomials in the form $\frac{P(x)}{Q(x)}$, where P(x) and Q(x) are polynomials in x and $Q(x) \neq 0$. If the degree of P(x) is less than the degree of Q(x), then the rational function is called proper, otherwise, it is called improper. The improper rational function can be reduced to the proper rational functions by long division process. Thus, if $\frac{P(x)}{Q(x)}$ is improper, then $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$, where T(x) is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is proper rational function. It is always possible to write the integrand as a sum of simpler rational functions by a method called partial fraction decomposition. After this, the integration can be carried out easily using the already known methods.

S. No.	Form of the rational function	Form of the partial fraction
1.	$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}$
2.	$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}+\dfrac{C}{x-b}$
3.	$\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\dfrac{A}{x-a}+\dfrac{Bx+C}{x^2+bx+c}$

where $x^2 + bx + c$ cannot be factorised further

Note :

In competitive exams, partial fraction are generally found by inspection by noting following fact :

$$\frac{1}{(x-\alpha)(x-\beta)} = \frac{1}{(\alpha-\beta)}\left(\frac{1}{x-\alpha} - \frac{1}{x-\beta}\right).$$

It can be applied to the case when x^2 or any other function is there in all places of x.

Example :

(1) $\dfrac{1}{(x^2+1)(x^2+3)} = \dfrac{1}{2}\left(\dfrac{1}{t+1} - \dfrac{1}{t+3}\right)$ {take $x^2 = t$}

(2) $\dfrac{1}{x^4(x^2+1)} = \dfrac{1}{x^2}\left(\dfrac{1}{x^2} - \dfrac{1}{x^2+1}\right) = \dfrac{1}{x^4} - \left(\dfrac{1}{x^2} - \dfrac{1}{x^2+1}\right)$

(3) $\dfrac{1}{x^3(x^2+1)} = \dfrac{1}{x}\left(\dfrac{1}{x^2} - \dfrac{1}{x^2+1}\right) = \dfrac{1}{x^3} - \dfrac{1}{x(x^2+1)}$