8 Integrals

BASIC CONCEPTS

1. **Antiderivative (or Primitive):** A function $\phi(x)$ is said to be antiderivative or primitive of a function $f(x)$ if $\phi'(x) = f(x)$ *i.e.*, $\frac{d}{dx}\{\phi(x)\} = f(x)$.

   For example, $\frac{x^2}{2}$ is primitive or antiderivative of x because

   $$\frac{d}{dx}\left(\frac{x^2}{2}\right) = \frac{1}{2}.2x = x$$

   Similarly, $$\frac{d}{dx}\left(\frac{x^2}{2}+1\right) = \frac{1}{2}.2x + 0 = x$$

   $$\vdots \qquad \vdots \qquad \vdots$$

   Similarly, $$\frac{d}{dx}\left(\frac{x^2}{2}+C\right) = \frac{1}{2}.2x + 0 = x$$

   In this way, we see that a function has infinitely many antiderivatives or primitives.

   i.e., if $\phi(x)$ be an antiderivative of $f(x)$, then $\phi(x) + C$ is also antiderivative of $f(x)$, where C is any constant.

   Because, $$\frac{d}{dx}\{\phi(x) + C\} = \phi'(x) + 0 = \phi'(x) = f(x)$$

   Indefinite Integrals: If $f(x)$ is a function, then the family of all its antiderivatives is called Indefinite Integral of $f(x)$. It is represented by $\int f(x)\,dx$ (read as indefinite integral of $f(x)$ with respect to x)

   For example, $$\int x^2 dx = \frac{x^3}{3} + C; \qquad \int x^3 dx = \frac{x^4}{4} + C$$

Why is it called Indefinite Integral?

It is called indefinite because it is not unique. Actually there exist infinitely many integrals of each function, which can be obtained by choosing C arbitrarily from the set of real numbers.

2. **Some Standard Integrals:**

   (*i*) $\int x^n dx = \frac{x^{n+1}}{n+1} + C\,(n \neq -1)$

   (*ii*) $\int \frac{dx}{x} = \log|x| + C$

   (*iii*) $\int dx = x + C$

   (*iv*) $\int \cos x\, dx = \sin x + C$

   (*v*) $\int \sin x\, dx = -\cos x + C$

   (*vi*) $\int \sec^2 x\, dx = \tan x + C$

(*vii*) $\int \text{cosec}^2 x\, dx = -\cot x + C$

(*viii*) $\int \sec x \tan x\, dx = \sec x + C$

(*ix*) $\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + C$

(*x*) $\int e^x dx = e^x + C$

(*xi*) $\int a^x dx = \dfrac{a^x}{\log a} + C$

(*xii*) (*a*) $\int \dfrac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

(*b*) $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$

(*xiii*) (*a*) $\int \dfrac{1}{1+x^2} dx = \tan^{-1} x + C$

(*b*) $\int \dfrac{1}{a^2+x^2} dx = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$

(*xiv*) $\int \dfrac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$

(*xv*) $\int -\dfrac{1}{x\sqrt{x^2-1}} dx = \text{cosec}^{-1} x + C$

(*xvi*) $\int -\dfrac{1}{a^2+x^2} dx = \dfrac{1}{a}\cot^{-1}\left(\dfrac{x}{a}\right) + C$

(*xvii*) $\int \dfrac{1}{x\sqrt{x^2-a^2}} dx = \dfrac{1}{a}\sec^{-1}\left(\dfrac{x}{a}\right) + C$

(*xviii*) $\int -\dfrac{1}{x\sqrt{x^2-a^2}} dx = \dfrac{1}{a}\text{cosec}^{-1}\left(\dfrac{x}{a}\right) + C$

3. **Methods of Integration:** It is not possible to integrate each integral with the help of following methods but a large number of various problems can be solved by these methods. So, we have the following methods of integration:
   (*i*) Integration by Substitution.
   (*ii*) Integration by Parts.
   (*iii*) Integration of Rational Algebraic Functions by Using Partial Fractions.
4. **Integration by Substitution:** The method of evaluating integrals of a function by suitable substitution is called Integration by substitution.

   We therefore give some of the fundamental integrals when x is replaced by $ax + b$.

(*i*) $\int (ax+b)^n dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$

(*ii*) $\int \dfrac{1}{ax+b} dx = \dfrac{1}{a}\log\,|\,(ax+b)\,| + C$

(*iii*) $\int e^{ax+b} dx = \dfrac{1}{a} e^{ax+b} + C$

(*iv*) $\int a^{bx+c} dx = \dfrac{1}{b}\cdot\dfrac{a^{bx+c}}{\log a} + C, a > 0 \text{ and } a \neq 1$

(*v*) $\int \sin(ax+b)\, dx = -\dfrac{1}{a}\cos(ax+b) + C$

(*vi*) $\int \cos(ax+b)\, dx = \dfrac{1}{a}\sin(ax+b) + C$

(*vii*) $\int \sec^2(ax+b)\, dx = \dfrac{1}{a}\tan(ax+b) + C$

(*viii*) $\int \text{cosec}^2(ax+b)\, dx = -\dfrac{1}{a}\cot(ax+b) + C$

(*ix*) $\int \sec(ax+b)\tan(ax+b)\, dx = \dfrac{1}{a}\sec(ax+b) + C$

(*x*) $\int \text{cosec}(ax+b)\cot(ax+b)\, dx = -\dfrac{1}{a}\text{cosec}(ax+b) + C$

(*xi*) $\int \tan(ax+b)\, dx = -\dfrac{1}{a}\log\,|\,\cos(ax+b)\,| + C$

(*xii*) $\int \cot(ax+b)\, dx = \dfrac{1}{a}\log\,|\,\sin(ax+b)\,| + C$

5. **More standard results:**

   $\int \tan x\, dx = -\log\,|\,\cos x\,| + C = \log\,|\,\sec x\,| + C$, provided x is not an odd multiple of $\dfrac{\pi}{2}$

   $\int \cot x\, dx = \log\,|\,\sin x\,| + C$

$$\int \sec x\, dx = \log|\sec x + \tan x| + C = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$$

$$\int \text{cosec}\, x\, dx = \log|\text{cosec}\, x - \cot x| + C = \log\left|\tan\frac{x}{2}\right| + C$$

6. Integration by Parts: To integrate the product of two functions, we use integration by parts. The method is as given below:

Let u and v be two functions of x then

$$\int u.v\, dx = u\int v\, dx - \int\left\{\frac{du}{dx}.\int v\, dx\right\}dx$$

Note:

(*i*) *To integrate the product of two functions we choose the 1st function according to word ILATE, where I stands for inverse function, L stands for logarithmic function, A stands for the algebraic functions, T stands for trigonometrical function and E stands for exponential function.*

(*ii*) *If the integrand has only one function then unity, i.e.,* 1 *is taken to be the second function.*

(*iii*) *Integration by parts is not applicable to product of functions in all cases. For example, the method does not work for* $\int \frac{1}{x}.\sin x\, dx$. *The reason is that there does not exist any function whose derivative is* $\frac{1}{x}.\sin x$.

(*iv*) *Observe that while finding the integral of the second function, we do not add a constant of integration on both the sides.*

7. Results of Some Special Integrals:

(*i*) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(*ii*) (*a*) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$; (*b*) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}.\log\left|\frac{a+x}{a-x}\right| + C$

(*iii*) $\int \frac{1}{\sqrt{a^2 + x^2}}dx = \log\left|\frac{x + \sqrt{x^2 + a^2}}{a}\right| + C$ or $\log|x + \sqrt{x^2 + a^2}| + C$

(*iv*) $\int \frac{1}{\sqrt{x^2 - a^2}}dx = \log\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right| + C$ or $\log|x + \sqrt{x^2 - a^2}| + C$

(*v*) (*a*) $\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\frac{x}{a} + C$; (*b*) $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

(*vi*) $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log|x + \sqrt{x^2 - a^2}| + C$

(*vii*) $\int \sqrt{a^2 + x^2}\,dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log|x + \sqrt{x^2 + a^2}| + C$

Theorem 1. *The indefinite integral of an algebraic sum of two or more functions is equal to the algebraic sum of their integrals,*

i.e., $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

Theorem 2. *A constant term may be taken outside from the integral sign i.e., if* k *is a constant then*

$$\int k\, f(x)dx = k\int f(x)dx$$

Theorem 3. *If the numerator in an integral is the exact derivative of denominator, then its integral is logarithmic of denominator,*

i.e., $$\int \frac{f'(x)dx}{f(x)} = \log|f(x)| + C$$

Theorem 4. *To integrate a function whose numerator is unity and denominator is a homogeneous function of* 1^{st} *degree in* cos *x and* sin *x i.e., the integrals of these forms:*

$$\int \frac{dx}{a+b\cos x}, \int \frac{dx}{a\sin x + b}, \int \frac{dx}{a+b\sin x}, \int \frac{dx}{a\cos x + b\sin x}, \int \frac{dx}{a\sin x + b\cos x}$$

i.e., when integrand is a rational function of sin x and cos x.

To find these, we can use following substitution.

(*i*) By putting $a = r\cos\alpha$, $b = r\sin\alpha$ respectively according to question

OR

(*ii*) By putting $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \cos x = \dfrac{\left(1-\tan^2\frac{x}{2}\right)}{\left(1+\tan^2\frac{x}{2}\right)}$

and putting $\tan\frac{x}{2} = t$ and then simplify.

Theorem 5. *To integrate a function whose numerator is* 1 *and denominator is a homogeneous function of the second degree in* cos *x and* sin *x or both, i.e.,*

$$\int \frac{dx}{a+b\sin^2 x}, \int \frac{dx}{a\sin^2 x + b}, \int \frac{dx}{a\cos^2 x + b}, \int \frac{dx}{a+b\cos^2 x}, \int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}$$

To evaluate such type of integrals we proceed as follows:

(*i*) Divide the numerator and denominator by $\cos^2 x$ and then

(*ii*) Putting $\tan x = z$ or $\cot x = z$ and then simplify.

Theorem 6. Integrals of the type $\int e^{f(x)}.f'(x)dx, \int f'(x)\cos[f(x)]dx, \int \sin[f(x)]f'(x)dx,$ $\int \log[f(x)]f'(x)dx.$

To evaluate these type of integrals, put f(x) = t so that f '(x) dx = dt and then integral converts to the standard forms for which the integrals are known.

Note: If the integrand is a rational function of e^x, then it always needs a replacement as the differentiation and integration of e^x is the same.

Thus, if on substituting denominator = t, the derivative of denominator is not present in the problem, then we need to generate it by multiplying and dividing by a suitable term containing the exponential function in numerator and denominator.

Integration by Partial Fractions

8. **Rational function:** Rational function is defined as the ratio of two polynomials in the form of $\dfrac{P(x)}{Q(x)}$

where $P(x)$ and $Q(x)$ are polynomials in x. If the degree of $P(x)$ is less than degree of $Q(x)$ then it is said to be Proper, otherwise it is called an Improper Rational Function.

Thus if $\dfrac{P(x)}{Q(x)}$ is improper, then by long division method it can be reduced to proper function *i.e.*,

$\dfrac{P(x)}{Q(x)} = T(x) + \dfrac{P_1(x)}{Q(x)}$, where $T(x)$ is a function of x and $\dfrac{P_1(x)}{Q(x)}$ is a proper rational function. Such fractions can be evaluated by breaking in factors given as follows:

S. No.	Form of the rational function	Form of the partial fraction
1.	$\dfrac{px+q}{(x-a)(x-b)}, a \neq b$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-b)}$
2.	$\dfrac{px+q}{(x-a)^2}$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2}$
3.	$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-b)} + \dfrac{C}{(x-c)}$
4.	$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2} + \dfrac{C}{(x-b)}$
5.	$\dfrac{px^2+qx+r}{(x-a)^3(x-b)}$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2} + \dfrac{C}{(x-a)^3} + \dfrac{D}{(x-b)}$
6.	$\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\dfrac{A}{(x-a)} + \dfrac{Bx+C}{x^2+bx+c}$, where x^2+bx+c cannot be factored further.

The constants A, B, C, etc. are obtained by equating coefficient of like terms from both sides or by substituting any value for x on both sides.

To find the integral of the form $\int \dfrac{dx}{ax^2+bx+c}$, we write

$$ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right] = a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$$

Now putting $x+\dfrac{b}{2a}=t$ so that $dx = dt$. Therefore, writing $\dfrac{c}{a}-\dfrac{b^2}{4a^2}=k$, and find the integral of reduced form $\dfrac{1}{|a|}\int \dfrac{dt}{(\pm t^2 \pm k)}$.

9. Integrals of the form $\int \dfrac{px+q}{ax^2+bx+c}dx$

Step I. The numerator $px + q$ is written in the form

$$px+q = A.\frac{d}{dx}(ax^2+bx+c)+B$$

$\Rightarrow$ $$px + q = A(2ax + b) + B$$

Step II. The value of A and B is obtained by equating the coefficients in the above equation.

Step III. $(px + q)$ is replaced by $A(2ax + b) + B$ and we write the given integral as

$$\int \frac{(px+q)}{ax^2+bx+c}dx = \int \frac{A(2ax+b)+B}{ax^2+bx+c}dx$$

10. Integrals of the form $\int \dfrac{px+q}{\sqrt{ax^2+bx+c}}dx$.

Step I. The numerator $px + q$ is written in the form

$$px+q = A\frac{d}{dx}(ax^2+bx+c)+B$$

$\Rightarrow$ $$px + q = A(2ax + b) + B$$

Step II. The values of A and B are obtained by equating the coeffcients in the above equation.

Step III. $(px + q)$ is replaced by A $(2ax + b) + B$ in given integration as

$$\int \frac{(px+q)}{\sqrt{ax^2+bx+c}}dx = \int \frac{A\{(2ax+b)+B\}}{\sqrt{ax^2+bx+c}}dx$$ and then solved.

11. **Integration of the form** $\int \frac{p(x)}{q(x)} dx$, where $p(x)$ and $q(x)$ are polynomials such that degree of $p(x) \geq$ degree $q(x)$.

Step I. $p(x)$ is divided by $q(x)$ and it is written as

$\frac{p(x)}{q(x)} = Q(x) + \frac{R(x)}{q(x)}$, where $Q(x)$ is quotient polynomial and $R(x)$ is remainder polynomial.

Step II. $\frac{p(x)}{q(x)}$ is replaced by $\left(Q(x) + \frac{R(x)}{q(x)}\right)$ as $\int \frac{p(x)}{q(x)} dx = \int \left(Q(x)) + \frac{R(x)}{q(x)}\right) dx$ and then solved.

12. **Integral of the form** $\int \sin^m x . \cos^n x \, dx$

(*i*) If the exponent of sin x is an odd positive integer, then put cos $x = t$.

(*ii*) If the exponent of cos x is an odd integer, then put sin $x = t$.

13. $\int e^x (f(x) + f'(x)) dx = f(x).e^x + C$

Definite Integrals

1. **Definition:** *If $F(x)$ is the integral of $f(x)$ over the interval $[a, b]$, i.e., $\int f(x) dx = F(x)$ then the definite integral of $f(x)$ over the interval $[a, b]$ is denoted by $\int_a^b f(x)$ is defined as*

$$\int_a^b f(x) dx = F(b) - F(a)$$

where 'a' is called the lower limit and 'b' is called the upper limit of integration and the interval $[a, b]$ is called the interval of integration.

2. **Integration as a Limit of Sum:** *If a function $f(x)$ is continuous in an interval $[a, b]$ then it is integrable on that interval.*

Therefore, we have $\int_a^b f(x) dx = \lim_{n \to \infty} S_n$.

Or, $\int_a^b f(x) dx = \lim_{n \to \infty} h[f(a) + f(a+h) + f(a+2h) + ... + f(a + \overline{n-1}h)]$

$\therefore \lim_{n \to \infty} S_n = \lim_{n \to \infty} h[f(a) + f(a+h) + f(a+2h) + ... + f(a + \overline{n-1}h)]$

Since when $n \to \infty$, *i.e.*, number of intervals is very large, then the width of the interval is very small which implies that $h \to 0$, so that $nh = b - a$ is a constant.

3. **Some Useful Results:** The following results will be useful in evaluating the definite integrals as the limit of sum.

(*i*) $\Sigma(n-1) = 1 + 2 + 3 + ... + (n-1) = \frac{n(n-1)}{2}$

(*ii*) $\Sigma(n-1)^2 = 1^2 + 2^2 + 3^2 + ... + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}$

(*iii*) $\Sigma(n-1)^3 = 1^3 + 2^3 + 3^3 + ... + (n-1)^3 = \left(\frac{n(n-1)}{2}\right)^2$

(*iv*) $a + ar + ar^2 + ... + ar^{n-1} = a\left(\frac{r^n - 1}{r-1}\right)(\text{if } r > 1)$ or $a\left(\frac{1-r^n}{1-r}\right)(\text{if } r < 1)$

(*v*) $\sin a + \sin(a+h) + \sin(a+2h) + ... + \sin\{a+(n-1)h\} = \dfrac{\sin\left\{a + \dfrac{(n-1)h}{2}\right\}\sin\dfrac{nh}{2}}{\sin\dfrac{h}{2}}$

(*vi*) $\cos a + \cos(a+h) + \cos(a+2h) + ... + \cos\{a+(n-1)h\} = \dfrac{\cos\left\{a + \dfrac{(n-1)h}{2}\right\}\sin\dfrac{nh}{2}}{\sin\dfrac{h}{2}}$

4. **Fundamental Properties of Definite Integrals:** There are certain properties of definite integrals which can be used while solving the definite integral.

(*i*) $\int_a^b f(x)dx = \int_a^b f(z)dz$ (Change of variable)

(*ii*) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ (Inter change the limits)

(*iii*) $\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)\,dx$, where $a < c < b$ (Change the limits)

(*iv*) (*a*) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ (*b*) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(*v*) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$, then following cases will occur:

(*a*) $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$, if $f(2a-x) = f(x)$ (*b*) $\int_0^{2a} f(x)dx = 0$, if $f(2a-x) = -f(x)$

(*vi*) $\int_0^{na} f(x)dx = n\int_0^a f(x)dx$, if $f(x) = f(a+x)$

(*vii*) (*a*) $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if *f* is an even function, *i.e.*, $f(-x) = f(x)$

(*b*) $\int_{-a}^{a} f(x)dx = 0$, if *f* is an odd function, *i.e.*, $f(-x) = -f(x)$

Selected NCERT Questions

1. Find $\int \dfrac{dx}{x-\sqrt{x}}$.

Sol. $I = \int \dfrac{dx}{x-\sqrt{x}} = \int \dfrac{dx}{\sqrt{x}(\sqrt{x}-1)}$

Let $\sqrt{x} - 1 = t \Rightarrow \dfrac{1}{2\sqrt{x}}dx = dt$

$\Rightarrow \dfrac{dx}{\sqrt{x}} = 2dt$

$\therefore$ $I = 2\int \dfrac{dt}{t} = 2\log|t| + C = 2\log|\sqrt{x}-1| + C$

2. Find : $\int \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}dx$ **[CBSE (F) 2011]**

Sol. Put $(e^{2x}+e^{-2x}) = t \Rightarrow (2e^{2x}-2e^{-2x})\,dx = dt \Rightarrow (e^{2x}-e^{-2x})\,dx = \frac{dt}{2}$

$$\therefore \int \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}dx = \frac{1}{2}\int \frac{1}{t}dt = \frac{1}{2}\log|t| + C = \frac{1}{2}\log|e^{2x}+e^{-2x}| + C$$

3. Find : $\int \frac{\sqrt{\tan x}}{\sin x \cos x}dx$

Sol. $\int \frac{\sqrt{\tan x}}{\sin x\cos x}dx = \int \frac{\sqrt{\tan x}}{\frac{\sin x}{\cos x}.\cos^2 x}dx = \int \frac{\sec^2 x}{\sqrt{\tan x}}dx$

Put $\tan x = t \quad \Rightarrow \sec^2 x\, dx = dt.$

$$\therefore \quad \int \frac{\sec^2 x}{\sqrt{\tan x}}dx = \int \frac{1}{\sqrt{t}}dt = \int t^{-1/2}\,dt = \frac{t^{1/2}}{1/2} + C = 2\sqrt{t} + C = 2\sqrt{\tan x} + C$$

4. Find : $\int \frac{(x+1)(x+\log x)^2}{x}dx$

Sol. $\int \frac{(x+1)(x+\log x)^2}{x}dx = \int \left(\frac{x+1}{x}\right)(x+\log x)^2 dx = \int \left(1+\frac{1}{x}\right)(x+\log x)^2 dx$

Put $(x+\log x) = t \;\Rightarrow\; \left(1+\frac{1}{x}\right)dx = dt$

$$\therefore \quad \int\left(1+\frac{1}{x}\right)(x+\log x)^2 dx = \int t^2 dt = \frac{t^3}{3} + C = \frac{1}{3}(x+\log x)^3 + C$$

5. Find : $\int \frac{x^3 \sin(\tan^{-1}x^4)}{1+x^8}dx$

Sol. $\int \frac{x^3\sin(\tan^{-1}x^4)}{1+x^8}dx$, put $\tan^{-1}(x^4) = t \Rightarrow \frac{4x^3}{1+x^8}dx = dt \Rightarrow \frac{x^3}{1+x^8}dx = \frac{dt}{4}$

$$\therefore \quad \int \frac{x^3\sin(\tan^{-1}x^4)}{1+x^8}dx = \frac{1}{4}\int \sin t\, dt = \frac{1}{4}(-\cos t) + C = \frac{-1}{4}\cos(\tan^{-1}x^4) + C$$

6. Find : $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}dx$ **(CBSE (AI) 2013)**

Sol. Let $\quad I = \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}dx = \int \frac{(2\cos^2 x - 1) - (2\cos^2\alpha - 1)}{\cos x - \cos\alpha}dx = 2\int \frac{\cos^2 x - \cos^2\alpha}{\cos x - \cos\alpha}dx$

$$= 2\int \frac{(\cos x - \cos\alpha)(\cos x + \cos\alpha)}{(\cos x - \cos\alpha)}dx = 2\int(\cos x + \cos\alpha)\,dx$$

$$= 2\int \cos x\, dx + 2\cos\alpha\int 1.dx \quad = 2\sin x + 2x\cos\alpha + C$$

7. Find : $\int \frac{1}{\sqrt{x^2+2x+2}}dx$

Sol. $\int \frac{1}{\sqrt{x^2+2x+2}}dx = \int \frac{1}{\sqrt{(x^2+2x+1)+1}}dx = \int \frac{1}{\sqrt{(x+1)^2+1}}dx$

Putting $x + 1 = t \;\Rightarrow\; dx = dt$

$$\therefore \quad \int \frac{1}{\sqrt{(x+1)^2+1}}dx = \int \frac{1}{\sqrt{(t)^2+(1)^2}}dt$$

$= \log |\ t + \sqrt{t^2+1}\ | + C = \log |\ (x+1) + \sqrt{(x^2+2x+1)+1}\ | + C$

$= \log |\ (x+1) + \sqrt{x^2+2x+2}\ | + C$

8. Find : $\int \frac{1}{9x^2+6x+5}dx$

Sol. $\int \frac{1}{9x^2+6x+5}dx = \frac{1}{9}\int \frac{1}{x^2+\frac{6}{9}x+\frac{5}{9}}dx$

$= \frac{1}{9}\int \frac{1}{x^2+\frac{2}{3}x+\frac{5}{9}+\left(\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)^2}dx = \frac{1}{9}\int \frac{1}{\left(x+\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2}dx$, putting $x+\frac{1}{3} = t \Rightarrow dx = dt$

$= \frac{1}{9}\int \frac{1}{t^2+\left(\frac{2}{3}\right)^2}dt = \frac{1}{9}.\frac{1}{\frac{2}{3}}\tan^{-1}\left(\frac{t}{2/3}\right) + C = \frac{1}{6}\tan^{-1}\left[\frac{3\left(x+\frac{1}{3}\right)}{2}\right] + C = \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right) + C$

9. Evaluate: $\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx$ **[*CBSE Delhi 2011; (AI) 2010*]**

Sol. We can express the N^r as $5x+3 = A\frac{d}{dx}(x^2+4x+10)+B$

$\Rightarrow \quad 5x+3 = A(2x+4)+B \quad \Rightarrow \quad 5x+3 = 2Ax + (4A+B)$

Equating the coefficients, we get

$2A = 5$ and $4A + B = 3$

$A = \frac{5}{2} \Rightarrow 4\times\frac{5}{2}+B = 3 \Rightarrow B = 3-10 = -7$

$\therefore \quad 5x+3 = \frac{5}{2}(2x+4)+(-7)$

$\therefore \quad I = \int \frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}\int\frac{(2x+4)}{\sqrt{x^2+4x+10}}dx - 7\int\frac{dx}{\sqrt{x^2+4x+10}}$

$I = \frac{5}{2}I_1 - 7I_2$ ...(*i*)

where $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$ and $I_2 = \int\frac{dx}{\sqrt{x^2+4x+10}}$

Now, $I_1 = \int\frac{(2x+4)}{\sqrt{x^2+4x+10}}dx$

Let $x^2+4x+10 = t \quad \Rightarrow \quad (2x+4)dx = dt$

$\therefore \quad I_1 = \int\frac{dt}{\sqrt{t}} = \int t^{-1/2}dt = \frac{t^{-1/2+1}}{-\frac{1}{2}+1} + C_1 = 2\sqrt{t}+C_1$

$I_1 = 2\sqrt{x^2+4x+10}+C_1$

Again, $I_2 = \int\frac{dx}{\sqrt{x^2+2.x.2+2^2-4+10}} = \int\frac{dx}{\sqrt{(x+2)^2+(\sqrt{6})^2}}$

$= \log|\ (x+2)+\sqrt{x^2+4x+10}\ | + C_2$

Putting the value of I_1 and I_2 in (*i*), we get

$$I = \frac{5}{2} \times 2\sqrt{x^2+4x+10} - 7\log|(x+2)+\sqrt{x^2+4x+10}| + \left(\frac{5}{2}C_1 - 7C_2\right)$$

$$= 5\sqrt{x^2+4x+10} - 7\log|(x+2)+\sqrt{x^2+4x+10}| + C$$

where $C = \left(\frac{5}{2}C_1 - 7C_2\right)$

10. Find : $\int \frac{1-x^2}{x(1-2x)}dx$

Sol. $\int \frac{1-x^2}{x(1-2x)}dx = \int \left[\frac{1}{2} + \frac{-\frac{1}{2}x+1}{x(1-2x)}\right]dx = \frac{1}{2}\int dx - \frac{1}{2}\int \frac{x-2}{x(1-2x)}dx = \frac{x}{2} - \frac{1}{2}I_1$ ...(*i*)

Now, $I_1 = \int \frac{x-2}{x(1-2x)}dx$

$\because$ $\frac{x-2}{x(1-2x)}$ is a proper rational function

$\therefore$ $\frac{x-2}{x(1-2x)} = \frac{A}{x} + \frac{B}{1-2x}$ ...(*ii*)

$\Rightarrow$ $x-2 = A(1-2x) + Bx$ $\Rightarrow$ $x-2 = (-2A+B)x + A$

$\Rightarrow$ $A = -2$ and $-2A + B = 1$ $\Rightarrow$ $B = 1 + 2A = 1 + 2(-2) = -3$

Putting values of A and B in (*ii*), we have

$$\frac{x-2}{x(1-2x)} = \frac{-2}{x} - \frac{3}{1-2x}$$

$$\therefore \int \frac{x-2}{x(1-2x)}dx = \int\left[\frac{-2}{x} - \frac{3}{1-2x}\right]dx$$

$$= -2\int \frac{1}{x}dx - 3\int \frac{1}{1-2x}dx = -2\log|x| - 3\frac{\log|1-2x|}{-2} + C_1$$

$$= -2\log|x| + \frac{3}{2}\log|1-2x| + C_1$$

Putting the value I_1 in (*i*), we have

$$\int \frac{1-x^2}{x(1-2x)}dx = \frac{x}{2} - \frac{1}{2}\left[-2\log|x| + \frac{3}{2}\log|1-2x| + C_1\right]$$

$$= \frac{x}{2} + \log|x| - \frac{3}{4}\log|1-2x| - \frac{C_1}{2}$$

$$= \frac{x}{2} + \log|x| - \frac{3}{4}\log|1-2x| + C, \text{ where } C = -\frac{C_1}{2}$$

11. Find : $\int \frac{x^3+x+1}{x^2-1}dx$

Sol. $\int \frac{x^3+x+1}{x^2-1}dx = \int\left[x + \frac{2x+1}{x^2-1}\right]dx = \int x\,dx + \int \frac{2x+1}{x^2-1}dx = \frac{x^2}{2} + \int \frac{2x}{x^2-1}dx + \int \frac{1}{x^2-1}dx$

Putting $x^2 - 1 = t \Rightarrow 2x\,dx = dt$ in second integral, we get

$$= \frac{x^2}{2} + \int \frac{1}{t}dt + \int \frac{1}{x^2-(1)^2}dx = \frac{x^2}{2} + \log|t| + \frac{1}{2(1)}\log\left|\frac{x-1}{x+1}\right| + C$$

$$= \frac{x^2}{2} + \log|x^2-1| + \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C$$

12. Evaluate: $\int \frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$ **[CBSE (F) 2014]**

Sol. Let $I = \int \frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

Put $\cos^{-1}x = z \Rightarrow -\frac{1}{\sqrt{1-x^2}}dx = dz \Rightarrow \frac{1}{\sqrt{1-x^2}}dx = -dz$

$$I = -\int \cos z.z\, dz = -z.\sin z + \int \sin z dz + C = -z\sin z - \cos z + C$$

$$I = -\cos^{-1}x.\sqrt{1-x^2} - x + C \qquad [\because x = \cos z \Rightarrow \sin z = \sqrt{1-x^2}]$$

$$I = -\sqrt{1-x^2}\cos^{-1}x - x + C$$

13. Find : $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$

Sol. Let $I = \int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx = \int e^x\left(\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)dx$

$$= \int e^x\left(\frac{1}{2\cos^2\frac{x}{2}} + \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)dx = \int e^x\left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right)dx$$

$$= \int \underset{\text{I}}{e^x}\underset{\text{II}}{\tan\frac{x}{2}}dx + \frac{1}{2}\int e^x\sec^2\frac{x}{2}dx$$

$$= \left(\tan\frac{x}{2}\right)e^x - \int\left(\frac{1}{2}\sec^2\frac{x}{2}\right)e^x dx + \frac{1}{2}\int e^x\sec^2\frac{x}{2}dx \Rightarrow \qquad I = e^x\tan\frac{x}{2} + C$$

14. Evaluate: $\int_1^4 (x^2 - x)\,dx$ **as limit of sums.** **[CBSE Delhi 2010; (F) 2011]**

Sol. $\int_1^4 (x^2 - x)\,dx$

We have to solve it by using limit of sums.

Here, $a = 1, b = 4, h = \frac{b-a}{n} = \frac{4-1}{n}$ *i.e.,* $nh = 3$

Limit of sum for $\int_1^4 (x^2 - x)dx$ is

$$= \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + ... + f\{1 + (n-1)h\}]$$

Now, $f(1) = 1 - 1 = 0$

$f(1+h) = (1+h)^2 - (1+h) = h^2 + h$

$f(1+2h) = (1+2h)^2 - (1+2h) = 4h^2 + 2h$

....

....

....

$f[1 + (n-1)h] = \{1 + (n-1)h\}^2 - \{1 + (n-1)h\} = (n-1)^2 h^2 + (n-1)\,h$

$$\therefore \quad \int_1^2 (x^2 - x)dx = \lim_{h\to 0} h[0 + h^2 + h + 4h^2 + 2h + ... (n-1)^2h^2 + (n-1)h]$$

$$= \lim_{h\to 0} h[h^2\{1 + 4 + .. + (n-1)^2\} + h\{1 + 2 + ... (n-1)\}]$$

$$= \lim_{h \to 0} h\left[h^2 . \frac{(n)(n-1)(2n-1)}{6} + h\frac{n(n-1)}{2}\right]$$

$$\left[\because 1 + 4 + ... + (n-1)^2 = \frac{n(n-1)(2n-1)}{6} \text{ and } 1 + 2 + ... + (n-1) = \frac{n(n-1)}{2}\right]$$

$$= \lim_{h \to 0}\left[\frac{nh(nh-h)(2nh-h)}{6} + \frac{nh(nh-h)}{2}\right]$$

$$= \lim_{h \to 0}\left[\frac{(3-h)(3)(6-h)}{6} + \frac{(3-h)(3)}{2}\right] = \left(\frac{3 \times 3 \times 6}{6}\right) + \left(\frac{3 \times 3}{2}\right) = 9 + \frac{9}{2} = \frac{27}{2}$$

15. Evaluate: $\int_{-5}^{5} |x+2|\, dx$ **[*CBSE (F) 2010*]**

Sol. Here, function is $|x+2|$ which is defined as

$$|x+2| = \begin{cases} (x+2), & \text{if } \quad x \geq -2 \\ -(x+2), & \text{if } \quad x < -2 \end{cases}$$

So, we have

$$\int_{-5}^{5} |x+2|\, dx = \int_{-5}^{-2} -(x+2)\,dx + \int_{-2}^{5} (x+2)\,dx \qquad \left[\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad a < c < b\right]$$

$$= \left[-\frac{x^2}{2} - 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$= -\frac{(-2)^2}{2} - 2(-2) + \frac{(-5)^2}{2} + 2 \times (-5) + \frac{(5)^2}{2} + 2 \times (5) - \frac{(-2)^2}{2} - 2 \times (-2)$$

$$= -2 + 4 + \frac{25}{2} - 10 + \frac{25}{2} + 10 - 2 + 4 = 29$$

16. Evaluate: $\int_0^{\pi/4} \log(1 + \tan x)\, dx$ **[*CBSE (AI) 2011*]**

Sol. Let $I = \int_0^{\pi/4} \log(1 + \tan x)\,dx \qquad \ldots(i)$

$\therefore \quad I = \int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - x\right)\right]dx \quad$ (By using property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$)

$$= \int_0^{\pi/4} \log\left[1 + \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}.\tan x}\right]dx$$

$$= \int_0^{\pi/4} \log\left[1 + \frac{1 - \tan x}{1 + \tan x}\right]dx = \int_0^{\pi/4} \log\left[\frac{1 + \tan x + 1 - \tan x}{1 + \tan x}\right]dx$$

$$I = \int_0^{\pi/4} \log\frac{2}{1 + \tan x}\,dx = \int_0^{\pi/4} [\log 2 - \log(1 + \tan x)]\,dx \qquad \ldots(ii)$$

Adding (*i*) and (*ii*), we get

$$2I = \int_0^{\pi/4} \log 2\, dx = \log 2 \int_0^{\pi/4} dx = \log 2\,[x]_0^{\pi/4}$$

$$2I = \frac{\pi}{4}\log 2 \qquad \Rightarrow \qquad I = \frac{\pi}{8}\log 2$$

17. Evaluate : $\int_{-1}^{3/2} |x \sin \pi x|\, dx$ **[*CBSE (F) 2010; Central 2016*]**

Sol. $\int_{-1}^{3/2} |x \sin \pi x|\, dx$

As we know

$$\sin\theta = 0 \quad \Rightarrow \quad \theta = n\pi,\ n \in Z$$

$$\therefore \quad \sin \pi x = 0 \qquad x = 0, 1, 2, \ldots$$

For $-1 < x < 0,\ x < 0, \sin \pi x < 0 \quad \Rightarrow \quad x \sin \pi x > 0$

For $0 < x < 1$

$x > 0, \sin \pi x > 0 \quad \Rightarrow \quad x \sin \pi x > 0$

For $1 < x < \frac{3}{2}, x > 0, \sin \pi x < 0 \quad \Rightarrow \quad x \sin \pi x < 0$

$$\therefore \quad \int_{-1}^{3/2} |\, x\sin\pi x \,|\, dx = \int_{-1}^{1} x\sin\pi x\, dx + \int_{1}^{3/2} (-x\sin\pi x)\, dx$$

$$= \left[x.\frac{(-\cos\pi x)}{\pi}\right]_{-1}^{1} - \int_{-1}^{1} 1.\frac{-\cos\pi x}{\pi}dx - \left[x.\frac{-\cos\pi x}{\pi}\right]_{1}^{3/2} + \int_{1}^{3/2} 1.\frac{-\cos\pi x}{\pi}dx$$

$$= \left[-\frac{x}{\pi}\cos\pi x + \frac{1}{\pi^2}\sin\pi x\right]_{-1}^{1} - \left[-\frac{x}{\pi}\cos\pi x + \frac{1}{\pi^2}\sin\pi x\right]_{1}^{3/2}$$

$$= \left[\frac{1}{\pi} + 0 + \frac{1}{\pi} - 0\right] - \left[0 - \frac{1}{\pi^2} - \frac{1}{\pi}\right] = \left[\frac{1}{\pi} + \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi}\right]$$

$$= \frac{1}{\pi^2} + \frac{3}{\pi} = \frac{1+3\pi}{\pi^2}$$

18. Find $\int e^{x}(\cos x - \sin x)\,\text{cosec}^2 x\, dx$. **[CBSE 2019(65/5/1)]**

Sol. Let $I = \int e^{x}(\cos x - \sin x)\,\text{cosec}^2 x\, dx = \int e^{x}(\cot x \,.\, \text{cosec}\, x - \text{cosec}\, x)\, dx$

$$= \int e^{x}\,\text{cosec}\, x \cot x\, dx - \int \underset{\text{II}}{e^{x}}\, \underset{\text{I}}{\text{cosec}\, x}\, dx$$

$$= \int e^{x}\,\text{cosec}\, x.\cot x - \text{cosec}\, x\, e^{x} + C + \int -\text{cosec}\, x \cot x\, e^{x} dx$$

[Using integration by parts for 2nd integral]

$$= \int e^{x}\,\text{cosec}\, x.\cot x\, dx - e^{x}\,\text{cosec}\, x + C - \int e^{x}\,\text{cosec}\, x.\cot x\, dx$$

$$= -e^{x}\text{cosec}\, x + C.$$

19. Evaluate: $\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$ **[CBSE Delhi 2010]**

Sol. We have,

$$I = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}dx \quad \Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1 - (\cos x - \sin x)^2}}dx$$

Let $t = (\cos x - \sin x) \quad \Rightarrow \quad dt = -(\sin x + \cos x)\, dx$

The limits are, when $x = \frac{\pi}{6} \quad \Rightarrow \quad t = \cos\frac{\pi}{6} - \sin\frac{\pi}{6} = \frac{\sqrt{3}-1}{2}$

and $x = \frac{\pi}{3} \Rightarrow t = \cos\frac{\pi}{3} - \sin\frac{\pi}{3} = \frac{1-\sqrt{3}}{2}$

$$\therefore \quad I = -\int_{(\sqrt{3}-1)/2}^{(1-\sqrt{3})/2} \frac{1}{\sqrt{1-t^2}}dt$$

$$= -[\sin^{-1} t]_{\frac{\sqrt{3}-1}{2}}^{\frac{1-\sqrt{3}}{2}} = -\left[\sin^{-1}\frac{1-\sqrt{3}}{2} - \sin^{-1}\frac{\sqrt{3}-1}{2}\right] = \left[-\sin^{-1}\frac{1-\sqrt{3}}{2} + \sin^{-1}\frac{\sqrt{3}-1}{2}\right]$$

$$\Rightarrow \quad I = 2\sin^{-1}\frac{\sqrt{3}-1}{2} \qquad [\because \sin^{-1}(-x) = -\sin^{-1} x]$$

20. Evaluate: $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$ **[CBSE (F) 2011, 2013, 2014]**

Sol. Let $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$

Here, we express denominator in terms of $\sin x - \cos x$ which is integral of the numerator.

We have, $(\sin x - \cos x)^2 = \sin^2 x + \cos^2 x - 2\sin x \cos x = 1 - \sin 2x$

$$\Rightarrow \quad \sin 2x = 1 - (\sin x - \cos x)^2$$

$$\therefore \quad I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\{1 - (\sin x - \cos x)^2\}} dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\sin x + \cos x}{25 - 16(\sin x - \cos x)^2} dx$$

Let $\sin x - \cos x = t \quad \Rightarrow \quad (\cos x + \sin x)dx = dt$

The limits are, when $x = 0 \Rightarrow t = \sin 0 - \cos 0 = -1$ and $x = \frac{\pi}{4} \Rightarrow t = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = 0$

$$\therefore \quad I = \int_{-1}^{0} \frac{dt}{25 - 16t^2}$$

$$\Rightarrow \quad I = \frac{1}{16}\int_{-1}^{0} \frac{dt}{\frac{25}{16} - t^2} = \frac{1}{16}\int_{-1}^{0} \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2} \quad \Rightarrow \quad I = \frac{1}{16}\cdot\frac{1}{2\left(\frac{5}{4}\right)}\left[\log\left|\frac{\frac{5}{4} + t}{\frac{5}{4} - t}\right|\right]_{-1}^{0}$$

$$\Rightarrow \quad I = \frac{1}{40}\left[\log 1 - \log\left(\frac{1/4}{9/4}\right)\right] \quad \Rightarrow \quad I = \frac{1}{40}\left[0 - \log\left(\frac{1}{9}\right)\right] = \frac{1}{40}\log 9$$

21. Evaluate: $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$ **[CBSE Delhi 2008, 2010, (AI) 2008, 2017, (F) 2010, 2013, 2014]**

Sol. Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx \quad \ldots(i)$

$$= \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx\right]$$

$$= \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x} dx \quad \ldots(ii)$$

By adding equations (*i*) and (*ii*), we get

$$2I = \pi\int_0^{\pi} \frac{\tan x}{\sec x + \tan x} dx$$

Multiplying and dividing by $(\sec x - \tan x)$, we get

$$2I = \pi\int_0^{\pi} \frac{\tan x(\sec x - \tan x)}{\sec^2 x - \tan^2 x} dx = \pi\int_0^{\pi} (\sec x \tan x - \tan^2 x)\,dx$$

$$= \pi\int_0^{\pi} \sec x \tan x\, dx - \pi\int_0^{\pi} \sec^2 x\, dx + \int_0^{\pi} dx$$

$$= \pi[\sec x]_0^{\pi} - \pi[\tan x]_0^{\pi} + \pi[x]_0^{\pi} = \pi(-1 - 1) - 0 + \pi(\pi - 0) = \pi(\pi - 2)$$

$$\Rightarrow \quad 2I = \pi(\pi - 2) \quad \Rightarrow \quad I = \frac{\pi}{2}(\pi - 2)$$

Multiple Choice Questions [1 mark]

Choose and write the correct option in the following questions.

1. $\int x^2 e^{x^3} dx$ is equal to **[CBSE 2020 (65/5/1)]**

(a) $\frac{1}{3}e^{x^3}+C$ (b) $\frac{1}{3}e^{x^4}+C$ (c) $\frac{1}{2}e^{x^3}+C$ (d) $\frac{1}{2}e^{x^2}+C$

2. $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx$ is equal to **[CBSE 2020 (65/3/1)]**

(a) $\tan(xe^x)+C$ (b) $\cot(xe^x)+C$ (c) $\cot(e^x)+C$ (d) $\tan[e^x(1+x)]+C$

3. $\int e^x(\cos x-\sin x)\,dx$ is equal to

(a) $e^x\cos x+C$ (b) $e^x\sin x+C$ (c) $-e^x\cos x+C$ (d) $-e^x\sin x+C$

4. $\int \frac{dx}{\sin^2 x\cos^2 x}$ is equal to **[NCERT Exemplar]**

(a) $\tan x+\cot x+C$ (b) $(\tan x+\cot x)^2+C$ (c) $\tan x-\cot x+C$ (d) $(\tan x-\cot x)^2+C$

5. If $\int \frac{3e^x-5e^{-x}}{4e^x+5e^{-x}}dx = ax+b\log|4e^x+5e^{-x}|+C$ then

(a) $a=\frac{-1}{8}, b=\frac{7}{8}$ (b) $a=\frac{1}{8}, b=\frac{7}{8}$ (c) $a=\frac{-1}{8}, b=\frac{-7}{8}$ (d) $a=\frac{1}{8}, b=\frac{-7}{8}$

6. $\int \frac{\cos 2x-\cos 2\theta}{\cos x-\cos\theta}dx$ is equal to **[NCERT Exemplar]**

(a) $2(\sin x+x\cos\theta)+C$ (b) $2(\sin x-x\cos\theta)+C$

(c) $2(\sin x+2x\cos\theta)+C$ (d) $2(\sin x-2x\cos\theta)+C$

7. $\int \frac{dx}{\sin(x-a)\sin(x-b)}$ is equal to **[NCERT Exemplar]**

(a) $\sin(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|+C$ (b) $\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|+C$

(c) $\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|+C$ (d) $\sin(b-a)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|+C$

8. $\int \tan^{-1}\sqrt{x}\,dx$ is equal to **[NCERT Exemplar]**

(a) $(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}+C$ (b) $x\tan^{-1}\sqrt{x}-\sqrt{x}+C$

(c) $\sqrt{x}-x\tan^{-1}\sqrt{x}+C$ (d) $\sqrt{x}-(x+1)\tan^{-1}\sqrt{x}+C$

9. $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$ is equal to **[NCERT Exemplar]**

(a) $\frac{e^x}{1+x^2}+C$ (b) $\frac{-e^x}{1+x^2}+C$ (c) $\frac{e^x}{(1+x^2)^2}+C$ (d) $\frac{-e^x}{(1+x^2)^2}+C$

10. $\int \frac{\sin^6 x}{\cos^8 x}dx$ is equal to

(a) $\frac{\tan^6 x}{5}+C$ (b) $\frac{\tan^7 x}{5}$ (c) $\frac{\tan^7 x}{7}+C$ (d) none of these

11. If $\int \frac{x^3}{\sqrt{1+x^2}}\,dx = a(1+x^2)^{3/2} + b\sqrt{1+x^2} + C$, then [NCERT Exemplar]

(a) $a=\frac{1}{3}, b=1$ (b) $a=-\frac{1}{3}, b=1$ (c) $a=\frac{-1}{3}, b=-1$ (d) $a=\frac{1}{3}, b=-1$

12. The integral $\int \frac{x^9}{(4x^2+1)^6}\,dx$ is equal to [NCERT Exemplar]

(a) $\frac{1}{5x}\left(4+\frac{1}{x^2}\right)^{-5}+C$ (b) $\frac{1}{5}\left(4+\frac{1}{x^2}\right)^{-5}+C$

(c) $\frac{1}{10x}(5)^{-5}+C$ (d) $\frac{1}{10}\left(\frac{1}{x^2}+4\right)^{-5}+C$

13. $\int_0^{\pi/8} \tan^2(2x)\,dx$ is equal to [CBSE 2020, (65/4/1)]

(a) $\frac{4-\pi}{8}$ (b) $\frac{4+\pi}{8}$ (c) $\frac{4-\pi}{4}$ (d) $\frac{4-\pi}{2}$

14. $\int_{a+c}^{b+c} f(x)\,dx$ is equal to

(a) $\int_a^b f(x-c)\,dx$ (b) $\int_a^b f(x+c)\,dx$ (c) $\int_a^b f(x)\,dx$ (d) $\int_{a-c}^{b-c} f(x)\,dx$

15. If f and g are continuous functions in [0, 1] satisfying $f(x) = f(a-x)$ and $g(x) + g(a-x) = a$, then $\int_0^a f(x).g(x)\,dx$ is equal to [NCERT Exemplar]

(a) $\frac{a}{2}$ (b) $\frac{a}{2}\int_0^a f(x)\,dx$ (c) $\int_0^a f(x)\,dx$ (d) $a\int_0^a f(x)\,dx$

16. $\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1}\,dx$ is equal to [NCERT Exemplar]

(a) log 2 (b) 2 log 2 (c) $\frac{1}{2}\log 2$ (d) 4 log 2

17. $\int_0^1 \frac{e^t}{1+t}\,dt = a$, then $\int_0^1 \frac{e^t}{(1+t)^2}\,dt$ is equal to

(a) $a-1+\frac{e}{2}$ (b) $a+1-\frac{e}{2}$ (c) $a-1-\frac{e}{2}$ (d) $a+1+\frac{e}{2}$

18. $\int_{-2}^{2} |x\cos\pi x|\,dx$ is equal to [NCERT Exemplar]

(a) $\frac{8}{\pi}$ (b) $\frac{4}{\pi}$ (c) $\frac{2}{\pi}$ (d) $\frac{1}{\pi}$

19. The integral value of $\int_0^{\frac{\pi}{2}} \frac{\tan x}{1+m^2\tan^2 x}\,dx$, $m > 0$, is

(a) $\frac{\log m}{(m^2-1)}$ (b) $\log\left(\frac{m^2-m}{2}\right)$ (c) log 3 m (d) 0

20. $\int_{-1}^{2} |x|\,dx$ is equal to

(a) 1 (b) $\frac{3}{2}$ (c) 2 (d) $\frac{5}{2}$

Answers

1. (*a*)	**2.** (*a*)	**3.** (*a*)	**4.** (*c*)	**5.** (*a*)	**6.** (*c*)
7. (*c*)	**8.** (*a*)	**9.** (*a*)	**10.** (*c*)	**11.** (*b*)	**12.** (*d*)
13. (*a*)	**14.** (*b*)	**15.** (*b*)	**16.** (*b*)	**17.** (*b*)	**18.** (*a*)
19. (*a*)	**20.** (*d*)				

Solutions of Selected Multiple Choice Questions

1. We have, $I = \int x^2 e^{x^3}\, dx$

$$x^3 = t \quad \Rightarrow 3x^2 dx = dt \quad \Rightarrow x^2 dx = \frac{dt}{3}$$

$$\therefore \quad I = \int e^t \frac{dt}{3} = \frac{1}{3}\int e^t dt = \frac{1}{3}e^t + C$$

$$= \frac{1}{3}e^{x^3} + C$$

2. Let $I = \int \frac{e^x(1+x)}{\cos^2(xe^x)}$

Let $xe^x = t \quad \Rightarrow \quad (xe^x + e^x)dx = dt$

$$\Rightarrow \quad (x+1)e^x dx = dt$$

$$\therefore \quad I = \int \frac{dt}{\cos^2 t}$$

$$\Rightarrow \quad I = \int \sec^2 t dt = \tan t + C = \tan(xe^x) + C$$

4. $I = \int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{(\sin^2 x + \cos^2 x)dx}{\sin^2 x \cos^2 x}$

$$= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx = \tan x - \cot x + C$$

7. $I = \int \frac{dx}{\sin(x-a)\sin(x-b)} = \frac{1}{\sin(b-a)}\int \frac{\sin(b-a)}{\sin(x-a)\sin(x-b)}dx$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin(x-a-x+b)}{\sin(x-a)\sin(x-b)}dx$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin\{(x-a)-(x-b)\}}{\sin(x-a)\sin(x-b)}dx$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin(x-a)\cos(x-b)-\cos(x-a)\sin(x-b))}{\sin(x-a)\sin(x-b)}dx$$

$$= \frac{1}{\sin(b-a)}\int [\cot(x-b)-\cot(x-a)]dx$$

$$= \frac{1}{\sin(b-a)}[\log|\sin(x-b)| - \log|\sin(x-a)|\,|] + C$$

$$= \operatorname{cosec}(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$$

13. $\int_0^{\pi/8} \tan^2(2x)dx = \int_0^{\pi/8} \left(\sec^2(2x) - 1\right)dx$

$$= \left[\frac{\tan(2x)}{2} - x\right]_0^{\pi/8}$$

$$= \frac{\tan\frac{\pi}{4}}{2} - \frac{\pi}{8} - 0$$

$$= \frac{1}{2} - \frac{\pi}{8} = \frac{4-\pi}{8}$$

15. $I = \int_0^a f(x).g(x)\,dx = \int_0^a f(a-x)g(a-x)\,dx = \int_0^a f(x)(a - g(x))\,dx$

$$= a\int_0^a f(x)\,dx - \int_0^a f(x).g(x)\,dx \Rightarrow I = a\int_0^a f(x)\,dx - I \Rightarrow I = \frac{a}{2}\int_0^a f(x)\,dx$$

16. $I = \int_{-1}^{1} \frac{x^3 + |x| + 1}{x^2 + 2|x| + 1}\,dx$

$$= \int_{-1}^{1} \frac{x^3}{x^2 + 2|x| + 1} + \int_{-1}^{1} \frac{|x| + 1}{x^2 + 2|x| + 1}\,dx = 0 + 2\int_0^1 \frac{|x|+1}{(|x|+1)^2}\,dx$$

[odd function + even function]

$$= 2\int_0^1 \frac{x+1}{(x+1)^2}\,dx = 2\int_0^1 \frac{1}{x+1}\,dx = 2\Big|\log|x+1|\Big|_0^1 = 2\log 2.$$

18. Since $I = \int_{-2}^{2} |x\cos\pi x|\,dx = 2\int_0^2 |x\cos\pi x|\,dx$

$$= 2\left\{\int_0^{\frac{1}{2}} |x\cos\pi x|\,dx + \int_{\frac{1}{2}}^{\frac{3}{2}} |x\cos\pi x|\,dx + \int_{\frac{3}{2}}^{2} |x\cos\pi x|\,dx\right\} = \frac{8}{\pi}$$

Fill in the Blanks [1 mark]

1. $\int_0^{\pi/2} \frac{\sin^n x\,dx}{\sin^n x + \cos^n x} =$ ______________ .

2. $\int_0^{\pi/2} \cos x\, e^{\sin x}\,dx =$ ______________ . [*NCERT Exemplar*]

3. $\int e^{\tan^{-1}x}\left(1 + \frac{x}{1+x^2}\right)dx =$ ______________ .

4. A primitive of $|x|$, when $x < 0$ is ______________ .

5. The value of $\int_{-\pi}^{\pi} \sin^3 x \cos^2 x\,dx =$ ______________ .

Answers

1. $\frac{\pi}{4}$ **2.** $e - 1$ **3.** $xe^{\tan^{-1}x} + C$ **4.** $-\frac{1}{2}x^2 + C$ **5.** 0

Solutions of Selected Fill in the Blanks

2. Let I, $= \int_0^{\pi/2} \cos x\, e^{\sin x}\,dx$

Let $\sin x = t \Rightarrow \cos x\,dx = dt$

When $x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{2} \Rightarrow t = 1$

$\therefore \quad I = \int_0^1 e^t dt = [e^t]_0^1 = e^1 - e^0 = e - 1$

3. Let $I = \int e^{\tan^{-1}x}\left(1 + \dfrac{x}{1+x^2}\right)dx$

Let $\tan^{-1}x = t \quad \Rightarrow \quad \dfrac{1}{1+x^2}dx = dt$

$$\therefore \quad I = \int e^{\tan^{-1}x}\left(\frac{1+x^2+x}{1+x^2}\right)dx$$

$$= \int e^t(1+\tan^2 t + \tan t)\,dx$$

$$= \int e^t(\sec^2 t + \tan t)dx = \int e^t(\tan t + \sec^2 t)\,dt$$

$$= e^t \tan t + C$$

$$= xe^{\tan^{-1}x} + C$$

5. Let $I = \int_{-\pi}^{\pi} \sin^3 x \cos^2 x \, dx$

Let $f(x) = \sin^3 x . \cos^2 x \quad \Rightarrow \quad f(-x) = -\sin^3 x \cos^2 x$

$\Rightarrow \quad f(-x) = -f(x)$

$\therefore \quad f(x)$ is an odd function

$\therefore \quad I = \int_{-\pi}^{\pi} \sin^3 x \cos^2 x \, dx = 0 \qquad \therefore \left(\int_{-a}^{a} f(x)dx = 0, \text{ when } f(x) \text{ is an odd function}\right)$

Very Short Answer Questions [1 mark]

1. Evaluate: $\int \dfrac{dx}{9+4x^2}$ **[*CBSE 2020 (65/3/1)*]**

Sol. $\int \dfrac{dx}{9+4x^2} = \dfrac{1}{4}\int \dfrac{dx}{\frac{9}{4}+x^2} = \dfrac{1}{4}\int \dfrac{dx}{\left(\frac{3}{2}\right)^2 + x^2}$

$$= \frac{1}{4}\int \frac{dx}{x^2 + \left(\frac{3}{2}\right)^2} = \frac{1}{4} \times \frac{2}{3}\tan^{-1}\left(\frac{x}{3/2}\right) + C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) + C$$

2. Find: $\int \dfrac{2^{x+1} - 5^{x-1}}{10^x}dx$ **[*CBSE 2020 (65/4/1)*]**

Sol. $\int \dfrac{2^{x+1} - 5^{x-1}}{10^x}dx = \int \dfrac{2^{x+1}}{(5\times 2)^x}dx - \int \dfrac{5^{x-1}}{(5\times 2)^x}dx$

$$= \int \frac{2^x \times 2}{5^x \times 2^x}dx - \int \frac{5^x}{5^x \times 2^x} \times \frac{1}{5}dx$$

$$= 2\int 5^{-x}dx - \frac{1}{5}\int 2^{-x}dx = \frac{-2\times 5^{-x}}{\log 5} - \frac{1}{5} \times \frac{-2^{-x}}{\log 2} + C$$

$$= \frac{-2}{\log 5}5^{-x} + \frac{1}{5} \times \frac{2^{-x}}{\log 2} + C$$

$$= \frac{1}{5\log 2} . \frac{1}{2^x} - \frac{2}{\log 5} . \frac{1}{5^x} + C$$

3. $\int \frac{(x^2+2)}{x+1}dx$ **[NCERT Exemplar]**

Sol. Let
$$I = \int \frac{x^2+2}{x+1}dx$$
$$= \int\left(x-1+\frac{3}{x+1}\right)dx$$
$$= \int (x-1)\,dx + 3\int \frac{1}{x+1}\,dx$$
$$= \frac{x^2}{2} - x + 3\log|x+1| + C$$

4. Evaluate: $\int \sec^2(7-x)dx$ **[CBSE Delhi 2009; (AI) 2010]**

Sol. $$\int \sec^2(7-x)dx = \frac{\tan(7-x)}{-1} + C = -\tan(7-x) + C$$

5. Evaluate: $\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}dx$ **[CBSE (AI) 2009]**

Sol. Let $\sqrt{x} = z \Rightarrow \frac{1}{2\sqrt{x}}dx = dz \Rightarrow \frac{dx}{\sqrt{x}} = 2dz$

$\therefore$ $$\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}dx = 2\int \sec^2 z\,dz = 2\tan z + C = 2\tan\sqrt{x} + C$$

6 Evaluate: $\int \frac{dx}{x + x\log x}$ **[CBSE (F) 2009]**

Sol. Let $$I = \int \frac{dx}{x(1+\log x)}$$

Put $1 + \log x = z, \Rightarrow \frac{1}{x}dx = dz$

$\therefore$ $$I = \int \frac{dz}{z} = \log|z| + C = \log|1+\log x| + C$$

7 If $\int_0^a \frac{dx}{1+4x^2} = \frac{\pi}{8}$**, then find the value of** a**.** **[CBSE 2020 (65/4/1)]**

Sol. We have, $$\int_0^a \frac{dx}{1+4x^2} = \frac{\pi}{8}$$

$$\Rightarrow \quad \frac{1}{4}\int_0^a \frac{dx}{\frac{1}{4}+x^2} = \frac{\pi}{8}$$

$$\Rightarrow \quad \frac{1}{4}\int_0^a \frac{dx}{(x)^2+\left(\frac{1}{2}\right)^2} = \frac{\pi}{8}$$

$$\Rightarrow \quad \frac{1}{4}\times 2\left[\tan^{-1}2x\right]_0^a = \frac{\pi}{8}$$

$$\Rightarrow \quad \tan^{-1}2a - \tan^{-1}0 = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}2a = \frac{\pi}{4} \quad \Rightarrow \quad 2a = \tan\frac{\pi}{4} = 1$$

$$\Rightarrow \quad a = \frac{1}{2}$$

8 **Find the value of $\int_1^4 |x-5|\, dx$** **[CBSE 2020 (65/5/1)]**

Sol. We have, $\int_1^4 |x-5|dx = \int_1^4 -(x-5)dx$

$$= -\left[\frac{x^2}{2}-5x\right]_1^4 = -\left[\frac{(4)^2}{2}-5\times 4-\frac{(1)^2}{2}+5\times 1\right]$$

$$= -\left[\frac{16}{2}-20-\frac{1}{2}+5\right] = -\left[\frac{15}{2}-15\right]$$

$$= \frac{15}{2}$$

9. **If $\int_0^1 (3x^2+2x+k)dx = 0$, then find the value of k.** **[CBSE Delhi 2009]**

Sol. Given, $\int_0^1 (3x^2+2x+k)dx = 0 \quad \Rightarrow \quad \left[\frac{3x^3}{3}+\frac{2x^2}{2}+kx\right]_0^1 = 0$

$\Rightarrow \quad [x^3+x^2+kx]_0^1 = 0 \quad \Rightarrow \quad (1+1+k)-(0) = 0 \quad \Rightarrow \quad k = -2$

Short Answer Questions-I [2 marks]

1. **Find $\int \frac{\tan^3 x}{\cos^3 x}dx$** **[CBSE 2020 (65/2/1)]**

Sol. We have,

$$I = \int \frac{\tan^3 x}{\cos^3 x}dx = \int \frac{\frac{\sin^3 x}{\cos^3 x}}{\cos^3 x}dx$$

$$\Rightarrow \quad I = \int \frac{\sin^3 x}{\cos^6 x}dx$$

Let $\cos x = t \quad \Rightarrow \quad -\sin x\, dx = dt$

$\Rightarrow \quad \sin x\, dx = -dt$

$$\therefore \quad I = \int \frac{\sin^2 x . \sin x\, dx}{\cos^6 x} = \int \frac{(1-\cos^2 x).\sin x}{\cos^6 x}dx$$

$$= -\int\left(\frac{1-t^2}{t^6}\right)dt = -\int \frac{1}{t^6}dt + \int \frac{t^2}{t^6}dt$$

$$= -\int t^{-6}dt + \int t^{-4}dt = \frac{t^{-5}}{5}+\frac{t^{-3}}{-3}+C$$

$$= \frac{1}{5}\times\frac{1}{t^5}-\frac{1}{3}\times\frac{1}{t^3}+C$$

$$= \frac{1}{5(\cos x)^5}-\frac{1}{3(\cos x)^3}+C = \frac{1}{5\cos^5 x}-\frac{1}{3\cos^3 x}+C$$

2. **Find $\int e^x \frac{\sqrt{1+\sin 2x}}{1+\cos 2x}dx$.** **[CBSE Sample Paper 2018]**

Sol. $I = \int e^x \frac{\sqrt{1+\sin 2x}}{1+\cos 2x}dx = \int e^x \frac{\sqrt{\sin^2 x+\cos^2 x+2\sin x.\cos x}}{1+\cos 2x}dx$

$$= \int e^x \frac{\sqrt{(\sin x+\cos x)^2}}{1+\cos 2x}dx = \int e^x \frac{\sin x+\cos x}{2\cos^2 x}dx = \frac{1}{2}\int e^x\left(\frac{\sin x}{\cos^2 x}+\frac{\cos x}{\cos^2 x}\right)dx$$

$$= \frac{1}{2} \int e^x (\sec x + \sec x \,.\, \tan x)\, dx$$

$$= \frac{1}{2} e^x . \sec x + C \qquad [\because \int e^x (f(x) + f'(x)) dx = e^x f(x) + C]$$

3. Find $\int \frac{x-1}{(x-2)(x-3)} dx.$ **[CBSE 2019 (65/5/1)]**

Sol. $\because \frac{x-1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$

where $A = \left.\frac{x-1}{x-3}\right|_{x=2}$ & $B = \left.\frac{x-1}{x-2}\right|_{x=3}$

$\therefore \quad A = \frac{1}{-1} = -1$ & $B = 2$

$$\Rightarrow \quad \frac{x-1}{(x-2)(x-3)} = \frac{-1}{(x-2)} + \frac{2}{(x-3)}$$

$$\Rightarrow \quad \int \frac{x-1}{(x-2)(x-3)} dx = -\int \frac{dx}{x-2} + 2\int \frac{dx}{x-3} = -\log(x-2) + 2\log(x-3) + C$$

$$\int \frac{x-1}{(x-2)(x-3)} dx = -\log(x-2) + \log(x-3)^2 + C = \log\frac{(x-3)^2}{(x-2)} + C$$

4. Find $\int_{-\pi/4}^{0} \frac{1+\tan x}{1-\tan x} dx$. **[CBSE 2019 (65/5/1)]**

Sol. $\int_{-\pi/4}^{0} \frac{1+\tan x}{1-\tan x} dx = \int_{-\pi/4}^{0} \tan\left(\frac{\pi}{4} + x\right) dx$

$= \log \sec\left(\frac{\pi}{4} + x\right)\Big|_{-\pi/4}^{0} = \log \sec\left(\frac{\pi}{4}\right) - \log \sec\left(\frac{\pi}{4} - \frac{\pi}{4}\right)$

$= \log(\sqrt{2}) - \log(\sec(0)) = \log(\sqrt{2}) - \log 1$

$= \log\sqrt{2} = \frac{1}{2}\log 2$

5. Find $\int \frac{dx}{\sqrt{5-4x-2x^2}}$. **[CBSE 2019 (65/4/1)]**

Sol. $\int \frac{dx}{\sqrt{5-4x-2x^2}} = \int \frac{dx}{\sqrt{7-2-4x-2x^2}}$

$= \int \frac{dx}{\sqrt{7-2(1+2x+x^2)}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{7}{2} - (x+1)^2}}$

$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2 - (x+1)^2}} = \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x+1}{\sqrt{\frac{7}{2}}}\right) + C = \frac{1}{\sqrt{2}} \sin^{-1}\left(\sqrt{\frac{2}{7}}(x+1)\right) + C$

6. Evaluate $\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$. **[CBSE Delhi 2011]**

Sol. Let $t = \tan^{-1} x \Rightarrow dt = \frac{1}{1+x^2} dx$

Also when, $x = 0$, $t = 0$ and when $x = 1$, $t = \frac{\pi}{4}$

$$\therefore \quad \int_0^1 \frac{\tan^{-1} x}{1+x^2} dx = \int_0^{\frac{\pi}{4}} t\,dt$$

$$= \left[\frac{t^2}{2}\right]_0^{\pi/4} = \frac{1}{2}\left[\frac{\pi^2}{16} - 0\right] = \frac{\pi^2}{32}$$

7. Evaluate: $\int_0^1 \frac{dx}{\sqrt{2x+3}}$. ***[CBSE (F) 2009]***

Sol. Let $I = \int_0^1 \frac{dx}{\sqrt{2x+3}} = \int_0^1 (2x+3)^{-1/2} dx$

$$= \left[\frac{(2x+3)^{-1/2+1}}{\left(-\frac{1}{2}+1\right)\times 2}\right]_0^1 = \left[\frac{(2x+3)^{1/2}}{\frac{1}{2}\times 2}\right]_0^1 = 5^{1/2} - 3^{1/2} = \sqrt{5} - \sqrt{3}$$

Short Answer Questions-II **[3 marks]**

1. Evaluate: $\int \frac{2x}{(x^2+1)(x^2+3)} dx$ ***[CBSE Delhi 2011]***

Sol. Let $x^2 = z \Rightarrow 2x\,dx = dz$

$$\therefore \quad \int \frac{2x}{(x^2+1)(x^2+3)} dx = \int \frac{dz}{(z+1)(z+3)}$$

Using partial fraction.

Let $\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$ ...(*i*)

$$\frac{1}{(z+1)(z+3)} = \frac{A(z+3)+B(z+1)}{(z+1)(z+3)}$$

$\Rightarrow \quad 1 = A(z+3) + B(z+1) \quad \Rightarrow \quad 1 = (A+B)z + (3A+B)$

Equating the coefficient of z and constant, we get

$A + B = 0$ ...(*ii*)

and $3A + B = 1$ ...(*iii*)

Subtracting (*ii*) from (*iii*), we get

$2A = 1 \quad \Rightarrow \quad A = \frac{1}{2} \quad$ and $\quad B = -\frac{1}{2}$

Putting the values of A and B in (*i*), we get

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$$\therefore \quad \int \frac{2x\,dx}{(x^2+1)(x^2+3)} = \int\left(\frac{1}{2(z+1)} - \frac{1}{2(z+3)}\right)dz = \frac{1}{2}\int\frac{dz}{z+1} - \frac{1}{2}\int\frac{dz}{z+3}$$

$$= \frac{1}{2}\log|z+1| - \frac{1}{2}\log|z+3| + C = \frac{1}{2}\log|x^2+1| - \frac{1}{2}\log|x^2+3| + C$$

$$= \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C \qquad \left[\begin{array}{l}\text{Note: } \log m + \log n = \log m\,.\,n \\ \text{and } \quad \log m - \log n = \log m/n\end{array}\right]$$

$$= \log\sqrt{\frac{x^2+1}{x^2+3}} + C$$

2. Evaluate: $\int \frac{x^2}{1-x^4}dx$ **[NCERT Exemplar]**

Sol. Let $I = \int \frac{x^2}{1-x^4}dx = \int \frac{\frac{1}{2}+\frac{x^2}{2}-\frac{1}{2}+\frac{x^2}{2}}{(1-x^2)(1+x^2)}dx$ $[\because a^2-b^2=(a+b)(a-b)]$

$$= \int \frac{\frac{1}{2}(1+x^2)-\frac{1}{2}(1-x^2)}{(1-x^2)(1+x^2)}dx = \int \frac{\frac{1}{2}(1+x^2)}{(1-x)^2(1+x^2)}dx - \frac{1}{2}\int \frac{(1-x)^2}{(1-x)^2(1+x)^2}dx$$

$$\frac{1}{2}\int \frac{1}{1-x^2}dx - \frac{1}{2}\int \frac{1}{1+x^2}dx = \frac{1}{2}\cdot\frac{1}{2}\log\left|\frac{1+x}{1-x}\right| + C_1 - \frac{1}{2}\tan^{-1}x + C_2$$

$$= \frac{1}{4}\log\left|\frac{1+x}{1-x}\right| - \frac{1}{2}\tan^{-1}x + C \quad [\because C = C_1 + C_2]$$

3. Evaluate: $\int \sin x \sin 2x \sin 3x\, dx$ **[CBSE (F) 2010, Delhi 2012, 2019 (65/5/3)]**

Sol. Let $I = \int \sin x \sin 2x \sin 3x\, dx$

$$= \frac{1}{2}\int 2\sin x.\sin 2x.\sin 3x dx = \frac{1}{2}\int \sin x.(2\sin 2x.\sin 3x)dx$$

$$= \frac{1}{2}\int \sin x.(\cos x - \cos 5x)dx \quad [\because 2\sin A \sin B = \cos(A-B) - \cos(A+B)]$$

$$= \frac{1}{2\times 2}\int 2\sin x.\cos x dx - \frac{1}{2\times 2}\int 2\sin x.\cos 5x dx \quad [\because 2\cos A\sin B = \sin(A+B) - \sin(A-B)]$$

$$= \frac{1}{4}\int \sin 2x dx - \frac{1}{4}\int (\sin 6x - \sin 4x)dx$$

$$= -\frac{\cos 2x}{8} + \frac{\cos 6x}{24} - \frac{\cos 4x}{16} + C$$

4. Evaluate: $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x.\cos^2 x}dx$ **[CBSE Delhi 2014]**

Sol. Let $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x.\cos^2 x}dx \quad \Rightarrow \quad I = \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x.\cos^2 x}dx$

$$\Rightarrow \quad I = \int \frac{(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x.\cos^2 x + \cos^4 x)}{\sin^2 x.\cos^2 x}dx$$

$$\Rightarrow \quad I = \int \frac{\sin^4 x - \sin^2 x.\cos^2 x + \cos^4 x}{\sin^2 x.\cos^2 x}dx = \int \tan^2 x dx - \int dx + \int \cot^2 x dx$$

$$\Rightarrow \quad I = \int (\sec^2 x - 1)\,dx - x + \int (\text{cosec}^2 x - 1)dx$$

$$\Rightarrow \quad I = \int \sec^2 x dx + \int \text{cosec}^2 x dx - x - x - x + C = \tan x - \cot x - 3x + C$$

5. Evaluate: $\int \frac{\sin(x-a)}{\sin(x+a)}dx$ **[CBSE Delhi 2013; (F) 2015]**

Sol. Let $I = \int \frac{\sin(x-a)}{\sin(x+a)}dx$

Let $x + a = t \quad \Rightarrow \quad x = t - a \quad \Rightarrow \quad dx = dt$

$$\therefore \quad I = \int \frac{\sin(t-2a)}{\sin t}dt = \int \frac{\sin t.\cos 2a - \cos t.\sin 2a}{\sin t}dt$$

$$= \cos 2a\int dt - \int \sin 2a.\cot t\, dt = \cos 2a.t - \sin 2a.\log|\sin t| + C$$

$$= \cos 2a\,.\,(x+a) - \sin 2a\,.\,\log|\sin(x+a)| + C$$

$$= x\cos 2a + a\cos 2a - (\sin 2a)\log|\sin(x+a)| + C$$

6. Evaluate: $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx$ **[CBSE Delhi 2009, 2019 (65/5/1)]**

Sol. Let $I = \int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx$

Put $e^x = t \quad \Rightarrow \quad e^x\,dx = dt$, we get

$\therefore \quad I = \int \frac{dt}{\sqrt{5-4t-t^2}} = \int \frac{dt}{\sqrt{-(t^2+4t-5)}} = \int \frac{dt}{\sqrt{-(t^2+2.t.2+2^2-9)}}$

$= \int \frac{dt}{\sqrt{3^2-(t+2)^2}} = \sin^{-1}\frac{t+2}{3} + C = \sin^{-1}\left(\frac{e^x+2}{3}\right) + C$

7. Evaluate: $\int e^x\left(\frac{\sin 4x - 4}{1-\cos 4x}\right)dx$ **[CBSE Delhi 2010]**

Sol. Let $I = \int e^x\left(\frac{\sin 4x-4}{1-\cos 4x}\right)dx$

$= \int e^x\left(\frac{2\sin 2x.\cos 2x - 4}{2\sin^2 2x}\right)dx \qquad [\because \sin 2x = 2\sin x \,.\, \cos x \text{ and } \cos 2x = 1-2\sin^2 x]$

$= \int e^x(\cot 2x - 2\operatorname{cosec}^2 2x)\,dx$

Let $f(x) = \cot 2x \quad \therefore \quad f'(x) = -2\operatorname{cosec}^2 2x$

$\therefore \quad I = \int e^x(f(x)+f'(x))\,dx$

$\Rightarrow \quad I = e^x \,.\, f(x) + C = e^x \,.\, \cot 2x + C \qquad [\because \int e^x(f(x)+f'(x))\,dx = e^x f(x) + C]$

8. Evaluate: $\int \frac{x+2}{\sqrt{x^2+5x+6}}dx$ **[CBSE (AI) 2014]**

Sol. Let $I = \int \frac{x+2}{\sqrt{x^2+5x+6}}dx$

Now, we can express as

$x+2 = A\frac{d}{dx}(x^2+5x+6) + B$

$\Rightarrow \quad x+2 = A(2x+5)+B \quad \Rightarrow \quad x+2 = 2Ax + (5A+B)$

Equating coefficients both sides, we get

$2A = 1,\ 5A+B = 2 \quad \Rightarrow \quad A = \frac{1}{2}, B = 2-\frac{5}{2} = -\frac{1}{2}$

$\therefore \quad x+2 = \frac{1}{2}(2x+5) - \frac{1}{2}$

Hence, $I = \int \frac{\frac{1}{2}(2x+5)-\frac{1}{2}}{\sqrt{x^2+5x+6}}dx = \frac{1}{2}\int\frac{2x+5}{\sqrt{x^2+5x+6}}dx - \frac{1}{2}\int\frac{dx}{\sqrt{x^2+5x+6}}$

$I = \frac{1}{2}.I_1 - \frac{1}{2}I_2 \qquad \ldots (i)$

where, $I_1 = \int \frac{2x+5}{\sqrt{x^2+5x+6}}dx,\ I_2 = \int\frac{dx}{\sqrt{x^2+5x+6}}$

Now, $I_1 = \int \frac{2x+5}{\sqrt{x^2+5x+6}}dx$

Let $x^2+5x+6 = z \quad \Rightarrow \quad (2x+5)dx = dz$

$\therefore \quad I_1 = \int \frac{dz}{\sqrt{z}} = \int z^{\frac{-1}{2}} dz = \frac{z^{-\frac{1}{2}+1}}{\frac{-1}{2}+1} + C_1 = 2\sqrt{z} + C_1 = 2\sqrt{x^2+5x+6} + C_1$

Again $I_2 = \int \frac{dx}{\sqrt{x^2+5x+6}} = \int \frac{dx}{\sqrt{x^2 + 2\times x \times \frac{5}{2} + \left(\frac{5}{2}\right)^2 - \frac{25}{4} + 6}}$

$= \int \frac{dx}{\left(x+\frac{5}{2}\right)^2 - \frac{1}{4}} = \int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$

$= \log \left| \left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6} \right| + C_2$

Putting the value of I_1 and I_2 in (*i*), we get

$I = \frac{1}{2}\{2\sqrt{x^2+5x+6} + C_1\} - \frac{1}{2}\left\{\log\left|\left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right| + C_2\right\}$

$= \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right| + \frac{1}{2}C_1 - \frac{1}{2}C_2$

$= \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right| + C \qquad \left[\text{Here, } C = \frac{1}{2}C_1 - \frac{1}{2}C_2\right]$

9. Evaluate: $\int \frac{(x^2-3x)}{(x-1)(x-2)}dx$ **[*CBSE (F) 2010*]**

Sol. Let $I = \int \frac{(x^2-3x)}{(x-1)(x-2)}dx = \int \frac{(x^2-3x)}{x^2-3x+2}dx$

$= \int \frac{x^2-3x+2-2}{x^2-3x+2}dx = \int dx - \int \frac{2dx}{x^2-3x+2}$

$= x - 2\int \frac{dx}{x^2 - 2.x.\frac{3}{2} + \frac{9}{4} - \frac{9}{4} + 2} = x - 2\int \frac{dx}{\left(x-\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$

$= x - 2\log\left|\frac{x-\frac{3}{2}-\frac{1}{2}}{x-\frac{3}{2}+\frac{1}{2}}\right| + C \qquad \left[\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$

$= x - 2\log\left|\frac{x-2}{x-1}\right| + C$

10. Find: $\int \sin^{-1}\sqrt{\frac{x}{a+x}}dx$ **[*NCERT Exemplar*]**

Sol. Let $I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}dx$

Put $x = a\tan^2\theta \quad \Rightarrow \quad dx = 2a\tan\theta\sec^2\theta\, d\,\theta$

$\therefore \quad I = \int \sin^{-1}\left(\sqrt{\frac{a\tan^2\theta}{a+a\tan^2\theta}}\right)(2a\tan\theta.\sec^2\theta)\,d\theta = 2a\int \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)\tan\theta.\sec^2\theta d\theta$

$= 2a\int \sin^{-1}(\sin\theta)\tan\theta.\sec^2\theta d\theta = 2a\int \underset{\text{I}}{\theta}.\underset{\text{II}}{\tan\theta\sec^2\theta} d\theta$

$= 2a\left[\theta.\int \tan\theta.\sec^2\theta d\theta - \int\left(\frac{d}{d\theta}\theta.\int\tan\theta.\sec^2\theta\,d\theta\right)d\theta\right]$

$= 2a\left[\theta.\frac{\tan^2\theta}{2} - \int \frac{\tan^2\theta}{2}d\theta\right]$

$$= a\theta \tan^2\theta - a\int (\sec^2\theta - 1)\, d\theta = a\theta . \tan^2\theta - a\tan\theta + a\theta + C$$

$$= a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1}\sqrt{\frac{x}{a}}\right] + C$$

11. Find : $\int \frac{dx}{\sin x + \sin 2x}$ **[*CBSE Delhi 2012*]**

Sol. Here, $I = \int \frac{1}{\sin x + \sin 2x} dx$

$\Rightarrow \quad I = \int \frac{1}{\sin x + 2\sin x \cos x} dx \qquad \Rightarrow \quad I = \int \frac{1}{\sin x (1 + 2\cos x)} dx$

$\Rightarrow \quad I = \int \frac{\sin x}{\sin^2 x (1 + 2\cos x)} dx \qquad \Rightarrow \quad I = \int \frac{\sin x}{(1 - \cos^2 x)(1 + 2\cos x)} dx$

Let $\cos x = z \quad \Rightarrow \; -\sin x \, dx = dz$

$\Rightarrow \quad I = \int \frac{-dz}{(1 - z^2)(1 + 2z)} \qquad \Rightarrow \quad I = -\int \frac{dz}{(1 + z)(1 - z)(1 + 2z)}$

Here, integrand is proper rational function. Therefore, by the form of partial fraction, we can write

$$\frac{1}{(1 + z)(1 - z)(1 + 2z)} = \frac{A}{1 + z} + \frac{B}{1 - z} + \frac{C}{1 + 2z} \qquad ...(i)$$

$$\Rightarrow \quad \frac{1}{(1 + z)(1 - z)(1 + 2z)} = \frac{A(1 - z)(1 + 2z) + B(1 + z)(1 + 2z) + C(1 + z)(1 - z)}{(1 + z)(1 - z)(1 + 2z)}$$

$$\Rightarrow \quad 1 = A(1 - z)(1 + 2z) + B(1 + z)(1 + 2z) + C(1 + z)(1 - z) \qquad ....(ii)$$

Putting the value of $z = -1$ in (*ii*), we get

$\Rightarrow \quad 1 = -2A + 0 + 0 \qquad \Rightarrow \quad A = -1/2$

Again, putting the value of z = 1 in (*ii*), we get

$\Rightarrow \quad 1 = 0 + B.2.(1 + 2) + 0 \quad \Rightarrow \quad 1 = 6B \quad \Rightarrow \quad B = \frac{1}{6}$

Similarly, putting the value of $z = -\frac{1}{2}$ in (*ii*), we get

$\Rightarrow \quad 1 = 0 + 0 + C\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) \quad \Rightarrow \quad 1 = \frac{3}{4}C \qquad \Rightarrow \quad C = \frac{4}{3}$

Putting the value of A, B, C in (*i*), we get

$$\frac{1}{(1 + z)(1 - z)(1 - 2z)} = \frac{-1}{2(1 + z)} + \frac{1}{6(1 - z)} + \frac{4}{3(1 + 2z)}$$

$$\therefore \quad I = -\int\left[-\frac{1}{2(1 + z)} + \frac{1}{6(1 - z)} + \frac{4}{3(1 + 2z)}\right]dz = \int\left[\frac{1}{2(1 + z)} - \frac{1}{6(1 - z)} - \frac{4}{3(1 + 2z)}\right]dz$$

$$\Rightarrow \quad I = \frac{1}{2}\log|1 + z| + \frac{1}{6}\log|1 - z| - \frac{4}{3\times 2}\log|1 + 2z| + C$$

Putting the value of z, we get

$$\Rightarrow \quad I = \frac{1}{2}\log|1 + \cos x| + \frac{1}{6}\log|1 - \cos x| - \frac{2}{3}\log|1 + 2\cos x| + C$$

12. Find: $\int \frac{x^2}{x^4 - x^2 - 12} dx$ **[*NCERT Exemplar*]**

Sol. Let $\quad I = \int \frac{x^2}{x^4 - x^2 - 12} dx = \int \frac{x^2}{x^4 - 4x^2 + 3x^2 - 12} dx$

$$= \int \frac{x^2 dx}{x^2(x^2 - 4) + 3(x^2 - 4)}$$

$$= \int \frac{x^2}{(x^2-4)(x^2+3)} dx$$

$\Rightarrow \quad \frac{t}{(t-4)(t+3)} = \frac{A}{t-4} + \frac{B}{t+3}$ [let $x^2 = t$] $\Rightarrow \quad t = A(t+3) + B(t-4)$

On comparing the coefficient of t on both sides, we get

$$A + B = 1$$

$\Rightarrow \quad 3A - 4B = 0 \quad \Rightarrow \quad 3(1-B) - 4B = 0$

$\Rightarrow \quad 3 - 3B - 4B = 0 \quad \Rightarrow \quad 7B = 3 \quad \Rightarrow \quad B = \frac{3}{7}$

If $B = \frac{3}{7}$, then $A + \frac{3}{7} = 1 \Rightarrow \quad A = 1 - \frac{3}{7} = \frac{4}{7}$

Now, $\frac{x^2}{(x^2-4)(x^2+3)} = \frac{4}{7(x^2-4)} + \frac{3}{7(x^2+3)}$

$$\therefore \quad = \frac{4}{7}\int \frac{1}{x^2-(2)^2}dx + \frac{3}{7}\int \frac{1}{x^2+(\sqrt{3})^2}dx$$

$$= \frac{4}{7}\cdot\frac{1}{2.2}\log\left|\frac{x-2}{x+2}\right| + \frac{3}{7}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

$$= \frac{1}{7}\log\left|\frac{x-2}{x+2}\right| + \frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

13. Evaluate : $\int \frac{x^2}{(x^2+4)(x^2+9)} dx$ **[*CBSE Delhi 2013; (F) 2015*]**

Sol. Let $I = \int \frac{x^2}{(x^2+4)(x^2+9)} dx$

Put $x^2 = t$, we get

$$\therefore \quad \frac{x^2}{(x^2+4)(x^2+9)} = \frac{t}{(t+4)(t+9)}$$

Now, $\frac{t}{(t+4)(t+9)} = \frac{A}{t+4} + \frac{B}{t+9} = \frac{A(t+9)+B(t+4)}{(t+4)(t+9)}$

$\Rightarrow \quad t = (A+B)t + (9A+4B)$

Equating the coefficients, we get

$$A + B = 1, \qquad 9A + 4B = 0$$

Solving above two equations, we get

$$A = -\frac{4}{5}, B = \frac{9}{5}$$

$$\therefore \quad \frac{x^2}{(x^2+4)(x^2+9)} = -\frac{4}{5(x^2+4)} + \frac{9}{5(x^2+9)}$$

$$\Rightarrow \quad \int \frac{x^2\,dx}{(x^2+4)(x^2+9)} = -\frac{4}{5}\int\frac{dx}{x^2+2^2} + \frac{9}{5}\int\frac{dx}{x^2+3^2} = -\frac{4}{5}\times\frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{9}{5}\times\frac{1}{3}\tan^{-1}\frac{x}{3} + C$$

$$= -\frac{2}{5}\tan^{-1}\frac{x}{2} + \frac{3}{5}\tan^{-1}\frac{x}{3} + C$$

14. Find: $\int \frac{(3\sin\theta - 2)\cos\theta}{5 - \cos^2\theta - 4\sin\theta} d\theta$ **[CBSE Delhi 2016]**

Sol. We have

$$I = \int \frac{(3\sin\theta - 2)\cos\theta}{5 - \cos^2\theta - 4\sin\theta} d\theta$$

Let $\sin\theta = z \Rightarrow \cos\theta\, d\theta = dz$

$$\therefore \quad I = \int \frac{(3z-2)dz}{5-(1-z^2)-4z}$$

$$= \int \frac{(3z-2)dz}{5-1+z^2-4z} = \int \frac{(3z-2)}{4-4z+z^2}dz = \int \frac{3z-2}{(z-2)^2}dz = \int \frac{3z}{(z-2)^2}dz - 2\int \frac{dz}{(z-2)^2}$$

Let $z - 2 = t \Rightarrow dz = dt$

$$= \int \frac{3(t+2)dt}{t^2} - 2\int \frac{dt}{t^2} = 3\int \frac{t.dt}{t^2} + 6\int \frac{dt}{t^2} - 2\int \frac{dt}{t^2} = 3\int \frac{dt}{t} + 4\int \frac{dt}{t^2} = 3\log|t| + 4\frac{t^{-2+1}}{-2+1} + C$$

$$= 3\log|t| - 4.\frac{1}{t} + C$$

Putting value of t in terms of z then z in terms of θ, we get

$$= 3\log|\sin\theta - 2| - \frac{4}{\sin\theta - 2} + C$$

15. Find: $\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$ **[CBSE Delhi 2016]**

Sol. We have $I = \int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx = \int \frac{x^{1/2}dx}{\sqrt{a^3-x^3}} = \int \frac{x^{1/2}dx}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}}$

Let $x^{3/2} = t \Rightarrow \frac{3}{2}x^{1/2}dx = dt \Rightarrow x^{1/2}dx = \frac{2}{3}dt$

$$I = \frac{2}{3}\int \frac{dt}{\sqrt{(a^{3/2})^2 - t^2}} \qquad [\because x^{3/2} = t \Rightarrow x^3 = t^2]$$

$$= \frac{2}{3}\sin^{-1}\left(\frac{t}{a^{3/2}}\right) + C = \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + C$$

$$= \frac{2}{3}\sin^{-1}\sqrt{\frac{x^3}{a^3}} + C$$

16. Find: $\int \frac{(2x-5)e^{2x}}{(2x-3)^3}dx$ **[CBSE (North) 2016]**

Sol. We have,

$$\int \frac{(2x-5)e^{2x}}{(2x-3)^3}dx = \int e^{2x}\left[\frac{(2x-3)-2}{(2x-3)^3}\right]dx$$

$$= \int e^3.e^{2x-3}\left[\frac{1}{(2x-3)^2} - \frac{2}{(2x-3)^3}\right]dx = e^3\int e^{2x-3}\left[\frac{1}{(2x-3)^2} - \frac{2}{(2x-3)^3}\right]dx$$

Let $2x - 3 = t \Rightarrow 2dx = dt \Rightarrow dx = \frac{dt}{2}$

$$\Rightarrow \quad I = \frac{e^3}{2}\int e^t\left[\frac{1}{t^2} - \frac{2}{t^3}\right]dt \qquad \Rightarrow \qquad I = \frac{e^3}{2}e^t.\frac{1}{t^2} + C$$

Putting $t = 2x - 3$

$$I = \frac{e^3}{2} e^{2x-3} \frac{1}{(2x-3)^2} + C \qquad \Rightarrow \qquad I = \frac{e^{2x}}{2(2x-3)^2} + C$$

17. Find : $\int (2x+5)\sqrt{10-4x-3x^2}\,dx$ **[*CBSE (F) 2016*]**

Sol. Let, $I = \int (2x+5)\sqrt{10-4x-3x^2}\,dx$

Let $(2x+5) = A\frac{d}{dx}(10-4x-3x^2) + B$

$\Rightarrow \quad 2x+5 = A(-4-6x) + B \qquad \Rightarrow \qquad 2x+5 = -4A - 6Ax + B$

Equating, we get

$-4A + B = 5 \quad ...(i) \quad$ and $\quad -6A = 2 \quad ...(ii)$

$(ii) \Rightarrow A = -\frac{1}{3}$

Now, from (i) $\frac{4}{3} + B = 5 \Rightarrow B = 5 - \frac{4}{3} = \frac{11}{3}$

$\therefore \quad 2x+5 = -\frac{1}{3}(-4-6x) + \frac{11}{3}$

Now, $\quad I = \int\left\{-\frac{1}{3}(-4-6x) + \frac{11}{3}\right\}\sqrt{10-4x-3x^2}\,dx$

$$= -\frac{1}{3}\int(-4-6x)\sqrt{10-4x-3x^2}\,dx + \frac{11}{3}\int\sqrt{10-4x-3x^2}\,dx$$

$$I = -\frac{1}{3}I_1 + \frac{11}{3}I_2 \qquad ...(iii)$$

where $\quad I_1 = \int(-4-6x)\sqrt{10-4x-3x^2}\,dx$ and $I_2 = \int\sqrt{(10-4x-3x^2)}\,dx$

Now, $\quad I_1 = \int(-4-6x)\sqrt{10-4x-3x^2}\,dx$

Let $\quad 10-4x-3x^2 = z \quad \Rightarrow \quad (-4-6x)\,dx = dz$

$$\therefore \quad I_1 = \int\sqrt{z}\,dz = \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C_1$$

$$= \frac{2}{3}(10-4x-3x^2)^{3/2} + C_1 \qquad ...(iv)$$

Again $\quad I_2 = \int\sqrt{10-4x-3x^2}\,dx = \int\sqrt{-3\left(x^2 + \frac{4}{3}x - \frac{10}{3}\right)}\,dx$

$$= \sqrt{3}\int\sqrt{-\left\{x^2 + 2x.\frac{2}{3} + \frac{4}{9} - \frac{4}{9} - \frac{10}{3}\right\}}\,dx = \sqrt{3}\int\sqrt{-\left\{\left(x+\frac{2}{3}\right)^2 - \frac{34}{9}\right\}}\,dx$$

$$= \sqrt{3}\int\sqrt{\frac{34}{9} - \left(x+\frac{2}{3}\right)^2}\,dx = \sqrt{3}\int\sqrt{\left(\frac{\sqrt{34}}{3}\right)^2 - \left(x+\frac{2}{3}\right)^2}\,dx$$

$$= \frac{\sqrt{3}}{2}\left(x+\frac{2}{3}\right)\sqrt{10-4x-3x^2} + \frac{\sqrt{3}}{2}\times\frac{34}{9}\sin^{-1}\left(\frac{x+\frac{2}{3}}{\frac{\sqrt{34}}{3}}\right) + C_2 \qquad ...(v)$$

Putting the value of I_1 and I_2 in (iii), we get

$$I = -\frac{2}{9}(10-4x-3x^2)^{3/2} + \frac{11}{2\sqrt{3}}\left(x+\frac{2}{3}\right)\sqrt{10-4x-3x^2} + \frac{187\sqrt{3}}{27}\sin^{-1}\left(\frac{3}{\sqrt{34}}(x+\frac{2}{3})\right) + C$$

18. Evaluate: $\int_0^1 x \log(1+2x)\,dx$ **[NCERT Exemplar]**

Sol. Let $I = \int_0^1 \overset{\text{II}}{x} \log(\overset{\text{I}}{1}+2x)\,dx$

$$= \left[\log(1+2x)\frac{x^2}{2}\right]_0^1 - \int \frac{1}{1+2x}.2.\frac{x^2}{2}dx = \frac{1}{2}[x^2\log(1+2x)]_0^1 - \int_0^1 \frac{x^2}{1+2x}dx$$

$$= \frac{1}{2}[1\log 3 - 0] - \left[\int_0^1\left(\frac{x}{2} - \frac{\frac{x}{2}}{1+2x}\right)dx\right] = \frac{1}{2}\log 3 - \frac{1}{2}\int_0^1 x\,dx + \frac{1}{2}\int_0^1 \frac{x}{1+2x}dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\int_0^1 \frac{\frac{1}{2}(2x+1-1)}{(2x+1)}dx = \frac{1}{2}\log 3 - \frac{1}{2}\left[\frac{1}{2}-0\right] + \frac{1}{4}\int_0^1 dx - \frac{1}{4}\int_0^1 \frac{1}{1+2x}dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4}[x]_0^1 - \frac{1}{8}[\log|(1+2x)|]_0^1 = \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4} - \frac{1}{8}[\log 3 - \log 1]$$

$$= \frac{1}{2}\log 3 - \frac{1}{8}\log 3 = \frac{3}{8}\log 3$$

19. Evaluate: $\int_0^{\pi} \frac{4x\sin x}{1+\cos^2 x}dx$ **[CBSE (AI) 2014]**

Sol. Let $I = \int_0^{\pi} \frac{4x\sin x}{1+\cos^2 x}dx$ ...(*i*)

$$= \int_0^{\pi} \frac{4(\pi-x).\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$$

$$I = \int_0^{\pi} \frac{4(\pi-x).\sin x}{1+\cos^2 x}dx \quad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$2I = \int_0^{\pi} \frac{4(x+\pi-x)\sin x}{1+\cos^2 x}dx \Rightarrow \quad 2I = 4\int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}dx$$

$$I = 2\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$

Let $\cos x = z \quad \Rightarrow \quad -\sin x\,dx = dz \quad \Rightarrow \quad \sin x\,dx = -dz$

The limits are, $x = 0 \quad \Rightarrow \quad z = 1$

$x = \pi \quad \Rightarrow \quad z = -1$

$\therefore \quad I = 2\pi\int_1^{-1} \frac{-dz}{1+z^2} = 2\pi[\tan^{-1} z]_{-1}^{1}$

$$= 2\pi[\tan^{-1}1 - \tan^{-1}(-1)] = 2\pi\left[\frac{\pi}{4}+\frac{\pi}{4}\right] = 2\pi \times \frac{\pi}{2}$$

$\Rightarrow \quad I = \pi^2.$

20. Evaluate: $\int_{-\pi}^{\pi} (\cos ax - \sin bx)^2 dx$ **[CBSE Delhi 2015]**

Sol. Here, $I = \int_{-\pi}^{\pi} (\cos ax - \sin bx)^2 dx$

$\Rightarrow \quad I = \int_{-\pi}^{\pi} (\cos^2 ax + \sin^2 bx - 2\cos ax \sin bx)\,dx$

$\Rightarrow \quad I = \int_{-\pi}^{\pi} \cos^2 ax\,dx + \int_{-\pi}^{\pi} \sin^2 bx\,dx - \int_{-\pi}^{\pi} 2\cos ax \sin bx\,dx$

$\Rightarrow \quad I = 2\int_0^{\pi} \cos^2 ax\,dx + 2\int_0^{\pi} \sin^2 bx\,dx - 0$ [First two integrands are even function while third is odd function.]

$\Rightarrow \quad I = \int_0^{\pi} 2\cos^2 ax\,dx + \int_0^{\pi} 2\sin^2 bx\,dx \Rightarrow I = \int_0^{\pi}(1+\cos 2ax)\,dx + \int_0^{\pi}(1-\cos 2bx)\,dx$

$\Rightarrow \quad I = \int_0^{\pi} dx + \int_0^{\pi}\cos 2ax\,dx + \int_0^{\pi} dx - \int_0^{\pi}\cos 2bx\,dx \Rightarrow I = 2\int_0^{\pi} dx + \int_0^{\pi}\cos 2\,ax\,dx - \int_0^{\pi}\cos 2\,bx\,dx$

$\Rightarrow \quad I = 2[x]_0^{\pi} + \left[\frac{\sin 2\,ax}{2a}\right]_0^{\pi} - \left[\frac{\sin 2\,bx}{2b}\right]_0^{\pi}$

$\Rightarrow \quad I = 2\pi + \frac{\sin 2\,a\pi}{2a} - \frac{\sin 2b\pi}{2b}$

21. Evaluate: $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$ **[CBSE Delhi 2017; (AI) 2012, CBSE 2020 (65/4/1)]**

Sol. Let $\quad I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx \qquad I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)\,dx}{1+\cos^2(\pi - x)}$

$= \int_0^{\pi} \frac{(\pi - x)\sin x\,dx}{1+\cos^2 x} = \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x} - I$

or $\quad 2I = \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x} \quad$ or $\quad I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x}$

Put $\cos x = t$ so that $-\sin x\,dx = dt$.

The limits are, when $x = 0$, $t = 1$ and $x = \pi$, $t = -1$, we get

$$I = \frac{-\pi}{2}\int_1^{-1} \frac{dt}{1+t^2} = \pi\int_0^1 \frac{dt}{1+t^2} \qquad \left[\because \int_a^{-a} f(x)\,dx = -\int_{-a}^{a} f(x)\,dx \text{ and } \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx\right]$$

$$= \pi[\tan^{-1}t]_0^1 = \pi[\tan^{-1}1 - \tan^{-1}0] = \pi\left[\frac{\pi}{4} - 0\right] = \frac{\pi^2}{4}$$

22. Find: $\int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$ **[CBSE (Allahabad) 2015]**

Sol. Let $I = \int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2\sin 2x}} = \int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2.2\sin x.\cos x}}$

$= \frac{1}{2}\int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{\frac{\sin x}{\cos x}.\cos^2 x}} = \frac{1}{2}\int_0^{\pi/4} \frac{dx}{\cos^4 x\sqrt{\tan x}}$

$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec^4 x\,dx}{\sqrt{\tan x}} = \frac{1}{2}\int_0^{\pi/4} \frac{\sec^2 x.\sec^2 x\,dx}{\sqrt{\tan x}}$

Let $\tan x = t \Rightarrow \sec^2 x\,dx = dt,\quad x = 0 \quad \Rightarrow \quad t = 0$ and $x = \frac{\pi}{4} \quad \Rightarrow \quad t = 1$

$\therefore \quad I = \frac{1}{2}\int_0^1 \frac{(1+t^2)dt}{\sqrt{t}} = \frac{1}{2}\int_0^1 (t^{-1/2} + t^{3/2})\,dt = \frac{1}{2}\left[\frac{t^{-1/2+1}}{-1/2+1}\right]_0^1 + \frac{1}{2}\left[\frac{t^{3/2+1}}{3/2+1}\right]_0^1$

$= \frac{1}{2}\times\frac{2}{1}[\sqrt{t}]_0^1 + \frac{1}{2}\times\frac{2}{5}[t^{5/2}]_0^1 = 1 + \frac{1}{5} = \frac{6}{5}$

23. Evaluate: $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}dx$ **[CBSE (F) 2015]**

Sol. Let $\quad I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}dx$

$= \int_{-\pi/2}^{0} \frac{\cos x}{1+e^x}dx + \int_0^{\pi/2} \frac{\cos x}{1+e^x}dx$

$= \int_{\pi/2}^{0} \frac{\cos t}{1+e^{-t}}(-dt) + \int_0^{\pi/2} \frac{\cos x}{1+e^x}dx$

In 1st Integrand

Let $x = -t$

$dx = -dt$

$x = -\pi/2 \Rightarrow t = \pi/2$

$x = 0 \quad \Rightarrow t = 0$

$$= \int_0^{\pi/2} \frac{\cos t}{1+\frac{1}{e^t}} dt + \int_0^{\pi/2} \frac{\cos x}{1+e^x} dx = \int_0^{\pi/2} \frac{e^t . \cos t}{1+e^t} dt + \int_0^{\pi/2} \frac{\cos x}{1+e^x} dx$$

$$= \int_0^{\pi/2} \frac{e^x . \cos x}{1+e^x} dx + \int_0^{\pi/2} \frac{\cos x}{1+e^x} dx \qquad \left[\text{By property } \int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right]$$

$$= \int_0^{\pi/2} \frac{(e^x+1) . \cos x}{1+e^x} dx = \int_0^{\pi/2} \cos x \, dx = [\sin x]_0^{\frac{\pi}{2}} = \sin \pi/2 - \sin 0$$

$= 1.$

24. Evaluate: $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ **[*CBSE (AI) 2011*]**

Sol. Let $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

$$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}} \qquad \left[\text{By using property } \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right] \text{...}(i)$$

$$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}} = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\frac{1}{\sqrt{\tan x}}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}} dx \qquad \text{...}(ii)$$

Adding (*i*) and (*ii*), we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{(1+\sqrt{\tan x})}{(1+\sqrt{\tan x})} dx$$

$$= \int_{\pi/6}^{\pi/3} dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$2I = \frac{\pi}{6} \qquad \text{or} \qquad I = \frac{\pi}{12}$$

25. Evaluate: $\int_1^3 [\,|x-1|+|x-2|+|x-3|\,]dx$ **[*CBSE Delhi 2013*]**

Sol. Let $I = \int_1^3 [\,|x-1|+|x-2|+|x-3|\,]dx = \int_1^3 |x-1|\,dx + \int_1^3 |x-2|\,dx + \int_1^3 |x-3|\,dx$

$$= \int_1^3 |x-1|\,dx + \int_1^2 |x-2|\,dx + \int_2^3 |x-2|\,dx + \int_1^3 |x-3|\,dx$$

[By property of definite integral]

$$= \int_1^3 (x-1)\,dx + \int_1^2 -(x-2)\,dx + \int_2^3 (x-2)\,dx + \int_1^3 -(x-3)\,dx$$

$$\left\{\begin{matrix} x-1 \ge 0, & if \ \ 1 \le x \le 3 \\ x-2 \le 0, & if \ \ 1 \le x \le 2 \\ x-2 \ge 0, & if \ \ 2 \le x \le 3 \\ x-3 \le 0, & if \ \ 1 \le x \le 3 \end{matrix}\right\}$$

$$= \left[\frac{(x-1)^2}{2}\right]_1^3 - \left[\frac{(x-2)^2}{2}\right]_1^2 + \left[\frac{(x-2)^2}{2}\right]_2^3 - \left[\frac{(x-3)^2}{2}\right]_1^3$$

$$= \left(\frac{4}{2}-0\right) - \left(0-\frac{1}{2}\right) + \left(\frac{1}{2}-0\right) - \left(-0-\frac{4}{2}\right) = 2 + \frac{1}{2} + \frac{1}{2} + 2 = 5$$

26. Evaluate: $\int_0^{\pi} \frac{x\tan x}{\sec x.\text{cosec}\,x}dx$ **[CBSE Delhi 2008, 2014; Chennai 2015]**

Sol. Let $I = \int_0^{\pi} \frac{x\tan x}{\sec x.\text{cosec}\,x}dx = \int_0^{\pi} \frac{x.\frac{\sin x}{\cos x}}{\frac{1}{\cos x}.\frac{1}{\sin x}}dx$

$$I = \int_0^{\pi} x\sin^2 x dx = \int_0^{\pi}(\pi - x)\sin^2(\pi - x)dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$I = \int_0^{\pi}\pi\sin^2 x dx - \int_0^{\pi} x\sin^2 x\,dx \Rightarrow 2I = \frac{\pi}{2}\int_0^{\pi} 2\sin^2 x\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi}(1-\cos 2x)dx = \frac{\pi}{2}\int_0^{\pi}dx - \frac{\pi}{2}\int_0^{\pi}\cos 2x dx = \frac{\pi}{2}[x]_0^{\pi} - \frac{\pi}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi}$$

$$\Rightarrow \quad 2I = \frac{\pi}{2}(\pi - 0) - \frac{\pi}{4}(\sin 2\pi - \sin 0)$$

$$\Rightarrow \quad 2I = \frac{\pi^2}{2} - 0 \Rightarrow I = \frac{\pi^2}{4}$$

27. Evaluate $\int \frac{dx}{\sin(x-a)\cos(x-b)}$. **[CBSE 2019(65/4/2)]**

Sol. Let $I = \int \frac{dx}{\sin(x-a)\cos(x-b)} = \frac{1}{\cos(a-b)}\int\frac{\cos(a-b)\,dx}{\sin(x-a)\cos(x-b)}$

$$= \frac{1}{\cos(a-b)}\int\frac{\cos(a-x+x-b)}{\sin(x-a)\cos(x-b)}dx = \frac{1}{\cos(a-b)}\int\frac{\cos((x-b)-(x-a))}{\sin(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\cos(a-b)}\int\frac{\cos(x-b)\cos(x-a)+\sin(x-b)\sin(x-a)}{\sin(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\cos(a-b)}\int\left\{\frac{\cos(x-a)}{\sin(x-a)}+\frac{\sin(x-b)}{\cos(x-b)}\right\}dx = \frac{1}{\cos(a-b)}\left[\int\frac{\cos(x-a)}{\sin(x-a)}dx + \int\frac{\sin(x-b)}{\cos(x-b)}dx\right]$$

$$= \frac{1}{\cos(a-b)}[\log|\sin(x-a)| - \log|\cos(x-b)|] + C$$

$$= \frac{1}{\cos(a-b)}\log\left|\frac{\sin(x-a)}{\cos(x-b)}\right| + C$$

28. Evaluate: $\int_0^1 \cot^{-1}(1-x+x^2)dx$ **[CBSE Delhi 2008; (AI) 2008; (South) 2016]**

Sol. Let $I = \int_0^1 \cot^{-1}(1-x+x^2)\,dx$

$$= \int_0^1 \tan^{-1}\frac{1}{1-x+x^2}dx \qquad \left[\because \cot^{-1}x = \tan^{-1}\frac{1}{x}\right]$$

$$= \int_0^1 \tan^{-1}\frac{x+(1-x)}{1-x(1-x)}dx$$

$$= \int_0^1\{\tan^{-1}x + \tan^{-1}(1-x)\}dx \qquad \left[\because \tan^{-1}(x+y) = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \int_0^1 \tan^{-1}x\,dx + \int_0^1\tan^{-1}(1-x)dx$$

$$= \int_0^1 \tan^{-1}x dx + \int_0^1 \tan^{-1}(1-(1-x))dx \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^1 \tan^{-1}x dx + \int_0^1\tan^{-1}x dx = 2\int_0^1\tan^{-1}x dx = 2\int_0^1 1.\tan^{-1}x\,dx$$

$$= 2\left\{[\tan^{-1}x.x]_0^1 - \int_0^1\frac{1}{1+x^2}.x.dx\right\} = 2\frac{\pi}{4} - \int_0^1\frac{2x\,dx}{1+x^2} = \frac{\pi}{2} - [\log|1+x^2|]_0^1$$

$$= \frac{\pi}{2} - [\log 2 - \log 1] = \frac{\pi}{2} - \log 2$$

29. Evaluate: $\int_0^1 x^2(1-x)^n dx$ **[CBSE (F) 2010, 2019 (65/4/3)]**

Sol. Let $I = \int_0^1 x^2(1-x)^n dx$

$\Rightarrow$ $I = \int_0^1 (1-x)^2[1-(1-x)]^n dx$ $\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$= \int_0^1 (1-2x+x^2)x^n dx = \int_0^1 (x^n - 2x^{n+1} + x^{n+2})dx$

$= \left[\frac{x^{n+1}}{n+1} - 2.\frac{x^{n+2}}{n+2} + \frac{x^{n+3}}{n+3}\right]_0^1 = \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3}$

$= \frac{(n+2)(n+3) - 2(n+1)(n+3) + (n+1)(n+2)}{(n+1)(n+2)(n+3)}$

$= \frac{n^2+5n+6-2n^2-8n-6+n^2+3n+2}{(n+1)(n+2)(n+3)} = \frac{2}{(n+1)(n+2)(n+3)}$

30. Evaluate: $\int_1^3 (2x^2+5x)dx$ as a limit of a sum. **[CBSE Delhi 2012]**

Sol. Let $f(x) = 2x^2 + 5x$

Here $a = 1, b = 3$ $\therefore$ $h = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$ $\Rightarrow$ $nh = 2$

Also, $n \to \infty \Leftrightarrow h \to 0$.

$\because$ $\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + ... + f\{a+(n-1)h\}]$

$\therefore$ $\int_1^3 (2x^2+5x)dx = \lim_{h\to 0} h[f(1) + f(1+h) + ... + f\{1+(n-1)h\}]$

$= \lim_{h\to 0} h[\{2\times 1^2 + 5\times 1\} + \{2(1+h)^2 + 5(1+h)\} + ... + \{2(1+(n-1)h)^2 + 5((1+(n-1)h\}]$

$= \lim_{h\to 0} h[(2+5) + \{2+4h+2h^2+5+5h\} + ... + \{2+4(n-1)h+2(n-1)^2h^2+5+5(n-1)h\}]$

$= \lim_{h\to 0} h[7 + \{7+9h+2h^2\} + ... + \{7+9(n-1)h+2(n-1)^2h^2\}]$

$= \lim_{h\to 0} h[7n + 9h\{1+2+...+(n-1)\} + 2h^2\{1^2+2^2+...+(n-1)^2\}]$

$= \lim_{h\to 0}\left[7nh + 9h^2\frac{(n-1).n}{2} + 2h^3\frac{(n-1).n(2n-1)}{6}\right]$

$= \lim_{h\to 0}\left[7(nh) + \frac{9(nh)^2.\left(1-\frac{1}{n}\right)}{2} + \frac{2(nh)^3.\left(1-\frac{1}{n}\right).\left(2-\frac{1}{n}\right)}{6}\right]$

$= \lim_{n\to\infty}\left[14 + \frac{36\left(1-\frac{1}{n}\right)}{2} + \frac{16\left(1-\frac{1}{n}\right).\left(2-\frac{1}{n}\right)}{6}\right]$ $[\because nh = 2]$

$= \lim_{n\to\infty}\left[14 + 18\left(1-\frac{1}{n}\right) + \frac{8}{3}\left(1-\frac{1}{n}\right).\left(2-\frac{1}{n}\right)\right]$

$= 14 + 18 + \frac{8}{3}\times 1\times 2 = 32 + \frac{16}{3} = \frac{96+16}{3} = \frac{112}{3}$

31. Evaluate : $\int_{-1}^{2} |x^3 - x|\, dx$ **[CBSE Delhi 2016; (AI)2012], [CBSE 2019 (65/5/3)]**

Sol. If $x^3 - x = 0$

$\Rightarrow$ $x(x^2-1) = 0$ $\Rightarrow$ $x = 0$ or $x^2 = 1$

$\Rightarrow$ $x = 0$ or $x = \pm 1$ $\Rightarrow$ $x = 0, -1, 1$

Hence [–1, 2] is divided into three sub intervals [–1, 0], [0, 1] and [1, 2] such that

$x^3 - x \geq 0$ on [–1, 0]

$x^3 - x \leq 0$ on [0, 1]

and $x^3 - x \geq 0$ on [1, 2]

Now, $\int_{-1}^{2} | x^3 - x | \, dx = \int_{-1}^{0} | x^3 - x | \, dx + \int_{0}^{1} | x^3 - x | \, dx + \int_{1}^{2} | x^3 - x | \, dx$

$$= \int_{-1}^{0} (x^3 - x)\, dx + \int_{0}^{1} -(x^3 - x)\, dx + \int_{1}^{2} (x^3 - x)\, dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$

$$= \left\{0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right\} - \left\{\left(\frac{1}{4} - \frac{1}{2}\right) - 0\right\} + \left\{(4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right)\right\}$$

$$= -\frac{1}{4} + \frac{1}{2} - \frac{1}{4} + \frac{1}{2} + 2 - \frac{1}{4} + \frac{1}{2} = \frac{3}{2} - \frac{3}{4} + 2 = \frac{11}{4}$$

32. Evaluate $\int_{0}^{\pi} e^{2x} . \sin\left(\frac{\pi}{4} + x\right) dx$. **[*CBSE Delhi 2016*]**

Sol. We have $I = \int_{0}^{\pi} e^{2x} . \sin\left(\frac{\pi}{4} + x\right) dx$

Integrating by part, we get

$$I = \left[\sin\left(\frac{\pi}{4} + x\right) . \frac{e^{2x}}{2}\right]_{0}^{\pi} - \int_0^{\pi} \cos\left(\frac{\pi}{4} + x\right) . \frac{e^{2x}}{2} dx$$

$$= \frac{1}{2}\left[\sin\frac{5\pi}{4} . e^{2\pi} - \sin\frac{\pi}{4}\right] - \frac{1}{2}\int_0^{\pi} e^{2x} . \cos\left(\frac{\pi}{4} + x\right) dx$$

$$= \frac{1}{2}\left(-\frac{e^{2\pi}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \frac{1}{2}\left[\left\{\cos\left(\frac{\pi}{4} + x\right) . \frac{e^{2x}}{2}\right]_{0}^{\pi} + \int_0^{\pi} \sin\left(\frac{\pi}{4} + x\right) . \frac{e^{2x}}{2} dx\right]$$

$$= -\frac{e^{2\pi} + 1}{2\sqrt{2}} - \frac{1}{2}\left[\cos\frac{5\pi}{4} . \frac{e^{2\pi}}{2} - \frac{1}{2}\cos\frac{\pi}{4}\right] - \frac{1}{4}\int e^{2x} . \sin\left(\frac{\pi}{4} + x\right) dx$$

$$I = -\frac{e^{2\pi} + 1}{2\sqrt{2}} - \frac{1}{4} . e^{2\pi} . \cos\frac{5\pi}{4} + \frac{1}{4\sqrt{2}} - \frac{1}{4} I$$

$$\frac{5I}{4} = -\frac{e^{2\pi} + 1}{2\sqrt{2}} + \frac{e^{2\pi}}{4\sqrt{2}} + \frac{1}{4\sqrt{2}} = -\frac{e^{2\pi} + 1}{2\sqrt{2}} + \frac{e^{2\pi} + 1}{4\sqrt{2}} = \frac{e^{2\pi} + 1}{4\sqrt{2}}(-2 + 1) = -\frac{e^{2\pi} + 1}{4\sqrt{2}}$$

$$I = -\frac{e^{2\pi} + 1}{5\sqrt{2}}$$

33. Evaluate : $\int_{-2}^{2} \frac{x^2}{1 + 5^x} dx$ **[*CBSE (North) 2016*]**

Sol. Let $I = \int_{-2}^{2} \frac{x^2}{1 + 5^x} dx$ ...(*i*)

$$= \int_{-2}^{2} \frac{(2 + (-2) - x)^2}{1 + 5^{(2 + (-2) - x)}} dx \qquad \left[\int_{a}^{b} f(x)\, dx = \int f(a + b - x)\, dx\right]$$

$$= \int_{-2}^{2} \frac{(-x)^2}{1 + 5^{-x}} dx = \int_{-2}^{2} \frac{x^2}{1 + \frac{1}{5^x}} dx$$

$$I = \int_{-2}^{2} \frac{5^x x^2}{1+5^x} dx \qquad \ldots(ii)$$

Adding (*i*) and (*ii*), we get

$$2I = \int_{-2}^{2} \frac{(1+5^x)x^2}{1+5^x} dx = \int_{-2}^{2} x^2 dx = \left[\frac{x^3}{3}\right]_{-2}^{2}$$

$$\Rightarrow \quad 2I = \frac{1}{3}[8-(-8)] \qquad \Rightarrow \qquad I = \frac{16}{3\times 2} = \frac{8}{3}$$

34. Find : $\int [\log(\log x) + \frac{1}{(\log x)^2}] dx$ **[*CBSE Bhubaneswar 2015, (South) 2016*]**

Sol. Let $\quad I = \int [\log(\log x) + \frac{1}{(\log x)^2}] dx$

Let $\log x = t \quad \Rightarrow \quad x = e^t \quad \Rightarrow \quad dx = e^t\, dt$

$$\therefore \quad I = \int \left\{\log t + \frac{1}{t^2}\right\} e^t dt$$

$$= \int \left\{\log t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2}\right\} e^t dt = \int \left(\log t + \frac{1}{t}\right)e^t + \left(-\frac{1}{t} + \frac{1}{t^2}\right)e^t dt$$

$$= e^t . \log t - \frac{1}{t}.e^t + C \qquad [\because \int (f(x) + f'(x))e^x dx = f(x)e^x + C]$$

$$= e^{\log x} \log(\log x) - \frac{1}{\log x} e^{\log x} + C \qquad [\text{Put } t = \log x]$$

$$= x. \log(\log x) - \frac{x}{\log x} + C$$

35. Find: $\int 5^{5^{5^x}} . 5^{5^x} . 5^x\, dx$ **[*HOTS*]**

Sol. Let $I = \int 5^{5^{5^x}} . 5^{5^x} . 5^x dx$

Putting $\; 5^x = t \Rightarrow 5^x . \log 5\, dx = dt$ or $5^x . dx = \frac{dt}{(\log 5)}$

Therefore, $I = \int 5^{5^{5^x}} . 5^{5^x} . 5^x . dx = \int 5^{5^t} . 5^t . \frac{dt}{(\log 5)} = \frac{1}{(\log 5)} \int 5^{5^t} . 5^t . dt$

Again, putting $5^t = u, \quad 5^t dt = \frac{du}{(\log 5)}$

Therefore, $\quad I = \frac{1}{(\log 5)} \int 5^u . \frac{du}{(\log 5)} = \frac{1}{(\log 5)^2} \int 5^u du = \frac{5^u}{(\log 5)^2 . (\log 5)} + C$

$$= \frac{5^u}{(\log 5)^3} + C = \frac{5^{5^t}}{(\log 5)^3} + C = \frac{5^{5^{5^x}}}{(\log 5)^3} + C$$

Long Answer Questions [5 marks]

1. Evaluate the following integral as the limit of sums: $\int_1^4 (x^2 - x)\, dx$ **[*CBSE 2020 65/5/1*]**

Sol. Let $\quad I = \int_1^4 (x^2 - x) dx$

$\therefore \quad f(x) = x^2 - x, \quad a = 1, \; b = 4$

$\because \quad h = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n} \quad \Rightarrow nh = 3$

As $n \to \infty, \quad h \to 0$

We know that

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h[f(a) + f(a+h) + \ldots + f(a + \overline{n-1}h)]$$

$$\therefore \qquad \int_a^b f(x)\,dx = \lim_{n \to \infty} h \sum_{r=0}^{n-1} f(a+rh) \qquad \ldots(i)$$

$$\therefore \qquad f(a+rh) = (a+rh)^2 - (a+rh)$$

$$\Rightarrow \qquad f(1+rh) = (1+rh)^2 - (1+rh) = r^2h^2 + rh$$

Using (*i*), we have

$$\int_1^4 (x^2 - x)\,dx = \lim_{n \to \infty} h \sum_{r}^{n-1} (r^2h^2 + rh)$$

$$\therefore \qquad I = \lim_{h \to 0} h \left\{ h^2 \sum_{r=0}^{n-1} r^2 + h \sum_{r=0}^{n-1} r \right\}$$

$$= \lim_{h \to 0} h \left\{ h^2 \times \frac{n(n-1)(2n-1)}{6} + h \times \frac{n(n-1)}{2} \right\}$$

$$= \lim_{h \to 0} \left[\frac{n^3h^3(1-\frac{1}{n})(2-\frac{1}{n})}{6} + \frac{n^2h^2\left(1-\frac{1}{n}\right)}{2} \right]$$

$$= \frac{(3)^3(1-0)(2-0)}{6} + \frac{(3)^2(1-0)}{2}$$

$$= 9 + \frac{9}{2} = \frac{27}{2}$$

2. Evaluate: $\int (\sqrt{\cot x} + \sqrt{\tan x})dx$ **[*CBSE (AI) 2014; Patna 2015*]**

Sol. Let $I = \int (\sqrt{\cot x} + \sqrt{\tan x})\,dx$

$$I = \int \left(\frac{\sqrt{\cos x}}{\sqrt{\sin x}} + \frac{\sqrt{\sin x}}{\sqrt{\cos x}} \right) dx = \int \frac{(\cos x + \sin x)}{\sqrt{\sin x . \cos x}} dx$$

Let $(\sin x - \cos x) = t \quad \Rightarrow \quad (\cos x + \sin x)\,dx = dt$

Also $(\sin x - \cos x)^2 = t^2 \quad \Rightarrow \quad \sin^2 x + \cos^2 x - 2 \sin x . \cos x = t^2$

$\Rightarrow \quad 1 - 2 \sin x . \cos x = t^2 \quad \Rightarrow \quad \sin x . \cos x = \frac{1-t^2}{2}$

Therefore, $$I = \int \frac{dt}{\sqrt{\frac{1-t^2}{2}}} = \sqrt{2} \int \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2} \sin^{-1} t + C = \sqrt{2} \sin^{-1}(\sin x - \cos x) + C$$

3. Evaluate: $\int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x} dx$ **[*CBSE (AI) 2014*]**

Sol. Let $I = \int \frac{1}{\sin^4 x + \sin^2 x . \cos^2 x + \cos^4 x} dx$

Dividing N^r and D^r by $\cos^4 x$, we get

$$I = \int \frac{\sec^4 x}{\tan^4 x + \tan^2 x + 1} dx$$

Put $z = \tan x \quad \Rightarrow \quad dz = \sec^2 x\,dx$

$$\therefore \qquad I = \int \frac{(1+z^2)\,dz}{z^4+z^2+1} = \int \frac{z^2\left(1+\frac{1}{z^2}\right)}{z^2\left\{z^2+\frac{1}{z^2}+1\right\}}dz = \int \frac{\left(1+\frac{1}{z^2}\right)}{\left(z-\frac{1}{z}\right)^2+3}dz = \int \frac{\left(1+\frac{1}{z^2}\right)}{\left(z-\frac{1}{z}\right)^2+(\sqrt{3})^2}dz$$

Again, let $z - \frac{1}{z} = t \;\Rightarrow\; \left(1+\frac{1}{z^2}\right)dz = dt$

$$\Rightarrow \qquad I = \int \frac{dt}{t^2+(\sqrt{3})^2} = \frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{t}{\sqrt{3}}\right)+C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{z-\frac{1}{z}}{\sqrt{3}}\right)+C \qquad \left[\because z-\frac{1}{z}=t\right]$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{z^2-1}{\sqrt{3}z}\right)+C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan^2 x-1}{\sqrt{3}\tan x}\right)+C$$

4. Evaluate: $\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx$ **[*CBSE Panchkula 2015; (South) 2016*]**

Sol. Let $\quad I = \int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx \qquad$ ..(*i*)

$$I = \int_0^{\pi/2}\frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$I = \int_0^{\pi/2}\frac{\cos^2 x}{\cos x+\sin x}dx \qquad \text{...}(ii)$$

Adding (*i*) and (*ii*), we get

$$2I = \int_0^{\pi/2}\frac{\sin^2 x+\cos^2 x}{\sin x+\cos x}dx = \int_0^{\pi/2}\frac{dx}{\sin x+\cos x}$$

$$= \int_0^{\pi/2}\frac{dx}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x\right)} = \frac{1}{\sqrt{2}}\int_0^{\pi/2}\frac{dx}{\cos x.\cos\frac{\pi}{4}+\sin x.\sin\frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/2}\frac{dx}{\cos\left(x-\frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}}\int_0^{\pi/2}\sec\left(x-\frac{\pi}{4}\right)dx \quad [\because \cos(A-B)=\cos A\cos B+\sin A\sin B]$$

$$= \frac{1}{\sqrt{2}}\left[\log\left\{\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right\}\right]_0^{\pi/2} \qquad [\because \int \sec x\,dx = \log(\sec x+\tan x)]$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right)-\log\left\{\sec\left(-\frac{\pi}{4}\right)+\tan\left(-\frac{\pi}{4}\right)\right\}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\log(\sqrt{2}+1)-\log\left(\sec\frac{\pi}{4}-\tan\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{\sqrt{2}}[\log(\sqrt{2}+1)-\log(\sqrt{2}-1)] = \frac{1}{\sqrt{2}}\log\left\{\frac{\sqrt{2}+1}{\sqrt{2}-1}\right\}$$

$$= \frac{1}{\sqrt{2}}\log\left\{\frac{(\sqrt{2}+1)^2}{2-1}\right\} = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)^2 = \frac{2}{\sqrt{2}}\log(\sqrt{2}+1)$$

Hence, $\quad I = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$

5. Evaluate: $\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x}dx$ **[CBSE Delhi 2011, 2014; Sample Paper 2017]**

Sol. Let $I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}dx$

$\Rightarrow$ $I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right). \sin\left(\frac{\pi}{2} - x\right). \cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)}dx$ $\left[\text{By Property } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$\Rightarrow$ $I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \cos x. \sin x}{\cos^4 x + \sin^4 x}dx$ $\left[\because \sin\left(\frac{\pi}{2} - x\right) = \cos x \text{ and } \cos\left(\frac{\pi}{2} - x\right) = \sin x\right]$

$\Rightarrow$ $I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\cos x. \sin x}{\sin^4 x + \cos^4 x}dx - \int_0^{\pi/2} \frac{x \sin x. \cos x}{\sin^4 x + \cos^4 x}dx$

$\Rightarrow$ $I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\cos x.\sin x}{\sin^4 x + \cos^4 x}dx - I \Rightarrow 2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x.\cos x}{\sin^4 x + \cos^4 x}dx$

$= \frac{\pi}{2}\int_0^{\pi/2} \frac{\frac{\sin x . \cos x}{\cos^4 x}dx}{\tan^4 x + 1}$ [Dividing numerator and denominator by $\cos^4 x$]

$= \frac{\pi}{2 \times 2}\int_0^{\pi/2} \frac{2 \tan x. \sec^2 x}{1 + (\tan^2 x)^2}dx$

Let $\tan^2 x = z;$ $2\tan x . \sec^2 x\, dx = dz$

The limits are, when $x = 0, z = 0;$ $x = \frac{\pi}{2}, z = \infty$

$\therefore$ $2I = \frac{\pi}{4}\int_0^{\infty} \frac{dz}{1 + z^2} = \frac{\pi}{4}[\tan^{-1} z]_0^{\infty} = \frac{\pi}{4}(\tan^{-1}\infty - \tan^{-1} 0)$

$\therefore$ $2I = \frac{\pi}{4}\left(\frac{\pi}{2} - 0\right)$ $\Rightarrow$ $I = \frac{\pi^2}{16}$

6. Evaluate: $\int_0^{\pi/2} 2\sin x\cos x\tan^{-1}(\sin x)\,dx$ **[CBSE Delhi 2011]**

Sol. Let $I = 2\int_0^{\pi/2} \sin x.\cos x.\tan^{-1}(\sin x)dx$

Put $\sin x = z$ $\Rightarrow$ $\cos x\, dx = dz$

The limits are, when $x = 0, z = \sin 0 = 0; x = \frac{\pi}{2}, z = \sin\frac{\pi}{2} = 1$

$\therefore$ $I = 2\int_0^1 z\tan^{-1}(z)dz = 2\left[\tan^{-1} z.\frac{z^2}{2}\right]_0^1 - 2\int_0^1 \frac{1}{1+z^2}.\frac{z^2}{2}dz$

$= 2\left[\frac{\pi}{4}.\frac{1}{2} - 0\right] - \frac{2}{2}\int_0^1 \frac{z^2}{1+z^2}dz = \frac{\pi}{4} - \int_0^1 \frac{1+z^2-1}{1+z^2}dz = \frac{\pi}{4} - \int_0^1 dz + \int_0^1 \frac{dz}{1+z^2}$

$= \frac{\pi}{4} - [z]_0^1 + [\tan^{-1} z]_0^1 = \frac{\pi}{4} - 1 + \left[\frac{\pi}{4} - 0\right] = \frac{\pi}{2} - 1$

7. Evaluate: $\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2\sin^2 x}\,dx$ **[CBSE (AI) 2009; (F) 2014]**

Sol. Let $I = \int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2\sin^2 x}dx$ …(*i*)

$$I = \int_0^{\pi} \frac{\pi - x}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}\, dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$I = \int_0^{\pi} \frac{\pi - x}{a^2\cos^2 x + b^2\sin^2 x}\,dx \qquad \ldots(ii)$$

Adding (*i*) and (*ii*), we get

$$2I = \int_0^{\pi} \frac{\pi}{a^2\cos^2 x + b^2\sin^2 x}\,dx \quad \Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

$$I = \frac{\pi}{2}\int_0^{\pi} \frac{\sec^2 x}{a^2 + b^2\tan^2 x}\,dx \qquad \text{[Divide numerator and denominator by } \cos^2 x]$$

$$= \pi\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{a^2 + b^2\tan^2 x}\,dx \qquad \left[\because \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx\right]$$

Put $b \tan x = t \quad \Rightarrow \quad b \sec^2 x\, dx = dt$

The limits are, when $x = 0$, $t = 0$ and $x = \frac{\pi}{2}, t = \infty$

$$I = \frac{\pi}{b}\int_0^{\infty} \frac{dt}{a^2 + t^2} = \frac{\pi}{b}\cdot\frac{1}{a}\tan^{-1}\frac{t}{a}\Big]_0^{\infty}$$

$$I = \frac{\pi}{ab}(\tan^{-1}\infty - \tan^{-1}0) = \frac{\pi}{ab}\cdot\frac{\pi}{2} \quad \Rightarrow \quad I = \frac{\pi^2}{2ab}$$

8. Solve: $\int\sqrt{\frac{(a+x)}{(a-x)}}\,dx$ **[*NCERT Exemplar*]**

Sol. Let $I = \int\sqrt{\frac{a+x}{a-x}}\,dx$

Put $x = a\cos 2\theta$

$\Rightarrow \quad dx = -a.\ \sin 2\theta.\ 2.\ d\theta$

$$\therefore \quad I = -2\int\sqrt{\frac{a + a\cos 2\theta}{a - a\cos 2\theta}}\,.\,a\sin 2\theta d\theta \qquad \left[\because \cos 2\theta = \frac{x}{a} \Rightarrow 2\theta = \cos^{-1}\frac{x}{a} \Rightarrow \theta = \frac{1}{2}\cos^{-1}\frac{x}{a}\right]$$

$$= -2a\int\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\sin 2\theta d\theta = -2a\int\sqrt{\frac{2\cos^2\theta}{2\sin^2\theta}}\sin 2\theta d\theta$$

$$= -2a\int \cot\theta.\ \sin 2\theta d\theta = -2a\int\frac{\cos\theta}{\sin\theta}2\sin\theta.\ \cos\theta d\theta = -4a\int\cos^2\theta d\theta = -2a\int(1+\cos 2\theta)\,d\theta$$

$$= -2a\left[\theta + \frac{\sin 2\theta}{2}\right] + C = -2a\left[\frac{1}{2}\cos^{-1}\frac{x}{a} + \frac{1}{2}\sqrt{1 - \frac{x^2}{a^2}}\right] + C$$

$$= -a\left[\cos^{-1}\left(\frac{x}{a}\right) + \sqrt{1 - \frac{x^2}{a^2}}\right] + C$$

9. Evaluate the following: $\int_0^{3/2} |\, x\cos\pi x \,|\, dx$ **[*CBSE (F) 2010; Patna 2015; (Central) 2016*]**

Sol. $\int_0^{3/2} |\, x\cos\pi x \,|\, dx$

As we know, $\cos x = 0 \quad \Rightarrow \quad x = (2n-1)\frac{\pi}{2}, n \in Z$

$\therefore \quad \cos \pi x = 0 \quad \Rightarrow \quad x = \frac{1}{2}, \frac{3}{2}$

For $0 < x < \frac{1}{2}$, $x > 0$ then $\cos \pi x > 0 \quad \Rightarrow \quad x\cos\pi x > 0$

For $\frac{1}{2} < x < \frac{3}{2}$, $x > 0$ then $\cos \pi x < 0 \Rightarrow x \cos \pi x < 0$

$$\therefore \int_0^{3/2} | x\cos\pi x \,|\, dx = \int_0^{1/2} x\cos\pi x\, dx + \int_{1/2}^{3/2} (-x\cos\pi x)\, dx \quad \ldots(i)$$

$$= \left[x \frac{\sin \pi x}{\pi}\right]_0^{1/2} - \int_0^{1/2} 1.\frac{\sin\pi x}{\pi} dx - \left[\frac{x\sin\pi x}{\pi}\right]_{1/2}^{3/2} + \int_{1/2}^{3/2} \frac{\sin\pi x}{\pi} dx$$

$$= \left[\frac{x}{\pi}\sin\pi x + \frac{1}{\pi^2}\cos\pi x\right]_0^{1/2} - \left[\frac{x}{\pi}\sin\pi x - \frac{1}{\pi^2}\cos\pi x\right]_{1/2}^{3/2}$$

$$= \left(\frac{1}{2\pi} + 0 - \frac{1}{\pi^2}\right) - \left(-\frac{3}{2\pi} - \frac{1}{2\pi}\right) = \frac{5}{2\pi} - \frac{1}{\pi^2}$$

10. Evaluate: $\int_0^{\pi} \frac{x}{1+\sin\alpha\sin x} dx$ **[*CBSE (F) 2016*]**

Sol. Let $I = \int_0^{\pi} \frac{x}{1+\sin\alpha\sin x} dx = \int_0^{\pi} \frac{\pi - x}{1+\sin\alpha.\sin(\pi - x)} dx$

$$= \int_0^{\pi} \frac{\pi}{1+\sin\alpha\sin x} dx - \int_0^{\pi} \frac{x}{1+\sin\alpha.\sin x} dx$$

$$I = \pi \int_0^{\pi} \frac{dx}{1+\sin\alpha.\sin x} - I$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{dx}{1+\sin\alpha.\sin x} = \pi \int_0^{\pi} \frac{dx}{1+\sin\alpha.\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$= \pi \int_0^{\pi} \frac{\left(1+\tan^2\frac{x}{2}\right)}{1+\tan^2\frac{x}{2} + 2\sin\alpha.\tan\frac{x}{2}} dx = \pi \int_0^{\pi} \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2} + 2\sin\alpha.\tan\frac{x}{2} + 1} dx$$

Let $\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2} dx = 2dt$; $x = 0 \Rightarrow t = 0$ and $x = \pi \Rightarrow t = \infty$

$$\therefore \quad 2I = 2\pi \int_0^{\infty} \frac{dt}{t^2 + 2\sin\alpha t + 1}$$

$$I = \pi \int_0^{\infty} \frac{dt}{t^2 + 2\sin\alpha t + \sin^2\alpha - \sin^2\alpha + 1}$$

$$= \pi \int_0^{\infty} \frac{dt}{(t+\sin\alpha)^2 + (1-\sin^2\alpha)} = \pi \int_0^{\infty} \frac{dt}{(t+\sin\alpha)^2 + \cos^2\alpha}$$

$$= \frac{\pi}{\cos\alpha}\left[\tan^{-1}\frac{t+\sin\alpha}{\cos\alpha}\right]_0^{\infty} = \frac{\pi}{\cos\alpha}\left[\tan^{-1}\frac{\tan\frac{x}{2}+\sin\alpha}{\cos\alpha}\right]_0^{\infty}$$

$$= \frac{\pi}{\cos\alpha}\left[\frac{\pi}{2} - \tan^{-1}(\tan\alpha)\right] = \frac{\pi}{\cos\alpha}\left(\frac{\pi}{2} - \alpha\right)$$

$$= \frac{\pi(\pi - 2\alpha)}{2\cos\alpha}$$

11. Find: $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$ **(*HOTS*)**

Sol. Let $I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$

Putting $\sqrt{x} = \cos\theta$, *i.e.*, $x = \cos^2\theta \Rightarrow \theta = \cos^{-1}\sqrt{x}$ and $dx = -2\cos\theta\sin\theta d\theta$, we get

$$I = \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-2\sin\theta\cos\theta)d\theta$$

$$= -2\int \sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}(\sin\theta\cos\theta)\,d\theta = -2\int \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\theta\right)d\theta$$

$$= -2\int 2\sin^2\frac{\theta}{2}\cos\theta d\theta = -2\int(1-\cos\theta)\cos\theta d\theta$$

$$= -2\int(1-\cos\theta)\cos\theta\,.\,d\theta = -2\int(\cos\theta - \cos^2\theta)\,.\,d\theta$$

$$= -2\int \cos\theta + \int 2\cos^2\theta.\,d\theta = -2\sin\theta + \int(1+\cos 2\theta).d\theta$$

$$= -2\sin\theta + \int 1.d\theta + \int\cos 2\theta.\,d\theta = -2\sin\theta + \theta + \frac{\sin 2\theta}{2} + C$$

$$= -2\sqrt{1-\cos^2\theta} + \theta + \frac{2\sqrt{1-\cos^2\theta}.\cos\theta}{2} + C = -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x} + C$$

12. Find: $\int \frac{x^2}{(x\sin x + \cos x)^2}dx$ **[*HOTS*]**

Sol. Let $I = \int \frac{x^2 dx}{(x\sin x + \cos x)^2}dx = \int \frac{x\cos x}{(x\sin x + \cos x)^2}.\frac{x}{\cos x}dx$

Integrating by parts, taking $\frac{x}{\cos x}$ as the first function and $\frac{x\cos x}{(x\sin x+\cos x)^2}$ as the second function, we get

$$I = \frac{x}{\cos x}\int \frac{x\cos x}{(x\sin x+\cos x)^2}.dx - \int\left[\frac{d}{dx}\left(\frac{x}{\cos x}\right)\int\left(\frac{x\cos x}{x\sin x+\cos x}\right).dx\right]dx$$

Now, let us first evaluate $\int \frac{x\cos x dx}{(x\sin x+\cos x)^2}$

Putting $(x\sin x + \cos x) = t$, then $(\sin x + x\cos x - \sin x)dx = dt$ *i.e.*, $x\cos x\,dx = dt$, we get

$$\int \frac{x\cos x}{(x\sin x+\cos x)^2}dx = \int\frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{(x\sin x+\cos x)}$$

Hence, $I = \frac{x}{\cos x}.\frac{-1}{(x\sin x+\cos x)} - \int\frac{\cos x + x\sin x}{\cos^2 x}\times\frac{-1}{(x\sin x+\cos x)}.dx$

$$= \frac{x}{\cos x}\times\frac{-1}{(x\sin x+\cos x)} + \int\sec^2 x.\,dx = \frac{-x}{\cos x(x\sin x+\cos x)} + \tan x + C$$

$$= \frac{-x + x\sin^2 x + \sin x\cos x}{\cos x(x\sin x+\cos x)} + C = \frac{-x(1-\sin^2 x) + \sin x\cos x}{\cos x(x\sin x+\cos x)} + C$$

$$= \frac{\cos x(\sin x - x\cos x)}{\cos x(x\sin x+\cos x)} + C$$

$$\int\frac{x^2 dx}{(x\sin x+\cos x)^2} = \frac{(\sin x - x\cos x)}{(x\sin x+\cos x)} + C$$

PROFICIENCY EXERCISE

■ **Objective Type Questions:** **[1 mark each]**

1. Choose and write the correct option in each of the following questions.

(*i*) $\int \frac{x^9}{(4x^2+1)^6}\,dx$ is equal to [*NCERT Exemplar*]

(*a*) $\frac{1}{5x}\left(4+\frac{1}{x^2}\right)^{-5}+C$ (*b*) $\frac{1}{5}\left(4+\frac{1}{x^2}\right)^{-5}+C$ (*c*) $\frac{1}{10x}(1+4)^{-5}+C$ (*d*) $\frac{1}{10}\left(\frac{1}{x^2}+4\right)^{-5}+C$

(*ii*) The integral of $\int \frac{x}{\sqrt{x}+1}\,dx$ is equal to

(*a*) $2\left[\frac{x\sqrt{x}}{3}-\frac{x}{2}+\sqrt{x}-\log\left|(\sqrt{x}+1)\right|\right]+C$ (*b*) $\frac{x\sqrt{x}}{3}+\frac{x}{2}-\sqrt{x}+\log(\sqrt{x}+1)+C$

(*c*) $\sqrt{x}-\log(\sqrt{x}+1)+C$ (*d*) None of these

(*iii*) $\int e^x[f(x)+f'(x)]\,dx = e^x\sin x + C$ then $f(x)$ is equal to

(*a*) $\sin x$ (*b*) $-\sin x$ (*c*) $\cos x - \sin x$ (*d*) $\sin x + \cos x$

(*iv*) $\int \frac{a^{\sqrt{x}}}{\sqrt{x}}\,dx$ is

(*a*) $a^{\sqrt{x}}\log a + C$ (*b*) $2a^{\sqrt{x}}\log_e a + C$ (*c*) $2a^{\sqrt{x}}\log_{10} a + C$ (*d*) $\frac{2a^{\sqrt{x}}}{\log_e a}+C$

(*v*) $\int_1^3 \frac{3\cos(\log x)}{x}\,dx$ is equal to

(*a*) sin (log 3) (*b*) cos (log 3) (*c*) 1 (*d*) $\frac{\pi}{4}$

(*vi*) $\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx$ is equal to

(*a*) 1 (*b*) $\frac{\pi^2}{4}$ (*c*) $\frac{\pi^2}{32}$ (*d*) none of these

2. Fill in the blanks. **[1 mark each]**

(*i*) $\int \frac{\sin x}{3+4\cos^2 x}\,dx =$ ______________ . [*CBSE 2020 (65/4/1)*]

(*ii*) $\int \frac{2dx}{\sqrt{1-4x^2}} =$ ______________ .

(*iii*) $\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is an ______________ function. [*CBSE 2020 (65/4/1)*]

(*iv*) $\int_3^6 2[x]\,dx =$ ______________ , where [*x*] is the greatest integer function.

■ **Very Short Answer Questions:** **[1 mark each]**

3. If $\int (e^{ax}+bx)\,dx = 4e^{4x}+\frac{3x^2}{2}$, find the values of *a* and *b*. [*CBSE (AI) 2008*]

4. If $\int_0^a 3x^2\,dx = 8$, write the value of '*a*'. [*CBSE (F) 2012, 2014*]

5. If $\int_0^1 (3x^2+2x+k)\,dx = 0$, find the value of *k*. [*CBSE Delhi 2009*]

6. If $f(x) = \int_0^x t \sin t \, dt$, then write the value of $f'(x)$. [*CBSE (AI) 2014*]

7. Write the antiderivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)$. [*CBSE Delhi 2014*]

8. If $\int (ax+b)^2 dx = f(x) + C$, find $f(x)$. [*CBSE (F) 2010*]

9. Evaluate: $\int_0^1 \frac{1}{\sqrt{2x+3}} dx$ [*CBSE (F) 2009*]

10. Evaluate: $\int \frac{dx}{\sin^2 x \cos^2 x}$ [*CBSE (F) 2014*]

11. Evaluate: $\int_0^{\pi/4} \tan x \, dx$ [*CBSE (F) 2014*]

12. Evaluate: $\int \cos^{-1}(\sin x) dx$ [*CBSE Delhi 2014*]

13. Evaluate : $\int_e^{e^2} \frac{dx}{x \log x}$ [*CBSE (AI) 2014*]

14. Evaluate: $\int \frac{dx}{\sqrt{1-x^2}}$ [*CBSE (AI) 2011*]

15. Write the value of $\int \frac{dx}{x^2+16}$. [*CBSE Delhi 2011*]

■ **Short Answer Questions–I:** **[2 marks each]**

16. Given $\int e^x (\tan x + 1) \sec x \, dx = e^x f(x) + C$ [*CBSE (AI) 2012*]
Write $f(x)$ satisfying the above.

17. Evaluate: $\int (1-x)\sqrt{x} \, dx$ [*CBSE Delhi 2012*]

18. If $\int \left(\frac{x-1}{x^2}\right) e^x dx = f(x) e^x + C$, find the value of $f(x)$. [*CBSE (F) 2012*]

19. Evaluate : $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ [*CBSE Delhi 2008*]

20. Show that $\int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x} \, dx) = \sqrt{2}\pi$. [*CBSE (AI) 2008*]

21. If $\int_0^a \frac{1}{4+x^2} dx = \frac{\pi}{8}$, find the value of a. [*CBSE (AI) 2014*]

22. Evaluate : $\int_0^{\pi/4} \left(\frac{\sin x + \cos x}{3 + \sin 2x}\right) dx$ [*CBSE (Ajmer) 2014*]

23. Find : $\int \frac{x \, dx}{1 + x \tan x}$ [*CBSE Bhubneshwar 2015*]

24. Evaluate : $\int_0^{\pi/2} \left(\frac{5 \sin x + 3 \cos x}{\sin x + \cos x}\right) dx$ [*CBSE Bhubneshwar 2015*]

25. Evaluate : $\int \frac{(x+3)e^x}{(x+5)^3} dx$ [*CBSE Punchkula 2015*]

26. Evaluate : $\int_0^{\pi/2} \frac{2^{\sin x}}{2^{\sin x} + 2^{\cos x}} dx$ [*CBSE Patna 2015*]

27. Evaluate : $\int_0^{\pi/2} e^x (\sin x - \cos x) dx$ [*CBSE Delhi 2014*]

28. Evaluate: $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$ [*CBSE* (*AI*) 2009]

29. Write the value of the following integral $\int_{-\pi/2}^{\pi/2} \sin^5 x\, dx$. [*CBSE* (*AI*) 2010]

30. Evaluate: $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$ [*CBSE* (*AI*) 2012, (*F*) 2016]

■ Short Answer Questions–II: [3 marks each]

31. Evaluate: $\int x \sin^{-1} x\, dx$ [*CBSE (AI) 2009; Chennai 2015*]

32. Evaluate: $\int \frac{dx}{x(x^5+3)}$ [*CBSE (AI) 2013*]

33. Evaluate: $\int x^2 . \cos^{-1} x\, dx$ [*CBSE (F) 2009*]

34. Evaluate: $\int \frac{dx}{x(x^3+8)}$ [*CBSE (AI) 2013*]

35. Evaluate: $\int \frac{3x+5}{\sqrt{x^2-8x+7}}dx$ [*CBSE (F) 2011*]

36. Evaluate: $\int \frac{1-x^2}{x(1-2x)}dx$ [*CBSE Delhi 2010, (F) 2013*]

37. Evaluate: $\int \frac{(x+2)}{\sqrt{(x-2)(x-3)}}dx$ [CBSE (AI) 2010]

38. Evaluate: $\int_0^{\pi} \frac{x}{1+\sin x}dx$ [*CBSE Delhi 2010*]

39. Evaluate: $\int_0^{4} (|x|+|x-2|+|x-4|)dx$ [*CBSE Delhi 2013*]

40. Evaluate the following indefinite integral : $\int \frac{\sin\phi}{\sqrt{\sin^2\phi+2\cos\phi+3}}d\phi$ [*CBSE Sample Paper 2016*]

41. Find : $\int \frac{x^2}{x^4+x^2-2}dx$ [*CBSE (Central) 2016*]

42. Find: $\int (3x+1)\sqrt{4-3x-2x^2}\,dx$ [*CBSE (Central) 2016*]

43. Evaluate : $\int_0^{\pi} \frac{x\sin x}{1+3\cos^2 x}dx$ [*CBSE (East) 2016*]

44. Find : $\int \frac{x^2+x+1}{(x^2+1)(x+2)}dx$ [*CBSE (North) 2016*]

45. Find : $\int (x+3))\sqrt{3-4x-x^2}\,dx$ [*CBSE (North) 2016*]

46. Find : $\int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx$ [*CBSE (F) 2016*]

47. Evaluate the following definite integral : [*CBSE Sample Paper 2016*]

$$\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}dx$$

48. Evaluate : $\int_{-\pi/2}^{\pi/2} e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$ [*CBSE Guwahati 2015*]

49. Evaluate : $\int_0^{\pi/2} \log \sin x\, dx$ [*CBSE (AI) 2008*]

50. Evaluate : $\int \frac{2x^2+3}{x^2+5x+6}dx$ [*CBSE (F) 2013*]

51. Evaluate : $\int e^{2x} . \sin(3x+1)\,dx$ **[CBSE (F) 2015]**

52. Evaluate : $\int \dfrac{1-\cos x}{\cos x\,(1+\cos x)}dx$ **[CBSE Chennai 2015]**

53. Find: $\int \dfrac{\sin 2x}{(\sin^2 x+1)(\sin^2 x+3)}dx$ **[CBSE 2019 (65/3/1)]**

■ Long Answer Questions: [5 marks each]

54. Evaluate: $\int_{\pi/6}^{\pi/3} \dfrac{dx}{1+\sqrt{\cot x}}$ **[CBSE Delhi 2014]**

55. Evaluate $\int_1^3 (e^{2-3x}+x^2+1)\,dx$ as a limit of a sum. **[CBSE Delhi 2015]**

56. Evaluate: $\int \dfrac{\sin x - x\cos x}{x(x+\sin x)}dx$ **[CBSE Ajmer 2015]**

57. Find: $\int_0^{\pi/4} \dfrac{dx}{\cos^3 x\sqrt{2\sin 2x}}$ **[CBSE Allahabad 2015]**

58. Evaluate : $\int_0^{\pi/2} \dfrac{\cos^2 x\,dx}{1+3\sin^2 x}$ **[CBSE Ajmer 2015]**

Answers

1. (*i*) (*d*) (*ii*) (*a*) (*iii*) (*a*) (*iv*) (*d*) (*v*) (*a*) (*vi*) (*c*)

2. (*i*) $-\dfrac{1}{2\sqrt{3}}\tan^{-1}\left(\dfrac{2\cos x}{\sqrt{3}}\right)+C$ (*ii*) $\sin^{-1}(2x)$ (*iii*) odd (*iv*) 24

3. a can't be determined, $b = 3$ **4.** $a = 2$ **5.** $k = -2$ **6.** $x\sin x$

7. $2x^{3/2}+2\sqrt{x}+C$ **8.** $\dfrac{(ax+b)^3}{3a}$ **9.** $\sqrt{5}-\sqrt{3}$ **10.** $\tan x - 1/\tan x + C$

11. $\dfrac{1}{2}\log 2$ **12.** $\dfrac{\pi x}{2}-\dfrac{x^2}{2}+C$ **13.** $\log 2$ **14.** $\sin^{-1} x + C$ **15.** $\dfrac{1}{4}\tan^{-1}\dfrac{x}{4}+C$

16. $f(x)=\sec x$ **17.** $\dfrac{2}{3}x^{3/2}-\dfrac{2}{5}x^{5/2}+C$ **18.** $f(x)=\dfrac{1}{x}$ **19.** $\dfrac{\pi^2}{2\sqrt{2}}$ **21.** $a = 2$

22. $\dfrac{1}{4}\log 3$ **23.** $\log|\cos x + x\sin x| + C$ **24.** 2π **25.** $e^x.\dfrac{1}{(x+5)^2}+C$

26. $\dfrac{\pi}{4}$ **27.** 1 **28.** $2\sin\sqrt{x}+C$ **29.** 0 **30.** $x-\sqrt{1-x^2}\sin^{-1}x+C$

31. $\dfrac{x^2}{2}\sin^{-1}x-\dfrac{1}{4}\sin^{-1}x+\dfrac{x}{4}\sqrt{1-x^2}+C$ **32.** $\dfrac{1}{15}\log\left|\dfrac{x^5}{x^5+3}\right|+C$

33. $\dfrac{x^3}{3}\cos^{-1}x-\dfrac{1}{3}\sqrt{1-x^2}+\dfrac{1}{9}(1-x^2)^{3/2}+C$ **34.** $\dfrac{1}{24}\log\left|\dfrac{x^3}{x^3+8}\right|+C$

35. $3\sqrt{x^2-8x+7}+17\log|(x-4)+\sqrt{x^2-8x+7}|+C$ **36.** $\dfrac{1}{2}x+\log|x|-\dfrac{3}{4}\log|2x-1|+C$

37. $\sqrt{x^2-5x+6}+\dfrac{9}{2}\log\left|\left(x-\dfrac{5}{2}\right)+\sqrt{x^2-5x+6}\right|+C$

38. π **39.** 20 **40.** $-\sin^{-1}\left(\dfrac{\cos\phi-1}{\sqrt{5}}\right)+$

41. $\dfrac{\sqrt{2}}{3}\tan^{-1}\left(\dfrac{x}{\sqrt{2}}\right)+\dfrac{1}{6\log}\left|\dfrac{x-1}{x+1}\right|+C$

42. $\frac{1}{2}(4-3x-2x^2)^{3/2}-\frac{5}{4\sqrt{2}}\left[x+\frac{3}{4}\right]\sqrt{2-\frac{3}{2}x-x^2}-\frac{205}{64\sqrt{2}}\sin^{-1}\left[\frac{4}{\sqrt{41}}\left(x+\frac{3}{4}\right)\right]+C$

43. $\frac{\sqrt{3}\pi^2}{9}$

44. $\frac{3}{5}\log|x+2|+\frac{1}{5}\log|x^2+1|+\frac{1}{5}\tan^{-1}x+C$

45. $-\frac{1}{3}(3-4x-x^2)^{3/2}+\frac{x+2}{2}\sqrt{3-4x-x^2}+\frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right)+C$

46. $\frac{2}{9}(10-4x-3x^2)^{3/2}+\frac{11}{2\sqrt{3}}\left(x+\frac{2}{3}\right)\sqrt{10-4x-3x^2}+\frac{17}{9}\sin^{-1}\left[\frac{3}{\sqrt{34}}\left(x+\frac{2}{3}\right)\right]+C$

47. π^2

48. $\frac{e^{\pi/2}}{2}$

49. $-\frac{\pi}{2}\log 2$

50. $2x + 11\log|x+2| - 21\log|x+3| + C$

51. $\frac{2e^{2x}.\sin(3x+1)}{13}-\frac{3e^{2x}.\cos(3x+1)}{13}+C$

52. $\log|\sec x+\tan x|-2\tan\frac{x}{2}+C$

53. $\frac{1}{2}\log(\sin^2 x+1)-\frac{1}{2}\log(\sin^2 x+3)+C$

54. $\frac{\pi}{12}$

55. $\frac{32+(e^{-1}-e^{-7})}{3}$ **Hint:** Apply $\int_a^b f(x)dx=\lim_{h\to 0} h\{f(a+h)+f(a+2h)+...+f(a+nh)\}$

Write $\int_1^3 (e^{2-3x}+x^2+1)dx=\int_1^3 e^{2-3x}dx+\int_1^3 (x^2+1)dx$

Get $e^2\int_1^3 e^{-3x}dx=\frac{e^2e^{-3}(1-e^{-6})}{3}$ and $\int_1^3 (x^2+1)dx=\frac{32}{3}$

56. $\log|x|-\log|x+\sin x|+C$

Hint: Write $\int\frac{\sin x-x\cos x}{x(x+\sin x)}dx=\int\frac{(x+\sin x)-x(1+\cos x)}{x(x+\sin x)}dx$

57. $\frac{6}{5}$; Hint: $\int_0^{\pi/4}\frac{dx}{\cos^3 x\sqrt{2\sin 2x}}=\frac{1}{2}\int_0^{\pi/4}\frac{dx}{\cos^3 x\sqrt{\frac{\sin x}{\cos x}.\cos^2 x}}=\frac{1}{2}\int_0^{\pi/2}\frac{dx}{\cos^4 x\sqrt{\tan x}}$ Then put $\tan x = t$

58. $\frac{\pi}{6}$

SELF-ASSESSMENT TEST

Time allowed: 1 hour **Max. marks: 30**

1. **Choose and write the correct option in the following questions.** **(4 × 1 = 4)**

(*i*) The value of $\int_{-1}^{0}\frac{dx}{x^2+2x+2}dx$ is

(*a*) 0 (*b*) $\frac{\pi}{4}$ (*c*) $\frac{-\pi}{4}$ (*d*) $\frac{\pi}{2}$

(*ii*) The value of $\int\frac{dx}{x\sqrt{x^4-1}}$ is

(*a*) $\frac{1}{2}\sec^{-1}(x^2)+C$ (*b*) $\sec^{-1}(x^2)+C$ (*c*) $\sec^{-1}x+C$ (*d*) $\tan^{-1}(x^2)+C$

(*iii*) If $x = \int_0^y \frac{dt}{\sqrt{1+9t^2}}$ and $\frac{d^2y}{dx^2} = ay$, then a is equal to

(*a*) 3 (*b*) 6 (*c*) 9 (*d*) 1

(*iv*) If $\int x^6 \sin(5x^7)dx = \frac{k}{5}\cos(5x^7), x \neq 0$ then k is equal to

(*a*) 7 (*b*) – 7 (*c*) $\frac{1}{7}$ (*d*) $-\frac{1}{7}$

2. Fill in the blanks. **(2 × 1 = 2)**

(*i*) $\int \frac{(\log x)^2}{x} dx =$ ____________ .

(*ii*) $\int_0^4 |x-2|dx =$ ____________ .

■ **Solve the following questions.** **(2 × 1 = 2)**

3. Find the antiderivative of $\int \sin 2x dx$.

4. Find: $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$

■ **Solve the following questions.** **(4 × 2 = 8)**

5. Find: $\int \frac{(x-3)e^x}{(x-1)^3} dx$

6. Evaluate: $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$

7. Show that $\int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x}\, dx) = \sqrt{2}\pi$.

8. If $\int_0^a \frac{1}{4+x^2} dx = \frac{\pi}{8}$, find the value of a.

■ **Solve the following questions.** **(3 × 3 = 9)**

9. Evaluate: $I = \int_0^{\pi} \frac{x \sin x}{1 + 3\cos^2 x} dx$

10. Find: $\int \frac{1}{x(x^4-1)} dx$

11. Evaluate: $I = \int_0^{\pi/2} (2\log \sin x - \log \sin 2x)\, dx$

■ **Solve the following question.** **(1 × 5 = 5)**

12. Evaluate: $\int_0^{\pi/2} \log(\sin x)\, dx$

Answers

1. (*i*) (*b*) (*ii*) (*a*) (*iii*) (*c*) (*iv*) (*d*) **2.** (*i*) $\frac{1}{3}(\log x)^3 + C$ (*ii*) 4

3. $-\frac{1}{2}\cos .2x$ **4.** $\frac{x^2}{2} - 2x + \log|x| + C$ **5.** $\frac{e^x}{(x-1)^2} + C$ **6.** $\frac{e^2(e^2-2)}{4}$

8. $a = 2$ **9.** $\frac{\sqrt{3}}{9}\pi^2$ **10.** $\frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right| + C$ **11.** $\frac{\pi}{2}\log\left(\frac{1}{2}\right)$ **12.** $-\frac{\pi}{2}\log 2$