Chapter 11

CUBES

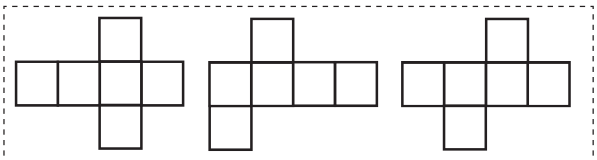
LEARNING Objectives

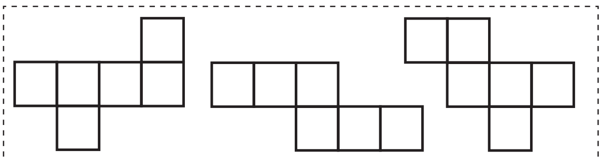
In this chapter, you will learn:

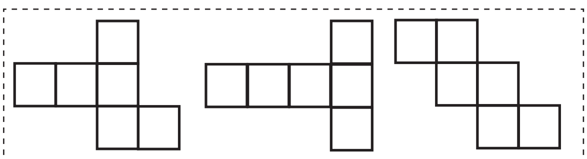
- Definitions of Edges, Faces and Corners
- Generation of new cubes through cutting the cube along different axis
- Questions based upon painting of the cubes
- Methods to solve the questions

A cube is a three–dimensional structure with the following features:

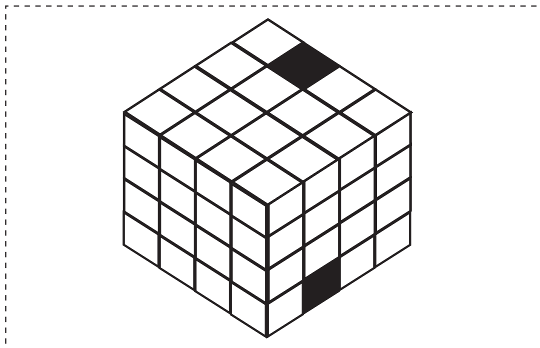
It has six faces, eight corners and twelve edges. Cube is composed of six square faces that meet each other at right angles. Let us see how the six different faces of a cube can be represented:







And finally the cube appears like:



Generally, the questions asked from Cubes in LR pertains to finding out the number of cubelets being formed from the original cube by cutting it into several pieces. However, sometimes we might be asked to find out the total number of cuts provided with the total cubelets being formed by cutting the original cube.

Cutting the Cubes

Before moving on to solve questions, we should be clear with the basics that what happens when we cut a cube:

i. One cut divides the cube into two parts.
ii. Second cut will divide the cube in either a total of 3 parts or 4 parts, depending upon the axis of cut.
iii. Third cut will divide the cube in either a maximum of 8 parts or a minimum of 4 parts.

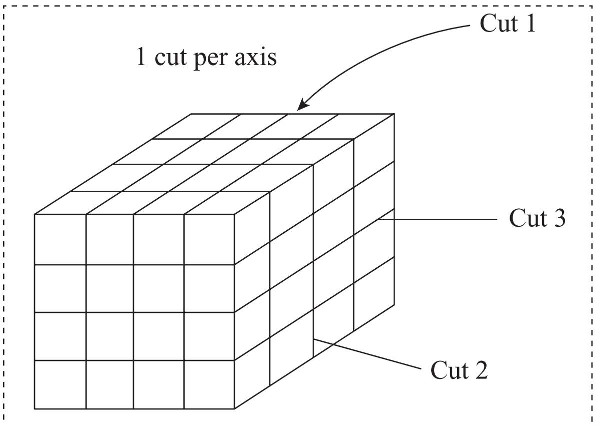
Example 1

A carpenter had a large wooden cube with side length 4 inches. He wanted to cut it into 64 smaller cubes with side length 1 inch. What is the least number of cuts required if (i) he can rearrange the pieces before each cut, (ii) the rearrangement of the pieces before/after making the cut is not allowed?

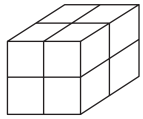
Solution

Let us understand the difference between the two questions first: (i) In first question, we are allowed to move the cube one over the other or we can stack the pieces of cube side by side or on top of each other, whereas (ii) in second question, we have to assume as if the cube is fixed on the horizontal surface and what we can do at best it to make cuts, along any of its surfaces.

(i) When rearrangement is allowed, the minimum is found by cutting each edge as nearly in half as possible, putting the pieces together and cutting as nearly in half again until we obtain a solid with a unit dimension. We would start here with making a cut midway on all the axis.



And this is what we will obtain now:



Now restack the solids into the 4 x 4 x 4 solid and repeat the procedure. After completing the three sides, we will have 1 x 1 x 1 cubes. Therefore, the sum of the cuts is the answer, which is six.

(ii) When rearrangement is not allowed:

We know that making n cuts along one axis divides the cube in (n + 1) parts. To obtain 64 cubelets by making minimum number of cuts, we should be making the cuts along all the axis.

Assume we have made n, m, p cuts along three axes.

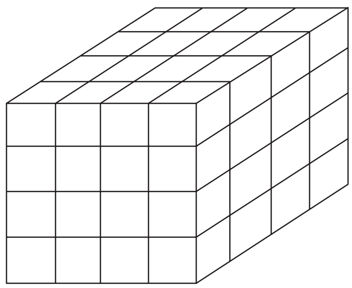
So, number of cubelets formed = $(n + 1)(m + 1)(p + 1) = 64$

To minimize the number of cuts, $(n + 1) = (m + 1) = (p + 1) = 4$

So, $n = m = p = 3$, hence a total of 9 cuts.

Alternatively, to make the number of cuts minimum, cuts should be made symmetrical.

Look at this figure:



It is having 64 cubelets of size $1 \times 1 \times 1$, and total number of cuts made = 9

Painting the cubes and then Cutting the Cubes

If we paint a cube of the dimension $n \times n \times n$ by any one colour, and then we cut it to have n^3 symmetric cubelets, then following is the number of cubelets with colour on different faces of it.

i. Cubelets with only one face painted $=6 (n–2)^2$
ii. Cubelets with two faces painted $=12 (n–2)$

iii. Cubelets with three faces painted = 8
iv. Cubelets with no face painted = $(n-2)^3$

So, if you add all the four types of cubelets given above (Cubelets with only one face painted + Cubelets with two faces painted + Cubelets with three faces painted + Cubelets with no face painted), it will be equal to total number of cubelets.

Formula wise:

$6(n-2)^2 + 12(n-2) + 8 + (n-2)^3 = n^3$

Examples

Directions for questions 2 to 5: ***Read the passage following and solve the questions based on it.***

64 symmetrical small cubes are put together to form a big cube. This cube is now coloured on all its surfaces by green colour.

2. How many of the smaller cubes have none of its faces coloured?

Solution

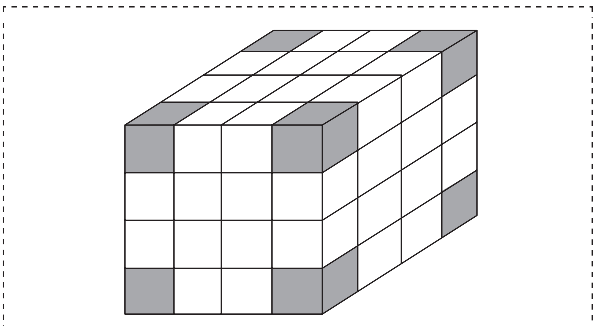
From the given $4 \times 4 \times 4$ cube, if we remove one layer from the top making it $2 \times 2 \times 2$ cube, it will not be coloured.

Hence 8 small cubes will not be coloured on any of its surfaces.

3. How many of the smaller cubes have exactly three faces coloured?

Solution

Look at the three faces coloured small cubes in the figure:



All the corner cubes (blackened) will be having exactly three faces coloured.

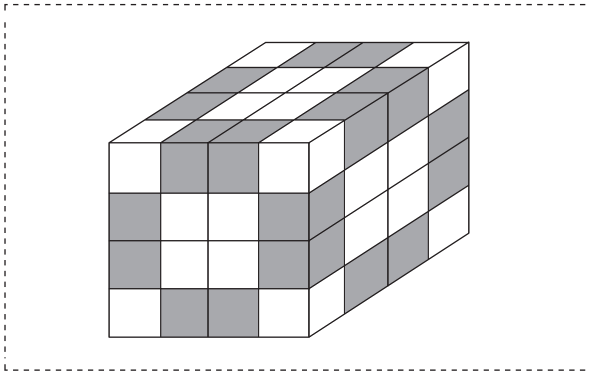
These are 8 in numbers.

Remember, for any $n \times n \times n$ dimension ($n \geq 2$), number of cubes having exactly three faces coloured = 8

4. How many of the smaller cubes have exactly two faces coloured?

Solution

Look at the two faces coloured small cubes in the figure:

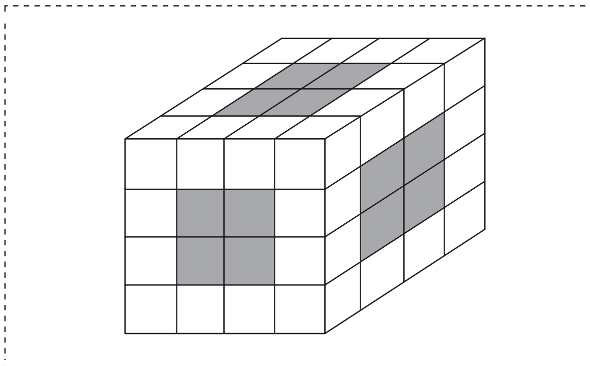


We can see that along every edge, there are two cubes painted with two colours.

So, total number of small cubes painted on exactly two of its faces = $2 \times 12 = 24$

5. How many of the smaller cubes have exactly one face coloured?

Solution



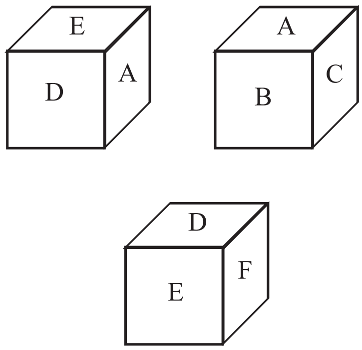
It can be seen from the above figure that total number of cubes coloured on only one of its faces = $4 \times 6 = 24$

Alternatively, total number of small cubes = Total no. of cubes painted on (one face + two faces + three faces + no face).

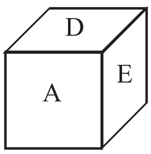
So, total number of cubes painted on only one of its faces = 64 – 8 – 8 – 24 = 24

Example 6

Directions for question 6: ***In this question, three views of a cube are given. If the same cube is rotated in a particular way, it will give rise to different views. Four such views are given in the options. However, out of the four options given, one of the options does not confirm to the original cube. Mark that option as your answer. (The letters used are only to mark the different faces of the cube.)***



(a)



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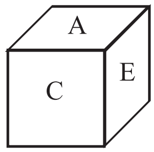

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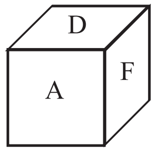

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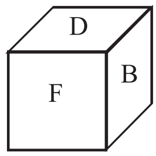


Solution

From the given figure, it can be inferred that the four faces adjacent to face A are—B, C, D and E. Hence face ‘F’ cannot be adjacent to face ‘A’.

Hence option c is wrong.

PRACTICE EXERCISE 1

Directions for questions 1 to 4: *Read the passage below and solve the questions based on it.*

A large cube is dipped into a tub filled with colour. Now the cube is taken out and it was observed that all its sides are painted. This large cube is now cut into 125 small but identical cubes.

1. How many of the smaller cubes have no face painted all?
(a) 27 (b) 64
(c) 8 (d) None of these

2. How many of the smaller cube have exactly one face painted?
(a) 49 (b) 54
(c) 64 (d) None of these

3. How many of the smaller cubes have exactly two faces painted?
(a) 25 (b) 16
(c) 36 (d) None of these

4. How many of the smaller cubes have exactly three faces painted?
(a) 4 (b) 8
(c) 9 (d) None of these

Directions for questions 5 to 9: *Read the passage below and solve the questions based on it.*

There is cube in which one pair of opposite faces is painted red; another pair of opposite faces is painted blue and the third pair of opposite faces is painted pink. This cube is now cut into 216 smaller but identical cubes.

5. How many small cubes will be there with no red paint at all?
(a) 121 (b) 144
(c) 169 (d) None of these

6. How many small cubes will be there with at least two different colours on their faces?
(a) 49 (b) 64
(c) 56 (d) 81

7. How many small cubes will be there without any face painted?
(a) 64 (b) 49
(c) 36 (d) None of these

8. How many small cubes will be there with only red and pink on their faces?
(a) 9 (b) 12
(c) 27 (d) 16

9. How many small cubes will be there showing only pink or only blue on their faces?
(a) 64 (b) 81
(c) 125 (d) None of these

Directions for questions 10 to 14: *Read the passage below and solve the questions based on it.*

There is cube in which one pair of adjacent faces is painted black; the second pair of adjacent faces is painted blue and third pair of adjacent faces is painted green. This cube is now cut into 216 smaller and identical cubes.

10. How many small cubes will be there with no black paint at all?
(a) 144 (b) 150
(c) 125 (d) None of these

11. How many small cubes will be there with at least two different colours on their faces?
(a) 64 (b) 54
(c) 33 (d) 44

12. How many small cubes will be there with one face painted black?
(a) 8 (b) 81
(c) 60 (d) 100

13. How many small cubes will be with both black and green on their faces?
(a) 8 (b) 12
(c) 16 (d) None of these

14. How many small cubes will be there showing only green or only blue on their faces?
(a) 64 (d) 72
(c) 81 (d) None of these

Directions for questions 15 to 18: *Read the passage below and solve the questions based on it.*

A large cube is painted on all its six faces. Now it is cut into a certain number of smaller identical cubes. It was found that among the smaller cubes, there were eight cubes which don't have any face painted.

15. How many smaller cubes was the original large cube cut into?
(a) 27 (b) 24
(c) 64 (d) None of these

16. How many small cubes have exactly one face painted?
(a) 12 (b) 24
(c) 16 (d) 32

17. How many small cubes have exactly two face painted?
(a) 6 (b) 12
(c) 18 (d) 24

18. How many small cubes have exactly three face painted?
(a) 0 (b) 8
(c) 27 (d) None of these

Directions for questions 19 to 25: ***Read the passage below and solve the questions based on it.***

Three different faces of a cube are coloured in three different colours—black, green and blue. This cube is now cut into 216 smaller but identical cubes.

19. What is the least number of the smaller cubes that will have exactly three faces coloured?
(a) 0 (b) 6
(c) 2 (d) 12

20. How many smaller cubes have exactly two face coloured?
(a) 12 (b) 15
(c) 16 (d) cannot be determined

21. What is the least numbers of small cubes that have only one face coloured?
(a) 86 (b) 81
(c) 64 (d) 75

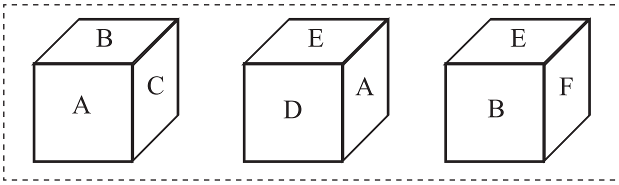
22. What is the largest number of small cubes that have only one face coloured?
(a) 86 (b) 64
(c) 72 (d) 84

23. What is the least number of small cubes that have exactly one face coloured black and no other face coloured?
(a) 12
(b) 18
(c) 24
(d) None of these

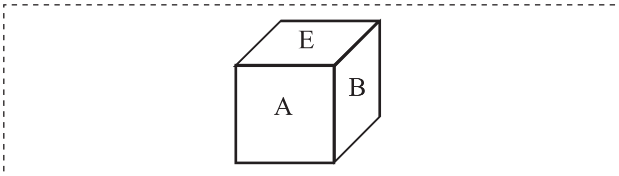
24. What is the maximum number of small cubes that have one face coloured green and one face blue and no other face coloured?
(a) 0 (b) 2
(c) 4 (d) 6

25. N is the number of cubes that is not coloured on any of its faces. Which of the following best describes the value of N?
(a) 125 < N< 130
(b) 120 ≤ N ≤ 125
(c) 115 ≤ N ≤ 120
(d) 100 < N < 125

26. In this question, three views of a cube are given. If the same cube is rotated in a particular way, it will give rise to different views. Four such views are given in the options. However, out of the four options given, one of the options does not confirm to the original cube. Mark that option as your answer. (The letters used are only to mark the different faces of the cube.)



(a)



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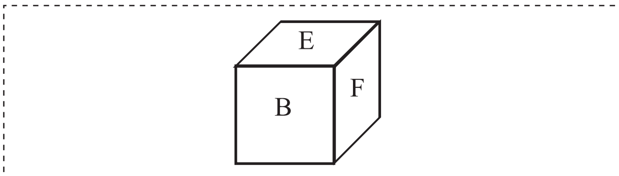

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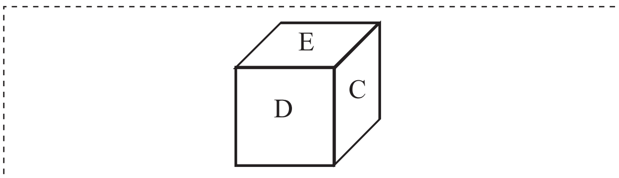

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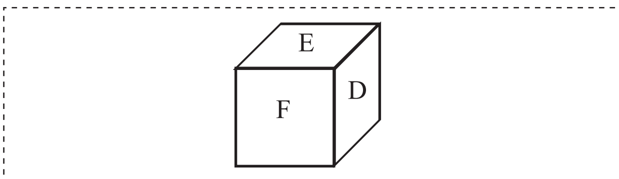

ANSWER KEYS

1. (a)	**2.** (b)	**3.** (c)	**4.** (b)	**5.** (b)	**6.** (c)	**7.** (a)	**8.** (d)	**9.** (a)	**10.** (b)
11. (d)	**12.** (c)	**13.** (c)	**14.** (d)	**15.** (c)	**16.** (b)	**17.** (d)	**18.** (b)	**19.** (a)	**20.** (d)
21. (d)	**22.** (d)	**23.** (c)	**24.** (d)	**25.** (b)	**26.** (c)				

HINTS AND EXPLANATIONS

1 to 4

There are 125 small identical cubes. So, $n^3 = 125$. Therefore $n = 5$.

1. The number of small cube with no painted face = $(n - 2)^3 = (5 - 2)^3 = 3^3 = 27$.

2. The number of small cube with exactly 1 painted face = $6(n - 2)^2 = 6(5 - 2)^2 = 6(9) = 54$.

3. The number of small cube with exactly 2 painted faces $=12(n - 2) = 12(5 - 2) = 12(3) = 36$.

4. The number of small cube with exactly 3 painted faces = 8.

5 to 9

There are 216 smaller identical cubes. So, $n^3 = 216$. Therefore $n = 6$. Now, the original cube has 2 opposite faces painted Red, Blue & Pink.

5. There are 2 faces which are painted red, so all the small cubes which are formed on these 2 opposite faces will have red colour at 1 of their faces. On each face there are $n^2 = 6^2 = 36$ small cubes. So, there must be total $2 \times 36 = 72$ small cubes with red colour in 1 of its face. There are total 216 small cubes. Therefore, the small cubes with no red colour at all = $216 - 72 = 144$.

6. The small cubes with at least 2 different colours on their faces = the small cubes with 2 coloured faces + small cubes with 3 coloured faces = $12(n - 2) + 8 = 12(6 - 2) + 8 = 56$.

7. The number of small cube with no painted face = $(n - 2)^3 = (6-2)^3 = 64$.

8. The small cubes with only red & pink on their faces = the small cubes with 1 face painted red & 1 (other) face painted pink = the small cubes with exactly 2 coloured faces with 1 face coloured as pink & other red + small cubes with 3 coloured faces (1 is coloured red, other is pink & 3rd face is blue).

The small cubes with exactly 2 coloured faces with red & pink colours are those cubes which are formed on 2 edges which are common to red & pink faces. There are such $2(n - 2)$ cubes.

So, the small cubes with only red & pink on their faces = $2(n - 2) + 8 = 16$.

9. The small cubes with only pink or only blue colour on their face = Small cubes with only pink colour + small cube with only blue colour.

There are 2 opposite faces which are coloured blue & 2 opposite faces are coloured pink. The cubes with only 1 coloured face on each face are $(n - 2)^2$. We have 4 such faces (2 which are coloured pink & 2 are blue coloured). So, there are total $4(n - 2)^2$ such faces.

Required number of small cubes = $4(n - 2)^2 = 4(6 - 2)^2 = 64$.

10 to 14

There are 216 smaller identical cubes. So, $n^3 = 216$. Therefore $n = 6$. Now, the original cube has 3 pairs of 2 adjacent faces painted Black, Blue & Green.

10. There are 2 adjacent faces which are painted black, so all the small cubes which are formed on these 2 adjacent faces will have black colour at 1 face or 2 faces (which are formed on edge where these adjacent black faces intersect). On each face there are $n^2 = 6^2 = 36$ small cubes. So, there must be total $2 \times 36 = 72$ small cubes with black colour in 1 face or 2 faces. But, 6 cubes (which are having 2 black coloured faces are counted twice). So, there are such $72 - 6 = 66$ small cubes. There are total 216 small cubes. Therefore, the small cubes with no black colour at all = $216 - 66 = 150$.

11. The small cubes with at least 2 different colours on their faces = the small cubes with 2 coloured faces + small cubes with 3 coloured faces– small cubes which have exactly 2 coloured faces & both have same colour = $12(n - 2) + 8 - 3(n - 2) = 12(6 - 2) + 8 - 3(6 - 2) = 44$.

12. The small cubes with 1 face painted black = All cube with at least 1 face coloured black- all cubes with 2 black coloured faces (as the cubes which have 2 black coloured faces are those which are formed on the edge where the adjacent black faces intersect, so we need to remove them from all possible cubes with black colour irrespective of fact whether or not it has any other coloured face).

There are 2 faces (which are adjacent) which are coloured black, so the cubes which are formed on any of these faces are n^2. But from these n^2 cubes there are n cubes which have both faces coloured black. So, there are total cubes with exactly 1 black coloured face = $2(n^2 – n) = 2(6^2 – 6) = 60$.

13. The small cubes with both black & green colours on their faces = Cubes with exactly 2 coloured faces which are black 7 green + cubes with 3 coloured faces (each face is black, blue & green coloured).

Each face intersects with 3 other faces. The 1st face must be of same colour, as adjacent faces are of same colour. The other 2 faces touching each face must be of other 2 colours. The 8 cubes which are formed at corner of original big cube must be cubes with 3 coloured face with 3 different colours. There will be 2 edges where black & green coloured faces will intersect, these cubes will have exactly 2 colours which are black & green.

So, required number of cubes= $2(n – 2) + 8 = 2(6 – 2) + 8 = 16$.

14. The small cubes with only green or only blue colour on their face = Small cubes with only green colour + small cube with only blue colour.

There are 2 adjacent faces which are coloured blue & 2 adjacent faces are coloured pink. The cubes with only 1 coloured face on each face are $(n – 2)^2$. We have 4 such faces (2 which are coloured green & 2 are blue coloured). So, there are total $4(n – 2)^2$ such faces. Further there are n – 2 cubes which have exactly 2 faces coloured such that both have green colour. Similarly there are n-2 cubes with exactly 2 coloured faces such that both faces are of blue colour. We know that 2 adjacent faces have same colour, so the cubes formed at common edge will have both faces of same colour, but we removed 2 as there 2 are formed at corner which has 3 coloured faces, so these will have 3 coloured faces with 3rd face coloured with other colour. From these 4 cubes 1 cube has 2 faces coloured blue & 3rd face coloured green & 1 cube has 2 faces coloured green & 3rd face is coloured blue.

Required number of small cubes = $4(n – 2)^2 + 2(n – 2) + 2 = 4(6 – 2)^2 + 2(6 – 2) + 2 = 74$

15 to 18

A large cube which was painted on all 6 faces is cut into certain number of identical smaller cubes. There are 8 smaller cubes which don't have any face painted. Let there are n^3 total number of smaller cubes.

15. The number of smaller cube is n^3. The number of cubes with no face painted = $(n – 2)^3 = 8$.

So, n – 2 = 4 & hence n = 4. So, total number of smaller cubes = $4^3 = 64$.

16. The number of small cube with exactly 1 painted face = $6(n – 2)^2 = 6(4 – 2)^2 = 24$.

17. The number of small cube with exactly 2 painted faces = $12(n – 2) = 12(4 – 2) = 24$.

18. The number of small cube with exactly 3 painted faces = 8.

19 to 25

There are 216 smaller identical cubes. So, $n^3 = 216$. Therefore n = 6. Now, the original cube has 3 different faces painted with different colours viz. black, green & Blue.

19. We can choose the 3 different coloured faces in such a way that no 3 intersect at any corner. So, it is possible to have no 3 face coloured for the smaller cubes. So, the required least value is 0.

20. We can choose 3 coloured faces in 2 different ways. 1st is such a way in which these 3 coloured faces won't intersect at any corner. In 2nd way all 3 coloured faces may intersect at a corner. We won't get the same answer by both ways. So, as it can have 2 possible answers, it cannot be determined.

21. We can get the least number of small cubes with just 1 coloured face if we choose 3 coloured faces such that all 3 coloured faces intersect at a corner as then we get more number of cubes with 2 coloured faces.

There are total $n^2 = 6^2 = 36$ faces on each face. But, among them we need to remove 1 on 2 sides as those cubes are having 2 coloured faces. So, for each face we will have $(n – 1)^2$ cubes with exactly 1 coloured face. So, there are total $3(n – 1)^2$ cubes required number of cubes. So, required number of cubes = $3(n – 1)^2 = 3(6 – 1)^2 = 75$.

22. We can get the largest number of small cubes with just 1 coloured face if we choose 3 coloured faces such that all 3 coloured faces do not intersect at a corner as then we get minimum number of cubes with 2 coloured faces.

There are total $n^2 = 6^2 = 36$ faces on each face. There are total 3 such faces. But, among them we need to remove the cubes which are formed on 2 edges where 2 faces intersect (there are 2 such faces). There will be 2n cubes on each edge (n for each of 2 faces which intersected on the edge).

Thus, required number of cubes = $3(n^2) – 2(2n) = 3(6^2) – 4(6) = 84$.

23. We need to assume the 3 coloured faces do not intersect at a corner & the black coloured face is adjacent to both green coloured face & blue coloured face (but blue & green coloured faces are not adjacent to each other). So, the smaller cubes formed at the 2 edges where black coloured face meet other coloured face will have 2 coloured faces & so these cubes must be removed. We get total n^2 small cubes in each face. So, the required number of cubes = $n^2 – 2n = 6^2 – 2(6) = 24$.

24. We need to assume the 3 coloured faces do not intersect at a corner & the green & blue coloured faces are adjacent to each other. So, the smaller cubes formed at the edge where these faces intersect will have exactly 2 coloured faces with 1 face coloured green & other coloured blue. In each edge 6 small cubes are formed. So, required number of cubes is 6.

25. The number of smaller cubes with no coloured face must be $(n – 1)^3$ as we need to subtract 1 due to each face. So, required number of cubes = $(n – 1)^3 = (6 – 1)^3 = 125$. Only option (b) includes 125.

26. This can be solved by visual inspection. By observing the given figures we can conclude that E & C must be opposite faces. In option (c), E & C are shown as adjacent faces. So it does not confirm to the original cube.