27. 3D GEOMETRY

1. COORDINATE OF A POINT IN SPACE

Let P be a point in the space. If a perpendicular from that point is dropped to the xy-plane, then the algebraic length of this perpendicular is considered as z-coordinate. From the foot of the perpendicular, drop a perpendicular to x and y axes, and algebraic lengths of perpendicular are considered as y and x coordinates, respectively.

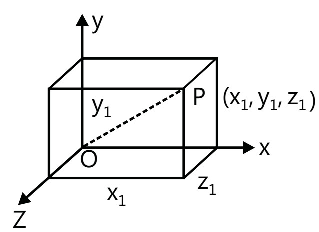


Figure 27.1

2. VECTOR REPRESENTATION OF A POINT IN SPACE

If (x, y, z) are the coordinates of a point P in space, then the position vector of the point P w.r.t. the same origin is $\vec{OP} = x\hat{i} + y\hat{i} + z\hat{k}$.

3. DISTANCE FORMULA

If (x_1, y_1, z_1) and (x_2, y_2, z_2) are any two points, then the distance between them can be calculated by the following formula: $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

3.1 Vector Method

If OA and OB are the position vectors of two points A (x_1, y_1, z_1) and B (x_2, y_2, z_2), then $AB = |\vec{OB} - \vec{OA}|$

$\Rightarrow AB = |(x_2 i + y_2 j + z_2 k) - (x_1 i + y_1 j + z_1 k)|$ $\quad \Rightarrow AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

3.2 Distance of a Point from Coordinate Axes

Let PA, PB and PC be the distances of the point P(x, y, z) from the coordinates axes OX, OY and OZ, respectively. Then $PA = \sqrt{y^2 + z^2}$, $PB = \sqrt{z^2 + x^2}$, $PC = \sqrt{x^2 + y^2}$

Illustration 1: Show that the points (0, 7, 10), (–1, 6, 6) and (–4, 9, 6) form a right-angled isosceles triangle.

(JEE MAIN)

Sol: By using distance formula we can find out length of sides formed by these points and if it satisfies Pythagoras theorem then these points form a right angled triangle.

Let A ≡ (0, 7, 10), B ≡ (–1, 6, 6), C ≡ (–4, 9, 6)$AB^2 = (0 + 1)^2 + (7 – 6)^2 + (10 – 6)^2 = 18$

∴ $AB = 3\sqrt{2}$ Similarly $BC = 3\sqrt{2}$ and AC = 6; Clearly $AB^2 + BC^2 = AC^2$ and AB = BC

Hence, ΔABC is isosceles right angled.

Illustration 2: Find the locus of a point which moves such that the sum of its distance from points A(0, 0, –α) and B(0, 0, α) is constant. **(JEE MAIN)**

Sol: Consider the point whose locus is required be P(x, y, z). As sum of its distance from point A and B is constant therefore PA + PB = constant = 2a.

Let P(x, y, z) be the variable point whose locus is required

Given that PA + PB = constant = 2a(say)

$$\therefore \quad \sqrt{(x-0)^2+(y-0)^2+(z+\alpha)^2}+\sqrt{(x-0)^2+(y-0)^2+(z-\alpha)^2}=2a$$

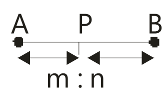
$$\Rightarrow \sqrt{x^2+y^2+(z+\alpha)^2}=2a-\sqrt{x^2+y^2+(z-\alpha)^2}$$

$$\Rightarrow x^2+y^2+z^2+\alpha^2+2z\alpha=4a^2+x^2+y^2+z^2+\alpha^2-2z\alpha-4a\sqrt{x^2+y^2+(z-\alpha)^2}$$

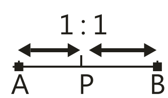
$$\Rightarrow 4z\alpha-4a^2=-4a\sqrt{x^2+y^2+(z-\alpha)^2} \Rightarrow \frac{z^2\alpha^2}{a^2}+a^2-2z\alpha=x^2+y^2+z^2+\alpha^2-2z\alpha \Rightarrow \frac{x^2+y^2}{a^2-\alpha^2}+\frac{z^2}{a^2}=1$$

4. SECTION FORMULA

If a point P divides the distance between the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the ratio of m:n, then the coordinates of P are $\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n},\frac{mz_2+nz_1}{m+n}\right)$



Note: Midpoint $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$



5. DIRECTION COSINES AND DIRECTION RATIOS

(a) Direction cosines: If α, β, γ are the angles which the line makes with the positive directions of the axes x, y and z coordinates, respectively, then cosα, cosβ, cosγ are called the direction cosines (d.c.s) of the line. The direction cosines are usually denoted by (*l*, m, n), where *l* = cosα, m = cosβ and n = cos γ.

(b) If ℓ, m, n are the direction cosines of a line, then $l^2 + m^2 + n^2 = 1$

(c) Direction ratios: If the intercepts a, b, c are proportional to the direction cosines ℓ,

(d) m, n, then a, b, c are called the direction ratios (d.r.s).

(e) If ℓ, m, n are the direction cosines and a, b, c are the direction ratios of a vector, then

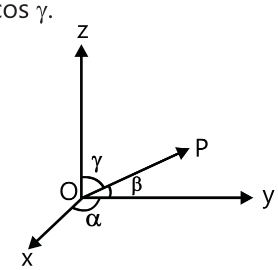


Figure 27.2

$$\ell=\frac{a}{\sqrt{a^2+b^2+c^2}}, m=\frac{b}{\sqrt{a^2+b^2+c^2}}, n=\frac{c}{\sqrt{a^2+b^2+c^2}} \quad \text{or}$$

$$\ell=\frac{-a}{\sqrt{a^2+b^2+c^2}}, m=\frac{-b}{\sqrt{a^2+b^2+c^2}}, n=\frac{-c}{\sqrt{a^2+b^2+c^2}}$$

(f) If OP = r, where O is the origin and *l*, m, n are the direction cosines of OP, then the coordinates of P are (*l*r, mr, nr) If direction cosines of the line AB are *l*, m, n, |AB| = r, and the coordinates of A is (x_1, y_1, z_1), then the coordinates of B are $(x_1 + rl, y_1 + rm, z_1 + rn)$

Illustration 3: Let α, β, γ be the angles made with the coordinate axes. Prove that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

(JEE ADVANCED)

Sol: Here line makes angles α, β, γ with the co-ordinates axes, hence by using its direction cosine we can prove given equation.

Since a line makes angles α, β, γ with the coordinates axes, $\cos\alpha, \cos\beta, \cos\gamma$, are direction cosines.

$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow (1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1 \Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Illustration 4: Find the direction cosines l, m, n of a line using the following relations: $l + m + n = 0$ and $2mn + 2ml - nl = 0$.

(JEE ADVANCED)

Sol: By solving these two equations simultaneously, we will be get $l : m : n$.

Given, $\ell + m + n = 0$ ... (i) $\quad 2mn + 2m\,l - n\,l = 0$ ...(ii)

From equation (i), $n = -(l + m)$

Substituting $n = -(l + m)$ in equation (ii), we get,

$-2m(l + m) + 2m\,l + (l + m)\,l = 0$

$\Rightarrow -2ml - 2m^2 + 2ml + l^2 + ml = 0$

$\Rightarrow l^2 + m\,l - 2m^2 = 0$

$\Rightarrow \left(\frac{\ell}{m}\right)^2 + \left(\frac{\ell}{m}\right) - 2 = 0 \Rightarrow \frac{l}{m} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = 1, -2$

Case I: When $\frac{\ell}{m} = 1$: In this case $m = \ell$ From equation (1), $2\,l + n = 0$

$\Rightarrow n = -2\,l$

$\therefore\ l : m : n = 1 : 1 : -2$ $\quad \therefore$ Direction ratios of the line are 1, 1, –2

$\therefore$ Direction cosines are $\pm\frac{1}{\sqrt{1^2+1^2+(-2)^2}}, \pm\frac{1}{\sqrt{1^2+1^2+(-2)^2}}, \pm\frac{-2}{\sqrt{1^2+1^2+(-2)^2}} = \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}$ or $-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$

Case II: When $\frac{\ell}{m} = -2$: In this case $\ell = -2m$

From equation (i), $-2m + m + n = 0$ Þ $n = m$ $\quad \therefore\ \ell : m : n = -2m : m : m$

$\therefore$ Direction ratios of the line are –2, 1, 1

$\therefore$ Direction cosines are given by

$$\frac{-2}{\sqrt{(-2)^2+1^2+1^2}}, \frac{1}{\sqrt{(-2)^2+1^2+1^2}}, \frac{1}{\sqrt{(-2)^2+1^2+1^2}} = \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \quad \text{or} \quad \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}$$

6. ANGLE BETWEEN TWO LINE SEGMENTS

If a_1, b_1, c_1 and a_2, b_2, c_2, are the direction ratios of any two lines, respectively, then $a_1i + b_1j + c_1k$ and $a_2i + b_2j + c_2k$ are the two vectors parallel to the lines, and the angle between them is given by the following formula:

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(a) The lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(b) The lines are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(c) Two parallel lines have same direction cosines, i.e. $l_1 = l_2$, $m_1 = m_2$, $n_1 = n_2$

Illustration 5: Prove that the lines, whose direction cosines given by the relations $a^2 l + b^2m + c^2n = 0$ and $mn + nl + lm = 0$, are perpendicular if $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=0$ and parallel, if $a \mp b \pm c = 0$ **(JEE ADVANCED)**

Sol: Here if two lines are perpendicular then, $\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$ and if they are parallel then,

$\ell_1 = \ell_2,\ m_1 = m_2,\ n_1 = n_2$

Given that, $a^2 l + b^2m + c^2n = 0$ ... (i)

and $mn + n l + l m = 0$ ... (ii)

Eliminating m from equations (i) and (ii), we have $-\frac{1}{b^2}(a^2\ell + c^2n)n + n\ell - \frac{1}{b^2}(a^2\ell + c^2n)\ell = 0$

$\Rightarrow a^2n\ell + c^2n^2 - b^2n\ell + a^2\ell^2 + c^2\ell n = 0 \qquad \Rightarrow a^2\frac{\ell}{n} + c^2 - b^2\frac{\ell}{n} + a^2\frac{\ell^2}{n^2} + c^2\frac{\ell}{n} = 0$

$\Rightarrow a^2\left(\frac{\ell}{n}\right)^2 + \left(a^2 - b^2 + c^2\right)\left(\frac{\ell}{n}\right) + c^2 = 0$ ... (iii)

Let $\frac{\ell_1}{n_1}, \frac{\ell_2}{n_2}$ be the roots of the equation (iii).

$\therefore$ Product of roots $\frac{\ell_1}{n_1}\cdot\frac{\ell_2}{n_2} = \frac{c^2}{a^2} \qquad \Rightarrow \frac{\ell_1\ell_2}{1/a^2} = \frac{n_1n_2}{1/c^2} \Rightarrow \frac{\ell_1\ell_2}{1/a^2} = \frac{m_1m_2}{1/b^2} = \frac{n_1n_2}{1/c^2}$ [By symmetry]

$\Rightarrow \frac{\ell_1\ell_2}{1/a^2} = \frac{m_1m_2}{1/b^2} = \frac{n_1n_2}{1/c^2} = \frac{\ell_1\ell_2 + m_1m_2 + n_1n_2}{1/a^2 + 1/b^2 + 1/c^2}$

For perpendicular lines $\quad \ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$

$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=0 \qquad$ Two lines are parallel if $\ell_1 = \ell_2, m_1 = m_2, n_1 = n_2$

$\Rightarrow \frac{\ell_1}{n_1} = \frac{\ell_2}{n_2} \qquad \Rightarrow$ roots of equation (iii) are equal $\Rightarrow (a^2 - b^2 + c^2)^2 - 4a^2c^2 = 0 \Rightarrow a^2 - b^2 + c^2 = \pm 2ac$

$\Rightarrow a^2 + c^2 \pm 2ac = b^2 \Rightarrow (a \pm c)^2 = b^2 \qquad \Rightarrow (a \pm c) = \pm b \qquad \Rightarrow a \mp b \pm c = 0$

Note: In the above result, the two signs are independent of each other. So, the total cases would be $(a+b+c=0, a+b-c=0, a-b+c=0, a-b-c=0)$.

7. PROJECTION OF A LINE SEGMENT ON A LINE

If (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of P and Q, respectively, then the projection of the line segments PQ on a line having direction cosines ℓ, m, n is $|\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$.

Vector form: Projection of a vector $\vec{a}$ on another vector $\vec{b}$ is $= \frac{\vec{a}\vec{b}}{|\vec{b}|}$. In the above case, we replace $2\sqrt{6}$ with $\overrightarrow{PQ}$ as $(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$ and $\Rightarrow (x-1)^2 = 11$ with $\ell\hat{i} + m\hat{j} + n\hat{k}$.

where $\ell|r|, m|r| \& n|r|$ are the projections of r in the coordinate axes OX, OY and OZ, respectively.

$r = |r|(\ell\hat{i} + m\hat{j} + n\hat{k})$

Illustration 6: Find the projection of the line joining the coordinates (1, 2, 3) and (–1, 4, 2) on line having direction ratios 2, 3, –6. **(JEE MAIN)**

Sol: Here projection of line joining (1, 2, 3) and (–1, 4, 2) on the line having direction ratios 2, 3, –6 is given by $\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$.

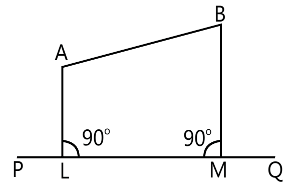


Figure 27.3

Let $A \equiv (1,2,3), B \equiv (-1,4,2)$. Direction ratios of the given line PQ are 2, 3, –6

$\therefore$ Direction cosines of PQ are $\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}$

Projection of AB on PQ = $\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

$$= \frac{2}{7}(-1-1) + \frac{2}{7}(4-2) - \frac{6}{7}(2-3) = \frac{-4+6+6}{7} = \frac{8}{7}$$

8. PLANE

If a line joining any two points on a surface entirely lies on it or if a line joining any two points on a surface is perpendicular to some fixed straight line, then the surface is called a plane. This fixed line is called the normal to the plane.

8.1 Equation of a Plane

(a) **Normal form:** The equation of a plane is given by lx + my + nz = p, where ℓ, m, n are the direction cosines of the normal to the plane and p is the distance of the plane from the origin.

(b) **General form:** The equation of a plane is given by ax + by + cz + d = 0, where a, b, c are the direction ratios of the normal to the plane.

(c) The equation of a plane passing through the point (x_1, y_1, z_1) is given by $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, where a, b, c are the direction ratios of the normal to the plane.

(d) **Plane through three points:** The equation of a plane through three noncollinear points is given by

$(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ is $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \equiv \begin{vmatrix} x - x_3 & y - y_3 & z - z_3 \\ x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \\ x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \end{vmatrix} = 0$

(e) **Intercept form:** The equation of a plane cutting the intercepts a, b, c on the axes is given by $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

(f) **Vector form:** The equation of a plane passing through a point having a position vector $\vec{a}$ and unit vector normal to plane is $(\vec{r} - \vec{a}).\hat{n} = 0 \Rightarrow \vec{r}.\hat{n} = \vec{a}.\hat{n}$

(g) The equation of any plane parallel to the given plane ax + by + cz + d = 0 is given by ax + by + cz + λ = 0 (same direction ratios), where λ is any scalar.

(h) The equation of a plane passing through a given point $\vec{a}$ and parallel to two vectors $\vec{b}$ and $\vec{c}$ is given by $\vec{r} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$ where $\vec{r}$ is a position vector of any point on the plane.

8.2 Plane Parallel to a Given Plane

The general equation of the plane parallel to the plane ax + by + cz + d = 0 is ax + by + cz + k = 0, where k is any scalar.

Distance between two parallel planes ax + by + cz + d_1 = 0 and ax + by + cz + d_2 = 0 is given by $\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$

Illustration 7: Find the distance between the planes 2x – y + 2z = 4 and 6x – 3y + 6z = 2. **(JEE MAIN)**

Sol: Here if two planes are parallel then the distance between them is equal to $\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$.

Given planes are $2x - y + 2z - 4 = 0$ ... (i)

and $6x - 3y + 6z - 2 = 0$ ... (ii)

We find that $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$. Hence, planes (i) and (ii) are parallel

Plane (ii) may be written as $2x - y + 2z - 2/3 = 0$ ... (iii)

$\therefore$ Required distance between the planes $= \frac{|4-(2/3)|}{\sqrt{2^2+(-1)^2+2^3}} = \frac{10}{3.3} = \frac{10}{9}$

8.3 Plane Passing Through the Line of Intersection of Planes

Let π_1 and π_2 be the two planes represented by equations $\vec{r}\cdot\hat{n}_1 = d_1$ and $\vec{r}\cdot\hat{n}_2 = d_2$, respectively. The position vector of any point on the line of intersection must satisfy both equations.

If $\vec{t}$ is the position vector of a point on the line, then $\vec{t}\cdot\hat{n}_1 = d_1$ and $\vec{t}\cdot\hat{n}_2 = d_2$

Therefore, for all real values of λ, we have $\vec{t}\cdot(\hat{n}_1 + \lambda\hat{n}_2) = d_1 + \lambda d_2$

Because $\vec{t}$ is arbitrary, it satisfies for any point on the line. Hence, the equation $\vec{r}\cdot(\hat{n}_1 + \lambda\hat{n}_2) = d_1 + \lambda d_2$ represents a plane π_3 which is such that if any vector X satisfies the equations of both the planes π_1 and π_2, it also satisfies the equation of plane π_3.

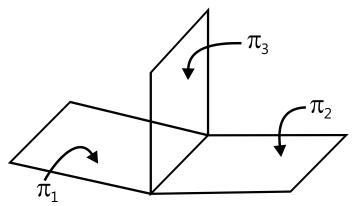


Figure 27.4

8.4 Cartesian Form

In a Cartesian system, let $n_1 = A_1\hat{i} + B_1\hat{j} + C_1\hat{k}$, $n_2 = A_2\hat{i} + B_2\hat{j} + C_2\hat{k}$ and $r = x\hat{i} + y\hat{j} + z\hat{k}$.

On substituting above values in vector equation we get,

$x(A_1 + \lambda A_2) + y(B_1 + \lambda B_2) + z(C_1 + \lambda C_2) = d_1 + \lambda d_2$ or $(A_1 x + B_1 y + C_1 z - d_1) + \ell(A_2 x + B_2 y + C_2 z - d_2) = 0$

Illustration 8: Show that the points (0, –1, 0), (2, 1, –1), (1, 1, 1), (3, 3, 0) are coplanar. **(JEE MAIN)**

Sol: Equation of any plane passing through (x_1, y_1, z_1) is given by $a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$, by using this formula we can obtain respective equation of plane.

Let A ≡ (0, –1, 0), B ≡ (2, 1, –1), C ≡ (1, 1, 1) and D ≡ (3, 3, 0)

Equation of a plane through A(0, –1, 0) is $a(x - 0) + b(y + 1) + c(z - 0) = 0$

$\Rightarrow ax + by + cz + b = 0$ ...(i)

If plane (i) passes through B(2, 1, –1) and C(1, 1, 1)

Then $2a + 2b - c = 0$ ...(2) and $a + 2b + c = 0$ ...(iii)

From equations (ii) and (iii), we have $\frac{a}{2+2} = \frac{b}{-1-2} = \frac{c}{4-2}$ or $\frac{a}{4} = \frac{b}{-3} = \frac{c}{2} = k$ (say)

Substituting values of a, b, c in equation (i), equation of the required plane is $4kx - 3k(y + 1) + 2kz = 0$

$\Rightarrow 4x - 3y + 2z - 3 = 0$ ...(iv)

Thus, point D(3, 3, 0) lies on plane (iv).

Because the points on the plane passes through A, B, C, the points A, B, C and D are coplanar.

Illustration 9: Find the equation of the plane upon which the length of normal from the origin is 10 and direction ratios of this normal are 3, 2, 6. **(JEE ADVANCED)**

Sol: Let p be the length of perpendicular from the origin to the plane and ℓ, m, n be the direction cosines of this normal. The equation is given by

$\ell x + my + nz = p$ ... (i)

From the data provided, p = 10 and the direction ratios of the normal to the plane are 3, 2, 6.

$\therefore$ Direction cosines of normal to the required plane are $\ell = \frac{3}{7}, m = \frac{2}{7}, n = \frac{6}{7}$

Substituting values of ℓ, m, n, p in equation (i), equation of the required plane is $\frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z = 10$

$\Rightarrow$ 3x + 2y + 6z = 70

Illustration 10: A point P moves on a plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$ A plane through P and perpendicular to OP meets the coordinate axes in A, B and C. If the planes through A, B and C parallel to the planes x = 0, y = 0, z = 0 intersect in Q, find the locus of Q. **(JEE ADVANCED)**

Sol: Similar to above problem.

Given plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ... (i)

Let P ≡ (h, k, ℓ), Then, $\frac{h}{a} + \frac{k}{b} + \frac{\ell}{c} = 1$ ... (ii)

(OP) = $\sqrt{h^2 + k^2 + \ell^2}$

Direction cosines of OP are $\frac{h}{\sqrt{h^2 + k^2 + \ell^2}}, \frac{k}{\sqrt{h^2 + k^2 + \ell^2}}, \frac{\ell}{\sqrt{h^2 + k^2 + \ell^2}}$

$\therefore$ Equation of the plane through P and normal to OP is

$$\frac{h}{\sqrt{h^2 + k^2 + \ell^2}}x + \frac{k}{\sqrt{h^2 + k^2 + \ell^2}}y + \frac{\ell}{\sqrt{h^2 + k^2 + \ell^2}} = \sqrt{h^2 + k^2 + \ell^2}$$

$$\Rightarrow hx + ky + \ell z = (h^2 + k^2 + \ell^2),\ A \equiv \left(\frac{h^2 + k^2 + \ell^2}{h}, 0, 0\right), B = \left(0, \frac{h^2 + k^2 + \ell^2}{k}, 0\right), C \equiv \left(0, 0 \frac{h^2 + k^2 + \ell^2}{\ell}\right)$$

$$\Rightarrow A = \left(\frac{h^2 + k^2 + \ell^2}{h}, 0, 0\right), B = \left(0, \frac{h^2 + k^2 + \ell^2}{k}, 0\right), C = \left(0, 0 \frac{h^2 + k^2 + \ell^2}{\ell}\right)$$

Let $Q \equiv (\alpha, \beta, \gamma)$, then $\alpha = \frac{h^2 + k^2 + \ell^2}{h}, \beta = \frac{h^2 + k^2 + \ell^2}{k}, \gamma = \frac{h^2 + k^2 + \ell^2}{\ell}$ ... (iii)

Now $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{h^2 + k^2 + \ell^2}{(h^2 + k^2 + \ell^2)^2} = \frac{1}{(h^2 + k^2 + \ell^2)}$ ... (iv)

From equation (iii), $h = \frac{h^2 + k^2 + \ell^2}{\alpha}$

$\therefore \frac{h}{a} = \frac{h^2 + k^2 + \ell^2}{a\alpha}$ Similarly $\frac{k}{b} = \frac{h^2 + k^2 + \ell^2}{b\beta}$ and $\frac{\ell}{c} = \frac{h^2 + k^2 + \ell^2}{c\gamma}$

$\therefore \frac{h^2 + k^2 + \ell^2}{a\alpha} + \frac{h^2 + k^2 + \ell^2}{b\beta} + \frac{h^2 + k^2 + \ell^2}{c\gamma} = \frac{h}{a} + \frac{k}{b} + \frac{\ell}{c} = 1$ [from equation (ii)] or, $\frac{1}{a\alpha}+\frac{1}{b\beta}+\frac{1}{c\gamma}=\frac{1}{h^2+k^2+\ell^2}=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$ [from equation (iv)]

$\therefore$ Required locus of Q(α, β, γ) is $\frac{1}{ax}+\frac{1}{by}=\frac{1}{cz}=\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}$.

8.5 Plane and a Point

(a) A plane divides the three-dimensional space into two equal segments. Two points A(x_1 y_1 z_1) and B(x_2 y_2 z_3) lie on the same sides of the plane ax + by + cz + d = 0 if the two expressions $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_3 + d$ are of same sign, and lie on the opposite sides of plane if both of these expressions are of opposite sign.

(b) Perpendicular distance of the point (x′, y′, z′) from the plane ax + by + cz + d = 0 is given by

$$\frac{ax'+by'+cz'+d}{\sqrt{a^2+b^2+c^2}}.$$

(c) The length of the perpendicular from the point having a position vector $\vec{a}$ to the plane $\vec{r}.\vec{n}=d$ is given by

$$P=\frac{|\vec{a}.\vec{n}-d|}{|\vec{n}|}$$

(d) The coordinates of the foot of the perpendicular from the point (x_1, y_1, z_1) to the plane $ax+by+cz+d=0$ are given by $\frac{x'-x_1}{a}=\frac{y'-y_1}{b}=\frac{z'-z_1}{c}=-\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

(e) If P′(x′, y′, z′) is the image of a point P(x_1, y_1, z_1) w.r.t. the plane ax + by + cz + d = 0, then

$$\frac{x'-x_1}{a}=\frac{y'-y_1}{b}=\frac{z'-z_1}{c}=-2\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$$

PLANCESS CONCEPTS

The distance between two parallel planes ax + by + cx + d = 0 and ax + by + cx + d′ = 0 is given by

$$\frac{|d-d'|}{\sqrt{a^2+b^2+c^2}}$$

If a variable point P moves so that $PA^2 - PB^2 = K$, where K is a constant and A and B are the two points, then the locus of P is a plane.

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Illustration 11: Show that the points (1, 2, 3) and (2, –1, 4) lie on the opposite sides of the plane x + 4y + z – 3 = 0. **(JEE MAIN)**

Sol: Substitute given points in to the given equation of plane, if their values are in opposite sign then the points are on opposite sides of the plane.

Since 1 + 4 × 2 + 3 – 3 = 9 and 2 – 4 + 4 – 3 = –1 are of opposite sign, the points are on opposite sides of the plane.

8.6 Angle Between Two Planes

Let us consider two planes ax + by + cz + d = 0 and a′x + b′y + c′z + d = 0. Angle between these planes is the angle between their normals. Let (a, b, c) and (a′, b′, c′) be the direction ratios of their normals of the two planes, respectively, and the angle θ between them is given by

$$\cos\theta = \frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}.$$

The planes are perpendicular if aa′ + bb′ + cc′ = 0 and the planes are parallel if $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$.

In vector form, if θ is the angle between the planes $\vec{r}.\vec{n}=d_1$ and $\vec{r}.\vec{n}_2=d_2$, then $\cos\theta=\frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1|\cdot|\vec{n}_2|}$. The planes are perpendicular if $\vec{n}_1\cdot\vec{n}_2=0$ and the planes are parallel if $\vec{n}_1=\lambda\vec{n}_2$.

8.6.1 Angle Bisectors

(a) Equations of the planes bisecting the angle between the two given planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

(b) Equation of bisector of the angle containing the origin is given by

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$ [where d_1 and d_2 are positive]

(c) In order to find the bisector of acute/obtuse angle, both the constant terms should be positive. If

$a_1 a_2 + b_1 b_2 + c_1 c_2 > 0$ Þ then the origin lies in the obtuse angle

$a_1 a_2 + b_1 b_2 + c_1 c_2 < 0$ Þ then the origin lies in the acute angle

now apply step (ii) according to the question.

8.7 Family of Planes

(a) The equation of any plane passing through the line of intersection of nonparallel planes or through the given line is

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, i.e. $P_1 = 0$ and $P_2 = 0$

$a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0$, i.e. $P_1+\lambda P_2=0$

(b) The equation of plane passing through the intersection of the planes $\vec{r}.\vec{n}_1=d_1$ & $\vec{r}.\vec{n}_2=d_2$ is $r.(n_1+\lambda n_2)=d_1+\lambda d_2$, where λ is an arbitrary scalar.

Illustration 12: The plane x – y – z = 4 is rotated through 90° about its line of intersection with the plane x + y + 2z = 4. Find its equation in the new position. **(JEE MAIN)**

Sol: As the required plane passes through the line of intersection of given planes, therefore its equation may be taken as x + y + 2z – 4 + k(x – y – z – 4) = 0

$\Rightarrow$ (1 + k) x + (1 – k) y + (2 – k) z – 4 – 4k = 0 ... (iii)

Thus, planes (i) and (iii) are mutually perpendicular.

$\therefore$ (1 + k) – (1 – k) – (2 – k) = 0 $\Rightarrow$ 1 + k – 1 + k – 2 + k = 0 $\Rightarrow$ k = 2/3

Substituting k = 2/3 in equation (iii), we get, 5x + y + 4z = 20. This is the required equation of the plane in its new position.

Illustration 13: Find the equation of the plane through the point (1, 1, 1) and passing through the line of intersection of the planes x + y + z = 6 and 2x + 3y + 4z + 5 = 0 **(JEE MAIN)**

Sol: Similar to above illustration.

Given planes are $x + y + z - 6 = 0$ ... (i)

and $2x + 3y + 4z + 5 = 0$ ... (ii)

Given point is P(1, 1, 1).

Equation of any plane passing through the line of intersection of the planes (i) and (ii) is

$x + y + z - 6 + k(2x + 3y + 4z + 5) = 0$ ... (iii)

If plane (iii) passes through a point P, then $1 + 1 + 1 - 6 + k(2 + 3 + 4 + 5) = 0$

$\Rightarrow k = \dfrac{3}{14}$

From equation (i), the required plane is $20x + 23y + 26z - 69 = 0$

Illustration 14: If the planes x – cy – bz = 0, cx – y + az = 0 and bx + ay – z = 0 pass through a straight line, then find the value of $a^2 + b^2 + c^2 + 2abc$. **(JEE ADVANCED)**

Sol: Here the plane passing through the line of intersection of planes x – cy – bz = 0 and cx – y + az = 0 is same as the plane bx + ay – z = 0. Hence by using family of planes we can obtain required result.

Given planes are $x - cy - bz = 0$ ... (i)

$cx - y + az = 0$ ... (ii)

$bx + ay - z = 0$ ... (iii)

Equation of any plane passing through the line of intersection of the planes (i) and (ii) may be written as

$x - cy - bz + \lambda(cx - y + az) = 0 \quad \Rightarrow x(1 + lc) - y(c + \lambda) + z(-b + a\lambda) = 0$ ... (iv)

If planes (3) and (4) are the same, then equations (iii) and (iv) will be identical.

$\therefore \underset{(i)}{\dfrac{1+c\lambda}{b}} = \underset{(ii)}{\dfrac{-(c+\lambda)}{a}} = \underset{(iii)}{\dfrac{-b+a\lambda}{-1}};$

From equations (i) and (ii), $a + ac\lambda = -bc - b\lambda$

$\Rightarrow \lambda = -\dfrac{(a+bc)}{(ac+b)}$ ... (v)

From equations (ii) and (iii), $c + \lambda = -ab + a^2\lambda$

$\Rightarrow \lambda = -\dfrac{(ab+c)}{1-a^2}$ ... (vi)

From equations (v) and (vi), we have, $\dfrac{-(a+bc)}{ac+b} = \dfrac{-(ab+c)}{(1-a^2)}$

$\Rightarrow a - a^3 + bc - a^2bc = a^2bc + ac^2 + ab^2 + bc \quad \Rightarrow 2a^2bc + ac^2 + ab^2 + a^3 - a = 0$

$\Rightarrow a^2 + b^2 + c^2 + 2abc = 1$

Illustration 15: Through a point P(h, k, ℓ), a plane is drawn at right angles to OP to meet the coordinate axes in A, B and C. If OP = p, show that the area of ΔABC is $\dfrac{p^5}{2hk\ell}$. **(JEE ADVANCED)**

Sol: Here line OP is normal to the plane, therefore ℓx + my nz = p. where ℓ, m and n are direction cosines of given plane.

$OP = \sqrt{h^2 + k^2 + \ell^2} = p$

Direction cosines of OP are $\frac{h}{\sqrt{h^2+k^2+\ell^2}}, \frac{k}{\sqrt{h^2+k^2+\ell^2}}, \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}$

Since OP is the normal to the plane, therefore, equation of the plane will be

$$\frac{h}{\sqrt{h^2+k^2+\ell^2}}x + \frac{k}{\sqrt{h^2+k^2+\ell^2}}y + \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}z = \sqrt{h^2+k^2+\ell^2}$$

$\Rightarrow$ $hx + ky + \ell z = h^2 + k^2 + \ell^2 = p^2$

$$\therefore\ A = \left(\frac{p^2}{h},0,0\right), B = \left(0,\frac{p^2}{k},0\right), C = \left(0,0,\frac{p^2}{\ell}\right)$$

Thus, area of $\Delta ABC = \frac{\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|}{2}$

$$= \frac{\left|\left(\frac{p^2}{h}\hat{i} - \frac{p^2}{k}\hat{j}\right)\times\left(\frac{p^2}{h}\hat{i} - \frac{p^2}{\ell}\hat{k}\right)\right|}{2} = \frac{\left|\left(\frac{p^4}{h\ell}\hat{j} + \frac{p^4}{kh}\hat{k} + \frac{p^4}{k\ell}\hat{i}\right)\right|}{2}$$

$$= \frac{1}{2}\sqrt{p^8\left(\frac{1}{h^2\ell^2} + \frac{1}{h^2k^2} + \frac{1}{k^2\ell^2}\right)} = \frac{1}{2}\sqrt{\frac{P^8}{h^2\ell^2k^2}\left(\ell^2 + h^2 + k^2\right)} = \frac{p^5}{2hk\ell}$$

9. TETRAHEDRON

Volume of a tetrahedron given the coordinates of its vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ can be calculated by

$$V = \frac{1}{6}\left\|\begin{matrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & x_4 & 1 \end{matrix}\right\| = \frac{1}{6}\left\|\begin{matrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{matrix}\right\|$$

PLANCESS CONCEPTS

Four points (x_r, y_r, z_r); r = 1, 2, 3, 4; will be coplanar if the volume of the tetrahedron with the points as vertices is zero. Therefore, the condition of coplanarity of the points

(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) is $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0.$

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Centroid of a Tetrahedron

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ be the vertices of a tetrahedron.

The coordinate of its centroid (G) is given as $\left(\frac{\Sigma x_i}{4}, \frac{\Sigma y_i}{4}, \frac{\Sigma z_i}{4}\right)$

Illustration 16: If two pairs of opposite edges of a tetrahedron are mutually perpendicular, show that the third pair will also be mutually perpendicular. **(JEE MAIN)**

Sol: If two lines are perpendicular then summation of product of their respective direction ratios is equals to zero.

Let OABC be the tetrahedron where O is the origin and coordinate of A, B, C be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), respectively.

Let $OA \perp BC$ and $OB \perp CA$. We have to prove that $OC \perp BA$

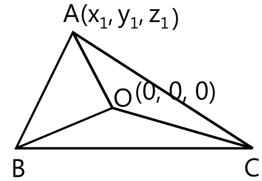

Figure 27.5

Direction ratios of OA are $x_1 - 0, y_1 - 0, z_1 - 0$ or x_1, y_1, z_1

Direction ratios of BC are $(x_3 - x_2), (y_3 - y_2), (z_3 - z_2)$

$OA \perp BC$

$\Rightarrow x_1(x_3 - x_2) + y_1(y_3 - y_2) + z_1(z_3 + z_2) = 0$ ... (i)

Similarly, $OB \perp CA$

$\Rightarrow x_2(x_1 - x_3) + y_2(y_1 - y_3) + z_2(z_1 - z_3) = 0$ ... (ii)

On adding equations (1) and (2), we obtain the following equation:

$x_3(x_1 - x_2) + y_3(y_1 - y_2) + z_3(z_1 - z_2) = 0$

$\therefore$ $OC \perp BA$ [$\because$ direction ratios of OC are x_3, y_3, z_3 and that of BA are $(x_1 - x_2), (y_1 - y_2), (z_1 - z_2)$]

10. LINE

10.1 Equation of a line

A straight line in space will be determined if it is the intersection of two given nonparallel planes and therefore, the equation of a straight line is present as a solution of the system constituted by the equations of the two planes, $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$. This form is also known as non-symmetrical form.

(a) The equation of a line passing through the point (x_1, y_1, z_1) with a, b, c as direction ratios is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r$. This form is called symmetrical form. A general point on the line is given by $(x_1 + ar, y_1 + br, z_1 + cr)$.

(b) Vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is a scalar.

(c) The equation of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_3) is

(d) $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$.

(e) Vector equation of a straight line passing through two points with position vectors $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

(f) Reduction of Cartesian form of equation of a line to vector form and vice versa is as

(g) $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \Leftrightarrow r = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$

PLANCESS CONCEPTS

Straight lines parallel to coordinate axes

Straight lines	Equations
(i) Through origin	y = mx, z = nx
(ii) x-axis	y = 0, z = 0
(iii) y-axis	x = 0, z = 0
(iv) z-axis	x = 0, y = 0
(v) Parallel to x-axis	y = p, z = q
(vi) Parallel to y-axis	x = h, z = q
(vii) Parallel to z-axis	x = h, y = p

The number of lines which are equally inclined to the coordinate axes are 4.

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Illustration 17: Find the equation of the line passing through the points (3, 4, –7) and (1, –1, 6) in vector form as well as in Cartesian form. **(JEE MAIN)**

Sol: Here line in vector form is given by $r = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$ and in Cartesian form is given by $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$.

Let $A \equiv (3, 4, -7)$, $B \equiv (1, -1, 6)$; Now, $\vec{a} = \overrightarrow{OA} = 3\hat{i} + 4\hat{j} - 7\hat{k}, \vec{b} = \overrightarrow{OB} = \hat{i} - \hat{j} + 6\hat{k}$

Equation (in vector form) of the line passing through A(a) and B($\vec{b}$) is $r = a + t(\vec{b} - \vec{a})$

$\Rightarrow \vec{r} = 3\vec{i} + 4\vec{j} - 7\vec{k} + t(-2\vec{i} - 5\vec{j} + 13\vec{k})$ ... (i)

Equation in Cartesian form is $\frac{x-3}{3-1} = \frac{y-4}{4+1} = \frac{z+7}{-7-6} \Rightarrow \frac{x-3}{2} = \frac{y-4}{5} = \frac{z+7}{-13}$

Illustration 18: Show that the two lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ intersect. Find also the point of intersection of these lines. **(JEE MAIN)**

Sol: The given lines will intersect if any point on respective lines coincide for some value of λ and r.

Given lines are $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ .... (i)

and $\frac{z-4}{5} = \frac{y-1}{2} = \frac{z-0}{1}$ .... (ii)

Any point on line (1) is P(2r + 1, 3r + 2, 4r + 3) and any point on line (2) is Q(5λ + 4, 2λ + 1, λ)

Lines (i) and (ii) will intersect if P and Q coincide for some value of λ and r.

$\Rightarrow 2r + 1 = 5\lambda + 4 \quad \Rightarrow 2r - 5\lambda = 3$ ... (iii)

$\Rightarrow 3r + 2 = 2\lambda + 1 \quad \Rightarrow 3r - 2\lambda = -1$ ... (iv)

$\Rightarrow 4r + 3 = \lambda \quad \Rightarrow 4r - \lambda = -3$ ... (v)

Solving equations (iii) and (iv), we get r = –1, λ = –1; these obtained values of r and λ clearly satisfy equation (v)

$\Rightarrow P \equiv (-1, -1, -1)$. Hence, lines (i) and (ii) intersect at (–1, –1, –1)

Illustration 19: Find the angle between the lines x – 3y – 4 = 0, 4y – z + 5 = 0 and x + 3y – 11 = 0, 2y – z + 6 = 0. **(JEE MAIN)**

Sol: $\frac{x-4}{3}=\frac{y-0}{1}=\frac{z-5}{4}$ ... (i)

$\frac{x-11}{-3}=\frac{y-0}{1}=\frac{z-6}{2}$ ... (ii)

a = 3, b = 1, c = 4 $\therefore$ $a^1=-3, b^1=1, c^1=2$

$aa^1+bb^1+cc^1=-9+1+8=0 \Rightarrow \cos\theta=0$ $\theta=90$

Illustration 20: Find the equation of the line drawn through point (1, 0, 2) to meet at right angle with the line $\frac{x+1}{3}=\frac{y-2}{-2}=\frac{z+1}{-1}$. **(JEE ADVANCED)**

Sol: If two lines are perpendicular then summation of product of their direction ratios are equal to zero. Hence by obtaining direction ratio of these line, we will be get the result.

Given line is $\frac{x+1}{3}=\frac{y-2}{-2}=\frac{z+1}{-1}$ ... (i)

Let P ≡ (1, 0, 2); coordinates of any point on line (i) may be written as Q≡ (3r – 1, –2r + 2, –r –1).

Direction ratios of PQ are 3r – 2, –2r + 2, –r –3

Direction ratios of the line $\frac{x+1}{3}=\frac{y-2}{-2}=\frac{z+1}{-1}$ are 3, –2, –1

Since PQ ⊥ to the line $\Rightarrow$ 3(3r – 2) – 2(–2r + 2) – 1 (–r – 3) = 0

$\Rightarrow$ 9r – 6 + 4r – 4 + r + 3 = 0 $\Rightarrow$ 14r = 7 $\Rightarrow$ r = $\frac{1}{2}$ $\therefore$ Direction ratios of PQ are $-\frac{1}{2},1,-\frac{7}{2}$

Hence, equation of the line PQ is $\frac{x-1}{-1}=\frac{y-0}{2}=\frac{z-2}{-7}$

Illustration 21: Find the equation of the line of intersection of the planes 4x + 4y – 5z = 12, 8x + 12y – 13z = 32 in the symmetric form **(JEE ADVANCED)**

Sol: Consider the line of intersection meet the xy-plane at P(α, β, 0), therefore obtain the value of α and β and direction ratios of line of intersection to solve the problem.

Given planes are 4x + 4y – 5z – 12 = 0 ... (i)

and 8x + 12y – 13z – 32 = 0 ... (ii)

Let ℓ, m, n be the direction ratios of the line of intersection. Then,

$4\ell + 4m - 5n = 0$ and $8\ell + 12m - 13n = 0 \Rightarrow \frac{\ell}{-52+60}=\frac{m}{-40+52}=\frac{n}{48-32} \Rightarrow \frac{\ell}{2}=\frac{m}{3}=\frac{n}{4}$

Direction ratios of the line of intersection are 2, 3, 4.

Let the line of intersection meet the xy-plane at P(α, β, 0)

Then P lies on planes (i) and (i) $\Rightarrow$ 4α + 4β – 12 = 0 $\Rightarrow$ α + β – 3 = 0

and 8α + 12β – 32 = 0 ... (v)

or 2α + 3β – 8 = 0 ... (vi)

Solving equations (v) and (vi), we get $\frac{\alpha}{-8+9}=\frac{\beta}{-6+8}=\frac{1}{3-2}$ $\Rightarrow$ α = 1, β = 2

Hence, equation of the line of intersection in symmetrical form is $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-0}{4}$

10.1 Coplanar Lines

Coplanar lines are lines that entirely lie on the same plane.

(i) If $\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and $\frac{x-\alpha'}{\ell'}=\frac{x-\beta'}{m'}=\frac{z-\gamma'}{n'}$, are the lines, then the condition for intersection/coplanarity is $\begin{vmatrix}\alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ \ell & m & n \\ \ell' & m' & n'\end{vmatrix}=0$ and the plane containing the aforementioned lines is $\begin{vmatrix}x-\alpha & y-\beta & z-\gamma \\ \ell & m & n \\ \ell' & m' & n'\end{vmatrix}=0.$

(ii) Condition of coplanarity if both lines are in general form.

Let the lines be ax + by + cz + d = 0 = a′x + b′y + c′z + d′ and ax + by + gz + δ = 0 = α'x+ β'y + γ'z + δ' = 0

If $\Delta=\begin{vmatrix}a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta'\end{vmatrix}=0$, then they are coplanar.

10.2 Skew Lines

Skew lines are two lines that do not intersect and are not parallel.

If $\Delta=\begin{vmatrix}\alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n'\end{vmatrix}\neq 0,$ then the lines are skew.

Shortest distance

Let the equation of the lines be $\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and $\frac{x-\alpha'}{\ell'}=\frac{x-\beta'}{m'}=\frac{z-y'}{n'}$

$$\text{S.D.}=\frac{(\alpha-\alpha')(mn'-m'n)+(\beta-\beta')(n\ell'-n'\ell)+(\gamma-\ell')(\ell m'-\ell' m)}{\sqrt{\sum(mn'-m'n)^2}}=\begin{vmatrix}\alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n'\end{vmatrix}\Bigg/\sqrt{\sum(nm'-m'n)^2}$$

Vector form

For lines $\vec{a}_1+\lambda\vec{b}_1$ and $\vec{a}_2+\lambda\vec{b}_2$ to be skew, the following condition should be satisfied: $(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_1-\vec{a}_2)\neq 0$

PLANCESS CONCEPTS

Shortest distance between two skew lines is perpendicular to both the lines.

Anvit Tawar (JEE 2009 AIR 9)

10.3 Intersecting Lines

Two or more lines that intersect at a point are called intersecting lines, and their shortest distance between the two lines is

zero, i.e. $\left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right| = 0 \Rightarrow (\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_1 - \vec{a}_2) = 0 \Rightarrow [\vec{b}_1\vec{b}_2(\vec{a}_2 - \vec{a}_1)] = 0 \Rightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & a_2 & c_2 \end{vmatrix} = 0$

10.4 Parallel Lines

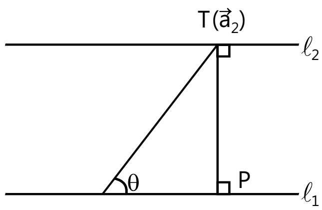


Figure 27.6

Parallel lines are lines that never intersect, and are coplanar.

Let $\left|-\frac{8}{3}, \frac{1}{3}, \frac{16}{3}\right|$ and $\left|\frac{8}{3}, \frac{1}{3}, \frac{16}{3}\right|$ be the two parallel lines.

Let the lines be given by $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ ... (i)

$\vec{r} = \vec{a}_2 + \mu\vec{b}$ ... (ii)

where $\vec{a}_1$ is the position vector of a point S on $\frac{x}{2} = \frac{y-1}{3} = \frac{z}{5}$ and $\vec{a}_2$ is the position vector of a point T on ℓ_2. As ℓ_1 and ℓ_2 are coplanar, if the foot of the perpendicular from T on the line ℓ_1 is P, then the distance between the lines ℓ_1 and ℓ_2 = |TP|. Let θ be the angle between the vectors $\vec{ST}$ and $\vec{b}$.

Then $\vec{b} \times \vec{ST} = (|\vec{b}||\vec{ST}|\sin\theta)\hat{n}$ ... (iii)

where $\hat{n}$ is the unit vector perpendicular to the plane of the lines $a\ell + bm + cn = 0$ and $ax_1 + by_1 + cz_1 + d \neq 0$. But $\vec{ST} = \vec{a}_2 - \vec{a}_1$

Therefore, from equation (iii), we get $\vec{b} \times (\vec{a}_2 - \vec{a}_1) = |\vec{b}|PT\hat{n}$ (as PT = ST sin θ)

$\Rightarrow |\vec{b} \times (\vec{a}_2 - \vec{a}_1)| = |\vec{b}|PT \cdot 1 (\text{as} |\hat{n}| = 1)$ Hence, the distance between the given parallel lines is $d = |\vec{PT}| = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$.

10.5 Angular Bisector

If $a(x-\alpha) + b(y-\beta)$, m_1, n_1 and $+c(z-\gamma) = 0$, , m_2, n_2 are the direction cosines of the two lines inclined to each other at an angle θ, then the direction cosines of the

(a) Internal bisectors of the angle between the lines are $\frac{\ell_1 + \ell_2}{2\cos(\theta/2)}, \frac{m_1 + m_2}{2\cos(\theta/2)}$ and $\frac{n_1 + n_2}{2\cos(\theta/2)}$.

(b) External bisectors of the angle between the lines are $\frac{\ell_1 - \ell_2}{2\sin(\theta/2)}, \frac{m_1 - m_2}{2\sin(\theta/2)}$ and $\frac{n_1 - n_2}{2\sin(\theta/2)}$.

10.6 Reduction to Symmetric Form

Let the line in nonsymmetrical form be represented as $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$.

To find the equation of the line in symmetrical form, (i) its direction ratios and (ii) coordinate of any point on it must be known.

(a) **Direction ratios:** Let ℓ, m, n be the direction ratios of the line. Since the line lies on both planes, it must be perpendicular to the normal of both planes. So $a_1\ell + b_1m + c_1n = 0$, $a_2\ell + b_2m + c_2n = 0$. From these equations, proportional values of ℓ, m, n can be found by using the method of cross-multiplication, i.e.

$$\frac{\ell}{b_1c_2 - b_2c_1} = \frac{m}{c_1a_2 - c_2a_1} = \frac{n}{a_1b_2 - a_2b_1}$$

Alternate method

The vector $\begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = i(b_1c_2 - b_2c_1) + j(c_1a_2 - c_2a_1) + k(a_1b_2 - a_2b_1)$ will be parallel to the line of intersection of the two given planes. Hence, $\ell : m : n = (b_1c_2 - b_2c_1) : (c_1a_2 - c_2a_1) : (a_1b_2 - a_2b_1)$.

(b) Coordinate of any point on the line: Note that as ℓ, m, n cannot be zero simultaneously, so at least one must be nonzero. Let $a_1b_2 - a_2b_1 \neq 0$, so that the line cannot be parallel to xy-plane, and will intersect it. Let it intersect xy-plane at the point $(x_1, y_1, 0)$. These $a_1x_1 + b_1y_1 + d_1 = 0$ and $a_2x_1 + b_2y_1 + d_2 = 0$. Solving these, we get a point on the line. Thus, we get the following equation:

$$\frac{x - x_1}{b_1c_2 - b_2c_1} = \frac{y - y_1}{c_1a_2 - c_2a_1} = \frac{z - 0}{a_1b_2 - a_2b_1} \text{ or } \frac{x - (b_1d_2 - b_2d_1 / a_1b_2 - a_2b_1)}{b_1c_2 - b_2c_1} = \frac{y - (d_1a_2 - d_2a_1 / a_1b_2 - a_2b_1)}{c_1a_2 - c_2a_1} = \frac{z - 0}{a_1b_2 - a_2b_1}$$

Note: If $\ell \neq 0$, take a point on yz-plane as $(0, y_1, z_1)$ and if $m \neq 0$, take a point on xz-plane as $(x_1, 0, z_1)$.

Alternate method:

If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then put z = 0 in both equations and solve the equation $a_1x + b_1y + d_1 = 0$, $a_2x + b_2y + d_2 = 0$ or put y = 0 and solve the equation $a_1x + c_1z + d_1 = 0$ and $a_2x + c_2z + d_2 = 0$.

10.7 Point and Line

Foot Length and Equation of Perpendicular from a Point to a Line

Cartesian form: Let equation of the line be $\frac{x - a}{\ell} = \frac{y - b}{m} = \frac{z - c}{n} = r\,(\text{say})$ ...(i)

and $A(\alpha, \beta, \gamma)$ be the point. Any point on line (i) is $P(lr + a, mr + b, nr + c)$. If it is the foot of the perpendicular from point A on the line, then AP is perpendicular to the line.

$\Rightarrow \ell(\ell r + a - \alpha) + m(mr + b - \beta) + n(nr + c - \gamma) = 0$, i.e. $r = [(\alpha - a)\ell + (\beta - b)m + (\gamma - c)n] / l^2 + m^2 + n^2$

Using this value of r, we get the foot of the perpendicular from point A on the given line. Because the foot of the perpendicular P is known, the length of the perpendicular $AP = \sqrt{(\ell r + a - \alpha^2) + (mr + b - \beta)^2 + (nr + c - \gamma)^2}$ is given by the equation of perpendicular as $\frac{x - \alpha}{\ell r + a - \alpha} = \frac{y - \beta}{mr + b - \beta} = \frac{z - \gamma}{nr + c - \gamma}$

Illustration 22: Find the coordinates of the foot of the perpendicular drawn from the point A(1, 2, 1) to the line joining B(1, 4, 6) and C(5, 4, 4). **(JEE ADVANCED)**

Sol: Using section formula we will get co-ordinates of the foot D, and as AD is perpendicular to BC therefore $\overrightarrow{AD}.\overrightarrow{BC} = 0$.

Let D be the foot of the perpendicular drawn from A on BC, and let D divide BC in the ratio k:1. Then, the coordinates of D are $\left(\frac{5k+1}{k+1}, \frac{4k+4}{k+1}, \frac{4k+6}{k+1}\right)$ ...(i)

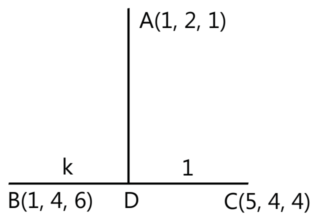


Figure 27.7

Now, $\overrightarrow{AD}$ = Position vector of D – Position vector of A

$$=\left(\frac{5k+1}{k+1}-1\right)\hat{i}+\left(\frac{4k+4}{k+1}-2\right)\hat{j}+\left(\frac{4K+6}{K+1}-1\right)\hat{K}=\left(\frac{4k}{k+1}\right)\hat{i}+\left(\frac{2k+2}{k+1}\right)\hat{j}+\left(\frac{3k+5}{k+1}\right)\hat{k}$$

and $\overrightarrow{BC}$ = Position vector of C – Position vector of B $=(5\hat{i}+4\hat{j}+4\hat{k})-(\hat{i}+4\hat{j}+6\hat{k})=4\hat{i}+0\hat{j}-2\hat{k}$

since $\overrightarrow{AD}\perp\overrightarrow{BC}\ \Rightarrow \overrightarrow{AD}.\overrightarrow{BC}=0\ \Rightarrow \left[\left(\frac{4k}{k+1}\right)\hat{i}+\left(\frac{2k+2}{k+1}\right)\hat{j}+\left(\frac{3k+5}{k+1}\right)\hat{k}\right].(4\hat{i}+0\hat{j}-2\hat{k})=0$

$$\Rightarrow 4\left(\frac{4k}{k+1}\right)+0\left(\frac{2k+2}{k+1}\right)-2\left(\frac{3k+5}{k+1}\right)=0 \quad \Rightarrow \frac{16k}{k+1}+0-2\frac{(3k+5)}{k+1}=0$$

$\Rightarrow \quad 16k-6k-10=0 \qquad \Rightarrow \quad k = 1$

Substituting k = 1 in equation (i), we obtain the coordinates of D as (3, 4, 5).

10.8 Vector Form

Equation of a line passing through a point having position vector $\bar{\alpha}$ and perpendicular to the lines $\vec{r}_1=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}_1=\vec{a}_2+\lambda\vec{b}_2$ is parallel to $\vec{b}_1\times\vec{b}_2$. So the vector equation of such line is $\vec{r}=\vec{\alpha}+\lambda(\vec{b}_1\times\vec{b}_2)$. The equation of the perpendicular passing through $\vec{\alpha}$ is $\vec{r}=\vec{\alpha}+\mu\left((\vec{a}-\vec{\alpha}-\left(\frac{(\vec{a}-\vec{\alpha})\vec{b}}{|\vec{b}|^2}\right)b.\right)$

10.9 Image w.r.t. the Line

Let $L\equiv\frac{x-x_2}{a}=\frac{y-y_2}{b}=\frac{z-z_2}{c}$ be the given line.

Let (x′, y′, z′) be the image of the point $P(x_1, y_1, z_1)$ w.r.t the line L. Then

(i) $a(x_1-x')+b(y_1-y')+c(z_1-z')=0$

(ii) $$\frac{\frac{x_1-x'}{2}-x_2}{a}=\frac{\frac{y_1-y'}{2}-y_2}{b}=\frac{\frac{z_1-z'}{2}-z_2}{c}=\lambda$$

From (ii), the value of x′, y′, z′ in terms of λ can be obtained as $x'=2a\lambda+2x_2-x_1$, $y'=$ $\lambda-2y_2-y_1$, $z'=2c\lambda+2z_2-z_1$

On substituting values of x′, y′, z′ in (i), we get λ and on re-substituting value of λ, we get (x′ y′ z′).

Illustration 23: Find the length of the perpendicular from P(2, –3, 1) to the line $\frac{x+1}{2}=\frac{y-3}{3}=\frac{z+2}{-1}$ **(JEE MAIN)**

Sol: Here Co-ordinates of any point on given line may be taken as Q ≡ (2r – 1, 3r + 3, –r – 2), therefore by using distance formula we can obtain required length.

Given line is $\frac{x+1}{2}=\frac{y-3}{3}=\frac{z+2}{-1}$ ...(i) P = (2, –3, 1)

Coordinates of any point on line (i) may be written as Q ≡ (2r – 1, 3r + 3, –r – 2)

Direction ratios of PQ are 2r – 3, 3r + 6, –r – 3.

Direction ratios of AB are 2, 3, –1.

Since PQ ⊥ AB $\Rightarrow$ 2(2r – 3) + 3(3r + 6) – 1(–r – 3) = 0 $\Rightarrow r=\frac{-15}{14}$

$$\Rightarrow Q=\left(\frac{-22}{7},\frac{-3}{14},\frac{-13}{14}\right) \quad \Rightarrow PQ=\frac{\sqrt{531}}{14}\text{units}$$

10.10 Plane Passing Through a Given Point and Line

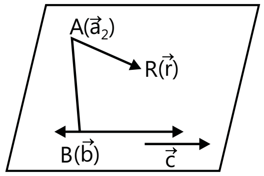


Figure 27.8

Let the plane pass through the given point A($\vec{a}$) and line $\vec{r} = \vec{b} + \lambda\vec{c}$. For any position of point R (r) on the plane, vectors $\overrightarrow{AB}, \overrightarrow{RA}$ and $\vec{c}$ are coplanar.

Then $\left[\vec{r} - \vec{a} \quad \vec{b} - \vec{a} \quad \vec{c}\right] = 0$, which is the required equation of the plane.

Angle between a plane and a line:

Angle between a line and a plane is complementary to the angle made by the line with the normal of plane. Hence, if θ is the angle between the line $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and the plane

ax + by + cz + d = 0, then $\sin\theta = \left(\frac{a\ell + bm + cn}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(\ell^2 + m^2 + n^2)}}\right)$

Vector form:

If θ is the angle between the line $\vec{r} = (\vec{a} + \lambda\vec{b})$ and $\vec{r}\cdot\vec{n} = d$ then $\sin\theta = \left[\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right]$

Line and plane are perpendicular if $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$, i.e. $\vec{b} \times \vec{n} = 0$.

Line and plane are parallel if aℓ + bm + cn = 0, i.e. $\vec{b}\cdot\vec{n} = 0$.

PLANCESS CONCEPTS

Condition for a Line to Lie on a Plane

(i) **Cartesian form:** Line $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ would lie on a plane ax + by + cz + d = 0 if,

$ax_1 + by_1 + cz_1 + d = 0$ and aℓ + bm + cn = 0

(ii) **Vector form:** Line $\vec{r} = \vec{a} + \lambda\vec{b} +$ would lie on the plane $\vec{r}\cdot\hat{n} = d$ if $\vec{b}\cdot\hat{n} = 0$ & $\vec{a}\cdot\hat{n} = d$.

The number of lines which are equally inclined to the coordinate axes is 4.

If ℓ, m, n are the d.c.s of a line, then the maximum value of lmn $= \frac{1}{3\sqrt{3}}$.

Akshat Kharaya (JEE 2009 AIR 235)

Illustration 24: Find the shortest distance and the vector equation of the lines of shortest distance between the lines given by $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda(3\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k})$. **(JEE ADVANCED)**

Sol: Consider LM is the shortest distance between given lines therefore LM is perpendicular to these lines, hence by obtaining their direction ratios and using perpendicular formula we will get the result.

Given lines are $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda\,(3\hat{i} - \hat{j} + \hat{k})$ ... (i)

and $\vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + \mu\,(-3\hat{i} + 2\hat{j} + 4\hat{k})$ ... (ii)

Equations of lines (i) and (ii) in Cartesian form are

$AB: \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = \lambda$ and $CD: \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} = \mu$

Let $L \equiv (3\lambda + 3, -\lambda + 8, \lambda + 3)$ and $M \equiv (-3\mu - 3,\ 2\mu - 7,\ 4\mu + 6)$

Direction ratios of LM are $3\lambda + 3\mu + 6, -\lambda - 2\mu + 15, \lambda - 4\mu - 3$ since LM $\perp$ AB

$\Rightarrow 3(3\lambda + 3\mu + 6) - 1(-\lambda - 2\mu + 15) + 1(\lambda - 4\mu - 3) = 0$

or, $11\lambda + 7\mu = 0$ ... (v)

Again $LM \perp CD$

$\therefore -3(3\lambda + 3\mu + 6) + 2(-\lambda - 2\mu + 15) + 4(\lambda - 4\mu - 3) = 0$

or, $-7\lambda - 29\mu = 0$ ... (vi)

Solving equations (v) and (vi), we get $\lambda = 0, \mu = 0 \quad \Rightarrow L \equiv (3, 8, 3), M \equiv (-3, -7, 6)$

Hence, the shortest distance LM = $\sqrt{(3+3)^2 + (8+7)^2 + (3-6)^2} = \sqrt{270} = 3\sqrt{30}$ units

Vector equation of LM is $\vec{r} = 3\vec{i} + 8\vec{j} + 3\vec{k} + t\,(-6\vec{i} + 15\vec{j} - 3\vec{k})$.

11. SPHERE

11.1 General Equation

The general equation of a sphere is given by $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, where $(-u, -v, -w)$ is the center and $\sqrt{u^2 + u^2 + w^2 - d}$ is the radius of the sphere.

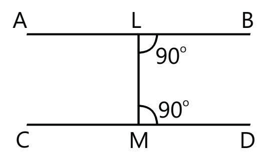


Figure 27.9

11.2 Diametric Form

If (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of the extremities of a diameter of a sphere, then its equation is given by $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$.

11.3 Plane and Sphere

If the perpendicular distance of the plane from the center of the sphere is equal to the radius of the sphere, then the plane touches the sphere. The plane $lx + my + nz = p$ touches the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, if $(u\ell + vm + wn + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$.

11.4 Intersection of Straight Line and a Sphere

Let the equations of the sphere and the straight line be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ... (i)

and $\dfrac{x - \alpha}{\ell} = \dfrac{y - \beta}{m} = \dfrac{z - \gamma}{n} = r$, (say) ... (ii)

Any point on the line (ii) is $(\alpha + lr, \beta + mr, \gamma + nr)$. If this point lies on the sphere (i), then we have

$$(\alpha + \ell r)^2 + (\beta + mr)^2 + (\gamma + nr)^2 + 2u(\alpha + \ell r) + 2v(\beta + mr) + 2w(\gamma + nr) + d = 0$$

$$\Rightarrow r^2[\ell^2 + m^2 + n^2] + 2r[\ell(u + \alpha) + m(v + \beta) + n(w + \gamma)] + (\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d) = 0 \quad \text{... (iii)}$$

This is a quadratic equation in r and thus two values of r are obtained. Therefore, the line (ii) intersects the sphere (i) at two points which may be real, coincident and imaginary, according to roots of (iii).

If ℓ, m, n are the actual d.c.s of the line, then $l^2 + m^2 + n^2 = 1$ and then the equation (iii) can be simplified.

11.5 Orthogonality of Two Spheres

Let the equation of the two spheres be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ... (i)

and $x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$ ... (ii)

If the sphere (i) and (ii) cut orthogonally, then $2uu' + 2vv' + 2ww' = d + d'$, which is the required condition. If the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ cut orthogonally, then $d = a^2$.

Two spheres of radii r_1 and r_2 cut orthogonally, then the radius of the common circle is $\frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$.

Illustration 25: A plane passes through a fixed point (a, b, c). Show that the locus of the foot of the perpendicular to it from the origin is the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$ **(JEE ADVANCED)**

Sol: Consider $P(\alpha, \beta, \gamma)$ be the foot of perpendicular from origin to, therefore by getting the direction ratios of OP we will get the required result.

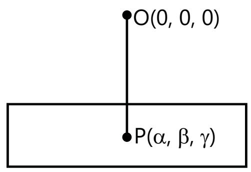


Figure 27.10

Let the equation of the variable plane be

$\ell x + my + nz + d = 0$ ... (i)

Plane passes through the fixed point (a, b, c)

$\therefore \ \ell a + mb + nc + d = 0$ ... (ii)

Let $P(\alpha, \beta, \gamma)$ be the foot of the perpendicular from the origin to plane (i)

Direction ratios of OP are $\alpha - 0, \beta - 0, \gamma - 0$, i.e. α, β, γ

From equation (i), it is clear that the direction ratios of the normal to the plane, i.e. OP are ℓ, m, n, and α, β, γ are the direction ratio of the same line OP $\therefore \ \frac{\alpha}{\ell} = \frac{\beta}{m} = \frac{\gamma}{n} = \frac{1}{k}$(say) ; $\ell = k\alpha, m = k\beta, n = k\gamma$ ... (iii)

Substituting values of ℓ, m, n from equation (iii) in equation (ii), we get, $ka\alpha + kb\beta + kc\gamma + d = 0$ ... (iv)

Since α, β, γ lies on plane (i) $\therefore \ \ell\alpha + m\beta + n\gamma + d = 0$ ... (v)

Substituting values of ℓ, m, n from equation (iii) in equation (v), we get $k\alpha^2 + k\beta^2 + k\gamma^2 + d = 0$ ... (vi)

[substituting value of d from equation (iv) in equation (vi)] or $\alpha^2 + \beta^2 + \gamma^2 - a\alpha - b\beta - c\gamma = 0$

Therefore, locus of foot of the perpendicular from the point $P(\alpha, \beta, \gamma)$ is $x^2 + y^2 + z^2 - ax - by - cz = 0$

Illustration 26: Find the equation of the sphere if it touches the plane $\vec{r} \cdot (2\vec{i} - 2\vec{j} - \vec{k}) = 0$ and the position vector of its center is $3\hat{i} + 6\hat{j} - 4\hat{k}$. **(JEE ADVANCED)**

Sol: Here equation of the required sphere is $|\vec{r} - \vec{c}| = a$ where a is the radius of the sphere.

Given plane is $\vec{r} \cdot (2\vec{i} - 2\vec{j} - \vec{k}) = 0$ ... (i)

Let H be the center of the sphere, then $\vec{OH} = 3\vec{i} + 6\vec{j} - 4\vec{k} = c$(say)

Radius of the sphere = length of perpendicular from H to plane (i)

$$= \frac{|c \cdot (2i - 2j - k)|}{|2\vec{i} - 2\vec{j} - \vec{k}|} = \frac{|(3i + 6j - 4k) \cdot (2i - 2j - k)|}{(2\vec{i} - 2\vec{j} - \vec{k})} = \frac{|6 - 12 + 4|}{3} = \frac{2}{3} = a \text{ (say)}$$

Equation of the required sphere is $|\vec{r} - \vec{c}| = a$

$$\Rightarrow |x\vec{i} + y\vec{j} + z\vec{k} - (3\vec{i} + 6\vec{j} - 4\vec{k})| = \frac{2}{3} \text{ or } |(x-3)\vec{i} + (y-6)\vec{j} + (z+4)\vec{k}|^2 = \frac{4}{9} \Rightarrow (x-3)^2 + (y-6)^2 + (z+4)^2$$

$= 4/9$ or $9(x^2 + y^2 + z^2 - 6x - 12y + 8z + 61) = 4 \Rightarrow 9x^2 + 9y^2 + 9z^2 - 54x - 108y + 72z + 545 = 0$

Illustration 27: Find the equation of the sphere with the points (1, 2, 2) and (2, 3, 4) as the extremities of a diameter. Find the coordinates of its center. **(JEE MAIN)**

Sol: Equation of the sphere having (x_1, y_1, z_1) and (x_2, y_2, z_2) as the extremities of a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$.

Let A ≡ (1, 2, 2), B ≡ (2, 3, 4)

Equation of the sphere having (x_1, y_1, z_1) and (x_2, y_2, z_2) as the extremities of a diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

Here $x_1 = 1, x_2 = 2, y_1 = 2, y_2 = 3, z_1 = 2, z_2 = 4$

$\therefore$ Required equation of the sphere is $(x - 1)(x - 2) + (y - 2)(y - 3) + (z - 2)(z - 4) = 0$ or

$x^2 + y^2 + z^2 - 3x - 5y - 6z + 16 = 0$

Center of the sphere is the midpoint of AB

$\therefore$ Center is $\left(\frac{3}{2}, \frac{5}{2}, 3\right)$.

Illustration 28: Find the equation of the sphere passing through the points (3, 0, 0), (0, –1, 0), (0, 0, –2) and whose center lies on the plane 3x + 2y + 4z = 1 **(JEE ADVANCED)**

Sol: Consider the equation of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

As the sphere passes through these given points hence these points will satisfy equation of sphere. ... (i)

Let A ≡ (3, 0, 0), B ≡ (0, –1, 0), C ≡ (0, 0, –2) since sphere (i) passes through A, B and C.

$\therefore 9 + 6u + d = 0$ ... (ii)

$1 - 2v + d = 0$ ... (iii)

$4 - 4w + d = 0$ ... (iv)

Since the center $(-u, -v, -w)$ of the sphere lies on plane $3x + 2y + 4z = 1 \therefore -3u - 2v - 4w = 1$ ....(v)

(ii) – (iii) $\Rightarrow 6u + 2v = -8$ ... (vi)

(iii) – (iv) $\Rightarrow -2v + 4w = 3$ ... (vii)

From (vi), $u = \frac{-2v - 8}{6}$ ... (viii)

From (vii), $4w = 3 + wv$

Substituting the values of u, v and w in equation (v), we get $\frac{2v + 8}{2} + 2v - 3 - 2v = 1$

$\Rightarrow 2v + 8 - 4v - 6 - 4v = 2 \Rightarrow v = 0$

From equation (viii), $u = \frac{0 - 8}{6} = -\frac{4}{3}$; From equation (ix), $4w = 3$ $\therefore w = 3/4$

From equation (iii), $d = 2v - 1 = 0 - 1 = -1$ From equation (i), the equation of the required sphere is

$x^2 + y^2 + z^2 - \frac{0 - 8}{6} - \frac{8}{3}x + \frac{3}{2}z - 1 = 0$ or $6x^2 + 6y^2 + 6z^2 - 16x + 9z - 6 = 0$

FORMULAE SHEET

(a) Distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

(b) Coordinates of the point dividing the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio m:n are

$$\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n},\frac{mz_2+nz_1}{m+n}\right)$$

(c) If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are vertices of a triangle, then its centroid is

$$\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right)$$

(d) If $A(x_1, y_1, z_1)$ and B (x_2, y_2, z_2) are the two points, the point which divides the line segment AB in ratio λ:1 is

$$\left(\frac{\lambda x_2+x_1}{\lambda+1},\frac{\lambda y_2+y_1}{\lambda+1},\frac{\lambda z_2+z_1}{\lambda+1}\right)$$

(e) If (x_1, y_1, z_1) and (x_2, y_2, z_2) are the two points on the line with x_2-x_1, y_2-y_1, z_2-z_1 as direction ratios, then their d.c.s are

$$\pm\frac{x_2-x_1}{\sqrt{\Sigma(x_2-x_1)^2}},\pm\frac{y_2-y_1}{\sqrt{\Sigma(x_2-x_1)^2}},\pm\frac{z_2-z_1}{\sqrt{\Sigma(x_2-x_1)^2}}$$

(f) If ℓ, m, n are d.c.s of a line, then $l^2+m^2+n^2=1$. Thus, if a line makes angles α, β, γ with axes, then $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ and $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$

(g) If a, b, c are the d.r.s of a line, then the d.c.s of the line are $\pm\frac{a}{\sqrt{\Sigma a^2}},\pm\frac{b}{\sqrt{\Sigma a^2}},\pm\frac{c}{\sqrt{\Sigma a^2}}$

(h) If p(x, y, z) is a point in space such that $\overrightarrow{OP}=\vec{r}$ has d.c.s ℓ, m, n, then

(a) $\ell|\vec{r}|, m|\vec{r}|, n|\vec{r}|$ are the projections on x-axis, y-axis and z-axis, respectively.

(b) $x=\ell|\vec{r}|, y=m|\vec{r}|, z=n|\vec{r}|$

(c) $\vec{r}=|\vec{r}|(l\hat{i}+m\hat{j}+n\hat{k})$ and $\hat{r}=\ell\hat{i}+m\hat{j}+n\hat{k}$

Moreover, if a, b, c are d.r.s of a vector $\vec{r}$, then $\vec{r}=\frac{|\vec{r}|}{\sqrt{a^2+b^2+c^2}}(a\hat{i}+b\hat{j}+c\hat{k})$.

(i) Length of projection of the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) on a line with d.c.s ℓ, m, n is $|\ell(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)|$

(j) If θ is the angle between two lines having direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 then

$$\cos\theta=\pm\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{\Sigma a_1^2}\sqrt{\Sigma a_2^2}}$$

(k) Two lines are parallel if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$ and two lines are perpendicular if $a_1a_2+b_1b_2+c_1c_2=0$

(l) Cartesian equations of a line passing through (x_1, y_1, z_1) and having direction ratios a, b, c are

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=t$$

(m) Vector equation of a line passing through the point $A(\vec{a})$ and parallel to vector $\vec{b}$ is $\vec{r}=\vec{a}+\lambda\vec{b}$ for scalar λ.

(n) Cartesian equation of a line passing through two points having coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\dfrac{x-x_1}{x_2-x_1}=\dfrac{y-y_1}{y_2-y_1}=\dfrac{z-z_1}{z_2-z_1}$.

(o) Vector equation of a line passing through two points having position vectors $\vec{a}$ and $\vec{b}$ is $\vec{r}=\vec{\alpha}+\lambda(\vec{b}-\vec{a})$

(p) Distance between the parallel lines $\vec{r}=\vec{a}_1+\lambda\vec{b}$ and $\vec{r}=\vec{a}_2+\mu\vec{b}$ is $\dfrac{|\vec{b}\times(\vec{a}_2-\vec{a}_1)|}{|\vec{b}|}$

(q) Shortest distance (S.D.) between two lines with equations; $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ is $\dfrac{|(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)|}{|\vec{b}_1\times\vec{b}_2|}$. If θ is the angle between the lines, then $\cos\theta=\dfrac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}$

(r) The length of perpendicular from the point (α,β,γ) to the line $\dfrac{x-x_1}{\ell}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$ (ℓ, m, n being d.cs)

is given by $\sqrt{(\alpha-x_1)^2+(\beta-y_1)^2+(\gamma-z_1)^2-[\ell(\alpha-x_1)+m(\beta-y_1)+n(\gamma-z_1)]^2}$

(s) If $\vec{a}$ and $\vec{b}$ are the unit vectors along the sides of an angle, then $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are the vectors, respectively, along the internal and external bisector of the angle. In fact, the bisectors of the angles between the lines, $\vec{r}=x\vec{a}$ and $\vec{r}=y\vec{b}$ are given by $\vec{r}=\lambda\left(\dfrac{\vec{a}}{|\vec{a}|}+\dfrac{\vec{b}}{|\vec{b}|}\right);\lambda\in R$

(t) Equation of plane passing through the point (x_1, y_1, z_1) is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$.

(u) Equation of plane passing through three points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

(v) Equation of a plane making intercepts a, b, c on axes is $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$

(w) Vector equation of a plane through the point $\vec{a}$ and perpendicular to the unit vector $\hat{n}$ is $(\vec{r}-\vec{a}).\hat{n}=0$

(x) If θ is the angle between the two planes $\vec{r}.\hat{n}_1=d_1$ and $\vec{r}\cdot\hat{n}_2=d_2$, then $\cos\theta=\dfrac{\hat{n}_1\cdot\hat{n}_2}{|\hat{n}_1||\hat{n}_2|}$

(y) Equation of a plane containing the line $\dfrac{x-x_1}{a}=\dfrac{y-y_1}{b}=\dfrac{z-z_1}{c}$ and passing through the point (x_2, y_2, z_2) not on the line is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a & b & c \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \end{vmatrix}=0$

(z) Equation of a plane through the line $\dfrac{x-x_1}{a_1}=\dfrac{y-y_1}{b_1}=\dfrac{z-z_1}{c_1}$ and parallel to the line $\dfrac{x-x_2}{a_2}=\dfrac{y-y_2}{b_2}=\dfrac{z-z_2}{c_2}$ is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$

(aa) If θ is the angle between the line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ and the plane $Ax+By+Cz+D=0$, then $\sin\theta=\frac{|aA+bB+cC|}{\sqrt{a^2+b^2+c^2}\sqrt{A^2+B^2+C^2}}$

(ab) Length of perpendicular from (x_1, y_1, z_1) to the plane $ax+by+cz+d=0$ is $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

(ac) The equation of a sphere with center at the origin and radius 'a' is $|\vec{r}|=a$ or $x^2+y^2+z^2=a^2$

(ad) Equation of a sphere with center (α,β,γ) and radius 'a' is $(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=a^2$

(ae) Vector equation of the sphere with center $\vec{c}$ and radius 'a' is $|\vec{r}-\vec{c}|=a$ or $(\vec{r}-\vec{c})\cdot(\vec{r}-\vec{c})=a^2$

(af) General equation of sphere is $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ whose center is $(-u,-v,-w)$ and radius is $\sqrt{u^2+v^2+w^2-d}$

(ag) Equation of a sphere concentric with $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is $x^2+y^2+z^2+2ux+2vy+2wz+\lambda=0$, where λ is a real number.

Solved Examples

JEE Main/Boards

Example 1: Find the coordinates of the point which divides the join of P(2, –1, 4) and Q(4, 3, 2) in the ratio 2 : 3 (i) internally (ii) externally

Sol: By using section formula we can obtain the result.

Let R(x, y, z) be the required point

(i) $x=\frac{2\times4+3\times2}{2+3}=\frac{14}{5}$; $y=\frac{2\times3+3\times(-1)}{2+3}=\frac{3}{5}$

$z=\frac{2\times2+3\times4}{2+3}=\frac{16}{5}$

So, the required point is $R\left(\frac{14}{5},\frac{3}{5},\frac{16}{5}\right)$

(ii) $x=\frac{2\times4-3\times2}{2-3}=-2$; $y=\frac{2\times3-3\times(-1)}{2-3}=-9$

$z=\frac{2\times2-3\times4}{2-3}=8$

Therefore, the required point is R(–2. –9, 8)

Example 2: Find the points on X-axis which are at a distance of $2\sqrt{6}$ units from the point (1, –2, 3)

Sol: Consider required point is P(x, 0, 0), therefore by using distance formula we can obtain the result.

Let P(x, 0, 0) be a point on X-axis such that distance of P from the point (1, –2, 3) is $2\sqrt{6}$

$\Rightarrow\sqrt{(1-x)^2+(-2-0)^2+(3-0)^2}=2\sqrt{6}$

$\Rightarrow(x-1)^2+4+9=24$ $\quad\Rightarrow(x-1)^2=11$

$\Rightarrow x-1=\pm\sqrt{11}$ $\quad\Rightarrow x=1\pm\sqrt{11}$

Example 3: If a line makes angles α,β,γ with OX, OY, OZ, respectively, prove that $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$

Sol: Same as illustration 2.

Let ℓ,m,n be the d.c.'s of the given line, then

$\ell=\cos\alpha, m=\cos\beta, n=\cos\gamma$

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$\Rightarrow(1-\sin^2\alpha)+(1-\sin^2\beta)+(1-\sin^2\gamma)=1$

$\Rightarrow\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$

Example 4: Projections of a line segment on the axes are 12, 4 and 3 respectively. Find the length and direction cosines of the line segment.

Sol: Let ℓ,m,n be the direction cosines and r be the length of the given segment, then $\ell r, mr, nr$ are the projections of the segment on the axes.

Let ℓ, m, n be the direction cosines and r be the length of the given segment, then $\ell r, mr, nr$ are the projections of the segment on the axes; therefore $\ell r = 12, mr = 4, nr = 3$

Squaring and adding, we get

$r^2(\ell^2 + m^2 + n^2) = 12^2 + 4^2 + 3^2 \Rightarrow r^2 = 169$

$\Rightarrow r = 13 \Rightarrow$ length of segment = 13

And direction cosines of segment are

$\ell = \frac{12}{r} = \frac{12}{13}, m = \frac{4}{r} = \frac{4}{13}$ and $n = \frac{3}{r} = \frac{3}{13}$

Example 5: Find the length of the perpendicular from the point (1, 2, 3) to the line through (6, 7, 7) and having direction ratios (3, 2, –2).

Sol: By using distance formula i.e. $|\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$, we can obtain required length.

Direction cosines of the line are

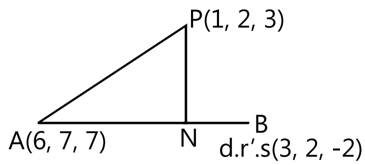
$$\frac{3}{\sqrt{3^2 + 2^2 + (-2)^2}}, \frac{2}{\sqrt{3^2 + 2^2 + (-2)^2}}, \frac{-2}{\sqrt{3^2 + 2^2 + (-2)^2}}$$

i.e. $\frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}$

$\therefore$ AN = Projection of AP on AB

$= (6-1)\frac{3}{\sqrt{17}} + (7-2)\frac{2}{\sqrt{17}} + (7-3)\frac{(-2)}{\sqrt{17}}$

$= \frac{15 + 10 - 8}{\sqrt{17}} = \frac{17}{\sqrt{17}} = \sqrt{17}$



Also, $AP = \sqrt{(6-1)^2 + (7-2)^2 + (7-3)^2}$

$= \sqrt{25 + 25 + 16} = \sqrt{66}$

$\therefore PN = \sqrt{AP^2 - AN^2} = \sqrt{66 - 17} = \sqrt{49}$ = 7 unit

Example 6: Find the equation of the plane through the points A(2, 2, –1), B(3, 4, 2) and C(7, 0, 6)

Sol: As we know, equation of a plane passing through the point (x_1, y_1, z_1) is given by

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

The general equation of a plane through (2, 2, –1) is

$a(x-2) + b(y-2) + c(z+1) = 0$ .... (i)

It will pass through B(3, 4, 2) and C(7, 0, 6) if

$a(3-2) + b(4-2) + c(2+1) = 0$ or

$a + 2b + 3c = 0$ .... (ii)

& $a(7-2) + b(0-2) + c(6+1) = 0$ or

$5a - 2b + 7c = 0$ .... (iii)

Solving (ii) and (iii) by cross multiplication, we get

$\frac{a}{14+6} = \frac{b}{15-7} = \frac{c}{-2-10}$ or $\frac{a}{5} = \frac{b}{2} = \frac{c}{-3} = \lambda$ (say)

$\Rightarrow a = 5\lambda, b = 2\lambda, c = -3\lambda$

Substituting the values of a, b and c in (i), we get $5\lambda(x-2) + 2\lambda(y-2) - 3\lambda(z+1) = 0$

or, $5(x-2) + 2(y-2) - 3(z+1) = 0$

$\Rightarrow 5x + 2y - 3z = 17$,

Which is the required equation of the plane.

Example 7: Find the angle between the planes x + y + 2z = 9 and 2x – y + z = 15

Sol: By using formula $\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

we can obtain the result.

The angle between x + y + 2z = 9 and 2x – y + z = 15 is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\cos\theta = \frac{(1)(2) + 1(-1) + (2)(1)}{\sqrt{1^2 + 1^2 + 2^2}\sqrt{2^2 + (-1)^2 + 1^2}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

Example 8: Find the distance between the parallel planes 2x – y + 2z + 3 = 0 and 4x – 2y + 4z + 5 = 0

Sol: By making the coefficient of x, y and z as unity we will be get required result.

Let $P(x_1, y_1, z_1)$ be any point on 2x – y + 2z + 3 = 0, then,

$2x_1 - y_1 + 2z_1 + 3 = 0$ ... (i)

The length of the perpendicular from

$P(x_1, y_1, z_1)$ to $4x - 2y + 4z + 5 = 0$ is

$$= \left|\frac{4x_1 - 2y_1 + 4z_1 + 5}{\sqrt{4^2 + (-2)^2 + 4^2}}\right| = \left|\frac{2(2x_1 - y_1 + 2z_1) + 5}{\sqrt{36}}\right|$$

$$= \left|\frac{2(-3) + 5}{6}\right| = \frac{1}{6}$$

Example 9: The equation of a line are 6x – 2 = 3y + 1 = 2z – 2. Find its direction ratios and its equation in symmetric form.

Sol: The given line is 6x – 2 = 3y + 1 = 2z – 2

$$\Rightarrow 6\left(x-\frac{1}{3}\right)=3\left(y+\frac{1}{3}\right)=2(z-1)$$

$$\Rightarrow \frac{x-\frac{1}{3}}{1}=\frac{y+\frac{1}{3}}{2}=\frac{z-1}{3}$$

[We make the coefficients of x, y and z as unity]

This equation is in symmetric form. Thus the direction ratios of the line are 1, 2 and 3 and this line passes through the point $\left(\frac{1}{3},-\frac{1}{3},1\right)$.

Example 10: Find the image of the point (3, –2, 1) in the plane 3x – y + 4z = 2.

Sol: Consider Q be the image of the point P(3, –2, 1) in the plane 3x – y + 4z = 2. Then PQ is normal to the plane hence direction ratios of PQ are 3, –1, 4.

Let Q be the image of the point P(3, –2, 1) in the plane 3x – y + 4z = 2. Then PQ is normal to the plane. Therefore direction ratios of PQ are 3, –1, 4. Since PQ passes through P(3, –2, 1) and has direction ratios 3, –1, 4. Therefore equation of PQ is

$$\frac{x-3}{3}=\frac{y+2}{-1}=\frac{z-1}{4}=r \qquad \text{(say)}$$

Let the coordination of Q be (3r + 3, –r, –2. 4r + 1). Let R be the mid-point of PQ. Then R lies on the plane 3x – y + 4z = 2. The coordinates of R are

$$\left(\frac{3r+3+3}{2},\frac{-r-2-2}{2},\frac{4r+1+1}{2}\right)$$

or $\left(\frac{3r+6}{2},\frac{-r-4}{2},2r+1\right)$

$$3\left(\frac{3r+6}{2}\right)-\left(\frac{-r-4}{2}\right)+4(2r+1)=2$$

$\Rightarrow$ 13r = –13 $\Rightarrow$ r = –1

So, the coordinates of Q are (0, –1, –3)

JEE Advanced/Boards

Example 1: Find the equations of the bisector planes of the angles between the planes 2x – y + 2z + 3 = 0 and 3x – 2y + 6z + 8 = 0 and specify the plane bisecting the acute angle and the plane bisecting obtuse angle.

Sol: As we know, Equation of the planes bisecting the angle between two given planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_1y + c_2z + d_2 = 0$ are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

The two given planes are

2x – y + 2z + 3 = 0 ... (i)

and 3x – 2y + 6z + 8 = 0 ... (ii)

The equations of the planes bisecting the angles between (i) and (ii) are

$$\frac{2x-y+2z+3}{\sqrt{2^2+(-1)^2+2^2}}=\pm\frac{3x-2y+6z+8}{\sqrt{3^2+(-2)^2+6^2}}$$

$$\Rightarrow 14x-7y+14z+21=\pm(9x-6y+18z+24)$$

Hence the two bisector planes are

$5x-y-4z-3=0$ ... (iii)

and $23x-13y+32z+45=0$ ... (iv)

Now we find angle θ between (i) & (iii)

We have,

$$\cos\theta=\frac{5(2)+(-1)(-1)+2(-4)}{\sqrt{2^2+(-1)^2+2^2}\sqrt{5^2+(-1)^2+(-4)^2}}=\frac{1}{\sqrt{42}}$$

Thus the angle between (i) & (iii) is more than $\frac{\pi}{4}$. Therefore, (iii) is the bisector of obtuse angle between (i) and (ii) and hence (iv) bisects acute angle between them.

Example 2: Find the distance of the point (1, –2, 3) from the plane x – y + z = 5 measured parallel to the line whose direction cosines are proportional to 2, 3, –6.

Sol: By using distance formula we can obtain required length,

Equation of line through (1, –2, 3) parallel to the line with d.r.'s 2, 3, –6 is

$$\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{-6}=r \qquad \text{...(i)}$$

Any point on it is $(1+2r,\,-2+3r,\,3-6r)$

Line (i) meets the plane x – y + z = 5.

If 1 + 2r – (–2 + 3r) + (3 – 6r) = 5 ; i.e. if $r=\frac{1}{7}$

$\therefore$ Point of intersection is $\left(\frac{9}{7}, -\frac{11}{7}, \frac{15}{7}\right)$

whose distance from (1, –2, 3) is

$$\sqrt{\left(\frac{9}{7}-1\right)^2+\left(-\frac{11}{7}+2\right)^2+\left(\frac{15}{7}-3\right)^2}$$

$$=\sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}}=\sqrt{\frac{49}{49}}=1$$

Example 3: Show that the lines

$$\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-1}{5} \quad \text{... (i)}$$

and $\frac{x+2}{4}=\frac{y-1}{3}=\frac{z+1}{-2}$ ... (ii)

do not intersect. Also find the shortest distance between them.

Sol: If $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} \neq 0$

then the lines do not intersect each other. And using distance formula we will be get required shortest distance.

Points on (i) and (ii) are (1, –1, 1) and (–2, 1, –1)

respectively and their d.c.'s are $\frac{3}{\sqrt{38}}, \frac{2}{\sqrt{38}}, \frac{5}{\sqrt{38}}$

and $\frac{4}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{-2}{\sqrt{29}}$ respectively.

$$\therefore \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} -2-1 & 1+1 & -1-1 \\ \frac{3}{\sqrt{38}} & \frac{2}{\sqrt{38}} & \frac{5}{\sqrt{38}} \\ \frac{4}{\sqrt{29}} & \frac{3}{\sqrt{29}} & \frac{-2}{\sqrt{29}} \end{vmatrix}$$

$$=\frac{1}{\sqrt{38}}\times\frac{1}{\sqrt{29}}\begin{vmatrix} -3 & 2 & -2 \\ 3 & 2 & 5 \\ 4 & 3 & -2 \end{vmatrix}$$

$$=\frac{1}{\sqrt{38}}\times\frac{1}{\sqrt{29}}, [-3(-4-15)+2(20+6)\ -2(9-8)] \neq 0$$

Hence the given lines do not intersect.

Any point P on (i) is $(1 + 3r_1, 2r_1, -1, 5r_1 + 1)$ and a point on (ii) is $Q(4r_2 - 2, 3r_2 + 1, -2r_2 - 1)$

$\therefore$ Direction ratios of PQ are

$(4r_2 - 3r_1 - 3, 3r_2 - 2r_1 + 2, -2r_2 - 5r_1 - 2)$

If PQ is perpendicular to (i) and (ii), we have

$3(4r_2 - 3r_1 - 3) + 2(3r_2 - 2r_1 + 2) + 5(-2r_2 - 5r_1 - 2) = 0$

& $4(4r_2 - 3r_1 - 3) + 3(3r_2 - 2r_1 + 2) - 2(-2r_2 - 5r_1 - 2) = 0$

i.e. $8r_2 - 38r_1 - 15 = 0$ & $29r_2 - 8r_1 - 2 = 0$

Solving them, $\frac{r_2}{76-120}=\frac{r_1}{-435+16}=\frac{1}{1038}$

$\Rightarrow r_2=-\frac{44}{1038}, r_1=-\frac{419}{1038}$

$\therefore$ Points P and Q are $\left(-\frac{1257}{1038}+1, -\frac{838}{1038}-1, -\frac{2095}{1038}+1\right)$

and $\left(-\frac{176}{1038}-2, -\frac{132}{1038}+1, \frac{88}{1038}-1\right)$

We can find the distance PQ by distance formula which is the shortest distance.

Example 4: Find the angle between the lines whose direction ratios satisfy the equations :

$3\ell+m+5n=0,\ 6mn-2n\ell+5\ell m=0$

Sol: Here, $\cos\theta=\frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}$.

The given equations are $3\ell+m+5n=0$ 3 ... (i)

and $6mn-2n\ell+5\ell m=0$ ... (ii)

From (i), we have $m=-3\ell-5n$ ... (iii)

Putting in (ii), we get

$6(-3\ell-5n)\,n-2n\ell+5\ell(-3\ell-5n)=0$

$\Rightarrow 30n^2+45\ell n+15\ell^2=0$

$\Rightarrow 2n^2+3\ell n+\ell^2=0 \quad \Rightarrow (n+\ell)(2n+\ell)=0$

$\Rightarrow$ Either $\ell=-n$ or $\ell=-2n$

If $\ell=-n$, then from (iii), $m = -2n$

If $\ell=-2n$, then from (iii), $m = n$

Thus the direction ratios of two lines are

–n, –2n, n and –2n, n, n

i.e. 1, 2, –1 and –2, 1, 1

$\therefore$ If θ is the angle between the lines, then

$$\cos\theta=\frac{1.(-2)+2.1+(-1).1}{\sqrt{1+4+1}.\sqrt{4+1+1}}=\frac{-2+2-1}{\sqrt{6}.\sqrt{6}}=\frac{-1}{6}.$$

Example 5: Find the equation of the plane through the intersection planes 2x + 3y + 4z = 5, 3x – y + 2z = 3 and parallel to the straight line having direction cosines (–1, 1, –1).

Sol: By using formula of family of plane, we will get the result.

Equation of plane through the given planes is $2x+3y+4z-5+\lambda(3x-y+2z-3)=0$

i.e. $(2+3\lambda)x+(3-\lambda)y+(4+2\lambda)z+(-5-3\lambda)=0$

This plane is parallel to the given straight line.

$\Rightarrow -1(2+3\lambda)+1(3-\lambda)+(-1)(4+2\lambda)=0$

$\Rightarrow -2-3\lambda+3-\lambda-4-2\lambda=0$

$\Rightarrow 6\lambda=-3 \qquad \Rightarrow \lambda=-\dfrac{1}{2}$

$\therefore$ Equation of required plane is

$\dfrac{1}{2}x+\dfrac{7}{2}y+3z-\dfrac{7}{2}=0 \qquad \Rightarrow x+7y+6z=7$

Example 6: Prove that the lines

$$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7} \text{ and } \frac{x-2}{1}=\frac{y-4}{4}=\frac{z-6}{7}$$

are coplanar. Also, find the plane containing these two lines.

Sol: As similar to example 3.

We know the lines $\dfrac{x-x_1}{\ell_1}=\dfrac{y-y_1}{m_1}=\dfrac{z-z_1}{n_1}$

and $\dfrac{x-x_2}{\ell_2}=\dfrac{y-y_2}{m_2}=\dfrac{z-z_2}{n_2}$ are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix}=0$$

and the equation of the plane containing these two lines is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix}=0$$

Here $x=-1, y_1=-3, z_1=-5, x_2=2, y_2=4, z_2=6,$

$\ell_1=3, m_1=5, n_1=7, \ell_2=1, m_2=4, n_2=7$

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix}=\begin{vmatrix} 3 & 7 & 11 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix}$$

$=21-98+77=0$

So, the given lines are coplanar, The equation of the plane containing the lines is

$$=\begin{vmatrix} x+1 & y+3 & z+5 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix}=0$$

or $(x+1)(35-28)-(y+3)(21-7)+(z+5)(12-5)=0$ or $x-2y+z=0$

Example 7: Find the equation of the plane passing through the lines of intersection of the planes 2x – y = 0 and 3z – y = 0 and perpendicular to the plane 4x + 5y – 3z = 8.

Sol: Here by using the family of plane and formula of two perpendicular plane we will get the result.

The plane $2x-y+k(3z-y)=0$

$\Leftrightarrow 2x-(1+k)y+3kz=0$ is perpendicular to the plane $4x+5y-3z=8$

$\Rightarrow 2.4-(1+k).5+3k(-3)=0 \qquad \Rightarrow 14k=3$

$\Rightarrow k=\dfrac{3}{14}$

Thus the required equation is

$$2x-y+\left(\frac{3}{14}\right)(3z-y)=0 \;\Leftrightarrow 28x-17y+9z=0$$

Example 8: Show that the lines

$$\frac{x+5}{3}=\frac{y+4}{1}=\frac{z-7}{-2}, 3x+2y+z-2=0=x-3y+2z-13$$

are coplanar and find the equation to the plane in which they lie.

Sol: By using the condition of coplanarity of line, we will get given lines are coplanar or not. And after that by using general equation of the plane we can obtain required equation of plane.

The general equation of the plane through the second line is

$3x+2y+z-2+k(x-3y+2z-13)=0$

$\Leftrightarrow x(3+k)+y(2-3k)+z(1+2k)-2-13k=0;$

K being the parameter

This contains the first line only if

$3(3+k)+(2-3k)-2(1+2k)=0 \;\Rightarrow k=\dfrac{9}{4}$

Hence the equation of the plane which contains the two lines is

$21x-19y+22x-125=0$

This plane clearly passes through the point $(-5,-4,7)$

JEE Main/Boards

Exercise 1

Q.1 Direction cosines of a line are $\frac{3}{7},\frac{-2}{7},\frac{6}{7}$, find its direction ratios.

Q.2 Find the direction ratios of a line passing through the points (2, 1, 0) and (1, –2, 3).

Q.3 Find the angle between the lines $\frac{x}{1}=\frac{y}{2}=\frac{z}{0}$ and $\frac{x-1}{3}=\frac{y+5}{2}=\frac{z-3}{1}$.

Q.4 Find the equation of a line parallel to the vector $3\hat{i}-\hat{j}-3\hat{k}$ and passing through the point $(-1,1,1)$.

Q.5 Write the vector equation of a line whose Cartesian equation is $\frac{x+3}{2}=\frac{y-1}{4}=\frac{z+1}{5}$.

Q.6 Write the Cartesian equation of a line whose vector equations is $\vec{r}=(3\hat{i}+2\hat{j}-5\hat{k})+\lambda(-2\hat{i}+\hat{j}+3\hat{k})$.

Q.7 Find the value of p, such that the line

$\frac{x}{1}=\frac{y}{3}=\frac{z}{2p}$ and $\frac{x}{-3}=\frac{y}{5}=\frac{z}{2}$

are perpendicular to each other.

Q.8 Write the Cartesian equation of the plane $\vec{r}(3\hat{i}+2\hat{j}+5\hat{k})=7$.

Q.9 Write the vector equation of plane $3x-y-4z+7=0$.

Q.10 Find the vector, normal to the plane $\vec{r}.(3\hat{i}-7\hat{k})+5=0$.

Q.11 Find the direction ratios of a line, normal to the plane 7x + y – 2z = 1.

Q.12 Find the angle between the line $\frac{x+2}{4}=\frac{y-1}{-5}=\frac{z}{7}$ and the plane $3x-2z+4=0$.

Q.13 Find the distance of the plane x + y + 3z + 7 = 0 from origin.

Q.14 Find the distance of the plane 3x – 3y + 3z = 0 from the point (1, 1, 1).

Q.15 Find the intercepts cut by the plane 3x – 2y + 4z – 12 = 0 on axes.

Q.16 Direction ratios of a line are 1, 3, –2. Find its direction cosines.

Q.17 Find the direction cosines of y-axis.

Q.18 Find the direction ratio of the line

$\frac{x+2}{1}=\frac{2y-1}{3}=\frac{3-z}{5}$.

Q.19 Find the angle between the planes

$r.(\hat{i}-2\hat{j}-2\hat{k})=1$ and $r.(3\hat{i}-6\hat{j}+2\hat{k})=0$.

Q.20 Find the angle between the line $\vec{r}=(2\hat{i}-\hat{j}+3\hat{k})+\lambda(3\hat{i}-6\hat{j}+2\hat{k})$ and the plane $\vec{r}.(\hat{i}+\hat{j}+\hat{k})+3$

Q.21 Find the direction cosines of the two lines which are connected by the relations $\ell-5m+3n=0$ and $7\ell^2+5m^2-3n^2=0$.

Q.22 Prove that, the line passing through the point (1, 2, 3) and (–1, –2, –3) is perpendicular to the line passing through the points (–2, 1, 5) and (3, 3, 2).

Q.23 Find the coordinates of the foot of the perpendicular drawn from the point (1, 2, 1) to the line joining the points (1, 4, 6) and (5, 4, 4).

Q.24 If a variable line in two adjacent positions has direction cosines ℓ, m, n and $\ell+\delta\ell, m+\delta m, n+\delta n$, prove that the small angle $\delta\theta$ between two position is given by $(\delta\theta)^2=(\delta\ell)^2+(\delta m)^2+(\delta n)^2$.

Q.25 Verify that $\frac{\ell_1+\ell_2+\ell_3}{\sqrt{3}},\frac{m_1+m_2+m_3}{\sqrt{3}},\frac{n_1+n_2+n_3}{\sqrt{3}}$ can be taken as direction cosines of a line equally inclined to three mutually perpendicular lines with direction cosines $\ell_1,m_1,n_1;\ell_2,m_2,n_2$ and ℓ_3,m_3,n_3

Q.26 Find the equations of line through the point (3, 0, 1) and parallel to the planes x + 2y = 0 and 3y – z = 0.

Q.27 Find the equations of the planes through the intersection of the planes x + 3y + 6 = 0 and 3x – y – 4z = 0 whose perpendicular distance from the origin is equal to 1.

Q.28 Find the equation of the plane through the points (–1, 1, 1) and (1, –1, 1) and perpendicular to the plane x + 2y + 2z = 5.

Q.29 Find the distance of the point (–1, –5, –10) from the plane x – y + z = 5 measured parallel to the line

$$\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}.$$

Q.30 Find the vector and Cartesian forms of the equation of the plane passing through (1, 2, –4) and parallel to the line $\vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$ and $\vec{r}=\hat{i}-3\hat{j}+5\hat{k}+\mu(\hat{i}+\hat{j}-\hat{k})$.

Q.31 If straight line having direction cosines given by $a\ell+bm+cn=0$ and $fmn+gn\ell+h\ell m=0$ are perpendicular, then prove that $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0.$

Q.32 Prove that, the lines x = ay + b, z = cy + d and x = a′y + b′, z = c′y + d′ are perpendicular to each other, if aa′ + cc′ + 1 = 0.

Q.33 Find the equation of the plane passing through the intersection of the planes 4x – y + z = 10 and x + y – z = 4 and parallel to the line with direction ratios 2, 1, 1. Find also the perpendicular distance of (1, 1, 1) from this plane.

Q.34 The foot of the perpendicular drawn from the origin to the plane is (2, 5, 7). Find the equation of plane.

Q.35 Find the equation of a plane through (–1, –1, 2) and perpendicular to the planes 3x + 2y – 3z = 1 and 5x – 4y + z = 5.

Q.36 Find the angle between the lines whose direction cosines are given by equations $\ell+m+n=0$; $\ell^2+m^2-n^2=0$

Q.37 Find the equation of the line which passes through (5, –7, –3) and is parallel to the line of intersection of the planes x – 3y – 5 = 0 and 9y – z + 16 = 0.

Q.38 Prove that, the plane through the points (1, 1, 1), (1, –1, 1) and (–7, 3, –5) is perpendicular to xz-plane.

Q.39 Find the length and coordinates of the foot of perpendicular from points (1, 1, 2) to the plane 2x – 2y + 4z + 5 = 0.

Q.40 Find the vector equation in the scalar product form, of the plane passing through the points (1, 0, –1), (3, 2, 2) and parallel to line

$$r=\hat{i}+\hat{j}+\lambda(\hat{i}-2\hat{j}+3\hat{k}).$$

Q.41 Find the distance between the parallel planes 2x – y + 3z – 4 = 0 and 6x – 3y + 9z + 13 =0.

Q.42 Prove that, the equation of a plane. Which meets the axes in A, B, and C and the given centroid of triangle ABC is the point (α,β,γ), is $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3.$

Q.43 Find the equation of the plane passing through the origin and the line of intersection of the planes x – 2y + 3z + 4 = 0 and x – y + z + 3 = 0.

Q.44 Prove that, the line 2x + 2y – z – 6 = 0, 2x + 3y – z – 8 = 0 is parallel to the plane y = 0. Find the coordinates of the point where this line meets the plane x = 0.

Q.45 Find the equation of the plane through the line ax + by + cz + d = 0, a′x + b′y + c′z + d′ = 0 and parallel to the line $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}.$

Q.46 Find the equation of a plane parallel to x-axis and has intercepts 5 and 7 on y and z-axis, respectively.

Q.47 A variable plane at a constant distance p from origin meets the coordinate axes in points A, B and C, respectively. Through these points, planes are drawn parallel to the coordinate planes, prove that locus of point of intersection is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}.$

Q.48 Find the value of λ, for which the points with position vectors $\hat{i}-\hat{j}+3\hat{k}$ and $3\hat{i}+\lambda\hat{j}+3\hat{k}$ are equidistant from the plane $\vec{r}.(5\hat{i}+2\hat{j}-7\hat{k})+9=0.$

Q.49 Find the equation of a plane which is at a distance of 7 units from the origin and which is normal to the vector $3\hat{i}+5\hat{j}-6\hat{k}$

Q.50 Find the vector equation of the plane, $r=\hat{i}-\hat{j}+\lambda(\hat{i}+\hat{j}+\hat{k})+\mu(4\hat{i}-2\hat{j}+3\hat{k})$ in the scalar product from.

Exercise 2

Single Correct Choice Type

Q.1 The sum of the squares of direction cosines of a straight line is

(A) Zero (B) Two

(C) 1 (D) None of these

Q.2 Which one of the following is best condition for the plane ax + by + cz + d = 0 to intersect the x and y axes at equal angle

(A) |a| = |b| (B) a = –b

(C) a = b (D) $a^2 + b^2 = 1$

Q.3 The equation of a straight line parallel to the x-axis is given by

(A) $\frac{x-a}{1}=\frac{y-b}{1}=\frac{z-c}{1}$ (B) $\frac{x-a}{0}=\frac{y-b}{1}=\frac{z-c}{1}$

(C) $\frac{x-a}{0}=\frac{y-b}{0}=\frac{z-c}{1}$ (D) $\frac{x-a}{1}=\frac{y-b}{0}=\frac{z-c}{0}$

Q.4 A straight line is inclined to the axes of x and z at angles 45° and 60° respectively, then the inclination of the line to the y-axis is

(A) 30° (B) 45° (C) 60° (D) 90°

Q.5 The coordinates of the point of intersection of the line $\frac{x+1}{1}=\frac{y+3}{3}=\frac{z+2}{-2}$ with the plane 3x + 4y + 5z = 5

(A) (5, 15, –14) (B) (3, 4, 5)

(C) (1, 3, –2) (D) (3, 12, –10)

Q.6 Perpendicular is drawn from the point (0, 3, 4) to the plane 2x – 2y + z = 10. The coordinates of the foot of the perpendicular are

(A) $\left|-\frac{8}{3},\frac{1}{3},\frac{16}{3}\right|$ (B) $\left|\frac{8}{3},\frac{1}{3},\frac{16}{3}\right|$

(C) $\left|\frac{8}{3},-\frac{1}{3},\frac{16}{3}\right|$ (D) $\left|\frac{8}{3},\frac{1}{3},-\frac{16}{3}\right|$

Q.7 The equation of the plane through the line of intersection of the planes 2x + y – z – 4 = 0 and 3x + 5z – 4 = 0 which cuts off equal intercepts from the x-axis and y-axis is

(A) 3x + 3y – 8z + 8 = 0 (B) 3x + 3y – 8z – 8 = 0

(C) 3x – 3y – 8z – 8 = 0 (D) x + y – 8z – 8 = 0

Q.8 The symmetric form of the equation of the line x + y – z = 1, 2x – 3y + z = 2 is

(A) $\frac{x+1}{3}=\frac{y-2}{-2}=\frac{z+1}{-1}$ (B) $\frac{x}{2}=\frac{y}{3}=\frac{z-1}{5}$

(C) $\frac{x}{2}=\frac{y-1}{3}=\frac{z}{5}$ (D) $\frac{x-1}{2}=\frac{y}{3}=\frac{z}{5}$

Q.9 The line $\frac{x-1}{1}=\frac{y-3}{2}=\frac{z-4}{3}$ is parallel to the plane

(A) 2x + y + 2z + 3 = 0 (B) 2x – y – 2z = 3

(C) 21x – 12y + z = 0 (D) 2x + y – 2z = 0

Q.10 The vertices of the triangle PQR are (2, 1, 1), (3, 1, 2) and (–4, 0, 1). The area of the triangle is

(A) $\frac{\sqrt{38}}{2}$ (B) $\sqrt{38}$ (C) 4 (D) 2

Q.11 Equation of straight line which passes through the point P(1, 0, –3) and Q(–2, 1, –4) is

(A) $\frac{x-2}{-3}=\frac{y+1}{1}=\frac{z-4}{-1}$ (B) $\frac{x-1}{3}=\frac{y}{1}=\frac{z+3}{1}$

(C) $\frac{x-1/2}{-3}=\frac{y-1}{1}=\frac{z+4}{-1}$ (D) $\frac{x-1}{-3}=\frac{y}{1}=\frac{z+3}{-1}$

Q.12 A point moves so that the sum of the squares of its distances from the six faces of a cube given by x =±1, y = ±1, z = ±1 is 10 units. The locus of the point is

(A) $x^2 + y + z^2 = 1$ (B) $x^2 + y^2 + z^2 = 2$

(C) x + y + z = 1 (D) x + y + z = 2

Q.13 The points (0, –1, –1), (–4, 4, 4), (4, 5, 1) and (3, 9, 4) are

(A) Collinear (B) Coplanar

(C) Forming a square (D) None of these

Q.14 The equation of the plane containing the line $\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ is $a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0$, where $a\ell+bm+cn$ is equal to

(A) 1 (B) –1 (C) 2 (D) 0

Q.15 The reflection of the plane 2 x + 3 y + 4 z − 3 = 0 in the plane x − y + z − 3 = 0 is the plane

(a) 4x− 3y + 2z −15 = 0 (b) x− 3 y + 2z − 15 = 0

(c) 4x + 3y− 2z + 15 = 0 (d) None of these

Previous Years' Questions

Q.1 The value of k such that $\frac{x-4}{1}=\frac{y-2}{1}=\frac{z-k}{2}$ lies in the plane
$2x-4y+z=7$, is ***(2003)***

(A) 7 (B) –7 (C) No real value (D) 4

Q.2 If the lines $\vec{r}=\vec{a}_2+\mu\vec{b}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect, then the value of k is ***(2004)***

(A) $\vec{a}_2$ (B) $\frac{9}{2}$ (C) $-\frac{2}{9}$ (D) $-\frac{3}{2}$

Q.3 A variable plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ at a unit distance from origin cuts the coordinate axes at A, B and C. Centroid (x, y, z) satisfies the equation $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=K$. The value of K is ***(2005)***

(A) 9 (B) 3 (C) 1/9 (D) 1/3

Fill in the Blanks for Q.4 and Q.5

Q.4 The area of the triangle whose vertices are A(1, –1, 2), B(2, 1, –1), C(3, –1, 2) is … ***(1983)***

Q.5 The unit vector perpendicular to the plane determined by P(1, –1, 2), Q(2, 0, –1) and R(0, 2, 1) is............. ***(1983)***

Q.6 A plane is parallel to two lines whose direction ratios are (1, 0, –1) and (–1, 1, 0) and it contains the point (1, 1, 1). If it cuts coordinate axes at A, B, C. Then find the volume of the tetrahedron OABC. ***(2004)***

Q.7 Find the equation of the plane containing the line 2x – y + z – 3 = 0, 3x + y + z = 5 and at a distance of $\frac{1}{\sqrt{6}}$ from the point (2, 1, –1). ***(2005)***

Q.8 If the line $\frac{x-3}{2}=\frac{y+2}{-1}=\frac{z+4}{3}$ lies in the plane, $\ell x+my-z=9$, then ℓ^2+m^2 is equal to: ***(2016)***

(A) 18 (B) 5 (C) 2 (D) 26

Q.9 The distance of the point $(1,-5,9)$ from the plane $x-y+z=5$ measured along the line $x=y=z$ is ***(2016)***

(A) $10\sqrt{3}$ (B) $\frac{10}{\sqrt{3}}$ (C) $\frac{20}{3}$ (D) $3\sqrt{10}$

Q.10 The distance of the point (1, 0, 2) from the point of intersection of the line $\frac{x-2}{3}-\frac{y+1}{4}-\frac{z-2}{12}$ and the plane $x-y+z=16$, is: ***(2015)***

(A) 8 (B) $3\sqrt{21}$ (C) 13 (D) $2\sqrt{14}$

Q.11 The equation of the plane containing the line $2x-5y+z+3; x+y+4z=5$ and parallel to the plane $x+3y+6z=1$ is ***(2015)***

(A) $x+3y+6z=-7$ (B) $x+3y+6z=7$

(C) $2x+6y+12z=-13$ (D) $2x+6y+12z=13$

Q.12 The number of common tangents to the circles ***(2015)***

(A) Meats the curve again in the second in the second quadrant

(B) Meats the curve again in the third quadrant

(C) Meets the curve again in the fourth quadrant

(D) Does not meet the curve again

Q.13 The image of the line $\frac{x-1}{3}=\frac{y-3}{1}=\frac{z-4}{-5}$ in the plane $2x-y+z+3$ is the ***(2014)***

(A) $\frac{x+3}{3}=\frac{y-5}{1}=\frac{z-2}{-5}$ (B) $\frac{x+3}{-3}=\frac{y-5}{-1}=\frac{z-2}{5}$

(C) $\frac{x+3}{3}=\frac{y-5}{1}=\frac{z-2}{-5}$ (D) $\frac{x-3}{-3}=\frac{y-5}{-1}=\frac{z-2}{-5}$

Q.14 The angle between the lines whose direction cosines satisfy the equations l+m+n=0 and $l^2=m^2+n^2$ is ***(2014)***

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{2}$

Q.15 If the lines
$\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$
are coplanar, then k have ***(2013)***

(A) Exactly one value (B) Exactly two value

(C) Exactly three values (D) Any value

Q.16 An equation of a plane parallel to the plane $x-2y+2z-5=0$ and at a unit distance from the origin is ***(2012)***

(A) $x-2y+2z-3=0$ (B) $x-2y+2z+1=0$

(C) $x-2y+2z-1=0$ (D) $x-2y+2z+5=0$

Q.17 If the angle between the line $x = \frac{y-1}{2} = \frac{z-3}{\lambda}$ and the plane $x + 2y + 3z = 4$ is $\cos^{-1}\left(\sqrt{\frac{5}{14}}\right)$, then λ equals ***(2011)***

(A) $\frac{3}{2}$ (B) $\frac{2}{5}$ (C) $\frac{5}{3}$ (D) $\frac{2}{3}$

Q.18 Statement–I: The point $A(1,0,7)$ is the mirror image of the point $B(1,6,3)$ in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$

Statement-II: The line: $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ bisects the line segment joining $A(1,0,7)$ and $B(1,6,3)$ ***(2011)***

(A) Statement-I is true, statement-II is true; statement-II is not a correct explanation for statement-I

(B) Statement-I is true, statement-II is false.

(C) Statement-I is false, statement-II is true

(D) Statement-I is true, statement-II is true, statement-II is a correct explanation for statement-I

JEE Advanced/Boards

Exercise 1

Q.1 Points X and Y are taken on the sides QR and RS respectively, of parallelogram PQRS, so that $QX = 4\overrightarrow{XR}$ and $\overrightarrow{RY} = 4\overrightarrow{YS}$. The line XY cuts the line PR at Z. prove that $\overrightarrow{PZ} = \left(\frac{21}{25}\right)\overrightarrow{PR}$.

Q.2 Given three points on the xy plane on O(0, 0), A(1, 0) and B(–1, 0). Point P is moving on the plane satisfying the condition $(\overrightarrow{PA}\cdot\overrightarrow{PB}) + 3(\overrightarrow{OA}\cdot\overrightarrow{OB}) = 0$. If the maximum and minimum values of $|\overrightarrow{PA}||\overrightarrow{PB}|$ are M and m, respectively then find the value of $M^2 + m^2$.

Instruction for questions 3 to 6.

Suppose the three vectors, $\vec{a}, \vec{b}, \vec{c}$ on a plane satisfy the condition that

$|\vec{a}| = |\vec{b}| = |\vec{c}| = |\vec{a} + \vec{b}| = 1; c$ is perpendicular to a and $\vec{b}\cdot\vec{c} > 0$, then

Q.3 Find the angle formed by $2\vec{a} + \vec{b}$ and $\vec{b}$.

Q.4 If the vector c is expressed as a linear combination $\lambda\vec{a} + \mu\vec{b}$ then find the ordered pair

$\frac{\ell_1 - \ell_2}{2\sin(\theta/2)}, \frac{m_1 - m_2}{2\sin(\theta/2)}$ and $\frac{n_1 - n_2}{2\sin(\theta/2)}$.

Q.5 For real number x, y the vector $\vec{p} = x\vec{a} + y\vec{c}$ satisfies the condition $0 \le \vec{p}\cdot\vec{a} \le 1$ and $0 \le \vec{p}\cdot\vec{b} \le 1$. Find the maximum value of $\vec{p}.\vec{c}$

Q.6 For the maximum value of x and y, find the linear combination of $\vec{p}$ in terms of $\vec{a}$ and $\vec{b}$.

Q.7 If O be the origin and the coordinates of P be (1, 2, -3), then find the equation of the plane passing through P and perpendicular to OP.

Q.8 Given non zero number x_1, x_2, x_3; y_1, y_2, y_3 and z_1, z_2 and z_3 (i) Can the given numbers satisfy

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0 \text{ and } \begin{cases} x_1x_2 + y_1y_2 + z_1z_2 = 0 \\ x_2x_3 + y_2y_3 + z_2z_3 = 0 \\ x_3x_1 + y_3y_1 + z_3z_1 = 0 \end{cases}$$

(ii) If $x_1 > 0$ and $y_1 < 0$ for all I = 1, 2, 3 and $P = (x_1, x_2, x_3)$; $Q(y_1, y_2, y_3)$ and O(0, 0, 0) can the triangle POQ be a right angled triangle?

Q.9 ABCD is a tetrahedron with pv's of its angular points as $A(-5,22,5); B(1,2,3); C(4,3,2)$ and $D(-1,2,-3)$. If the area of the triangle AEF where the quadrilaterals ABDE and ABCF are parallelogram is S, then find the value of S.

Q.10 If x, y are two non-zero and non-collinear vectors satisfying $[(a-2)\alpha^2+(b-3)\alpha+c]x+[(a-2)\beta^2+(b-3)\beta+c]y$ $+[(a-2)\gamma^2+(b-3)\gamma+c](x\times y)=0$

where α,β,γ are three distinct real numbers, then find the value of $(a^2+b^2+c^2)$.

Q.11 Find the distance of the point (-1, -5, -10) from the point of intersection of the line $\vec{r}=2\hat{i}-\hat{j}+2\hat{k}+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)$ and the plane $\vec{r}.\left(\hat{i}-\hat{j}+\hat{k}\right)=5$

Q.12 Find the equations of the straight line passing through the point (1, 2, 3) to intersect the straight line $x+1=2(y-2)=x+4$ and parallel to the plane $x+5y+4z=0$.

Q.13 Find the equations of the two lines through the origin which intersect the line $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$ at an angle of $\frac{\pi}{3}$.

Exercise 2

Single Correct Choice Type

Q.1 If P(2, 3, –6) and Q(3, –4, 5) are two points, the direction cosines of line PQ are

(A) $-\frac{1}{\sqrt{171}},-\frac{7}{\sqrt{171}},-\frac{11}{\sqrt{171}}$ (B) $\frac{1}{\sqrt{171}},-\frac{7}{\sqrt{171}},\frac{11}{\sqrt{171}}$

(C) $\frac{1}{\sqrt{171}},\frac{7}{\sqrt{171}},-\frac{11}{\sqrt{171}}$ (D) $-\frac{1}{\sqrt{171}},-\frac{7}{\sqrt{171}},\frac{11}{\sqrt{171}}$

Q.2 The ratio in which yz-plane divide the line joining the points A(3, 1, –5) and B(1, 4, –6) is

(A) –3 : 1 (B) 3 : 1 (C) –1 : 3 (D) 1 : 3

Q.3 The value of λ for which the lines $3x+2y+z+5=0=x+y-2z-3$ and $2x-y-\lambda z=0$ $=7x+10y-8z$ are perpendicular to each other is

(A) –1 (B) –2 (C) 2 (D) 1

Q.4 The ratio in which yz-plane divides the line joining (2, 4, 5) and (3, 5, 7)

(A) –2 : 3 (B) 2 : 3 (C) 3 : 2 (D) –3 : 2

Q.5 A line makes angle $\alpha,\beta,\gamma,\delta$ with the four diagonals of a cube then $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta$ is equal to

(A) 1 (B) 4/3 (C) ¾ (D) 4/5

Q.6 A variable plane passes through a fixed point (a, b, c) and meets the coordinate axes in A, B, C. The locus of the point common to plane through A, B, C parallel to coordinate planes is

(A) ayz + bzx + cxy = xyz (B) axy + byz + czx = xyz

(C) axy + byz + czx = abc (D) bcx + acy + abz = abc

Q.7 The equation of the plane bisecting the acute angle between the planes

2x – y + 2z + 3 = 0 and 3x – 2y + 6z + 8 = 0

(A) 23x – 13y + 32z + 45 = 0

(B) 5x – y – 4z = 3

(C) 5x – y – 4z + 45 = 0

(D) 23x – 13y + 32z + 3 = 0

Q.8 The shortest distance between the two straight lines $\frac{x-4/3}{2}=\frac{y+6/5}{3}=\frac{z-3/2}{4}$ and $\frac{5y+6}{8}=\frac{2z-3}{9}=\frac{3x-4}{5}$ is

(A) $\sqrt{29}$ (B) 3 (C) 0 (D) $6\sqrt{10}$

Q.9 The equation of the straight line through the origin parallel to the line (b + c)x + (c + a)y + (a + b) z = k = (b – c)x + (c – a)y + (a – b)z is

(A) $\frac{x}{b^2-c^2}=\frac{y}{c^2-a^2}=\frac{z}{a^2-b^2}$

(B) $\frac{x}{b}=\frac{y}{c}=\frac{z}{a}$

(C) $\frac{x}{a^2-bc}=\frac{y}{b^2-ca}=\frac{z}{c^2-ab}$

(D) None of these

Assertion Reasoning Type

Q.10 Consider the following statements

Assertion: The plane y + z + 1 = 0 is parallel to x-axis.

Reason: Normal to the plane is parallel to x-axis.

(A) Both A and R are true and R is the correct

(B) Both A and R are true and R is not a correct explanation of A

(C) A is true but R is false

(D) A is false but R is true

Previous Years' Questions

Q.1 A plane passes through (1, –2, 1) and is perpendicular to two planes 2x – 2y + z = 0 and x – y + 2z = 4, then the distance of the plane from the point (1, 2, 2) is ***(2006)***

(A) 0 (B) 1 (C) $\sqrt{2}$ (D) $2\sqrt{2}$

Q.2 Let P(3, 2, 6) be a point in space and Q be a point on the line $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(-3\hat{i} + \hat{j} + 5\hat{k})$. Then the value of μ for which the vector $\overrightarrow{PQ}$ is parallel to the plane x – 4y + 3z = 1 is ***(2009)***

(A) $\frac{1}{4}$ (B) $-\frac{1}{4}$ (C) $\frac{1}{8}$ (D) $-\frac{1}{8}$

Q.3 A line with positive direction cosines passes through the point P(2, –1, 2) and makes equal angles with the coordinate axes. The line meets the plane 2x + y + z = 9 at point Q. The length of the line segment PQ equals ***(2009)***

(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2

For the following question, choose the correct answer from the codes (A), (B), (C) and (D) defined as follows.

(A) Statement-I is true, statement-II is also true; statement-II is the correct explanation of statement-I

(B) Statement-I is true, statement-II is also true; statement-II is not the correct explanation of statement-I.

(C) Statement-I is true; statement-II is false.

(D) Statement-I is false; statement-II is true

Q.4 Consider the planes 3x – 6y – 2z = 15 and 2x + y –2z = 5.

Statement-I: The parametric equations of the line of intersection of the given planes are x = 3 + 14t, y = 1 + 2t, z =15t.

Statement-II: The vectors $14\hat{i} + 2\hat{j} + 15\hat{k}$ is parallel to the line of intersection of the given planes. ***(2007)***

Q.5 Consider three planes

$AB: \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = \lambda$

$CD: \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} = \mu$ and

$L \equiv (3\lambda + 3, -\lambda + 8, \lambda + 3)$

Let L_1, L_2, L_3 be the lines of intersection of the planes P_2 and P_3, P_3 and P_1, P_1 and P_2, respectively.

Statement-I : At least two of the lines L_1, L_2 and L_3 are non-parallel.

Statement-II : The three planes do not have a common point ***(2008)***

Paragraph for Q.6 to Q.8

Read the following passage and answer the questions. Consider the lines

$L_1: \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}, L_2: \frac{x-2}{1} = \frac{y+2}{2} = \frac{z-3}{3}$

(2008)

Q.6 The unit vector perpendicular to both L_1 and L_2 is

(A) $\frac{-\hat{i} + 7\hat{j} + 7\hat{k}}{\sqrt{99}}$ (B) $\frac{-\hat{i} - 7\hat{j} + 5\hat{k}}{5\sqrt{3}}$

(C) $\frac{-\hat{i} + 7\hat{j} + 5\hat{k}}{5\sqrt{3}}$ (D) $\frac{7\hat{i} - 7\hat{j} - \hat{k}}{\sqrt{99}}$

Q.7 The shortest distance between L_1 and L_2 is

(A) 0 (B) $\frac{17}{\sqrt{3}}$ (C) $\frac{41}{5\sqrt{3}}$ (D) $\frac{17}{5\sqrt{3}}$

Q.8 The distance of the point (1, 1, 1) from the plane passing through the point (-1, -2, -1) and whose normal is perpendicular to both the lines L_1 and L_2 is

(A) $\frac{2}{\sqrt{75}}$ (B) $\frac{7}{\sqrt{75}}$ (C) $\frac{13}{\sqrt{75}}$ (D) $\frac{23}{\sqrt{75}}$

Match the Columns

Match the condition/expression in column I with statement in column II.

Q.9 Consider the following linear equations ax + by + cz = 0, bx + cy + az = 0, cx + ay + bz =0 ***(2007)***

Column I	Column II
(A) $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	(p) The equations represent planes meeting only at a single point

(B) $a+b+c=0$ and $a^2+b^2+c^2 \neq ab+bc+ca$	(q) The equation represent the line x = y = z
(C) $a+b+c \neq 0$ and $a^2+b^2+c^2 \neq ab+bc+ca$	(r) The equations represent identical planes
(D) $a+b+c=0$ and $a^2+b^2+c^2\ ab+bc+ca$	(s) The equations represent the whole of the three dimensional space

Q.10 (i) Find the equation of the plane passing through the points (2, 1, 0), (5, 0, 1) and (4, 1, 1).

(ii) If P is the point (2, 1, 6), then the point Q such that PQ is perpendicular to the plane in (a) and the mid point of PQ lies on it. ***(2003)***

Q.11 T is a parallelepiped in which A, B, C and D are vertices of one face and the face just above it has corresponding vertices A', B', C', D', T is now compressed to S with face ABCD remaining same and A', B', C', D' shifted to A", B", C", D" in S. the volume of parallelepiped S is reduced to 90% of T. Prove that locus of A" is a plane. ***(2003)***

Q.12 Consider a pyramid OPQRS located in the first octant $(x \geq 0, y \geq 0, z \geq 0)$ with O as origin, and OP and OR along the x-axis and the y-axis, respectively. The base OPQR of the pyramid is a square with $OP = 3$. The point S is directly above the mid-point T of diagonal OQ such that $TS = 3$. Then ***(2016)***

(A) The acute angle between OQ and OS is $\frac{\pi}{3}$

(B) The equation of the plane containing the triangle OQS is $x - y = 0$

(C) The length of the perpendicular from p to the plane containing the triangle OQS is $\frac{3}{\sqrt{2}}$

(D) The perpendicular distance from O to the straight line containing RS is $\sqrt{\frac{15}{2}}$

Q.13 Let P be the image of the point $(3,1,7)$ with respect to the plane $x - y + x = 3$. Then equation of the plane passing through P and containing the straight line $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$ is ***(2016)***

(A) $x + y - 3z = 0$ (B) $3x + z = 0$

(C) $x - 4y + 7z = 0$ (D) $2x - y = 0$

Q.14 In R^3, consider the planes $P_1 : y=0$ and $P_2 : x+z= 1$. Let P_3 be a plane, different from P_1 and P_2, which passes through the intersection of P_1 and P_2. If the distance of the distance of the point (0, 1, 0) from P_3 is 1 and the distance a point (α, β, γ) from p_3 is 2, then which of the following relations is (are) true? ***(2015)***

(A) $2\alpha + \beta + 2\gamma + 2 = 0$ (B) $2\alpha - \beta + 2\gamma + 4 = 0$

(C) $2\alpha + \beta - 2\gamma - 10 = 0$ (D) $2\alpha - \beta + 2\gamma - 8 = 0$

Q.15 In R^3 let L be a straight line passing through the origin. Suppose that all the points on L are at a constant distance from the two planes $P_1 : x + 2y - z + 1 = 0$ and $P_2 : 2x - y + z - 1 = 0$. Let M be the locus of the feet of the perpendiculars drawn from the points on L to the plane P_1. Which of the following points lie (s) on M? ***(2015)***

(A) $\left(0, -\frac{5}{6}, -\frac{2}{3}\right)$ (B) $\left(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}\right)$

(C) $\left(-\frac{5}{6}, 0, \frac{1}{6}\right)$ (D) $\left(-\frac{1}{3}, 0, \frac{2}{3}\right)$

Q.16 From a point $p(\lambda, \lambda, \lambda)$ perpendiculars PQ and PR are drawn respectively on the lines $y = x, z = 1$ and $y = -x, z = -1$. If p is such that $\angle QPR$ is a right angle, then the possible value(s) of λ is(are) ***(2014)***

(A) $\sqrt{2}$ (B) 1 (C) -1 (D) $-\sqrt{2}$

Q.17 Perpendiculars are drawn from points on the line $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z}{3}$ to the plane $x + y + z = 3$. The feet of perpendiculars lie on the line **(2013)**

(A) $\frac{x}{5} = \frac{y-1}{8} = \frac{z-2}{-13}$ (B) $\frac{x}{2} = \frac{y-1}{3} = \frac{z-2}{-5}$

(C) $\frac{x}{4} = \frac{y-1}{3} = \frac{z-2}{-7}$ (D) $\frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}$

Q.18 Two lines $L_1 : x = 5, \frac{y}{3-\alpha} = \frac{z}{-2}$ and $L_2 : x = \alpha, \frac{y}{-1} = \frac{z}{2-\alpha}$ are coplanar.

The α can take value(s) ***(2013)***

(A) 1 (B) 2 (C) 3 (D) 4

Q.19 Consider the lines $L_1 : \frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1}$, $L_2 : \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2}$ and the planes

$P_1: 7x + y + 2z = 3, P_2: 3x + 5y - 6z = 4.$

Let $ax + by + cz = d$ be the equation of the plane passing through the point of intersection of lines L_1 and L_2.and perpendicular to planes P_1 and P_2

Match List I with List II and select the correct answer using the code given below the list: ***(2013)***

List I	List II
p. a=	1. 13
q. b=	2. -3
r. c=	3. 1
s. d=	4. -2

Codes:

	p	q	r	s
(A)	3	2	4	1
(B)	1	3	4	2
(C)	3	2	1	4
(D)	2	4	1	3

Q.20 The point P is the intersection of the straight line joining the point $Q(2,3,5)$ and $R(1,-1,4)$ with the plane $5x - 4y - z = 1$. If S is the foot of the perpendicular drawn from the point $T(2,1,4)$ to QR, then the length of the line segment PS is ***(2012)***

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$ (C) 2 (D) $2\sqrt{2}$

Q.21 The equation of a plane passing through the line of intersection of the planes $x + 2y + 3z = 2$ and $x - y + z = 3$ and at a distance $\frac{2}{\sqrt{3}}$ from the point $(3,1,-1)$ is ***(2012)***

(A) $5x - 11y + z = 17$ (B) $\sqrt{2}x + y = 3\sqrt{2} - 1$

(C) $x + y + z = \sqrt{3}$ (D) $x - \sqrt{2}y = 1 - \sqrt{2}$

Q.22 If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)dt$ for all $x \in (0,\infty)$ then ***(2012)***

(A) f has a local maximum at $x = 2$

(B) f is decreasing on (2, 3)

(C) There exists some $c \in (0,\infty)$ such that f'(c)=0

(D) f has local minimum at $x = 3$

Q.23 If the distance between the plane $Ax - 2y + z = d$ and the plane containing the lines

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is $\sqrt{6}$, then $|d|$ is

(2010)

Q.24 If the distance of the point $P(1,-2,1)$ from the plane $x + 2y - 2z = \alpha$, where $\alpha > 0$, is 5, then the foot of the perpendicular from P to the plane is ***(2010)***

(A) $\left(\frac{8}{3},\frac{4}{3},-\frac{7}{3}\right)$ (B) $\left(\frac{4}{3},-\frac{4}{3},\frac{1}{3}\right)$

(C) $\left(\frac{1}{3},\frac{2}{3},\frac{10}{3}\right)$ (D) $\left(\frac{2}{3},-\frac{1}{3},\frac{5}{2}\right)$

Q.25 Two adjacent sides of a parallelogram ABCD are given by $\overrightarrow{AB} = 2\hat{i} + 10\hat{j} + 11\hat{k}$ and $\overrightarrow{AD} = -\hat{i} + 2\hat{j} + 2\hat{k}$

The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD'. If AD' makes a right angle with the side AB, then the cosine of the angle a is given by ***(2010)***

(A) $\frac{8}{9}$ (B) $\frac{\sqrt{17}}{9}$ (C) $\frac{1}{9}$ (D) $\frac{4\sqrt{5}}{9}$

PlancEssential Questions

JEE Main/Boards

Exercise 1

Q.5	Q.10	Q.23
Q.29	Q.36	Q.40
Q.42	Q.47	Q.49
Q.50		

Exercise 2

Q.2	Q.8	Q.12
Q.13	Q.14	

Previous Years' Questions

Q.3	Q.6

JEE Advanced/Boards

Exercise 1

Q.2	Q.5	Q.8
Q.10	Q.13	

Exercise 2

Q.2	Q.5	Q.6
Q.7	Q.9	

Previous Years' Questions

Q.3	Q.5	Q.6
Q.9	Q.11	

Answer Key

JEE Main/Boards

Exercise 1

Q.1 <3, –2, 6>

Q.2 <1, 3, –3>

Q.3 $\cos^{-1}\left(\frac{7}{\sqrt{70}}\right)$

Q.4 $r = (-\hat{i} + \hat{j} + \hat{k}) + \lambda(3\hat{i} - \hat{j} - 3\hat{k})$

Q.5 $r = (-3\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} + 4\hat{j} + 5\hat{k})$

Q.6 $\frac{x-3}{-2} = \frac{y-2}{1} = \frac{z+5}{3}$

Q.7 –3

Q.8 3x + 2y +5z = 7

Q.9 $r(3\hat{i} - \hat{j} - 4\hat{k}) + 7 = 0$

Q.10 $3\hat{i} - 7\hat{j}$

Q.11 <7, 1, –2>

Q.12 $\sin^{-1}\left(\frac{-2}{\sqrt{90}\sqrt{13}}\right)$

Q.13 $\frac{7}{\sqrt{11}}$

Q.14 $\frac{1}{\sqrt{3}}$

Q.15 4, –6, 3

Q.16 $\left(\frac{1}{\sqrt{14}}\frac{3}{\sqrt{14}}\frac{-2}{\sqrt{14}}\right)$

Q.17 <0, 1, 0>

Q.18 <2, 3, –10>

Q.19 $\cos^{-1}\left(\frac{11}{21}\right)$ **Q.20** $\sin^{-1}\left(\frac{-1}{7\sqrt{3}}\right)$ **Q.21** $\left(\frac{1}{\sqrt{14}}.\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}}\right),\left(\frac{-1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}}\right)$

Q.23 (3, 4, 5) **Q.26** $\frac{x-3}{-2}=\frac{y}{1}=\frac{z-1}{3}$ **Q.27** $x - 2y - 2x - 3 = 0$; $2x + y - 2z + 3 = 0$

Q.28 $2x + 2y - 3z + 3 = 0$ **Q.29** 13 units **Q.30** $9x - 8y + z + 11 = 0$

Q.33 $5y-5z-6=0, \frac{3\sqrt{2}}{5}$ **Q.34** $2x + 5y + 7z = 78$ **Q.35** $5x + 9y + 11z - 8 = 0$

Q.36 $\frac{\pi}{4}$ **Q.37** $\frac{x-5}{3}=\frac{y+7}{1}=\frac{z+3}{9}$ **Q.39** $\left(-\frac{1}{12},\frac{25}{12},-\frac{1}{6}\right),\frac{13\sqrt{6}}{12}$

Q.40 $r(4\hat{i}-\hat{j}-2\hat{k})=6$ **Q.41** $\frac{25\sqrt{14}}{42}$ **Q.43** $x + 2y - 5z = 0$

Q.44 (0, 2, –2) **Q.45** $(ax + by + cz + d) - \frac{a\ell+bm+cn}{(a'1+b'm+c'm)}(a'x+b'y+c'z+a')=0$

Q.46 $7y + 5z = 35$ **Q.48** $\lambda = 3, -6$ **Q.49** $r(3\hat{i}+5\hat{j}-6\hat{k})-7\sqrt{70}=0$

Q.50 $r(5\hat{i}+\hat{j}-6\hat{k})=4$

Exercise 2

Single Correct Choice Type

Q.1 C **Q.2** A **Q.3** D **Q.4** C **Q.5** A **Q.6** B
Q.7 B **Q.8** D **Q.9** C **Q.10** A **Q.11** D **Q.12** B
Q.13 B **Q.14** D **Q.15** A

Previous Years' Questions

Q.1 A **Q.2** B **Q.3** A **Q.4** $\sqrt{13}$ sq. units **Q.5** $\pm\frac{(2\hat{i}+\hat{j}+\hat{k})}{\sqrt{6}}$ **Q.6** $\frac{9}{2}$ cu unit
Q.7 $2x - y + z - 3 = 0$ and $62x + 29y + 19z - 105 = 0$ **Q.8** C **Q.9** A **Q.10** C
Q.11 B **Q.12** B **Q.13** A **Q.14** A **Q.15** B **Q.16** A
Q.17 D **Q.18** B

JEE Advanced/Boards

Exercise 1

Q.2 34 **Q.3** $\frac{\pi}{2}$ **Q.4** $\left(\frac{1}{\sqrt{3}},\frac{2}{\sqrt{3}}\right)$ **Q.5** $\sqrt{3}$ **Q.6** $p=2(\vec{a}+\vec{b})$

Q.7 $x + 2y - 3z = 14$ **Q.8** No, No **Q.9** $\sqrt{110}$ **Q.10** 13 **Q.11** 13

Q.12 $\frac{x-1}{2}=\frac{y-2}{2}=\frac{z-3}{-3}$ **Q.13** $\frac{x}{1}=\frac{y}{2}=\frac{z}{-1}$ or $\frac{x}{-1}=\frac{y}{1}=\frac{z}{-2}$

Exercise 2

Single Correct Choice Type

Q.1 B **Q.2** A **Q.3** D **Q.4** A **Q.5** B **Q.6** A

Q.7 A **Q.8** C **Q.9** C

Assertion Reasoning Type

Q.10 C

Previous Years' Question

Q.1 D **Q.2** A **Q.3** C **Q.4** D **Q.5** D **Q.6** B

Q.7 D **Q.8** C

Q.9 $A \to r; B \to q; C \to p; D \to s$ **Q.10** (a) $x + y - 2z = 3$ (b) Q(6, 5, –2) **Q.12** B, C, D

Q.13 C **Q.14** B, D **Q.15** A, B **Q.16** C **Q.17** D **Q.18** A, D

Q.19 A **Q.20** A **Q.21** A **Q.22** B, C **Q.23** 6 **Q.24** A

Q.25 B

Solutions

JEE Main/Boards

Exercise 1

Sol 1: $l = \frac{3}{7}$ $m = \frac{-2}{7}$ $n = \frac{6}{7}$

Direction ratios are <3, –2, 6>

Sol 2: [2, 1, 0] & [1, –2, 3]

Direction ratios = 2 – 1, 1 + 2, 0 – 3 = <1, 3, –3>

Sol 3: $\frac{x}{1} = \frac{y}{2} = \frac{z}{0}$ and $\frac{x-1}{3} = \frac{y+5}{2} = \frac{z-3}{1}$

<1, 2, 0> and <3, 2, 1>

$$\cos\theta = \frac{1.3+2.2+0.1}{\sqrt{5}\sqrt{14}} = \frac{7}{\sqrt{70}} \Rightarrow \theta = \cos^{-1}\left(\frac{7}{\sqrt{70}}\right)$$

Sol 4: $\frac{x+1}{3} = \frac{y-1}{-1} = \frac{z-1}{-3} = t$

$r = \hat{i} + \hat{j} + \hat{k} + t\left(3\hat{i} - \hat{j} - 3\hat{k}\right)$

Sol 5: $r = -3\hat{i} + \hat{j} - \hat{k} + \lambda(2\hat{i} + 4\hat{j} + 5\hat{k})$

Sol 6: $x\hat{i} + y\hat{j} + z\hat{k} = (3 - 2\lambda)\hat{i} + (2 + \lambda)\hat{j} + (-5 + 3\lambda)\hat{k}$

$\frac{x-3}{-2} = \frac{y-2}{1} = \frac{z+5}{3}$

Sol 7: $\cos\theta = 0 = -3.1 + 3.5 + 2p.2$

$\Rightarrow 12 + 4p = 0 \Rightarrow p = -3$

Sol 8: (xi + yj + zk) . (3i + 2j + 5k) = 7

3x+ 2y + 5z = 7

Sol 9: 3x – y – 4z = –7; r(3i – j – 4k) = –7

$r(3\hat{i} - \hat{j} - 4\hat{k}) + 7 = 0$

Sol 10: 3x – 7y = –5

Direction ratios of normal to plane are (3, –7, 0) the vector along that normal is $3\hat{i} - 7\hat{j}$.

Sol 11: 7x + y – 2z = 1

Direction ratios of vector normal to the plane are

7i + j – 2k = 0

(7, 1, -2)

Sol 12: Direction ratios of line <4, -5, 7>

Direction ratio of line perpendicular to plane <3, 0, –2>

$$\sin\theta = \frac{4\times 3 + (-5)\times(0) + 7\times(-2)}{\sqrt{16+25+49}\sqrt{9+0+4}} = \frac{-2}{\sqrt{90}\sqrt{13}}$$

Sol 13: x + y + 3z + 7 = 0

Distance from origin is $\frac{0+0+3(0)+7}{\sqrt{1+1+9}} = \frac{7}{\sqrt{11}}$

Sol 14: 3x – 3y + 3z = 0

Distance from (1, 1, 1) is $\frac{3(1)-3(1)+3(1)}{\sqrt{9+9+9}} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$

Sol 15: 3x – 2y + 4z = 12

Intercept on x-axis (y, z = 0, 0)x = 4

Intercept on y-axis (z, z = 0, 0)y = –6

Intercept on z-axis (x, y = 0, 0)z = 3

Sol 16: <a, b, c> = <1, 3, –2>

$$< l, m, n> = \left(\frac{1}{\sqrt{1+9+4}}, \frac{3}{\sqrt{1+9+4}}, \frac{-2}{\sqrt{1+9+4}}\right)$$

$$= \left(\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}\right)$$

Sol 17: Direction cosines of y-axis = <0, 1, 0>

Sol 18: $\frac{x+2}{1} = \frac{2y-1}{3} = \frac{3-z}{5}$

$$\Rightarrow \frac{x+2}{1} = \frac{y-\frac{1}{2}}{\frac{3}{2}} = \frac{z-3}{-5}$$

Direction ratio are $\left\langle 1, \frac{3}{2}, -5\right\rangle$ or $\langle 2, 3, -10\rangle$

Sol 19: $\vec{r}$. (i – 2j – 2k) = 1; $\vec{r}$. (3i – 6j + 2k) = 0

$$\cos\theta = \frac{3.1 + (-6).(-2) + (2)(-2)}{\sqrt{1+4+4}\sqrt{9+36+4}} = \frac{3+12-4}{3.7} = \frac{11}{21}$$

$$\theta = \cos^{-1}\left(\frac{11}{21}\right)$$

Sol 20: $\vec{r}$ = 2i – j + 3k + λ(3i – 6j + 2k) and Plane $\vec{r}$. (i + j + k) = 3

$$\text{Sin}\theta = \frac{3.1 - 6.1 + 2.1}{\sqrt{3}\sqrt{9+36+4}} = \frac{-1}{7\sqrt{3}}$$

Sol 21: l - 5m + 3n = 0 ; $7l^2 + 5m^2 - 3n^2 = 0$

$l = 5m - 3n$

$\Rightarrow 7(25m^2 + 9n^2 - 30mn) + 5m^2 - 3n^2 = 0$

$\Rightarrow 180m^2 + 60n^2 - 210mn = 0$

$\Rightarrow 6m^2 - 7mn + 2n^2 = 0$

$\Rightarrow 6m^2 - 4mn - 3mn + 2n^2 = 0$

$\Rightarrow$ 2m (3m – 2n) – n(3m – 2n) = 0

$\Rightarrow m = \frac{n}{2}$ or $m = \frac{2n}{3}$

If $m = \frac{n}{2}$, l = –m, if $m = \frac{2n}{3}$, $l = \frac{m}{2}$

The following ratio are

$$\left\langle \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right\rangle \text{ or } \left\langle \frac{+1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right\rangle$$

Sol 22: Line through the points

$$\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-3}{6} = \lambda$$

$$\frac{x-3}{5} = \frac{y-3}{2} = \frac{z-3}{-3} = \lambda$$

$$\cos\theta = \frac{2.5 + 4.2 + 6.(-3)}{\sqrt{56}\sqrt{38}} = 0$$

$\theta = 90°$

Sol 23: $\frac{x-5}{4} = \frac{y-4}{0} = \frac{z-4}{-2}$

equation of line $= \lambda$

Let foot of ⊥ is (α, β, γ)

$\alpha = 5 + 4\lambda$; $\beta = 4$; $\gamma = 4 - 2\lambda$

$\Rightarrow (\alpha - 1).4 + (\beta - 2) . 0 + (\gamma - 1) . (-2) = 0$

$\Rightarrow (4 + 4\lambda)4 - 2(3 - 2\lambda) = 0 \Rightarrow 20\lambda + 10 = 0 \Rightarrow \lambda = \frac{-1}{2}$

$\Rightarrow \alpha = 5 + 4\left(\frac{-1}{2}\right) = 3$ β = 4 = 4

$\Rightarrow \gamma = 4 - 2\left(\frac{-1}{2}\right) = 5$

(3, 4, 5)

Sol 24: cos $(\delta\theta)$

$$= \frac{l\,.\,(l+\delta l) + m\,.\,(m+\delta m) + n\,.\,(n+\delta n)}{\sqrt{l^2+m^2+n^2}\sqrt{(l+\delta l)^2+(m+\delta m)^2+(n+\delta n)^2}}$$

$\begin{bmatrix} \text{neglecting} \\ \delta l^2, \delta m^2, \delta n^2 \end{bmatrix}$

$$= \frac{l^2+m^2+n^2+l\delta l+m\delta m+n\delta n}{\sqrt{(l^2+m^2+n^2)}\sqrt{(l^2+2l\delta l+m^2+2l\delta m+n^2+2l\delta n)}}$$

$$\frac{1-(\delta\theta)^2}{2} = \frac{1+l\delta l+m\delta m+n\delta n}{1}$$

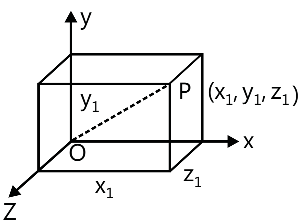
$(\delta\theta)^2 = -2(l\delta l+m\delta m+n\delta n)$ ... (i)

$l^2+m^2+n^2 = (l+\delta l)^2+(m+\delta m)^2+(n+\delta n)^2$

$\Rightarrow\ (\delta l)^2+(\delta m)^2+(\delta n)^2 = -2l\delta l-2m\delta m-2n\delta n$ ... (ii)

$(\delta\theta)^2 = (\delta l)^2+(\delta m)^2+(\delta m)^2$

Sol 25:



$$\left(\frac{l_1+l_2+l_3}{\sqrt{3}}\right)l_1 + \left(\frac{m_1+m_2+m_3}{\sqrt{3}}\right)m_1 + \left(\frac{n_1+n_2+n_3}{\sqrt{3}}\right).n_1$$

$$= \frac{1+l_1l_2+l_1l_3+m_1m_2+m_3m_1+n_1m_2+n_1n_3}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Similarly dot product with l_2 and l_3 gives $\frac{1}{\sqrt{3}}$ as result

i.e. it makes same angle with (l_1, m_1, n_1) (l_2, m_2, n_2) and (l_3, m_3, n_3)

Sol 26: x + 2y = 0 ... (i)

3y – z = 0

$2y - \frac{2z}{3} = 0$ ... (ii)

The line will be across $(a_1, b_1, c_1) \times (a_2, b_2, c_2)$

(1 2 0) × (0 3 –1)

$\begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 0 & 3 & -1 \end{vmatrix}$ = i (–2) –j (–1) + k(3) = –2i + j + 3k

Equation of line will be $\frac{x-3}{-2} = \frac{y-0}{1} = \frac{z-1}{3}$

Sol 27: x + 3y + 6 = 0, 3x – y – 4z = 0

x + 3y + 6 + λ (3x – y – 4z) = 0

x(1 + 3λ) + y (3 – λ) + z (-4λ) + 6 = 0

Distance from origin = $\frac{6}{\sqrt{(1+3\lambda)^2+(4\lambda)^2+(3-\lambda)^2}} = 1$

$36 = 1+9\lambda^2+6\lambda+16\lambda^2+9+\lambda^2-6\lambda$

$36 = 26\lambda^2+10$

$\lambda = \pm 1$

Planes are $\Rightarrow$ 4x + 2y – 4z + 6 = 0 $(\lambda = 1)$

$-2x+4y+4z+6=0\ \ (\lambda=-1)$

Sol 28: ax + by + cz = 1 ... (i)

(–1, 1, 1) lies on (1)

$\Rightarrow$ – a + b + c = 1

(1, –1, 1) lies on (1)

$\Rightarrow$ +a – b + c = 1 $\Rightarrow$ c = 1

If $\perp$ to $x+2y+2z=5$

a • 1 + b • 2 + 2 • c = 0

a + 2b = –2

a – b = 0

a = b = $\frac{-2}{3}$

Equation of plane is–2x – 2y + 3z = 3.

Sol 29: P (–1 + r cos α, – 5 + r cos β, – 10 + r cos γ)

are coordinates of point at distance r from (–1, –5, –10) along $\langle \alpha, \beta, \gamma \rangle$

Point P lies on the given plane

x – y + z = 5

– 1 + r cos α + 5 – r cos β + r cos γ – 10 = 5

r cos α – r cos β + r cos γ = 11

$$r = \frac{11}{\frac{3-4+12}{13}} = \frac{11.13}{11} = 13 \text{ units}$$

Sol 30: ax + by + cz = 1

(1, 2, –4)

a + 2b – 4c = 1 ... (i)

This plane is parallel

r_1 = i + 2j + 4k + λ (2i + 3j + 6k)

r_2 = i – 3j + 5k + λ (i + j – k)

$\Rightarrow 2a + 3b + 6c = 0$

$\Rightarrow a + b - c = 0$

$\Rightarrow b = -8c$

$\Rightarrow a = 9c$

$\Rightarrow 9c - 16c - 4c = 1$

$\Rightarrow c = \frac{-1}{11}, b = \frac{+8}{11}, a = \frac{-9}{11}$

Equation of plane is $-9x + 8y - z = 11$ or

$\Rightarrow \vec{r} \cdot (-9i + 8j - k) = 11$

Sol 31: $al + bm + cn = 0$ ... (i)

and $fmn + gnl + hlm = 0$ ... (ii)

$\Rightarrow \frac{f}{l} + \frac{g}{m} + \frac{h}{n} = 0$ ... (iii)

Comparing (i) and (iii)

$\frac{a}{f} l^2 = \frac{b}{g} m^2 = \frac{c}{h} n^2 = \lambda$

$\Rightarrow l^2 = \frac{f}{a} \lambda \Rightarrow l = \pm\sqrt{\frac{f}{a}\lambda}$

Similarly

$m^2 = \frac{g}{b}\lambda \Rightarrow m = \pm\sqrt{\frac{g}{h}\lambda}$

$n^2 = \frac{h}{c}\lambda \Rightarrow n = \pm\sqrt{\frac{h}{c}\lambda}$

Since, lines are $\perp$

$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$-\frac{f}{a}\lambda - \frac{g}{b}\lambda - \frac{h}{c}\lambda = 0$

$\Rightarrow \lambda\left(\frac{f}{a} + \frac{g}{b} + \frac{h}{c}\right) = 0 \Rightarrow \frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$

Sol 32: $\frac{x-b}{a} = y = \frac{z-d}{c}$ ... (i)

$\frac{x-b'}{a'} = y = \frac{z-d'}{c'}$ ... (ii)

These 2 are perpendicular if $aa' + cc' + 1 = 0$

Sol 33: $4x - y + z - 10 + \lambda(x + y - z - 4) = 0$

$\Rightarrow x(4+\lambda) + y(-1+\lambda) + z(1-\lambda) = 10 + 4\lambda$

$\Rightarrow (4+\lambda)\cdot 2 + (\lambda - 1)\cdot 1 + (1-\lambda)\cdot 1 = 0$

$\Rightarrow 8 - 1 + 1 + 2\lambda = 0 \Rightarrow \lambda = -4$

$\Rightarrow -5y + 5z = -6$, equation of plane

Distance from (1, 1, 1)

$= \frac{-5+5+6}{\sqrt{25+25}} = \frac{6}{5\sqrt{2}} = \frac{3\sqrt{2}}{5}$

Sol 34: Ratios of line perpendicular to plane is {(2 – 0), (5 – 0), (7 – 0)}

Equation of plane is $2x + 5y + 7z = k$

(2, 5, 7) lies on the plane

$2.2 + 5.5 + 7.7 = k = 78$

$2x + 5y + 7z = 78$

Sol 35: Direction ratios of line $\perp$ to the given planes

$3x + 2y - 3y = 1$; $5x - 4y + z = 5$

$\begin{vmatrix} i & j & k \\ 3 & 2 & -3 \\ 5 & -4 & 1 \end{vmatrix} = i(2-12) - j(3+15) + k(-12-10)$

$= -10i - 18j - 22k$

Plane will be $10x + 18y + 22z = k$

Passes through (–1, –1, 2)

$2 \cdot (22) - 28 = k \quad \therefore K = +16$

$10x + 18y + 22z - 16 = 0$

$\Rightarrow 5x + 9y + 11z - 8 = 0$

Sol 36: $l + m + n = 0$ and $l^2 + m^2 = n^2$

$\Rightarrow n = -(l+m)$

$\Rightarrow l^2 + m^2 = (l+m)^2 = l^2 + m^2 = l^2 + m^2 + 2m.n$

$\Rightarrow m.n = 0$

$\Rightarrow m = 0$ or $n = 0$

$\Rightarrow (l,m,n) \equiv \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ or $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

$\Rightarrow$ Angle $= \frac{\pi}{4}$

Sol 37:

$\begin{vmatrix} i & j & k \\ 1 & -3 & 0 \\ 0 & +9 & -1 \end{vmatrix} = i(3) - j(-1) + k(9) = 3i + j + 9k$

$\frac{x-5}{3} = \frac{y+7}{1} = \frac{z+3}{9}$

Sol 38: $ax + by + cz = 1$

$a + b + c = 1 \Rightarrow a + c = 1 - b$

a – b + c = 1 $\Rightarrow$ b = 0

–7a + 3b – 5 +5a = 1

b = 6 + 2a/3,a = –3,c = 4

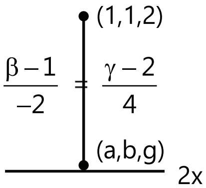
–3x + 4z = 1 $\rightarrow$ ratio $\rightarrow$ [-3, 0, 4]

xz plane $\rightarrow$ ratio $\rightarrow$ [0, 1, 0]

–3.0 + 0.1 + 4.0 = 0

Hence given plane is perpendicular to xz plane.

Sol 39: $\frac{\alpha-1}{2} = \frac{\beta-1}{-2} = \frac{\gamma-2}{4}$



$= \frac{-(2-2+8+5)}{4+4+16} = \frac{-13}{24}$

$\alpha = 1 - \frac{13}{12} = -\frac{1}{12}$, $\beta = 1 + \frac{13}{12} = \frac{25}{12}$

$\gamma = 2 - \frac{13}{6} = \frac{-1}{6}$

Length $= \frac{2-2+8+5}{\sqrt{24}} = \frac{13}{\sqrt{24}}$

Sol 40: ax + by + cz = 1

(1, 0, –1) $\Rightarrow$ a – c = 1

(3, 2, 2) $\Rightarrow$ 3a + 2b + 2c = 1

It is parallel to $\langle 1,-2,3 \rangle$

$\Rightarrow$ a – 2b + 3c = 0

$\Rightarrow$ 4a + 5c = 1

$\Rightarrow$ 4 + 4c + 5c = 1

$\Rightarrow c = \frac{-1}{3}$

$\Rightarrow a = \frac{2}{3}$

$\Rightarrow \frac{2}{3} - \frac{3}{3} = +2b$

$b = \frac{-1}{6}$

Eq. of plane $2x - \frac{y}{2} - z = 3$

4x – y – 2z = 6

$\Rightarrow \vec{r} \cdot (4\hat{i} - \hat{j} - 2\hat{k}) = 6$

Sol 41: distance between 2x – y + 3z = 4

$2x - y + 3z = \frac{-13}{3}$

Distance, $d = \frac{4 + \frac{13}{3}}{\sqrt{4+1+9}} = \frac{25}{3\sqrt{14}} = \frac{25\sqrt{14}}{42}$

Sol 42: ax + by + cz = 1

$A\left(\frac{1}{a},0,0\right), B\left(0,\frac{1}{b},0\right), C\left(0,0,\frac{1}{c}\right)$

$\frac{1}{a} = 3\alpha$, $\frac{1}{b} = 3\beta$, $\frac{1}{c} = 3\gamma$

$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$

Sol 43: $x - 2y + 3z + 4 + \lambda(x - y + z + 3) = 0$

Through origin $3\lambda + 4 = 0$; $\lambda = \frac{-4}{3}$

$\Rightarrow x\left(1 - \frac{4}{3}\right) + y\left(-2 + \frac{4}{3}\right) + z\left(3 - \frac{4}{3}\right) = 0$

$\Rightarrow \vec{r}\,\frac{-x}{3} - \frac{2y}{3} + \frac{5z}{3} = 0$

$\Rightarrow$ x + 2y – 5z = 0

Sol 44: $2x + 2y - z - 6 + \lambda(2x + 3y - z - 8) = 0$

$x(2+2\lambda) + y(2+3\lambda) + z(-1-\lambda) - 6 - 8\lambda = 0$ equation of plane

xz plane $\langle 0,1,0 \rangle$ any point on the line is $(\alpha, 2, 2\alpha - 2)$

Direction ratios of line

$\begin{vmatrix} i & j & k \\ 2 & 2 & -1 \\ 2 & 3 & -1 \end{vmatrix} = i(-2+3) - j(-2+2) + k(6-4)$

= i + 2k = $\langle 1,0,2 \rangle$

This is parallel to plane y = 0 as

(1, 0, 2) . (0, 1, 0) = 0

$\alpha = 0$ i.e. (0, 2, –2)

Sol 45: The equation of Plane

$ax + by + cz + d + \lambda\left(a'x + b'y + c'z + d'\right) = 0$ ... (i)

$\Rightarrow \left(a + \lambda a'\right)x + \left(b + \lambda b\right)y + \left(c + \lambda c'\right)z + d + \lambda d' = 0$

Which parallel to line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

$\Rightarrow (a+\lambda a')l+(b+\lambda b')m+(c+\lambda c')n=0$

$\Rightarrow -\dfrac{al+bm+cn}{a'l+b'm+c'n}=\lambda$

Substituting in (i)

$(ax+by+cz+d)-\dfrac{al+bm+cn}{a'l+b'm+c'n}(a'x+b'y+c'z+d)=0$

Sol 46: ax + by + cz = 1

$\dfrac{1}{b}$ = 5, $\dfrac{1}{c}$ = 7{given intercepts}

<a, b, c> • <1, 0, 0> = 0

a = 0

$\dfrac{y}{5}+\dfrac{z}{7}=1$;   7y + 5z = 35

Sol 47: ax + by + cz = 1

$\dfrac{1}{\sqrt{a^2+b^2+c^2}}$ = P   ......(i)

$A\left(\dfrac{1}{a},0,0\right)$; $B\left(0,\dfrac{1}{b},0\right)$; $C\left(0,0,\dfrac{1}{c}\right)$

x = $\dfrac{1}{a}$, y = $\dfrac{1}{b}$, c = $\dfrac{1}{z}$

$\dfrac{1}{P^2}$ = $a^2+b^2+c^2$ = $\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}$ from (i)

Sol 48: i – j + 3k from 5x + 2y – 7z + 9 = 0

$\Rightarrow\left|\dfrac{5-2-21+9}{\sqrt{49+4+25}}\right|=\dfrac{9}{\sqrt{78}}$

$\Rightarrow (3i+\lambda j+3k)$ from 5x + 2y – 7z + 9 =0

$\Rightarrow\left|\dfrac{15+2\lambda-21+9}{\sqrt{49+4+25}}\right|=\left|\dfrac{3+2\lambda}{\sqrt{78}}\right|\ \Rightarrow|3+2\lambda|=9$

$\Rightarrow \lambda=3$ or –6

Sol 49: Normal to vector 3i + 5j – 6k

3x +5y – 6z = k

at 7 units from origin

$\left|\dfrac{k}{\sqrt{36+25+9}}\right|=7$;   k = $\pm 7\sqrt{70}$

$\vec{r}\cdot(3\hat{i}-5\hat{j}-6\hat{k})$ = $\pm 7\sqrt{70}$

Sol 50: r = $i-j+\lambda(i+j+k)+\mu(4i-2j+3k)$

B = $\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 4 & -2 & 3 \end{vmatrix}$ = i(5) – j(–1) + k(– 2 – 4) = 5i + j – 6 k

Plane pass through (1, –1, 0)

Equation of plane $\vec{r}$ • (5i + j – 6k) = z

5(1) + 1(–1) – 6 (0) – z = 4

The equation of plane ⇒ $\vec{r}$ • (5i + j – 6k) = 4

Exercise 2

Single Correct Choice Type

Sol 1: (C) $l^2+m^2+n^2=\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

Sol 2: (A) ax + by + cz + d = 0 to intersect x and y axis at equal angle

$|\tan\alpha|=|\tan\beta|\ \Rightarrow\ |a|=|b|$

Sol 3: (D) Parallel to x-axis i.e. <1, 0, 0>

$\dfrac{x-a}{1}=\dfrac{y-b}{0}=\dfrac{z-c}{0}$

Sol 4: (C) $\cos\alpha=\dfrac{1}{\sqrt{2}}$ $\cos\gamma=\cos 60^\circ=\dfrac{1}{2}$

$l^2+m^2+n^2=1$

$\Rightarrow\dfrac{1}{2}+\dfrac{1}{4}+m^2=1\ \ \Rightarrow m^2=\dfrac{1}{4}$

$\Rightarrow m=\dfrac{1}{2}$ = $\cos\beta$

$\Rightarrow\beta=60^\circ$

Sol 5: (A) $\dfrac{x+1}{1}=\dfrac{y+3}{3}=\dfrac{z+2}{-2}=\lambda$

$\Rightarrow(-1+\lambda,\ -3+3\lambda,\ -2-2\lambda)$

$\Rightarrow 3(-1+\lambda)+4(3\lambda-3)+5(-2-2\lambda)=5$

$\Rightarrow 5\lambda-3-12-10=5\ \ \Rightarrow 5\lambda=30$

⇒ x = 6

(5, 15, –14)

Sol 6: (B)

$\dfrac{x-0}{2}=\dfrac{y-3}{-2}=\dfrac{z-4}{1}=-\dfrac{(0-6+4-10)}{9}$

$\Rightarrow \frac{x}{2} = \frac{y-3}{-2} = z-4 = \frac{12}{3\times3} = \frac{4}{3}$

$\Rightarrow x = \frac{8}{3}, y = 3 - \frac{8}{3}, z = 4 + \frac{4}{3}$

$\Rightarrow \frac{8}{3}, \frac{1}{3}, \frac{16}{3}$

Sol 7: (B) $2x + y - z - 4 + \lambda(3x + 5z - 4) = 4$

$2 + 3\lambda = 1 \Rightarrow \lambda = \frac{-1}{3}$

$\Rightarrow 2x - x + y - z - \frac{5}{3}z - 4 + \frac{4}{3} = 0$

$\Rightarrow 3x + 3y - 8z - 8 = 0$

Sol 8: (D) $\begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 2 & -3 & 1 \end{vmatrix} = i(-2) - j(3) + k(-3-2)$

$= -2i - 3j - 5k$

It passes through (1, 0, 0)

Equation of line is $\frac{x-1}{2} = \frac{y}{3} = \frac{z}{5}$

Sol 9: (C) Line $\frac{x-1}{1} = \frac{y-3}{2} = \frac{z-4}{3}$ is parallel to plane

$ax + by + cz = 1$

If $a + 2b + 3c = 0$

Only C satisfies the condition

Sol 10: (A) $a = \sqrt{49+1+1} = \sqrt{51}$;

$b = \sqrt{1+0+1} = \sqrt{2}; c = \sqrt{36+1+0} = \sqrt{37}$

$s = \frac{\sqrt{2}+\sqrt{51}+\sqrt{37}}{2}$

$s(s-a)(s-b)(s-c)$

$\Rightarrow \frac{\sqrt{51}+\sqrt{37}+\sqrt{2}}{2}\left[\frac{\sqrt{2}+\sqrt{37}-\sqrt{51}}{2}\right]$

$\left[\frac{\sqrt{2}+\sqrt{51}-\sqrt{37}}{2}\right]\left[\frac{\sqrt{37}+\sqrt{51}-\sqrt{2}}{2}\right]$

$= \frac{\left[37+2+2\sqrt{74}-51\right]\left[31-(37+2-2\sqrt{74}\right]}{16}$

$= \frac{\left[2\sqrt{74}-12\right]\left[12+2\sqrt{74}\right]}{16}$

$= \frac{4\times74-144}{16} = \frac{296-144}{16} = \frac{152}{16} = \frac{38}{4}$

$\Delta = \frac{\sqrt{38}}{2}$

Sol 11: (D) $\frac{x+2}{3} = \frac{y-1}{-1} = \frac{z+4}{1}$

or $\frac{x-1}{3} = \frac{y}{-1} = \frac{z+3}{1}$

Sol 12: (B) Let P(x, y, z) be any point on the locus, then the distances from the six faces are

$|x + 1|, |x - 1|, |y + 1|, |y - 1|, |z + 1|, |z - 1|$

According to the given condition

$|x + 1|^2 + |x - 1| + |y + 1|^2 + |y - 1|^2 + |z + 1|^2 + |z - 1|^2 = 10$

$\Rightarrow \quad 2(x^2 + y^2 + z^2) = 10 - 6 = 4$

$\Rightarrow \quad x^2 + y^2 + z^2 = 2$

Sol 13: (B) If $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ and (x_4, y_4, z_4) are coplanar, then

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} -4-0 & 4+1 & 4+1 \\ 4-0 & 5+1 & 1+1 \\ 3-0 & 9+1 & 4+1 \end{vmatrix} \Rightarrow = \begin{vmatrix} -4 & 5 & 5 \\ 4 & 6 & 2 \\ 3 & 10 & 5 \end{vmatrix} = 0$$

$= -4(30-20) - 5(20-6) + 5(40-18) = -40 - 70 + 110 = 0$

Sol 14: (D) The plane $y + z + 1 = 0$

Since the plane does not have any intercepts on x-axis, therefore it is parallel to x-axis.

Then normal to plane can not be parallel to x-axis.

Sol 15: (A) Using the fact that reflection of $a'x + b'y + c'z + d' = 0$ in the plane $ax + by + cz + d = 0$ is given by $2(aa' + bb' + cc')(ax + by + cz + d)$

$= (a^2 + b^2 + c^2)(a'x + b'y + c'z + d')$

We get the required equation as

$2(2 + 3 + 4)(x - y + z - 3) = (1 + 1 + 1)(2x - 3y + 4z - 3)$

$6(x - y + z - 3) = 2x - 3y + 4z - 3$

$4x - 3y + 2z - 15 = 0$

Previous Years' Questions

Sol 1: (A) Given equation of straight line

$$\frac{x-4}{1}=\frac{y-2}{1}=\frac{z-k}{2}$$

Since, the line lies in the plane 2x – 4y + z = 7

∴ Point (4, 2, k) must satisfy the plane.

⇒ 8 – 8 + k = 7 ⇒ k = 7

Sol 2: (B) Since, the lines intersect they must have a point in common

i.e., $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}=\lambda$

and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}=\mu$

$\Rightarrow x=2\lambda+1, y=3\lambda-1, z=4\lambda+1$

and $x=\mu+3, y=2\mu+k, z=\mu$ are same

$\Rightarrow 2\lambda+1=\mu+3,\ 3\lambda-1=2\mu+k,\ 4\lambda+1=\mu$

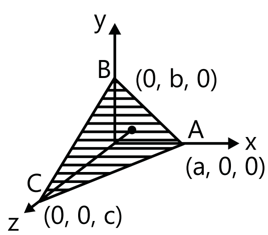
On solving Ist and IIIrd terms, we get,

$\lambda=-\frac{3}{2}$ and $\mu=-5$

$\therefore k=3\lambda-2\mu-1 \Rightarrow k=3\left(-\frac{3}{2}\right)-2(-5)-1=\frac{9}{2}$

$\therefore k=\frac{9}{2}$

Sol 3: (A) Since, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ cuts the coordinate axes at A(a, 0, 0), B(0, b, 0), C(0, 0, c) and its distance from origin = 1



$$\therefore \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=1$$

or $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=1$ ... (i)

where P is centroid of triangle

$\therefore x=\frac{a}{3}, y=\frac{b}{3}, z=\frac{c}{3}$ ... (ii)

∴ From Eqs. (i) and (ii), we get

$$\frac{1}{9x^2}+\frac{1}{9y^2}+\frac{1}{9z^2}=1 \quad \text{or} \quad \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=9=K$$

$\therefore K=9$

Sol 4: Area of $\Delta ABC=\frac{1}{2}(\vec{AB}\times\vec{AC})$, where

$\vec{AB}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{AC}=2\hat{i}+0\hat{j}+0\hat{k}$

$$\therefore \vec{AB}\times\vec{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -3\\ 2 & 0 & 0\end{vmatrix}=2(-3\hat{j}-2\hat{k})$$

$\Rightarrow$ Area of triangle $=\frac{1}{2}(\vec{AB}\times\vec{AC})$

$=\frac{1}{2}.2.\sqrt{9+4}=\sqrt{13}$ sq. units

Sol 5: A unit vector perpendicular to the plane determined by P, Q, R $=\pm\frac{(\vec{PQ}\times\vec{PR})}{|\vec{PQ}\times\vec{PR}|}$

where $\vec{PQ}=[\hat{i}+\hat{j}]-3\hat{k}$ and $\vec{PR}=-\hat{i}+3\hat{j}-\hat{k}$

$$\therefore \vec{PQ}\times\vec{PR}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & -3\\ -1 & 3 & -1\end{vmatrix}$$

$=\hat{i}(-1+9)-\hat{j}(-1-3)+\hat{k}(3+1)=8\hat{i}+4\hat{j}+4\hat{k}$

$\Rightarrow |\vec{PQ}\times\vec{PR}|=4\sqrt{4+1+1}=4\sqrt{6}$

$\therefore$ Unit vector $=\pm\frac{(\vec{PQ}\times\vec{PR})}{|\vec{PQ}\times\vec{PR}|}=\pm\frac{4(2\hat{i}+\hat{j}+\hat{k})}{4\sqrt{6}}=\pm\frac{(2\hat{i}+\hat{j}+\hat{k})}{\sqrt{6}}$

Sol 6: Let the equation of plane through (1, 1, 1) having a, b, c as DR's of normal to plane, $a(x-1)+b(y-1)+c(z-1)=0$ and plane is parallel to straight line having DR's.

(1, 0, –1) and (–1, 1, 0)

⇒ a – c = 0 and –a + b = 0

⇒ a = b = c

∴ Equation of plane is x – 1 + y – 1 + z – 1 = 0

or $\frac{x}{3}+\frac{y}{3}+\frac{z}{3}=1$. Its intercept on coordinate axes are

A(3, 0, 0), B(0, 3, 0), C(0, 0, 3)

Hence, the volume of tetrahedron OABC

$$=\frac{1}{6}[\vec{a}\vec{b}\vec{c}]=\frac{1}{6}\begin{vmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{vmatrix}=\frac{27}{6}=\frac{9}{2}\text{cu units}$$

Sol 7: Equation of plane containing the lines

$2x - y + z - 3 = 0$ and $3x + y + z = 5$ is

$(2x - y + z - 3) + \lambda(3x + y + z - 5) = 0$

$\Rightarrow \ (2 + 3\lambda)x + (\lambda - 1)y + (\lambda + 1)z - 3 - 5\lambda = 0$

Since, distance of plane from (2, 1, –1) to above plane is $1/\sqrt{6}$.

$\therefore \left| \dfrac{6\lambda + 4 + \lambda - 1 - \lambda - 1 - 3 - 5\lambda}{\sqrt{(3\lambda+2)^2 + (\lambda-1)^2 + (\lambda+1)^2}} \right| = \dfrac{1}{\sqrt{6}}$

$\Rightarrow 6(\lambda - 1)^2 = 11\lambda^2 + 12\lambda + 6$

$\Rightarrow \lambda = 0, -\dfrac{24}{5}$

$\therefore$ Equations of planes are $2x - y + z - 3 = 0$

and $62x + 29y + 19z - 105 = 0$

Sol 8: (C) The line $\dfrac{x-3}{2} = \dfrac{y+2}{-1} = \dfrac{z+4}{3}$ lies in the plane,

then point $(3, -2, -4)$ lies on the plane

$\Rightarrow 3\ell - 2m = 5$ ...(i)

And line is $\perp$ to normal of plane

$\Rightarrow 2\ell - m = 3$ ...(ii)

From (i) and (ii)

$\ell = 1$ and $m = -1$

$\Rightarrow \ell^2 + m^2 = 1^2 + (-1)^2 = 2$

Sol 9: (A) The eq of line passes through $(1, -5, 9)$

along $x = y = z$ is

$\dfrac{x-1}{1} = \dfrac{y+5}{1} = \dfrac{z-9}{1} = r$

The point on line $(r+1, r-5, r+9)$

This point also lies on the given plane

$r + 1 - r + 5 + r + 9 = 5$

$r = -10$

The point in $(-9, -15, -1)$

Distance between $(1, -5, 9)$ and $(-9, -15, -1)$

$= \sqrt{10^2 + (-10)^2 + (10)^2} = 10\sqrt{3}$ unit

Sol 10: (C) $\dfrac{x-2}{3} = \dfrac{y+1}{4} = \dfrac{z-2}{12} = r$

The point of interstation $(3r+2, 4r-1, 12r+2)$

Lies on plane, then

$3r + 2 - 4r + 1 + 12r + 2 - 16 = 0$

$\Rightarrow 11r - 11 = 0$

$\Rightarrow r = 1$

The point in $(5, 3, 14)$

Distance $= \sqrt{(5-1)^2 + (3-0)^2 + (14-2)^2}$

$= \sqrt{16 + 9 + 144}$

$= \sqrt{169} = 13$

Sol 11: (B) Let the two lines in a same plane interest at $P(x, y, 0)$, then $2x - 5y = 3$ and $x + y = 5$

On solving, we get $P \equiv (4, 1, 0)$

Any plane || to $x + 3y + 6z = 1$ is

$x + 3y + 6z = \lambda$

$P(4, 1, 0)$ must satisfies it, then

$4 + 3 + 0 = \lambda \Rightarrow \lambda = 7$

The eq. to required plane

$\Rightarrow x + 3y + 6z = 7$

Sol 12: (B) The parallel planes $2x + y + 2z = 8$

and $4x + 2y + 4z = -5$

Distance $= \dfrac{|-8 \times 2 - 5|}{\sqrt{16 + 4 + 16}} = \dfrac{21}{\sqrt{36}} = \dfrac{21}{6} = \dfrac{7}{2}$

Sol 13: (A) Image of point $(1, 3, 4)$ is

$\dfrac{x-1}{2} = \dfrac{y-3}{-1} = \dfrac{z-4}{1} = \dfrac{-2(2 - 3 + 4 + 3)}{4 + 1 + 1} = -2$

$\Rightarrow (-3, 5, 2)$

Since line is parallel to plane direction, ratio will not change

Eq. of imaged line $\dfrac{x+3}{3} = \dfrac{y-5}{1} = \dfrac{z-2}{1}$

Sol 14: (A) $\ell + m + n = 0 \ \Rightarrow n = -(\ell + m)$

Substituting in $\ell^2 = m^2 + n^2$

$\ell^2 = m^2 + (\ell + m)^2$

$\Rightarrow \ell^2 = m^2 + \ell^2 + m^2 + 2m$

$\Rightarrow 2m^2 + 2m = 0$

$\Rightarrow 2m(m + 1) = 0$

$\Rightarrow m = 0, 1$

if $m = 0, \ell = \frac{-1}{\sqrt{2}}, n = \frac{1}{\sqrt{2}}$

if $m = 1, \ell = 0 = n$ (not possible)

Therefore direction cosine

$$\left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right) \text{or} \left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$$

$$\cos\phi = \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)+(0)\left(\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}\right)(0)=\frac{1}{2}$$

$$\Rightarrow \phi = \frac{\pi}{3}$$

Sol 15: (B) The lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanars, then

$$\begin{vmatrix} 1 & 1 & -k \\ k & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow k(k+3)=0$$

$$\Rightarrow K = 0, -3$$

Two values exist.

Sol 16: (A) Eq. of plane parallel to $x-2y+2z-5=0$ is $x-2y+2z=\lambda$

$\perp$ distance from origin is 1,

then $\frac{|0-0+0-\lambda|}{\sqrt{1+4+4}}=1 \Rightarrow \frac{|\lambda|}{3}=1 \Rightarrow \lambda = \pm 3$

Eq. of plane $x-2y+2z=\pm 3$

Sol 17: (D) $\sin\theta = \frac{1+4+3\lambda}{\sqrt{1+4+9}\sqrt{1+4+\lambda}}$

$$= \frac{5+3\lambda}{\sqrt{14}\sqrt{5+\lambda}} \qquad \text{... (i)}$$

Given $\cos\theta = \sqrt{\frac{5}{14}}$

$$\Rightarrow \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{5}{14}} = \frac{3}{\sqrt{14}}$$

From (i) $\frac{3}{\sqrt{14}} = \frac{5+3\lambda}{\sqrt{14}\sqrt{5+\lambda}}$

$$\Rightarrow 3\sqrt{5+\lambda^2} = (5+3\lambda) \Rightarrow 9(5+\lambda^2) = 25+9\lambda^2+30\lambda$$

$$\Rightarrow 30\lambda = 20 \Rightarrow \lambda = \frac{2}{3}$$

Sol 18: (B) Statement-I: Since mid point of A(1, 0, 7) and B(1, 6, 3) is which lies on the line, therefore point B is image of A about line

Statement-II: Since it given that the line only bisects the line joining A and B, therefore not the correct explanation.

$$\left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right) \text{or} \left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$$

$$\cos\theta = \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right)+(0)\left(\frac{1}{\sqrt{2}}\right)+(0)$$

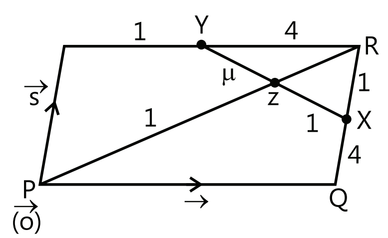
$$=\frac{1}{2} \Rightarrow 9 = \frac{\pi}{3}$$

JEE Advanced/Boards

Exercise 1

Sol 1: Let point P be taken as origin and $\vec{q},\vec{s}$ are the position vectors of Q and S points respectively.

$\Rightarrow \overrightarrow{PR} = \vec{q}+\vec{s}$



P.V. of X $= \frac{\vec{q}+4(\vec{q}+\vec{s})}{5} = \frac{5\vec{q}+4\vec{s}}{5}$

P.V. of Y $= \frac{4\vec{s}+\vec{q}+\vec{s}}{5} = \frac{\vec{q}+5\vec{s}}{5}$

Let, $\frac{PZ}{ZR}=\frac{1}{\lambda}$ and $\frac{YZ}{ZX}=\mu$

P.V. of P $= \frac{\vec{q}+\vec{s}}{\lambda+1} = \frac{\mu\left(\vec{q}+\frac{4}{5}\vec{s}\right)+\left(\frac{\vec{q}}{5}+\vec{s}\right)}{\mu+1}$

$$\Rightarrow \frac{1}{\lambda+1} = \frac{\mu+\frac{1}{5}}{\mu+1} \qquad \text{... (i)}$$

$$\Rightarrow \frac{1}{\lambda+1} = \frac{\mu+\frac{1}{5}}{\mu+1} \qquad \text{... (ii)}$$

From (i) & (ii), we get

$\mu = 4, \lambda = \dfrac{4}{21}$

$\Rightarrow \dfrac{PZ}{ZR} = \dfrac{21}{4} \Rightarrow \dfrac{PZ}{PR} = \dfrac{21}{25}$

Sol 2: p (x ,y)

$\overrightarrow{PA} \bullet \overrightarrow{PB} + 3\overrightarrow{OA} \bullet \overrightarrow{OB} = 0$

$(x-1)(x+1) + y^2 + 3(-1) = 0$

$x^2 + y^2 = 4$

$\left|\overrightarrow{PA}\right| \cdot \left|\overrightarrow{PB}\right|$

$\sqrt{(x-1)^2 + y^2} \quad \sqrt{(x+1)^2 + y^2}$

$\sqrt{(5-2x)}\sqrt{(5+2x)} = \sqrt{25-4x^2}$

Max is $\sqrt{25} = 5 = M$

Min = $\sqrt{9} = 3$

$M^2 + m^2 = 34$

Sol 3: $|\vec{a}| = |\vec{b}| = |\vec{c}| = |\vec{a}+\vec{b}| = 1$

$\Rightarrow \; |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b} = 1; \; \vec{a}\cdot\vec{b} = \dfrac{-1}{2}$

$\theta = 120°$

$\angle\left(2\vec{a} + \vec{b} \;\&\; \vec{b}\right)$

$\Rightarrow$ (2a + b) (b) = |2a + b| |b| cos θ_1

$\Rightarrow \; 2a \,.\, b + |b|^2 = \sqrt{4a^2 + b^2 + 4a\,.\,b} \; |b| \; \cos\theta_1$

$\Rightarrow \; -1 + 1 = \cos\theta_1 \times k$

$\Rightarrow \; \cos\theta_1 = 0$

$\Rightarrow \theta_1 = \dfrac{\pi}{2}$

Sol 4: $c = \lambda\vec{a} + \mu\vec{b}$

$|c|^2 = \lambda^2 + \mu^2 + 2\lambda\mu\,\vec{a}\,.\,\vec{b} = 1$

$\Rightarrow \lambda^2 + \mu^2 - \lambda\mu = 1$

$\Rightarrow \vec{c}.\vec{a} = \lambda + \mu\,(\vec{a}\,.\,\vec{b}) = 0$

$\Rightarrow \lambda - \dfrac{\mu}{2} = 0$

$\Rightarrow \lambda = \dfrac{\mu}{2} \Rightarrow u = 2\lambda$

$\Rightarrow \lambda^2 + 4\lambda^2 - 2\lambda^2 = 1$

$\Rightarrow \lambda = \dfrac{1}{\sqrt{3}}, \mu = \dfrac{2}{\sqrt{3}} \Rightarrow \left(\dfrac{1}{\sqrt{3}}, \dfrac{2}{\sqrt{3}}\right)$

Sol 5: $\vec{P} = x\vec{a} + y\vec{c} \Rightarrow \vec{p} = y\vec{c}$

$0 \le \vec{p}.\vec{a} = x \le 1 \; x \in [0,1]$

$0 \le \vec{p}.\vec{b} = x\vec{a}.\vec{b} \le 1 \; x \in [-2,0] \Rightarrow x = 0$

$\vec{p}.\vec{c} = y$

$\vec{c} = \dfrac{\vec{a}}{\sqrt{3}} + \dfrac{2\vec{b}}{\sqrt{3}}$

$\vec{p}.\vec{c} = \dfrac{\vec{p}.\vec{a}}{\sqrt{3}} + \dfrac{2\vec{p}.\vec{b}}{\sqrt{3}}$

$\vec{p} = x\vec{a} + y\left(\dfrac{\vec{a} + 2\vec{b}}{\sqrt{3}}\right)$

Sol 6: For max. x and y; $x + \dfrac{y}{\sqrt{3}} = \dfrac{2y}{\sqrt{3}} \Rightarrow \; y = \sqrt{3}\,x$

$x = 1; \; y = \sqrt{3}$

$\Rightarrow \vec{p} = 2\vec{a} + 2\vec{b}$

Sol 7: The coordinates of the points, O and P, are (0, 0, 0) and (1, 2, -3) respectively.

Therefore, the direction ratios of OP are (1 - 0) = 1, (2 - 0) = 2 and (-3-0) = -3

It is knows that the equation of the plane passing through the point $\left(x_1\, y_1\, z_1\right)$ is

$a\left(x - x_1\right) + b\left(y - y_1\right) + c\left(z - z_1\right) = 0$, where a, b and c are the direction ratio of normal.

Here, the direction ratios of normal are 1, 2 and -3 and the point P is (1, 2, -3).

Thus, the equation of the required plane is

$1\left(x-1\right) + 2\left(y-2\right) - 3\left(z+3\right) = 0$

$\Rightarrow x + 2y - 3z - 14 = 0$

Sol 8: (i) $\vec{A} = [x_1 y_1 z_1]; \vec{B} = [x_2 y_2 z_2]; C = [x_3 y_3 z_3]$

$\vec{A}\left(\vec{B} \times \vec{C}\right) = 0$ all are coplanar

$\vec{A} \times \vec{B} = 0 = \vec{B} \times \vec{C} = \vec{C} \times \vec{A}$ i.e. all are mutually $\perp$ which simultaneously is not possible.

(ii) P = (x_1, y_2, x_3) Q = $(y_1 y_2 y_3)$ O (0,0,0)

In ΔPOQ; OP = $x_1 i + x_2 j + x_3 k$

OQ = $y_1 i + y_2 j + y_3 k$

OP . OQ = $x_1 y_1 + x_2 y_2 + x_3 y_3 \; x_1 > 0 \; y_1 > 0$

OP . OQ < 0 [i.e. it can never be zero]

Sol 9: $A=(-5,22,5)$; $B=(1,2,3)$; $C=(4,3,2)$

$D=(-1,2,-3)$

and $\Delta AEF = \sqrt{S}$

$\overrightarrow{DE}=\overrightarrow{DA}+\overrightarrow{DB}$

$\overrightarrow{OE}-\overrightarrow{OD}=\overrightarrow{OA}-\overrightarrow{OD}+\overrightarrow{OB}-\overrightarrow{OD}$

$\overrightarrow{OE} = -5i+22j+5k+i+2j+3k-(-i+2j-3k)$

$\overrightarrow{OE} = -3i+22j+11k$

$\overrightarrow{BF}=\overrightarrow{BA}+\overrightarrow{BC}$

$\overrightarrow{OF}=\overrightarrow{OA}+\overrightarrow{OC}-\overrightarrow{OB}$

$\overrightarrow{OF}=-5i+22j+5k+4i+3j+2k-i-2j-3k$

$OF = -2i+23j+4k$

Area $=\frac{1}{2}\left[\overrightarrow{AE}\times\overrightarrow{AF}\right]=\frac{1}{2}\left[(2i+6k)\times(3i+j-k)\right]$

$$\frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 0 & 6\\ 3 & 1 & -1\end{vmatrix}=\frac{1}{2}\hat{i}(-6)-\hat{j}(-2-18)+\hat{k}(2)=-6\hat{i}+20\hat{j}+2\hat{k}$$

$$=\frac{1}{2}\sqrt{36+400+4}=\sqrt{\frac{440}{4}}=\sqrt{110}\quad\Rightarrow S=\sqrt{110}$$

Sol 10: $\left((a-2)\alpha^2+(b-3)\alpha+c\right)x+$

$\Rightarrow\left((a-2)\beta^2+(b-3)\beta+c\right)y+$

$\Rightarrow\left((a-2)\gamma^2+(b-3)\gamma+c\right)\left(x\times y\right)=0$

$\Rightarrow a-2=b-3=c=0$

$\Rightarrow a=2$; $b=3$; $c=0$

$\Rightarrow a^2+b^2+c^2=13$

Sol 11: The equation of the given line is

$\vec{r}=2\hat{i}-\hat{j}+2\hat{k}+\lambda(3\hat{i}+4\hat{j}+2\hat{k})$ ... (i)

The equation of the given plane is

$\vec{r}.\left(\hat{i}-\hat{j}+\hat{k}\right)=5$ ... (ii)

Substituting the value of from equation (i) in equation (ii), we obtain.

$$\left[2\hat{i}-\hat{j}+2\hat{k}+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)\right]\left(\hat{i}-\hat{j}+\hat{k}\right)=5$$

$$\Rightarrow\left[(3\lambda+2)\hat{i}+(4\lambda-1)\hat{j}(2\lambda+2)\hat{k}\right].\left(\hat{i}-\hat{j}+\hat{k}\right)=5$$

$$\Rightarrow(3\lambda+2)-(4\lambda-1)+(2\lambda+2)=5$$

$$\Rightarrow\lambda=0$$

Substituting this value in equation (i), we obtain the equation of the line as $\vec{r}=2\hat{i}-\hat{j}+2\hat{k}$

This means that the position vector of the point of intersection of the line and the plane is $\vec{r}=2\hat{i}-\hat{j}+2\hat{k}$

This shows that the point of intersection of the given line and plane is given by the coordinates (2, -1, 2). The point is (-1, -5, -10).

The distance d between the points, (2, -1, 2) and (-1, -5, -10), is

$$d=\sqrt{(-1-2)^2+(-5+1)^2+(-10-2)^2}$$
$$=\sqrt{9+16+144}=\sqrt{169}=13$$

Sol 12: $\frac{x-1}{a}=\frac{y-2}{b}=\frac{z-3}{c}$

Parallel to the plane $x+5y+4z=0$

$\Rightarrow a+5b+4c=0$

$\Rightarrow\left(-1+2\lambda,\ 2+\lambda,-4+2\lambda\right)=\Rightarrow\left(1+ka,2+kb,3+kc\right)$

$\Rightarrow\frac{ka+2}{2}=\frac{2kb}{2}=\frac{7+kc}{2}$

$\Rightarrow\frac{2}{2b-a}=\frac{7}{2b-c}=\frac{5}{a-c}$

$\Rightarrow 10b=7a-2c$ ....(i)

$\Rightarrow\frac{7a-10b}{2}=\frac{-a-5b}{4}$

$\Rightarrow a=b$; $c=\frac{-3a}{2}$ $a=2b=2c=-3$

Sol 13: $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\lambda$

$\lambda_a=3+2k$; $\lambda_b=3+k$; $\lambda_c=k$

$$\frac{1}{2}=\left|\frac{2a+b+c}{\sqrt{6\sqrt{a^2+b^2+c^2}}}\right|\quad 3(a^2+b^2+c^2)=2(2a+b-c)^2$$

$$\frac{3+2k}{a}=\frac{3+k}{b}=\frac{k}{c}$$

$$\frac{3a-3b}{2b-a}=\frac{3c}{b-c}=\frac{3c}{a-2c}$$

$\Rightarrow a=1, b=2, c=-1$ or $a=-1$ $b=1, c=-2$

Exercise 2

Single Correct Choice Type

Sol 1: (B) Direction cosines of PQ (2, 3, –6) (3, –4, 5)

Ratios = 2 – 3, 3 + 4, –6 – 5 = –1, 7, –11

Direction cosines = $\frac{-1}{\sqrt{171}}, \frac{7}{\sqrt{171}}, \frac{-11}{\sqrt{171}}$

or $\frac{1}{\sqrt{171}}, \frac{-7}{\sqrt{171}}, \frac{11}{\sqrt{171}}$

Sol 2: (A) $\alpha = \frac{m+3}{m+1}$, $\beta = \frac{4m+1}{m+1}$, $\gamma = \frac{-6m-5}{m+1}$

As $\alpha = 0 \Rightarrow$ m = –3[A]

Sol 3: (D) 3x + 2y + z + 5 = x + y – 2z – 3

2x – y – λz = 7x + 10y – 8z are $\perp$ to each other

1. $\begin{vmatrix} i & j & k \\ 3 & 2 & 1 \\ 1 & 1 & -2 \end{vmatrix}$ = i(–5) – j (–6 –1) + k(3–2) = –5i + 7j + k

2. $\begin{vmatrix} i & j & k \\ 2 & -1 & -\lambda \\ 7 & +10 & -8 \end{vmatrix}$ = i$(8+10\lambda)$ –j (–16 + 7λ) + k (+20

+ 7) –40 – 50λ + 112 – 49λ – 127 = 0$\Rightarrow$ λ = 1

Sol 4: (A) (2,4,5) —m— (a,b,g) —1— (3,5,7)

$\alpha = \frac{3m+2}{m+1} = 0 \Rightarrow$ m = $\frac{-2}{3}$

Sol 5: (B) $\cos\alpha = \frac{1}{\sqrt{3}}$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3} = \frac{4}{3}$

Sol 6: (A) $\alpha(x-a)+\beta(y-b)+\gamma(z-c)=0$

$A\left(\frac{\beta b+\gamma c+\alpha a}{\alpha}, 0, 0\right)$

$(h,k,l) = \left(\frac{\alpha a+\beta b+\gamma c}{\alpha}, \frac{\alpha a+\beta b+\gamma c}{\beta}, \frac{\alpha a+\beta b+\gamma c}{\gamma}\right)$

(h – a)α = $\beta b+\gamma c$

(k – b)β = $\alpha a+\gamma c$

$(l - c)\ \gamma$ = $\alpha a+\beta b$

$\begin{vmatrix} h-a & -b & -c \\ -a & k-b & -c \\ -a & -b & l-c \end{vmatrix} = 0$

$(h-\alpha)\left[(k-b)(l-c)-bc\right] + b\left[-al+ac-ac\right] - c$ [ab + ak – ab] = 0

$(h-\alpha)$ $[kl-kc-bl]$ $-bal-cak=0$

$hkl-hkc-hbl-akl+akc+abl-bal-cak=0$

$ayz+bzx+cxy=xyz$

Sol 7: (A) $\frac{2x-y+2z+3}{3} = -\frac{(3x-2y+6z+8)}{7}$

$p_1p_2+q_1q_2+r_1r_2 = 6+2+12>0$

+ve $\rightarrow$ acute

23x – 13y + 32z + 45 = 0[C]

Sol 8: (C) $\frac{x-4/3}{2} = \frac{y+6/5}{3} = \frac{z-3/2}{4} = \lambda$

$\left(\frac{4}{3}+2\lambda, \frac{-6}{5}+3\lambda, \frac{3}{2}+4\lambda\right)$

$\left(\frac{5}{3}, \frac{8}{5}, \frac{9}{2}\right) = \left(\frac{4}{3}+5\lambda, \frac{-6}{5}+8\lambda, \frac{3}{2}+9\lambda\right)$

Both passes through $\left(\frac{+4}{3}, \frac{-6}{5}, \frac{3}{2}\right)$

Minimum distance is zero.

Sol 9: (C) $\frac{x}{2}+\frac{y}{\beta}=\frac{z}{\gamma}$

$\Rightarrow \alpha(b+c)+\beta(a+c)+\gamma(a+b)=0$

$\Rightarrow \alpha(b-c)+\beta(c-a)+\gamma(a-b)=0$

$\Rightarrow \alpha b+\beta c+\gamma a=0$; $\alpha c+\beta a+\gamma b=0$

$\Rightarrow \alpha = a^2-bc$

$\Rightarrow \beta = b^2-ac$

$\Rightarrow \gamma = c^2-ab$

$\Rightarrow \frac{x}{a^2-bc} = \frac{\gamma}{b^2-ac} = \frac{z}{c^2-ab}$

Assertion Reasoning Type

Sol 10: (C) y + z + 1 = 0 [0, 1, 1]

x-axis [1, 0, 0]

$\sin\theta = 0$

R is wrong.

Previous Years' Questions

Sol 1: (D) Let the equation of plane be

$a(x-1) + b(y+2) + c(z-1) = 0$ which is perpendicular to $2x - 2y + z = 0$ and

$x - y + 2z = 4$.

$\Rightarrow 2a - 2b + c = 0$ and $a - b + 2c = 0$

$\Rightarrow \frac{a}{-3} = \frac{b}{-3} = \frac{c}{0} \Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{0}$.

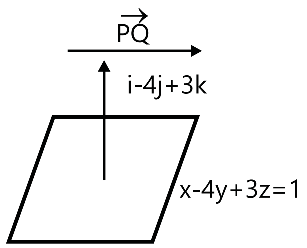
So, the equation of plane is

$x - 1 + y + 2 = 0$ or $x + y + 1 = 0$

Its distance from the point (1, 2, 2) is $\frac{|1+2+1|}{\sqrt{2}} = 2\sqrt{2}$.

Sol 2: (A) Given $\vec{OQ} - (1-3\mu)\hat{i} + (\mu - 1)\hat{j} + (5\mu + 2)\hat{k}$,

$\vec{OP} = 3\hat{i} + 2\hat{j} + 6\hat{k}$ (Where O is origin)



Now, $\vec{PQ} = (1 - 3\mu - 3)\hat{i} + (\mu - 1 - 2)\hat{j} + (5\mu + 2 - 6)\hat{k}$

$= (-2 - 3\mu)\hat{i} + (\mu + 3)\hat{j} + (5\mu - 4)\hat{k}$

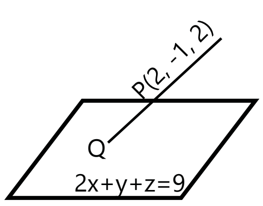
$\because$ $\vec{PQ}$ is parallel to the plane

$x - 4y + 3z = 1$

$\therefore -2 - 3\mu - 4\mu + 12 + 15\mu - 12 = 0$

$\Rightarrow 8\mu = 2 \Rightarrow \mu = \frac{1}{4}$

Sol 3: (C) Since, $\ell = m = n = \frac{1}{\sqrt{3}}$



$\therefore$ Equation of line are $\frac{x-2}{1/\sqrt{3}} = \frac{y+1}{1/\sqrt{3}} = \frac{z-2}{1/\sqrt{3}}$

$\Rightarrow x - 2 = y + 1 = z - 2 = r$

$\therefore$ Any point on the line is $Q \equiv (r+2, r-1, r+2)$

$\because$ Q lies on the plane $2x + y + z = 9$

$\Rightarrow 4r + 5 = 9 \Rightarrow r = 1$

$2(x+2) + (r-1) + (r+2) = 9$

$\therefore Q(3,0,3)$

$\therefore PQ = \sqrt{(3-2)^2 + (0+1)^2 + (3-2)^2} = \sqrt{3}$

Sol 4: (D) Given planes are $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ For $z = 0$, we get $x = 3$,

$y = -1$

Direction ratios of planes are <3, –6, –2> and <2, 1, –2>

then the DR's of line of intersection of planes is < 14, 2, 15> and line is

$\frac{x-3}{14} = \frac{y+1}{2} = \frac{z-0}{15} = \lambda$ (say)

$\Rightarrow x = 14\lambda + 3, y = 2\lambda - 1, z = 15\lambda$

Hence, statement I is false.

But statement II is true.

Sol 5: (D) Given three planes are

$P_1 : x - y + z = 1$ ... (i)

$P_2 : x + y - z = -1$ ... (ii)

and $P_3 : x - 3y + 3z = 2$ ... (iii)

Solving Eqs. (i) and (ii), we have $x = 0, z = 1 + y$

which does not satisfy Eq. (iii)

As, $x - 3y + 3z = 0 - 3y + 3(1 + y) = 3(\neq 2)$

$\therefore$ Statement-II is true.

Next, since we know that direction ratio's of line of intersection of planes $a_1x + b_1y + c_1z + d_1 = 0$

and $a_2x + b_2y + c_2z + d_2 = 0$ is

$b_1c_2 - b_2c_1, c_1a_2 - a_1c_2, a_1b_2 - a_2b_1$

Using above result.

Direction ratio's of lines L_1, L_2 and L_3 are 0, 2, 2 ; 0; – 4, –4; 0, –2, –2

Respectively

$\Rightarrow$ All the three lines L_1, L_2, and L_3 are parallel pairwise.

$\therefore$ Statement-I is false.

Sol 6: (B) The equation of given lines in vector form may be written as

$L_1 : \vec{r} = (-\hat{i} - 2\hat{j} - \hat{k}) + \lambda(3\hat{i} + \hat{j} + 2\hat{k})$

and $L_2 : \vec{r} = (2\hat{i} - 2\hat{j} + 3\hat{k}) + \mu(\hat{i} + 2\hat{j} + 3\hat{k})$

Since, the vector perpendicular to both L_1 and L_2.

$$\therefore \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -\hat{i} - 7\hat{j} + 5\hat{k}$$

$\therefore$ Required unit vector

$$= \frac{(-\hat{i} - 7\hat{j} + 5\hat{k})}{\sqrt{(-1)^2 + (-7)^2 + (5)^2}} = \frac{1}{5\sqrt{3}}(-\hat{i} - 7\hat{j} + 5\hat{k})$$

Sol 7: (D) The shortest distance between L_1 and L_2 is

$$\left| \frac{((2-(-1)\}\hat{i} + (2-2)\hat{j} + (3-(-1))\hat{k}) \cdot (-\hat{i} - 7\hat{j} + 5\hat{k})}{5\sqrt{3}} \right|$$

$$= \left| \frac{(3\hat{i} + 4\hat{k}) \cdot (-\hat{i} - 7\hat{j} + 5\hat{k})}{5\sqrt{3}} \right| = \frac{17}{5\sqrt{3}} \text{unit.}$$

Sol 8: (C) The equation of the plane passing through the point (–1, –2, –1) and whose normal is perpendicular to both the given lines L_1 and L_2 may be written as

$$(x+1) + 7(y+2) - 5(z+1) = 0$$

$$\Rightarrow x + 7y - 5z + 10 = 0$$

$$= \left| \frac{1 + 7 - 5 + 10}{\sqrt{1 + 49 + 25}} \right| = \frac{13}{\sqrt{75}} \text{units.}$$

Match the Columns

Sol 9: $A \to r$; $B \to q$; $C \to p$; $D \to s$

$$\text{Let } \Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= -\frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

(A) If $a + b + c \neq 0$ and $a^2 + b^2 + c^2$

$\Rightarrow \Delta = 0$ and $a = b = c \neq 0$

The equations represent identical planes.

(B) $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$

$\Rightarrow \Delta = 0$

$\Rightarrow$ the equations have infinitely many solutions.

$ax + by = (a+b)z$

$bx + cy = (b+c)z$

$\Rightarrow (b^2 - ac)y = (b^2 - ac)z \Rightarrow y = z$

$\Rightarrow ax + by + cy = 0$

$\Rightarrow ax = ay$

$\Rightarrow x = y = z.$

(C) $a + b + c \neq 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$

$\Rightarrow \Delta \neq 0$

$\Rightarrow$ The equation represent planes meeting at only one point.

(D) $a + b + c = 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$

$\Rightarrow a = b = c = 0$

$\Rightarrow$ The equations represent whole of the three dimensional space.

Sol 10: (i) Equations of a plane passing through (2, 1, 0) is

$a(x-2)+b(y-1)+c(z)=0$

It also passes through (5, 0, 1) and (4, 1, 1)

$3a - b + c = 0$ and $2a + 0b + c = 0$

On solving, we get $\frac{a}{-1} = \frac{b}{-1} = \frac{c}{2}$

$\therefore$ Equation of plane is

$-(x-2) - (y-1) + 2(z-0) = 0$

$\Rightarrow -x + 2 - y + 1 + 2z = 0$

$\Rightarrow x + y - 2z = 3$

(ii) Let the coordinate of Q (α, β, γ)

Equation of line $PQ = \frac{x-2}{1} = \frac{y-1}{1} = \frac{z-6}{-2}$

Since, mid point of P and Q is $\left(\frac{\alpha+2}{2}, \frac{\beta+1}{2}, \frac{\gamma+6}{2} \right)$.

Which lies in a line P

$$\Rightarrow \frac{\frac{\alpha+2}{2} - 2}{1} = \frac{\frac{\beta+1}{2} - 1}{1} = \frac{\frac{\gamma+6}{2} - 6}{-2}$$

$$= \frac{1\left(\frac{\alpha+2}{2} - 2\right) + 1\left(\frac{\beta+1}{2} - 1\right) - 2\left(\frac{\gamma+6}{2} - 6\right)}{1 \cdot 1 + 1 \cdot 1 + (-2)(-2)} = 2$$

$$\left\{ \text{since,} \left(\frac{\alpha+2}{2}\right) + 1\left(\frac{\beta+1}{2}\right) - 2\left(\frac{\gamma+6}{2}\right) = 3 \right\}$$

$\Rightarrow \alpha = 6, \beta = 5, \gamma = -2$

$\Rightarrow$ Q(6, 5, –2)

Sol 11: Let the equation of the plane ABCD be ax + by + cz + d = 0, the point A″ be (α,β,γ) and the height of the parallelepiped ABCD be h.

$$\Rightarrow \frac{|a\alpha+b\beta+c\gamma+d|}{\sqrt{a^2+b^2+c^2}}=90\%h$$

$$\Rightarrow a\alpha+b\beta+c\gamma+d=\pm 0.9h\sqrt{a^2+b^2+c^2}$$

$\therefore$ Locus is, ax + by + cz + d = $\pm 0.9h\sqrt{a^2+b^2+c^2}$

$\therefore$ Locus of A″ is a plane parallel to the plane ABCD.

Sol 12: (B, C, D) According to given data, we have

$$P(3,0,0),Q(3,3,0),R(0,3,0),S\left(\frac{3}{2},\frac{3}{2},3\right)$$

$$\overrightarrow{OQ}=3\hat{i}+3\hat{j}$$

$$\overrightarrow{OS}=\frac{3}{2}\hat{i}+\frac{3}{2}\hat{j}+3\hat{k}$$

$$\overrightarrow{OQ}.\overrightarrow{OS}=\left|\overrightarrow{OQ}\right|\left|\overrightarrow{OS}\right|\cos\phi$$

$$\frac{9}{2}+\frac{9}{2}=9\sqrt{2}\times\frac{3\sqrt{3}}{\sqrt{2}}\cos\phi \Rightarrow 9=9\sqrt{3}\cos\phi$$

$$\Rightarrow \phi=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

The equation of plane containing ΔOQR is $x-y=0$

The $\perp$ distance of point (3, 0, 0) from the plane $x-y=0$ is given by

$$=\left|\frac{3-0}{\sqrt{2}}\right|=\frac{3}{\sqrt{2}}$$

The equation of RS

Direction ratios of RS $<\frac{-3}{2},\frac{3}{2},-3>$ or $<1,-1,2>$

Equation of line RS $\frac{x}{1}=\frac{y-3}{-1}=\frac{z}{2}=r$

$\Rightarrow$ point on line $(r,3-r,2r)$

$$r+(3-r)(-1)+2(2r)=0 \quad \Rightarrow r-3+r+4r=0$$

$$\Rightarrow r=\frac{1}{2} \Rightarrow \text{point}\left(\frac{1}{2},\frac{5}{2},1\right)$$

Perpendicular distance

$$=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5}{2}\right)^2+1}=\sqrt{\frac{1}{4}+\frac{25}{4}+1}=\sqrt{\frac{30}{4}}=\sqrt{\frac{15}{2}}$$

Sol 13: (C) Let $P^1(3,1,7)$

The imago of P′ given by

$$\frac{x-3}{1}=\frac{y-1}{-1}=\frac{z-7}{1}=-\frac{2(3-1+7-3)}{3}=-4$$

$$\Rightarrow P(x,y,z)\equiv(-1,5,3)$$

Any plane passing through $P(-1,5,3)$ and containing line $\frac{x}{1}=\frac{y}{2}=\frac{z}{1}$

$$\begin{vmatrix} x & y & z \\ -1 & 5 & 3 \\ 1 & 2 & 1 \end{vmatrix}=0$$

$$x(5-6)-y(-1-3)+z(-2-5)=0$$

$$\Rightarrow x-4y+72=0$$

Sol 14: (B, D) Any plane passes through point of intersection of plane P_1 and P_2 is $x+z-1+\lambda y=0$

Given:

$$\left|\frac{0+0-1+\lambda}{\sqrt{1+1+\lambda^2}}\right|=1 \Rightarrow |\lambda-1|=\sqrt{\lambda^2+2} \Rightarrow \lambda=-\frac{1}{2}$$

$\Rightarrow P_3$ is $2x-y+2z=2$

Now, distance of P_3 from (α,β,γ) is 2.

$$\Rightarrow \left|\frac{2\alpha-\beta+2\gamma-2}{\sqrt{4+4+1}}\right|=2$$

$\Rightarrow 2\alpha-\beta+2\gamma=8$ and $2\alpha-\beta+2\gamma=-4$

Sol 15: (A, B) Since all the points on L are at same distance from planes P_1 and P_2 implies that line L is parallel to line of intersection of P_1 and P_2.

Let direction ratio of line L be α, β, γ then

$\alpha+2\beta-\gamma=0$ and $2\alpha-\beta+\gamma=0$

$$\Rightarrow \alpha:\beta:\gamma\equiv 1:-3:-5$$

Eq .of line L passes through origin

$$\frac{x-0}{1}=\frac{y-0}{-3}=\frac{z-0}{-5}=r$$

Foot of perpendicular from origin to the plane $P_1\equiv x+2y-z+1=0$ can be obtained as

$$\frac{x-0}{1}=\frac{y-0}{2}=\frac{z-0}{-1}=\frac{-(0+0-0+1)}{1+4+1}=\frac{-1}{6}$$

$\Rightarrow\left(\frac{-1}{6},\frac{-1}{3},\frac{1}{6}\right)$

Now equation of perpendicular from any point on L is

$$\frac{x+\frac{1}{6}}{1}=\frac{y+\frac{1}{3}}{-3}=\frac{z-\frac{1}{6}}{-5}=\lambda$$

Any point on line $\left(\lambda-\frac{1}{6},-3\lambda-\frac{1}{3},-5\lambda+\frac{1}{6}\right)$

Point $\left(0,\frac{-5}{6},\frac{-2}{3}\right)$ and $\left(\frac{-1}{6},\frac{-1}{3},\frac{1}{6}\right)$ satisfy the line.

Sol 16: (C) Given: $P\equiv(\lambda,\lambda,\lambda)$

$L_1\equiv\frac{x-0}{1}=\frac{y-0}{1}=\frac{z-1}{0}=m$

$L_2\equiv\frac{x-0}{1}=\frac{y-0}{1}=\frac{z-1}{0}=n$

$\Rightarrow Q\equiv(m,m,1)$

$\Rightarrow R\equiv(n,-n,-1)$

$\overrightarrow{PQ}=(\lambda-m)\hat{i}+(\lambda-m)\hat{j}+(\lambda-1)\hat{k}$

Since $\overrightarrow{PQ}$ is perpendicular to L_1

$\Rightarrow\lambda-m+\lambda-m+0=0\ \ \Rightarrow\lambda=m\ \Rightarrow Q(\lambda,\lambda,1)$

Similarly, $R=(0,0,-1)$

Now, $PQ\perp PR$

$\Rightarrow(\lambda-m).(\lambda-n)+(\lambda-m).(\lambda+n)+(\lambda-1).(\lambda+1)=0$

$\Rightarrow 0+0+(\lambda-1).(\lambda+1)=0\ \Rightarrow\lambda=\pm1$

Negotiating $\lambda=1$ became points p and Q will coincide.
$\lambda=-1$

Sol 17: (D) Let any point P on the line $\frac{x+2}{2}=\frac{y-1}{-1}=\frac{z}{3}$ be $(2\gamma-2,-\gamma-1,3\gamma)$

P lies on the plane $x+y+2=3$

$\Rightarrow 2\gamma-2(-\gamma-1)+3\gamma=3\ \Rightarrow 4\gamma=6\ \ \Rightarrow\gamma=\frac{3}{2}$

$P\equiv\left(1,\frac{-5}{2},\frac{9}{2}\right)$

Point $(-2,-1,0)$ lies on the line, the feet of perpendicular Q is given by

$\frac{x+2}{1}=\frac{y+1}{1}=\frac{z-0}{1}=-\frac{(-2-1+0-3)}{1^2+1^2+1^2}$

$\Rightarrow Q\equiv(0,1,2)$

Direction ratio of line PQ joining feets of perpendicular are $\left(1,\frac{-7}{2},\frac{5}{2}\right)$

Equation of PQ $\frac{x}{2}=\frac{y-1}{-7}=\frac{z-2}{5}$

Sol 18: (A, D) Given lines

$L_1\equiv\frac{x-5}{0}=\frac{y-0}{3-\alpha}=\frac{z-0}{-2}$

$L_2\equiv\frac{x-\alpha}{0}=\frac{y}{-1}=\frac{z}{2-\alpha}$

L_1 and L_2 will be co-planar, then

$$\begin{vmatrix}0 & 3-\alpha & 2\\ 0 & -1 & 2-\alpha\\ 5-\alpha & 0 & 0\end{vmatrix}=0$$

$\Rightarrow(5-\alpha)\left[(3-\alpha)(2-\alpha)+2\right]=0$

$\Rightarrow(5-\alpha)(\alpha-1)(\alpha-4)=0$

$\Rightarrow\alpha=1,4,5$

Sol 19: (A) $L_1\equiv\frac{x-1}{2}=\frac{y}{-1}=\frac{z+3}{1}=r_1$

$L_2\equiv\frac{x-y}{1}=\frac{y+3}{1}=\frac{z+3}{2}=r_2$

For point of intersection of L_1 and L_2

$2r_1+1=r_2+4\Rightarrow 2r_1-r_1=3$ ... (i)

$-r_1=r_2-3$ ... (ii)

and $r_1-3=2r_3-3$ ... (iii)

Form (i), (ii), (iii), we get $r_1=2,\ r_2=1$

The point of intersection $(5,-2,-1)$

Now, direction ratio of plane $\perp$ to P_1 and P_2 given by

$$\begin{vmatrix}i & j & k\\ 3 & 5 & -6\\ 7 & 1 & 2\end{vmatrix}$$

$=i(10+6-j(6+42)+k(3-35)$

$=16i-48j-32k$

Any plane passes through $(5,-2,-1)$ and having direction ratio of normal

$16(x-5)-48(y+2)-32(z+1)=0$

$\Rightarrow(x-5)-3(y+2)-2(z+1)=0$

$\Rightarrow x-3y-2z=13$

$\Rightarrow a=1,b=-3,c=-2$ and $d=13$

Sol 20: (A) Given: $Q(2,3,5)$ $R(1,-1,4)$

Direction ratio of line QR is $(1,4,1)$

The eq. of QR

$$\frac{x-2}{1}=\frac{y-3}{4}=\frac{z-5}{1}=r$$

Any point on it $P(r+2,\ 4r+3,\ r+5)$

P lies on the plane $5x-4y-z=1$

$$5(r+2)-4(4r+3)-(r+5)=1$$

$$\Rightarrow 5r+10-16r-12-r-5=1$$

$$\Rightarrow r=-\frac{8}{12}=-\frac{2}{3}$$

$$\Rightarrow P\equiv\left(\frac{4}{3},\frac{1}{3},\frac{13}{3}\right)$$

$$PT=\sqrt{\left(\frac{4}{3}-2\right)^2\left(\frac{1}{3}-1\right)^2+\left(\frac{13}{3}-4\right)^2}$$

Now, direction ration of PT is $(2,\ 2,-1)$

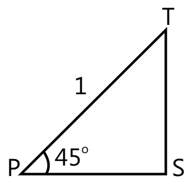
Angle between PT and QR

$$\cos\theta=\frac{1\times2+4\times2+1\times-1}{\sqrt{1+16+1}\sqrt{4+4+1}}$$

$$=\frac{9}{3\sqrt{2}\times3}=\frac{1}{\sqrt{2}}$$

$$\Rightarrow\theta=45^0$$

$$TS=PS=\frac{1}{\sqrt{2}}$$



Sol 21: (A) Eq of plane

$$x+2y+3z-2+k(x-y+z-3)=0$$

$$\Rightarrow x(1+k)+y(2-k)+z(3+k)-2-3k=0$$

Distend from point $(3,1,-1)$ is $\frac{2}{\sqrt{3}}$

$$\left|\frac{3(1+k)+1(2-k)-1(3+k)-(2+3k)}{\sqrt{(1+k)^2+(2-k)^2+(3+k)^2}}\right|$$

On solving, we get $k=\frac{-7}{3}$ $=\frac{2}{\sqrt{3}}$

Eq. of plane is $5x-11y+z=17$

Sol 22: (B, C) The lines $\frac{x-1}{2}=\frac{y+1}{k}=\frac{z}{2}$

and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{k}$

are coplanar, then $\begin{vmatrix}2&0&0\\2&k&2\\5&2&k\end{vmatrix}=0$

$$2(k^2-4)=0\ \Rightarrow k=\pm2$$

For $k=2$, the lines are

$$\frac{x-1}{2}=\frac{y+1}{2}=\frac{z}{2}\ \text{and}\ \frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{2}$$

Clearly plane $y+1=z$ contains both the lines

For $k=-2$, the lines are

$$\frac{x-1}{2}=\frac{y+1}{2}=\frac{z}{2}\ \text{and}\ \frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{-2}$$

From options plane $y+z+1=0$ also contains both the lines.

Sol 23: Let the direction ratio of plane containing both the given lines are a, b, c then

$$2a+3b+4c=0 \text{ and } 3a+4b+5c=0$$

$$\Rightarrow\frac{a}{-1}=\frac{b}{2}=\frac{c}{-1}$$

Now, the equation of plane is

$$a(x-2)+b(y-3)+c(z-4)=0$$

$$\Rightarrow-(x-2)+2(y-3)-(z-4)=0$$

$$\Rightarrow-x+2+2y-6-z+4=0$$

$$\Rightarrow-x+2y-z=0$$

$$\Rightarrow x-2y+z=0$$

Distance between planes

$$\frac{|d-0|}{\sqrt{1+4+1}}=\sqrt{6}\ \Rightarrow|d|=6$$

Sol 24: (A) Distance of point $p(1,-2,1)$

From plane $x+2y-2z=\alpha$ is 5, then

$$\left|\frac{1+2x-2-2\times1-\alpha}{\sqrt{1+4+4}}\right|=5$$

$$|-5-\alpha|=5$$

$$\Rightarrow\alpha=10$$

For foot of perpendicular is M, then

$$\frac{x-1}{1}=\frac{y+2}{2}=\frac{z-1}{-2}=\lambda$$

$M \equiv (\lambda+1,\ 2\lambda-2, -2\lambda+1)$ lies on plane

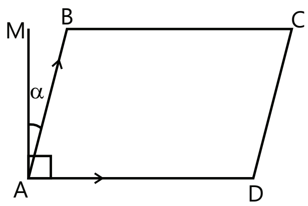
$$\Rightarrow \lambda+1+2(2\lambda-2)-2(2\lambda+1)-10=0$$

$$\Rightarrow \lambda+1+4\lambda-4+4\lambda-2-10=0$$

$$\Rightarrow 9\lambda=15$$

$$\Rightarrow \lambda=\frac{5}{3} \quad \Rightarrow Q\left(\frac{8}{3},\frac{4}{3},\frac{-7}{3}\right)$$

Sol 25: (B) Angle between $\overrightarrow{AB}$ and $\overrightarrow{AD}$



$$\cos\theta\frac{(2i+10j+11k).(-i+2j+2k)}{\sqrt{4+100+121}\sqrt{1+4+4}}=\frac{40}{15\times 3}=\frac{8}{9}$$

$$\Rightarrow \alpha=\frac{\pi}{2}-\cos-1\left(\frac{8}{9}\right)=\sin^{-1}\left(\frac{8}{9}\right)=\cos^{-1}\left(\frac{\sqrt{17}}{9}\right)$$

$$\Rightarrow \cos\alpha=\left(\frac{\sqrt{17}}{9}\right)$$