11 Trigonometrical Ratios and Equations

QUICK LOOK

Relation between an Arc and an Angle

If s is the length of an arc of a circle of radius r, then the angle θ (in radians) subtended by this arc at the centre of the circle is given by $\theta = \frac{s}{r}$ or $s = r\theta$ *i.e.*, arc = radius × angle in radians

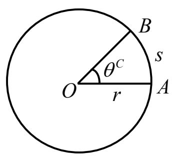


Figure: 11.1

Sectorial Area: Let OAB be a sector having central angle θ^C and radius r. Then area of the sector OAB is given by $\frac{1}{2}r^2\theta$.

Note

- The angle between two consecutive digits in a clock is 30º (= $\pi/6$ radians). The hour hand rotates through an angle of 30º in one hour.
- The minute hand rotate through an angle of 6º in one minute.

Trigonometrical Ratios or Functions

In the right angled triangle OMP, we have base = OM = x, perpendicular = PM = y and hypotenues = OP =r. We define the following trigonometric ratio which are also known as trigonometric function.

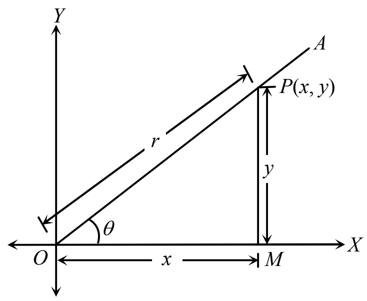


Figure: 11.2

$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenues}} = \frac{y}{r}$ $\cos\theta = \frac{\text{Base}}{\text{Hypotenues}} = \frac{x}{r}$

$\tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{y}{x}$ $\cot\theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{x}{y}$,

$\sec\theta = \frac{\text{Hypotenues}}{\text{Base}} = \frac{r}{x}$ $\text{cosec}\,\theta = \frac{\text{Hypotenues}}{\text{Perpendicular}} = \frac{r}{y}$

Relation between Trigonometric Ratio (function)

- $\sin\theta.cosec\,\theta = 1$
- $\tan\theta.\cot\theta = 1$
- $\cos\theta.\sec\theta = 1$
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\cot\theta = \frac{\cos\theta}{\sin\theta}$

sin x, cos x and tan x are (trigonometrical) circular functions. Their reciprocals cosec x, sec x and cot x are also circular functions. These functions are related by the following identities.

- $\sin^2 x + \cos^2 x = 1$
- $\sin^2 x = 1 - \cos^2 x$
- $\cos^2 x = 1 - \sin^2 x$
- $\sec^2 x - \tan^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $\sec^2 x - 1 = \tan^2 x$
- $\text{cosec}^2 x - \cot^2 x = 1$
- $1 + \cot^2 x = \text{cosec}^2 x$
- $\text{cosec}^2 x - 1 = \cot^2 x$
- $\sec x + \tan x = \frac{1}{\sec x - \tan x}$
- $\text{cosec}\,x + \cot x = \frac{1}{\cos ec\,x - \cot x}$

Sign of Trigonometrical Ratios or Functions: Their signs depends on the quadrant in which the terminal side of the angle lies.

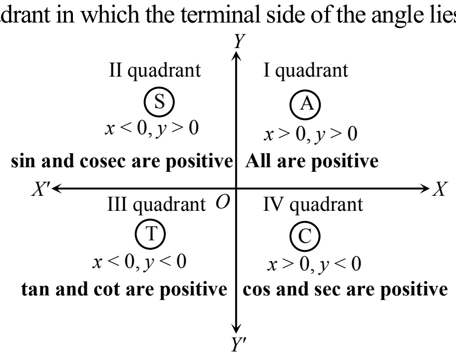


Figure: 11.3

In First Quadrant

$x > 0, y > 0 \Rightarrow \sin\theta = \frac{y}{r} > 0, \cos\theta = \frac{x}{r} > 0, \tan\theta = \frac{y}{x} > 0,$

$\text{cosec}\,\theta = \frac{r}{y} > 0,$ $\sec\theta = \frac{r}{x} > 0$ and $\cot\theta = \frac{x}{y} > 0.$

Thus, in the first quadrant all trigonometric functions are positive.

In Second Quadrant

$x<0, y>0 \Rightarrow \sin\theta=\frac{y}{r}>0, \cos\theta=\frac{x}{r}<0,\ \tan\theta=\frac{y}{x}<0,$

$\operatorname{cosec}\theta=\frac{r}{y}>0,\ \sec\theta=\frac{r}{x}<0$ and $\cot\theta=\frac{x}{y}<0.$

Thus, in the second quadrant sin and cosec function are positive and all others are negative.

In Third Quadrant

$x<0, y<0 \Rightarrow \sin\theta=\frac{y}{r}<0, \cos\theta=\frac{x}{r}<0, \tan\theta=\frac{y}{x}>0,$

$\operatorname{cosec}\theta=\frac{r}{y}<0,\ \sec\theta=\frac{r}{x}<0$ and $\cot\theta=\frac{x}{y}>0.$

Thus, in the third quadrant all trigonometric functions are negative except tangent and cotangent.

In fourth quadrant

$x>0, y<0 \Rightarrow \sin\theta=\frac{y}{r}<0,\quad \cos\theta=\frac{x}{r}>0,\quad \tan\theta=\frac{y}{x}<0,$

$\operatorname{cosec}\theta=\frac{r}{y}<0,\ \sec\theta=\frac{r}{x}>0$ and $\cot\theta=\frac{x}{y}<0.$

Thus, in the fourth quadrant all trigonometric functions are negative except cos and sec.

In brief: A crude aid to memorise the signs of trigonometrical ratio in different quadrant. "Add Sugar To Coffee".

Note

- First determine the sign of the trigonometric function.
- If θ is measured from $X'OX$ i.e., $\{(\pi \pm \theta,\ 2\pi-\theta)\}$ then retain the original name of the function.
- If θ is measured from $Y'OY$ i.e., $\left\{\frac{\pi}{2}\pm\theta, \frac{3\pi}{2}\pm\theta\right\}$, then change sine to cosine, cosine to sine, tangent to cotangent, cot to tan, sec to cosec and cosec to sec.

Variations in Values of Trigonometric Functions in Different Quadrants: Let $X'OX$ and YOY' be the coordinate axes. Draw a circle with centre at origin O and radius unity.

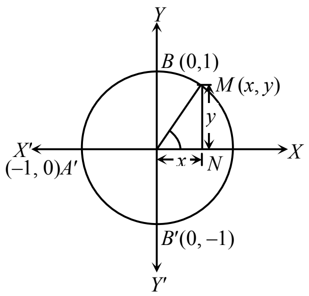


Figure: 11.4

Let $M(x, y)$ be a point on the circle such that $\angle AOM=\theta$ then $x=\cos\theta$ and $y=\sin\theta$; $-1\le\cos\theta\le 1$ and $-1\le\sin\theta\le 1$ for all values of θ.

Table 11.1: Values of Trigonometric Functions

II-Quadrant (S)	**I-Quadrant (A)**
$\sin\theta\rightarrow$ decreases from 1 to 0	$\sin\theta\rightarrow$ increases from 0 to 1
$\cos\theta\rightarrow$ decreases from 0 to – 1	$\cos\theta\rightarrow$ decreases from 1 to 0
$\tan\theta\rightarrow$ increases from $-\infty$ to 0	$\tan\theta\rightarrow$ increases from 0 to ∞
$\cot\theta\rightarrow$ decreases from 0 to $-\infty$	$\cot\theta\rightarrow$ decreases from ∞ to 0
$\sec\theta\rightarrow$ increases from $-\infty$ to – 1	$\sec\theta\rightarrow$ increases from 1 to ∞
$\operatorname{cosec}\theta\rightarrow$ increases from 1 to ∞	$\operatorname{cosec}\theta\rightarrow$ decreases from ∞ to 1
III-Quadrant (T)	**IV-Quadrant (C)**
$\sin\theta\rightarrow$ decreases from 0 to – 1	$\sin\theta\rightarrow$ increases from – 1 to 0
$\cos\theta\rightarrow$ increases from – 1 to 0	$\cos\theta\rightarrow$ increases from 0 to 1
$\tan\theta\rightarrow$ increases from 0 to ∞	$\tan\theta\rightarrow$ increases from $-\infty$ to 0
$\cot\theta\rightarrow$ decreases from ∞ to 0	$\cot\theta\rightarrow$ decreases from 0 to $-\infty$
$\sec\theta\rightarrow$ decreases from – 1 to $-\infty$	$\sec\theta\rightarrow$ decreases from ∞ to 1
$\operatorname{cosec}\theta\rightarrow$ increases from $-\infty$ to – 1	$\operatorname{cosec}\theta\rightarrow$ decreases from – 1 to $-\infty$

Note

$+\infty$ and $-\infty$ are two symbols. These are not real number. When we say that $\tan\theta$ increases from 0 to ∞ for as θ varies from 0 to $\frac{\pi}{2}$ it means that $\tan\theta$ increases in the interval $\left(0, \frac{\pi}{2}\right)$ and it attains large positive values as θ tends to $\frac{\pi}{2}$. Similarly for other trigonometric functions.

Table 11.2: Domain and Range of Circular Function

Function	**Domain**	**Range**
$\sin x$	R	[–1, 1]
$\cos x$	R	[–1, 1]
$\tan x$	$R-\left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\right\}$	$(-\infty, \infty)$
$\operatorname{cosec} x$	$R-\{0, \pi, 2\pi, 3\pi, \ldots\}$	$(-\infty, -] \cup [1, +\infty)$
$\sec x$	$R-\left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\right\}$	$(-\infty, -1) \cup [1, +\infty)$
$\cot x$	$R-\{0, \pi, 2\pi, 3\pi, \ldots\}$	$(-\infty, \infty)$

Note

For values of circular functions, the angle $x=-\frac{\pi}{2}$ and the angle $x=\frac{3\pi}{2}$ are the same angle.

Sign of Value

- $\sin x \ge 0$ if $0 \le x \le \pi$
- $\sin x < 0$ if $\pi < x < 2\pi$

- $\cos x \geq 0 \; if \; -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
- $\cos x < 0 \; if \; \frac{\pi}{2} < x < \frac{3\pi}{2}$
- $\tan x \geq 0 \; if \; 0 \leq x < \frac{\pi}{2} \; or \; \pi \leq x < \frac{3\pi}{2}$
- $\tan x < 0 \; if \; \frac{\pi}{2} < x < \pi \; or \; \frac{3\pi}{2} < x < 2\pi$

Table 11.3: Trend of values of circular functions in [0, 2π] or [−π, π]

Function	Domain of Gradual Increase	Domain of Gradual Decrease
$\sin x$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
	i.e., $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$	
$\cos x$	$[-\pi, 0]$ i.e., $[\pi, 2\pi]$	$[0, \pi]$
$\tan x$	$\left[0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right]$	No gradual decrease; sudden fall of value $\left[0, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, 2\pi\right]$ at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$

Pictorically the result can be remembered as follows:

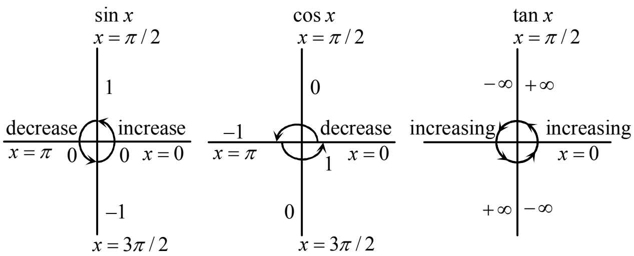


Figure: 11.5

Conversion of Circular Functions of Complementary Angles, Supplementary Angles, etc

- If $f(x)$ is a circular function then $f(2n\pi + x) = f(x)$ where n is an integer. That is why all circular functions are periodic functions.
- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = -\cos\theta$
- $\tan(-\theta) = -\tan\theta$
- $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$
- $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$
- $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$
- $\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$
- $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$
- $\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$
- $\sin(\pi - \theta) = \sin\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\tan(\pi - \theta) = -\tan\theta$
- $\sin(\pi + \theta) = -\sin\theta$
- $\cos(\pi + \theta) = -\cos\theta$
- $\tan(\pi + \theta) = \tan\theta$

Table 11.4: Trigonometric Ratios and Allied Angles

Allied angles	Trigo. Ratio		
	sinθ	cosθ	tan θ
$(-\theta)$	– sinθ	cosθ	– tanθ
(90 – θ) or $\left(\frac{\pi}{2} - \theta\right)$	cosθ	sinθ	cotθ
$(90 + \theta)$ or $\left(\frac{\pi}{2} + \theta\right)$	cosθ	– sinθ	– cotθ
$(180 - \theta)$ or $(\pi - \theta)$	sinθ	– cosθ	– tanθ
$(180 + \theta)$ or $(\pi + \theta)$	– sinθ	– cosθ	tanθ
$(270 - \theta)$ or $\left(\frac{3\pi}{2} - \theta\right)$	– cosθ	– sinθ	cotθ
$(270 + \theta)$ or $\left(\frac{3\pi}{2} + \theta\right)$	– cos θ	sinθ	– cotθ
$(360 - \theta)$ or $(2\pi - \theta)$	– sinθ	cosθ	– tanθ

Note

- $\sin n\pi = 0, \; \cos n\pi = (-1)^n$
- $\sin(n\pi + \theta) = (-1)^n \sin\theta, \; \cos(n\pi + \theta) = (-1)^n \cos\theta$
- $\sin\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n-1}{2}} \cos\theta$, if n is odd

  $= (-1)^{n/2} \sin\theta$, if n is even
- $\cos\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n+1}{2}} \sin\theta$, if n is odd

  $= (-1)^{n/2} \cos\theta$, if n is even

Table 11.5: Trigonometrical Ratios for Various Angles

θ	0	π/6	π/4	π/3	π/2	π	3π/2	2π
sinθ	0	½	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	–1	0
cosθ	1	$\sqrt{3}/2$	$1/\sqrt{2}$	½	0	–1	0	1
tanθ	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	0	∞	0

Table 11.6: Trigonometrical Ratios in Terms of Each Other

	sinθ	**cosθ**	**tanθ**
sinθ	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$
cosθ	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\frac{1}{\sqrt{1+\tan^2\theta}}$
tanθ	$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$
cot θ	$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$
secθ	$\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$
cosecθ	$\frac{1}{\sin\theta}$	$\frac{1}{\sqrt{1-\cos^2\theta}}$	$\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$
	cotθ	**secθ**	**cosecθ**
sinθ	$\frac{1}{\sqrt{1+\cot^2\theta}}$	$\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\frac{1}{cosec\theta}$
cosθ	$\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{\text{cosec}^2\theta-1}}{\text{cosec}\theta}$
tanθ	$\frac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\text{cosec}^2\theta-1}}$
cot θ	$\cot\theta$	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\text{cosec}^2\theta-1}$
secθ	$\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\frac{\text{cosec}\theta}{\sqrt{\text{cosec}^2\theta-1}}$
cosecθ	$\sqrt{1+\cot^2\theta}$	$\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\text{cosec}\theta$

Note

Values for some standard angles

- $\sin 15^\circ = \cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$; $\cos 15^\circ = \sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$;
- $\tan 15^\circ = \cot 75^\circ = 2-\sqrt{3}$;
- $\sin 18^\circ = \cos 72^\circ = \frac{\sqrt{5}-1}{4}$;
- $\cos 36^\circ = \sin 54^\circ = \frac{\sqrt{5}+1}{4}$;
- $\tan 75^\circ = \cot 15^\circ = 2+\sqrt{3}$
- $\sin 22\frac{1}{2}^\circ = \cos 67\frac{1}{2}^\circ = \frac{\sqrt{2-\sqrt{2}}}{2}$,
- $\cos 22\frac{1}{2}^\circ = \sin 67\frac{1}{2}^\circ = \frac{\sqrt{2+\sqrt{2}}}{2}$;
- $\cot\ 22\frac{1}{2}^\circ = \tan 67\frac{1}{2}^\circ = \sqrt{2}+1$
- $\tan 22\frac{1}{2}^\circ = \cot 67\frac{1}{2}^\circ = \sqrt{2}-1$

Formulae for the Trigonometric Ratios of Sum and Differences of Two Angles

- $\sin(A+B) = \sin A\cos B + \cos A\sin B$
- $\sin(A-B) = \sin A\cos B - \cos A\sin B$
- $\cos(A+B) = \cos A\cos B - \sin A\sin B$
- $\cos(A-B) = \cos A\cos B + \sin A\sin B$
- $\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}$
- $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$
- $\cot(A+B) = \frac{\cot A\cot B - 1}{\cot A + \cot B}$
- $\cot(A-B) = \frac{\cot A\cot B + 1}{\cot B - \cot A}$
- $\sin(A+B).\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
- $\cos(A+B).\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
- $\tan A \pm \tan B = \frac{\sin A}{\cos A} \pm \frac{\sin B}{\cos B}$

  $= \frac{\sin A\cos B \pm \cos A\sin B}{\cos A\cos B} = \frac{\sin(A\pm B)}{\cos A.\cos B}$

  $\left(A \neq n\pi + \frac{\pi}{2}, B \neq m\pi\right)$
- $\cot A \pm \cot B = \frac{\sin(B\pm A)}{\sin A.\sin B}\left(A \neq n\pi, B \neq m\pi + \frac{\pi}{2}\right)$

Formulae for the Trigonometric Ratios of Sum and Differences of Three Angles

- $\sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C +$

  $\cos A\cos B\sin C - \sin A\sin B\sin C$

or $\sin(A+B+C) =$

  $\cos A\cos B\cos C(\tan A + \tan B + \tan C - \tan A.\tan B.\tan C)$
- $\cos(A+B+C) = \cos A\cos B\cos C - \sin A\sin B\cos C -$

  $\sin A\cos B\sin C - \cos A\sin B\sin C$

  $\cos(A+B+C) = \cos A\cos B\cos$

  $C(1-\tan A\tan B - \tan B\tan C - \tan C\tan A)$
- $\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1-\tan A\tan B - \tan B\tan C - \tan C\tan A}$
- $\cot(A+B+C) = \frac{\cot A\cot B\cot C - \cot A - \cot B - \cot C}{\cot A\cot B + \cot B\cot C + \cot C.\cot A - 1}$

In General

- $\sin(A_1 + A_2 + ... + A_n) =$

  $\cos A_1 \cos A_2 .....\cos A_n (S_1 - S_3 + S_5 - S_7 + ...)$

- $\cos(A_1 + A_2 + ... + A_n) =$
  $\cos A_1 \cos A_2 ... \cos A_n (1 - S_2 + S_4 - S_6 ....)$
- $\tan(A_1 + A_2 + ... + A_n) = \dfrac{S_1 - S_3 + S_5 - S_7 + ...}{1 - S_2 + S_4 - S_6 + ...}$

  Where; $S_1 = \tan A_1 + \tan A_2 + ... + \tan A_n =$
  The sum of the tangents of the separate angles
  $S_2 = \tan A_1 \tan A_2 + \tan A_1 \tan A_3 + ... =$
  The sum of the tangents taken two at a time
  $S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + ... =$
  Sum of tangents three at a time, and so on
  If $A_1 = A_2 = ... = A_n = A$, then $S_1 = n \tan A$,
  $S_2 = {}^nC_2 \tan^2 A,\ S_3 = {}^nC_3 \tan^3 A, ...$
- $\sin nA = \cos^n A({}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A - ....)$
- $\cos nA = \cos^n A(1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A - ...)$
- $\tan nA = \dfrac{{}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A - ....}{1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A - {}^nC_6 \tan^6 A + ...}$
- $\sin nA + \cos nA = \cos^n A(1 + {}^nC_1 \tan A - {}^nC_2 \tan^2 A - {}^nC_3 \tan^3 A + {}^nC_4 \tan^4 A + {}^nC_5 \tan^5 A - {}^nC_6 \tan^6 A - .....)$
- $\sin nA - \cos nA = \cos^n A(-1 + {}^nC_1 \tan A + {}^nC_2 \tan^2 A - {}^nC_3 \tan^3 A - {}^nC_4 \tan^4 A + {}^nC_5 \tan^5 A + {}^nC_6 \tan^6 A ...)$
- $\sin(\alpha) + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + ... + \sin(\alpha + (n-1)\beta)$
  $= \dfrac{\sin\{\alpha + (n-1)(\beta/2)\}.\sin(n\beta/2)}{\sin(\beta/2)}$
- $\cos(\alpha) + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + ... + \cos(\alpha + (n-1)\beta)$
  $= \dfrac{\cos\left\{\alpha + (n-1)\left(\frac{\beta}{2}\right)\right\}.\sin\left\{n\left(\frac{\beta}{2}\right)\right\}}{\sin\left(\frac{\beta}{2}\right)}$

Trigonometric Ratio of Multiple and Sub-multiple of an Angle

Trigonometric Ratio of Multiple

- $\sin 2A = 2 \sin A \cos A = \dfrac{2 \tan A}{1 + \tan^2 A}$
- $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$
  $= \cos^2 A - \sin^2 A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$; where $A \neq (2n+1)\dfrac{\pi}{4}$.
- $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$
- $\sin 3A = 3\sin A - 4\sin^3 A = 4\sin(60^o - A).\sin A.\sin(60^o + A)$
- $\cos 3A = 4\cos^3 A - 3\cos A$
  $= 4\cos(60^o - A).\cos A.\cos(60^o + A)$
- $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} = \tan(60^o - A).\tan A.\tan(60^o + A)$,
  where $A \neq n\pi + \pi/6$
- $\sin 4\theta = 4\sin\theta.\cos^3\theta - 4\cos\theta\sin^3\theta$
- $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$
- $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$
- $\sin 5A = 16\sin^5 A - 20\sin^3 A + 5\sin A$
- $\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A$

Trigonometric Ratio of Sub-Multiple of an Angle

- $\left|\sin\dfrac{A}{2} + \cos\dfrac{A}{2}\right| = \sqrt{1 + \sin A}$

or $\sin\dfrac{A}{2} + \cos\dfrac{A}{2} = \pm\sqrt{1 + \sin A}$

i.e., $\begin{cases} +, \text{If } 2n\pi - \pi/4 \leq A/2 \leq 2n\pi + \dfrac{3\pi}{4} \\ -, \text{otherwise} \end{cases}$

- $\left|\sin\dfrac{A}{2} - \cos\dfrac{A}{2}\right| = \sqrt{1 - \sin A}$

or $(\sin\dfrac{A}{2} - \cos\dfrac{A}{2}) = \pm\sqrt{1 - \sin A}$

i.e., $\begin{cases} +, \text{If } 2n\pi + \pi/4 \leq A/2 \leq 2n\pi + \dfrac{5\pi}{4} \\ -, \text{otherwise} \end{cases}$

- $\tan\dfrac{A}{2} = \dfrac{\pm\sqrt{\tan^2 A + 1} - 1}{\tan A} = \pm\sqrt{\dfrac{1 - \cos A}{1 + \cos A}} = \dfrac{1 - \cos A}{\sin A}$,
  where $A \neq (2n+1)\pi$
- $\cot\dfrac{A}{2} = \pm\sqrt{\dfrac{1 + \cos A}{1 - \cos A}} = \dfrac{1 + \cos A}{\sin A}$,
  where $A \neq 2n\pi$
- $\tan^2\dfrac{A}{2} = \dfrac{1 - \cos A}{1 + \cos A}$;
  where $A \neq (2n+1)\pi$
- $\cot^2\dfrac{A}{2} = \dfrac{1 + \cos A}{1 - \cos A}$;
  where $A \neq 2n\pi$

  The ambiguities of signs are removed by locating the quadrants in which $\dfrac{A}{2}$ lies or you can follow the following figure,

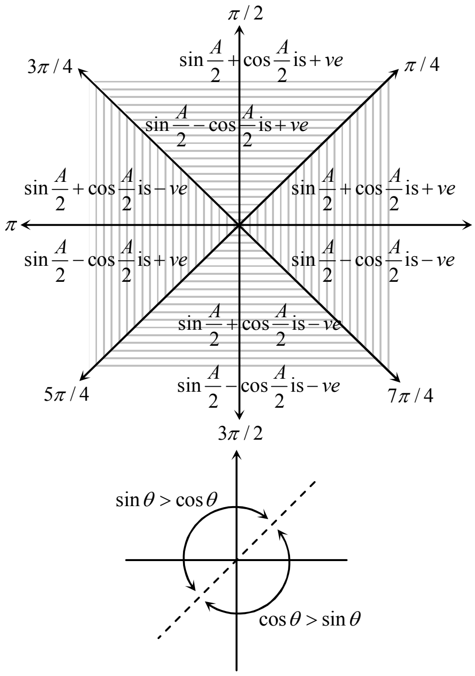


Figure: 11.6

Note

- Any formula that gives the value of $\sin\frac{A}{2}$ in terms of sin A shall also give the value of sine of $\frac{n\pi+(-1)^n A}{2}$.
- Any formula that gives the value of $\cos\frac{A}{2}$ in terms of cos A shall also give the value of cos of $\frac{2n\pi \pm A}{2}$.
- Any formula that gives the value of $\tan\frac{A}{2}$ in terms of tan A shall also give the value of tan of $\frac{n\pi \pm A}{2}$.

Maximum and Minimum Value of *a* cosθ+ *b* sin θ

Let $a = r\cos\alpha$ . . . (*i*)

and $b = r\sin\alpha$ . . . (*ii*)

Squaring and adding (*i*) and (*ii*), then $a^2+b^2=r^2$

or $r=\sqrt{a^2+b^2}$

$\therefore$ $a\sin\theta+b\cos\theta = r(\sin\theta\cos\alpha+\cos\theta\sin\alpha)$

$= r\sin(\theta+\alpha)$

But $-1\le\sin\theta<1$

So, $-1\le\sin(\theta+\alpha)\le 1$;

Then $-r\le r\sin(\theta+\alpha)\le r$

Hence, $-\sqrt{a^2+b^2}\le a\sin\theta+b\cos\theta\le\sqrt{a^2+b^2}$

Then the greatest and least values of $a\sin\theta+b\cos\theta$ are respectively $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$.

- $\sin^2 x+\text{cosec}^2 x\ge 2$, for every real x.
- $\cos^2 x+\sec^2 x\ge 2$, for every real x.
- $\tan^2 x+\cot^2 x\ge 2$, for every real x.

Note: Use of Σ (Sigma) and ∏ (Pie) notation

- $\sin(A+B+C)=\Sigma\sin A\cos B\cos C-\Pi\sin A$,

  $\cos(A+B+C)=\Pi\cos A-\Sigma\cos A\sin B\sin C$,

  $\tan(A+B+C)=\frac{\Sigma\tan A-\Pi\tan A}{1-\Sigma\tan A\tan B}$. ($\because$ Σ denotes summation)
- $\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+.........n$ terms

  ($\because$ ∏ denotes product)

  $=\frac{\sin[\alpha+(n-1)\beta/2]\sin[n\beta/2]}{\sin(\beta/2)}$

  or $\sum_{r=1}^{n}\sin(A+\overline{r-1}B)=\frac{\sin\left(A+\frac{n-1}{2}B\right)\sin\frac{nB}{2}}{\sin\frac{B}{2}}$.
- $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+.........n$ terms $=\frac{\cos[\alpha+(n-1)\beta/2]\sin[n\beta/2]}{\sin[\beta/2]}$

  or $\sum_{r=1}^{n}\cos(A+\overline{r-1}B)=\frac{\cos\left(A+\frac{n-1}{2}B\right)\sin\frac{nB}{2}}{\sin\frac{B}{2}}$
- $\sin A/2\pm\cos A/2=\sqrt{2}\sin\left[\pi/4\pm A\right]=\sqrt{2}\cos\left[A\mp\pi/4\right]$.
- $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)$

  $=4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$.
- $\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)$

  $=4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}$
- $\tan\alpha+2\tan 2\alpha+4\tan 4\alpha+8\cot 8\alpha=\cot\alpha$.

Conditional Trigonometrical Identities

We have certain trigonometric identities.

Like, $\sin^2\theta+\cos^2\theta=1$ and $1+\tan^2\theta=\sec^2\theta$ etc.

Such identities are identities in the sense that they hold for all value of the angles which satisfy the given condition among them and they are called conditional identities.

If *A, B, C* denote the anlges of a triangle *ABC*, then the relation *A* + *B* + *C* = π enables us to establish many important identities involving trigonometric ratios of these angles.

- If $A+B+C=\pi$, then $A+B=\pi-C, B+C=\pi-A$ and $C+A=\pi-B$.
- If $A+B+C=\pi$, then $\sin(A+B)=\sin(\pi-C)=\sin C$

  Similarly, $\sin(B+C)=\sin(\pi-A)=\sin A$

  and $\sin(C+A)=\sin(\pi-B)=\sin B$
- If $A+B+C=\pi$, then $\cos(A+B)=\cos(\pi-C)=-\cos C$

  Similarly, $\cos(B+C)=\cos(\pi-A)=-\cos A$

  and $\cos(C+A)=\cos(\pi-B)=-\cos B$
- If $A+B+C=\pi$, then $\tan(A+B)=\tan(\pi-C)=-\tan C$

  Similarly, $\tan(B+C)=\tan(\pi-A)=-\tan A$

  and $\tan(C+A)=\tan(\pi-B)=-\tan B$
- If $A+B+C=\pi$, then $\frac{A+B}{2}=\frac{\pi}{2}-\frac{C}{2}$

  and $\frac{B+C}{2}=\frac{\pi}{2}-\frac{A}{2}$ and $\frac{C+A}{2}=\frac{\pi}{2}-\frac{B}{2}$

  $\sin\left(\frac{A+B}{2}\right)=\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cos\left(\frac{C}{2}\right)$,

  $\cos\left(\frac{A+B}{2}\right)=\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)=\sin\left(\frac{C}{2}\right)$,

  $\tan\left(\frac{A+B}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cot\left(\frac{C}{2}\right)$

  All problems on conditional identities are broadly divided into the following three types

Case (i): **Identities involving sine and cosine of the multiple or sub-multiple of the angles involved**

Working Method

Step (i): Use $C \pm D$ formulae.

Step (ii): Use the given relation $(A+B+C=\pi)$ in the expression obtained in step-(i) such that a factor can be taken common after using multiple angles formulae in the remaining term.

Step (iii): Take the common factor outside.

Step (iv): Again use the given relation $(A+B+C=\pi)$ within the bracket in such a manner so that we can apply $C \pm D$ formulae.

Step (v): Find the result according to the given options.

Case (ii): **Identities involving squares of sine and cosine of multiple or sub-multiples of the angles involved**

Working Method

Step (i): Arrange the terms of the identity such that either $\sin^2 A-\sin^2 B=\sin(A+B).\sin(A-B)$

or $\cos^2 A-\sin^2 B=\cos(A+B).\cos(A-B)$ can be used.

Step (ii): Take the common factor outside.

Step (iii): Use the given relation $(A+B+C=\pi)$ within the bracket in such a manner so that we can apply $C \pm D$ formulae.

Step (iv): Find the result according to the given options.

Case (iii): **Identities for tangent and cotangent of the angles**

Working Method

Step (i): Express the sum of the two angles in terms of third angle by using the given relation $(A+B+C=\pi)$.

Step (ii): Taking tangent or cotangent of the angles of both the sides.

Step (iii): Use sum and difference formulae in the left hand side.

Step (iv): Use cross multiplication in the expression obtained in the step (iii).

Step (v): Arrange the terms as per the result required.

Trigonometrical Equations and General Solution of Standard Equations

An equation involving one or more trigonometrical ratio of an unknown angle is called a trigonometrical equation *i.e.*, $\sin x+\cos 2x=1$, $(1-\tan\theta)(1+\sin 2\theta)=1+\tan\theta$;

$$|\sec\left(\theta+\frac{\pi}{4}\right)|=2 \text{ etc.}$$

A trigonometric equation is different from a trigonometrical identities. An identity is satisfied for every value of the unknown angle *e.g.*, $\cos^2 x=1-\sin^2 x$ is true $\forall x \in R$ while a trigonometric equation is satisfied for some particular values of the unknown angle.

Roots of Trigonometrical Equation: The value of unknown angle (a variable quantity) which satisfies the given equation is called the root of an equation *e.g.*, $\cos\theta=\frac{1}{2}$, the root is $\theta=60°$ or $\theta=300°$ because the equation is satisfied if we put $\theta=60°$ or $\theta=300°$.

Solution of Trigonometrical Equations: A value of the unknown angle which satisfies the trigonometrical equation is called its solution.

Since all trigonometrical ratios are periodic in nature, generally a trigonometrical equation has more than one solution or an infinite number of solutions. There are basically three types of solutions:

- **Particular Solution:** A specific value of unknown angle satisfying the equation.

- **Principal Solution:** Smallest numerical value of the unknown angle satisfying the equation (Numerically smallest particular solution.)
- **General Solution:** Complete set of values of the unknown angle satisfying the equation. It contains all particular solutions as well as principal solutions.
  When we have two numerically equal smallest unknown angles, preference is given to the positive value in writing the principal solution.

e.g., $\sec\theta = \frac{2}{\sqrt{3}}$ has $\frac{\pi}{6}, -\frac{\pi}{6}, \frac{11\pi}{6}, -\frac{11\pi}{6}, \frac{23\pi}{6}, -\frac{23\pi}{6}$ etc.

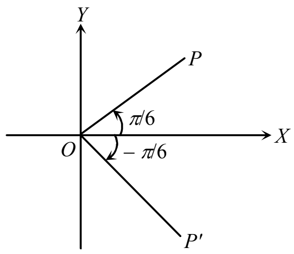


Figure: 11.7

As its particular solutions out of these, the numerically smallest are $\frac{\pi}{6}$ and $-\frac{\pi}{6}$ but the principal solution is taken as $\theta = \frac{\pi}{6}$ to write the general solution we notice that the position on P or P' can be obtained by rotation of OP or OP' around O through a complete angle (2π) any number of times and in any direction (clockwise or anticlockwise)

$\therefore$ The general solution is $\theta = 2k\pi \pm \frac{\pi}{6},\ k \in Z$.

General Solution of the Equation sin θ= sinα:

If $\sin\theta = \sin\alpha$ or $\sin\theta - \sin\alpha = 0$

or, $2\sin\left(\frac{\theta-\alpha}{2}\right)\cos\left(\frac{\theta+\alpha}{2}\right) = 0$

$\Rightarrow \sin\left(\frac{\theta-\alpha}{2}\right) = 0$ or $\cos\left(\frac{\theta+\alpha}{2}\right) = 0$

or, $\frac{\theta-\alpha}{2} = m\pi; m \in I$

or $\frac{\theta+\alpha}{2} = (2m+1)\frac{\pi}{2}; m \in I$

$\Rightarrow \theta = 2m\pi + \alpha; m \in I$

or $\theta = (2m+1)\pi - \alpha; m \in I$

$\Rightarrow \theta$= (any even multiple of π) + α

or θ= (any odd multiple of π) – α

$\theta = n\pi + (-1)^n \alpha;\ n \in I$

- The equation $\text{cosec}\,\theta = \text{cosec}\,\alpha$ s equivalent to $\sin\theta = \sin\alpha$. So these two equation having the same general solution.

General solution of the equation cosθ = cosα :

If $\cos\theta = \cos\alpha$

$\Rightarrow \cos\theta - \cos\alpha = 0$

$\Rightarrow -2\sin\left(\frac{\theta+\alpha}{2}\right).\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Rightarrow \sin\left(\frac{\theta+\alpha}{2}\right) = 0$ or $\sin\left(\frac{\theta-\alpha}{2}\right) = 0,$

$\Rightarrow \frac{\theta+\alpha}{2} = n\pi; n \in I$ or $\frac{\theta-\alpha}{2} = n\pi; n \in I$

$\Rightarrow \theta = 2n\pi - \alpha; n \in I$ or $\theta = 2n\pi + \alpha; n \in I.$

For the general solution of $\cos\theta = \cos\alpha$, combine these two result which gives $\theta = 2n\pi \pm \alpha;\ n \in I$

- The equation $\sec\theta = \sec\alpha$ is equivalent to $\cos\theta = \cos\alpha$, so the general solution of these two equations are same.

General solution of the equation tan θ = tan α

If $\tan\theta = \tan\alpha \Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$

$\Rightarrow \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0$

$\Rightarrow \sin(\theta - \alpha) = 0$

$\Rightarrow \theta - \alpha = n\pi; n \in I$

$\Rightarrow \theta = n\pi + \alpha;\ n \in I$

- The equation $\cot\theta = \cot\alpha$ is equivalent to $\tan\theta = \tan\alpha$ so these two equations having the same general solution.

General Solution of Some Particular Equations

- $\sin\theta = 0 \Rightarrow \theta = n\pi,\ \cos\theta = 0$

$\Rightarrow \theta = (2n+1)\frac{\pi}{2}$ or $n\pi + \frac{\pi}{2},\ \tan\theta = 0$

$\Rightarrow \theta = n\pi$

- $\sin\theta = 1 \Rightarrow \theta = (4n+1)\frac{\pi}{2}$

or $2n\pi + \frac{\pi}{2},\ \cos\theta = 1$

$\Rightarrow \theta = 2n\pi,\ \tan\theta = 1$

$\Rightarrow \theta = (4n+1)\frac{\pi}{4}$ or $n\pi + \frac{\pi}{4}$

- $\sin\theta = -1 \Rightarrow \theta = (4n+3)\frac{\pi}{2}$

or $2n\pi + \frac{3\pi}{2},\ \cos\theta = -1$

$\Rightarrow \theta = (2n+1)\pi,\ \tan\theta = -1$

$\Rightarrow \theta = (4n-1)\frac{\pi}{4}$ or $n\pi - \frac{\pi}{4}$

- $\tan\theta =$ not defined

$\Rightarrow\ \theta = (2n+1)\frac{\pi}{2},\ \cot\theta =$ not defined

$\Rightarrow\ \theta = n\pi\ \ \text{cosec}\,\theta =$ not defined

$\Rightarrow\ \theta = n\pi,\ \ \sec\theta =$ not defined

$\Rightarrow\ \theta = (2n+1)\frac{\pi}{2}.$

Note

- For equations involving two multiple angles, use multiple and sub-multiple angle formulas, if necessary.
- For equations involving more than two multiple angles (i) Apply $C \pm D$ formula to combine the two. (ii) Choose such pairs of multiple angle so that after applying the above formulae we get a common factor in the equation.

Solutions in the Case of Two Equations are given

We may divide the problem into two categories. (1) Two equations in one 'unknown' satisfied simultaneously. (2) Two equations in two 'unknowns' satisfied simultaneously.

- **Two Equations is One 'unknown':** Two equations are given and we have to find the values of variables θ which may satisfy with the given equations.

  $\cos\theta = \cos\alpha$ and $\sin\theta = \sin\alpha$, so the common solution is $\theta = 2n\pi + \alpha,\ n \in I$

  $\sin\theta = \sin\alpha$ and $\tan\theta = \tan\alpha$, so the common solution is $\theta = 2n\pi + \alpha,\ n \in I$

  $\cos\theta = \cos\alpha$ and $\tan\theta = \tan\alpha$, so the common solution is $\theta = 2n\pi + \alpha,\ n \in I$

- **System of Equations (Two Equations in Two Unknowns)** Let $f(\theta,\phi) = 0, g(\theta,\phi) = 0$ be the system of two equations in two unknowns.

  Step (i): Eliminate any one variable, say ϕ. Let $\theta = \alpha$ be one solution.

  Step (ii): Then consider the system $f(\alpha,\phi) = 0,\ g(\alpha,\phi) = 0$ and use the method of two equations in one variable.

- It is preferable to solve the system of equations quadrant wise.

Particular Equations

- **General Solution of the form $a\cos\theta + b\sin\theta = c$:** In $a\cos\theta + b\sin\theta = c$, put $a = r\cos\alpha$ and $b = r\sin\alpha$ where $r = \sqrt{a^2+b^2}$ and $|c| \le \sqrt{a^2+b^2}$

Then, $r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = c$

$\Rightarrow\ \cos(\theta-\alpha) = \frac{c}{\sqrt{a^2+b^2}} = \cos\beta$ (say) $\quad$ . . . (*i*)

$\Rightarrow\ \theta-\alpha = 2n\pi \pm \beta \Rightarrow \theta = 2n\pi \pm \beta + \alpha$, where $\tan\alpha = \frac{b}{a}$, is the general solution Alternatively, putting $a = r\sin\alpha$ and $b = r\cos\alpha$ where $r = \sqrt{a^2+b^2}$

$\Rightarrow\ \sin(\theta+\alpha) = \frac{c}{\sqrt{a^2+b^2}} = \sin\gamma$ (say) $\Rightarrow\ \theta+\alpha = n\pi + (-1)^n\gamma$

$\Rightarrow\ \theta = n\pi + (-1)^n\gamma - \alpha$, where $\tan\alpha = \frac{a}{b}$, is the general solution.

Note

- $(-\sqrt{a^2+b^2}) \le a\cos\theta + b\sin\theta \le (\sqrt{a^2+b^2})$
- The general solution of $a\cos x + b\sin x = c$ is

  $x = 2n\pi + \tan^{-1}\left(\frac{b}{a}\right) \pm \cos^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right).$

- **Equation of the form:** $a_0\sin^2 x + a_1\sin^{n-1}x\cos x + a_2 \sin^{n-2}x\cos^2 x + ... + a_n\cos^n x = 0$ Here $a_0, a_1 ..., a_n$ are real numbers and the sum of the exponents in sin *x* and cos *x* in each term is equal to *n*, are said to be homogeneous with respect to sin*x* and cos*x*. For $\cos x \neq 0$, above equation can be written as, $a_0\tan^n x + a_1\tan^{n-1}x + ... + a_n = 0$.

- **A trigonometric equation of the form $R(\sin kx, \cos nx, \tan mx, \cot lx) = 0$:** Here *R* is a rational function of the indicated arguments and (*k, l, m, n* are natural numbers) can be reduced to a rational equation with respect to the arguments $\sin x, \cos x, \tan x,$ and $\cot x$ by means of the formulae for trigonometric functions of the sum of angles (in particular, the formulas for double and triple angles) and then reduce equation of the given form to a rational equation with respect to the unknown, $t = \tan\frac{x}{2}$ by means of the formulas, $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}},$

  $\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}, \cot x = \frac{1-\tan^2\frac{x}{2}}{2\tan\frac{x}{2}}$

- **Equation of the form $R(\sin x + \cos x, \sin x.\cos x) = 0$:** where *R* is rational function of the arguments in brackets,

  Put $\sin x + \cos x = t$ $\quad$ . . . (*i*)

  and use the following identity:

$(\sin x+\cos x)^2=\sin^2 x+\cos^2 x+2\sin x\cos x=1+2\sin x\cos x$

$\Rightarrow \sin x\cos x=\dfrac{t^2-1}{2}$ . . . (*ii*)

Taking (*i*) and (*ii*) into account, we can reduce given equation into; $R\left(t,\dfrac{t^2-1}{2}\right)=0$. Similarly, by the substitution (sin *x* – cos *x*) = *t*, we can reduce the equation of the form; *R*(sin*x* –cos *x*, sin *x* cos *x*) =0 to an equation; $R\left(t,\dfrac{1-t^2}{2}\right)=0.$

Method for Finding Principal Value: Suppose we have to find the principal value of θ satisfying the equation $\sin\theta=-\dfrac{1}{2}$.

Since $\sin\theta$ is negative, θ will be in 3rd or 4th quadrant. We can approach 3rd or 4th quadrant from two directions. If we take anticlockwise direction the numerical value of the angle will be greater than π. If we approach it in clockwise direction the angle will be numerically less than π. For principal value, we have to take numerically smallest angle. So for principal value

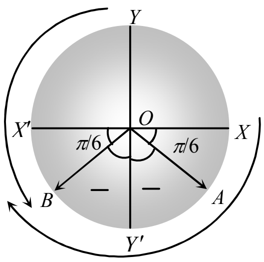


Figure: 11.8

Principal value always lies in the first circle (*i.e.*, in first rotation). On the above criteria, θ will be $-\dfrac{\pi}{6}$ or $-\dfrac{5\pi}{6}$. Among these two $-\dfrac{\pi}{6}$ has the least numerical value. Hence $-\dfrac{\pi}{6}$ is the principal value of θ satisfying the equation $\sin\theta=-\dfrac{1}{2}$. The method for finding principal value can be summed up as follows:

- First draw a trigonometrical circle and mark the quadrant, in which the angle may lie.
- Select anticlockwise direction for 1st and 2nd quadrants and select clockwise direction for 3rd and 4th quadrants.
- Find the angle in the first rotation.
- Select the numerically least angle. The angle thus found will be principal value.
- In case, two angles one with positive sign and the other with negative sign qualify for the numerically least angle, then it is the convention to select the angle with positive sign as principal value.

Properties of Triangle

Angles of a triangle: In a triangle ABC, three angles are A, B and C.

- $A+B+C=\pi$
- $\sin(B+C)=\sin(\pi-A)=\sin A$
- $\cos(C+A)=\cos(\pi-B)=-\cos B$
- $\sin\dfrac{A+B}{2}=\sin\left(\dfrac{\pi}{2}-\dfrac{C}{2}\right)=\cos\dfrac{C}{2}$
- $\cos\dfrac{B+C}{2}=\cos\left(\dfrac{\pi}{2}-\dfrac{A}{2}\right)=\sin\dfrac{A}{2}$

Trigonometrical Relations between Sides and Angles

For any ΔABC we have

- $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=2R$ where R = circumradius (Sine-rule)
- $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$, etc. (Cosine formulae)
- $a\cos B+b\cos A=c$, etc.
- $\sin\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$, etc., where $2s=a+b+c$
- $\cos\dfrac{A}{2}=\sqrt{\dfrac{s(s-a)}{bc}}$ etc.
- $\tan\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}=\dfrac{(s-b)(s-c)}{\Delta}=\dfrac{\Delta}{s(s-a)}$, etc., where $\Delta=$ area of DABC
- $\cot\dfrac{A}{2}=\dfrac{(s-a)}{\Delta}$ *etc.*
- $\tan\dfrac{B-C}{2}=\dfrac{b-c}{b+c}\cot\dfrac{A}{2}$, etc.

Area of a Triangle: If the area of a ΔABC is denoted by Δ then

- $\Delta=\dfrac{1}{2}bc\sin A$, etc.
- $\sin A=\dfrac{2\Delta}{bc}$, etc.
- $\Delta=\sqrt{s(s-a)(s-b)(s-c)}$

Ratio Formula: In the $\Delta ABC, AD$ divides BC in the ratio m: n at D and $\angle BAC$ in two parts $\angle BAD=\alpha$, $\angle CAD=\beta$. If $\angle ADB=\theta$ then

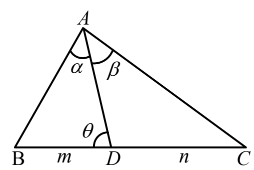


Figure: 11.9

- $(m+n)\cot\theta = n\cot\beta - m\cot\alpha$
- $(m+n)\cot\theta = m\cot C - n\cot B$

Some Important identities for Angles of a Triangle: In a ΔABC, we have

- $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$
- $1 + \cos 2A + \cos 2B + \cos 2C = -4\cos A \cos B \cos C$
- $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
- $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}$
- $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$
- $\tan A + \tan B + \tan C = \tan A\cdot\tan B\cdot\tan C,$

i.e., $\cot B\cdot\cot C + \cot C\cdot\cot A + \cot A\cdot\cot B = 1$

- $\sum\tan\frac{A}{2}\cdot\tan\frac{B}{2} = 1$

i.e., $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$

Circumrarius, Inradius and Exradii: In the ΔABC, let the circumradius = R, inradius = r and the three exradii corresponding to the vertices A, B and C be r_1, r_2 and r_3 respectively. Then

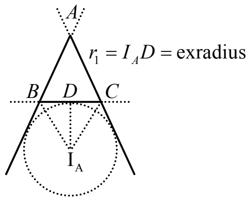


Figure: 11.10

- $R = \frac{a}{2\sin A} = \frac{abc}{4\Delta}$
- $R = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2} = 4R\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}$
- $r_1 = \frac{\Delta}{s-a} = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
- $r_2 = \frac{\Delta}{s-b} = s\tan\frac{B}{2} = 4R\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2}$
- $r_3 = \frac{\Delta}{s-c} = s\tan\frac{C}{2} = 4R\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$

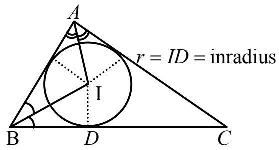


Figure: 11.11

Regular polygon: A regular polygon of n sides will have its vertices on a circle. If O be the centre and r be the radius of the circle, and a be the length of each side then clearly, in the $\Delta OA_1, A_2, \angle A_1OA_2 = \frac{2\pi}{n}$ because ΔOA_1A_2 will be one of the n equal triangles with a vertex at O.

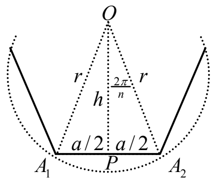


Figure: 11.12

- Also, $\frac{\frac{a}{2}}{h} = \tan\frac{1}{2}\left(\frac{2\pi}{n}\right), \frac{\frac{a}{2}}{r} = \sin\frac{1}{2}\left(\frac{2\pi}{n}\right)$, etc.
- Each interior angle of the regular polygon $= \frac{2(n-2)}{n}\times 90°$.
- Each exterior angle of the regular polygon $= \frac{2\pi}{n}$

Trigonometrical Inequalities and Inequalities

Basic inequalities

- $-1 \le \sin x \le 1$
- $-1 \le \cos x \le 1$
- $\sec x \ge 1$ or $\sec x \ge -1$
- $\operatorname{cosec} x \ge 1$ or $\operatorname{cosec} x \le -1$
- For positive quantities $AM \ge GM$, equally holding if all the quantities are equal.

Table 11.7: Solutions of Trigonometrical In-equations

Inequation	**Solution in $[0,2\pi]$ or $[-\pi,\pi]$**	**General Solution** $(n \in Z)$
$\sin x > k (= \sin\alpha)$	$x \in (\alpha, \pi-\alpha)$	$x \in (2n\pi+\alpha, \overline{2n+1}\pi-\alpha)$
$\sin x < k (= \sin\alpha)$	$x \in [0,\alpha] \cup (\pi-\alpha, 2\pi)$	$x \in [2n\pi, 2n\pi+\alpha) \cup (\overline{2n+1}\pi-\alpha, \overline{2n+1}\pi+\pi)$
$\cos x > k (= \cos\alpha)$	$x \in (-\alpha, \alpha)$	$x \in (2n\pi-\alpha, 2n\pi+\alpha)$
$\cos x < k (= \cos\alpha)$	$x \in (\alpha, 2\pi-\alpha)$	$x \in (2n\pi+\alpha, \overline{2n+1}\pi-\alpha)$
$\tan x > k (= \tan\alpha)$	$x \in \left(\alpha, \frac{\pi}{2}\right) \cup \left(\pi+\alpha, \frac{3\pi}{2}\right)$	$x \in \left(n\pi+\alpha, n\pi+\frac{\pi}{2}\right)$
$\tan x < k (= \tan\alpha)$	$x \in \left(\frac{\pi}{2}, \pi+\alpha\right) \cup \left(-\frac{\pi}{2}, \alpha\right)$	$x \in \left(n\pi-\frac{\pi}{2}, n\pi+\alpha\right)$

MULTIPLE CHOICE QUESTIONS

System of Measurement of Angles and Relation

1. The circular wire of radius 7 cm is cut and bend again into an arc of a circle of radius 12 cm. The angle subtended by an arc at the centre of the circle is:

a. 50^o **b.** 210^o **c.** 100^o **d.** 60^o

2. The degree measure corresponding to the given radian $\left[\frac{2\pi}{15}\right]^c$:

a. 21^o **b.** 22^o **c.** 23^o **d.** 24^o

3. The minute hand of a clock is 10 *cm* long. How far does the tip of the hand move in 20 minutes?

a. $\frac{10\pi}{3}$ **b.** $\frac{20\pi}{3}$ **c.** $\frac{30\pi}{3}$ **d.** $\frac{40\pi}{3}$

4. The angle subtended at the centre of radius 3 *metres* by the arc of length 1 *metre* is equal to:

a. 20^o **b.** 60^o **c.** 1/3 radian **d.** 3 radian

Trigonometrical Ratios or Functions

5. If $\tan A + \cot A = 4$, then $\tan^4 A + \cot^4 A$ is equal to:

a. 110 **b.** 191 **c.** 80 **d.** 194

6. If $\sin x + \cos x = \frac{1}{5}$, then $\tan 2x$ is:

a. $\frac{25}{17}$ **b.** $\frac{7}{25}$ **c.** $\frac{25}{7}$ **d.** $\frac{24}{7}$

7. If $\sin x = -\frac{24}{25}$, then the value of $\tan x$ is:

a. $\frac{24}{25}$ **b.** $\frac{-24}{7}$ **c.** $\frac{25}{24}$ **d.** None of these

8. For $0 < \phi < \frac{\pi}{2}$, if $x = \sum_{n=0}^{\infty} \cos^{2n}\phi, y = \sum_{n=0}^{\infty} \sin^{2n}\phi,$

$z = \sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi$, then:

a. $xyz = xz + y$ **b.** $xyz = xy + z$ **c.** $xyz = x + y + z$ **d.** Both (b) and (c)

Trigonometrical Ratios of Allied Angles

9. $\sin 75^o =$:

a. $\frac{2-\sqrt{3}}{2}$ **b.** $\frac{\sqrt{3}+1}{2\sqrt{2}}$ **c.** $-\frac{\sqrt{3}-1}{2\sqrt{2}}$ **d.** $\frac{\sqrt{3}-1}{2\sqrt{2}}$

10. If $4\sin\theta = 3\cos\theta$ then $\frac{\sec^2\theta}{4[1-\tan^2\theta]}$ equals to:

a. $\frac{25}{16}$ **b.** $\frac{25}{28}$ **c.** $\frac{1}{4}$ **d.** 1

Formulae to Transform the Product into Sum or Difference

11. $\frac{\cos 12^o - \sin 12^o}{\cos 12^o + \sin 12^o} + \frac{\sin 147^o}{\cos 147^o} = ?$

a. 2 tan 33° **b.** 1 **c.** – 1 **d.** 0

12. If $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$, then $\cos\theta_1 + \cos\theta_2 + \cos\theta_3 = ?$

a. 3 **b.** 2 **c.** 1 **d.** 0

13. $\frac{\sin^2 A - \sin^2 B}{\sin A \cos A - \sin B \cos B} = ?$

a. $\tan(A-B)$ **b.** $\tan(A+B)$ **c.** $\cot(A-B)$ **d.** $\cot(A+B)$

14. $\sin 47^o + \sin 61^o - \sin 11^o - \sin 25^o = ?$

a. $\sin 36^o$ **b.** $\sin 7^o$ **c.** $\cos 36^o$ **d.** $\cos 7^o$

15. $\sin 163^o \cos 347^o + \sin 73^o \sin 167^o = ?$

a. 0 **b.** $\frac{1}{2}$ **c.** 1 **d.** None of these

16. The value of $\cot 70^o + 4\cos 70^o$ is:

a. $\frac{1}{\sqrt{3}}$ **b.** $\sqrt{3}$ **c.** $2\sqrt{3}$ **d.** $\frac{1}{2}$

Trigonometric Ratio of Multiple and Sub-multiple of an Angle

17. If $\sin\alpha = \frac{-3}{5}$ where $\pi < \alpha < \frac{3\pi}{2}$, then $\cos\frac{\alpha}{2}$ equal to:

a. $\frac{1}{\sqrt{10}}$ **b.** $-\frac{1}{\sqrt{10}}$ **c.** $\frac{3}{\sqrt{10}}$ **d.** $\frac{-3}{\sqrt{10}}$

18. $2\sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta)$ equal to:

a. $\sin 2\alpha$ **b.** $\cos 2\beta$ **c.** $\tan A/2$ **d.** $\sin 2\beta$

19. $\frac{\cot^2 15^\circ - 1}{\cot^2 15^\circ + 1} = ?$

a. $\frac{1}{2}$ **b.** $\frac{\sqrt{3}}{2}$ **c.** $\frac{3\sqrt{3}}{4}$ **d.** $\sqrt{3}$

20. If $\sin 2\theta + \sin 2\phi = \frac{1}{2}$ and $\cos 2\theta + \cos 2\phi = \frac{3}{2}$, then $\cos^2(\theta - \phi)$ equal to:

a. $\frac{3}{8}$ **b.** $\frac{5}{8}$ **c.** $\frac{3}{4}$ **d.** $\frac{5}{4}$

Maximum and Minimum Value of $a\cos\theta + b\sin\theta$

21. If $x = y\cos\frac{2\pi}{3} = z\cos\frac{4\pi}{3}$, then $xy + yz + zx =$:

a. -1 **b.** 0
c. 1 **d.** 2

22. If $\tan\theta = \frac{a}{b}$, then $\frac{\sin\theta}{\cos^8\theta} + \frac{\cos\theta}{\sin^8\theta}$ equal to:

a. $\pm\frac{(a^2+b^2)^4}{\sqrt{a^2+b^2}}\left[\frac{a}{b^8} + \frac{b}{a^8}\right]$ **b.** $\pm\frac{(a^2+b^2)^4}{\sqrt{a^2+b^2}}\left[\frac{a}{b^8} - \frac{b}{a^8}\right]$

c. $\pm\frac{(a^2-b^2)^4}{\sqrt{a^2+b^2}}\left[\frac{a}{b^8} + \frac{b}{a^8}\right]$ **d.** $\pm\frac{(a^2-b^2)^4}{\sqrt{a^2+b^2}}\left[\frac{a}{b^8} - \frac{b}{a^8}\right]$

Conditional Trigonometrical Identities

23. If $A + B + C = \pi$, then $\cos^2 A + \cos^2 B - \cos^2 C$ equal to:

a. $1 - 2\sin A\sin B\cos C$ **b.** $1 - 2\cos A\cos B\sin C$
c. $1 + 2\sin A\sin B\cos C$ **d.** $1 + 2\cos A\cos B\sin C$

24. If $\alpha + \beta + \gamma = 2\pi$, then:

a. $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

b. $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1$

c. $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = -\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

d. None of these

25. If $\tan\beta = \cos\theta.\tan\alpha$ then $\tan^2\frac{\theta}{2}$ equal to:

a. $\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}$ **b.** $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$

c. $\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}$ **d.** $\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}$

Trigonometrical Equations and General Solution of Standard Equations

26. The general solution of $\tan 3x = 1$ is :

a. $n\pi + \frac{\pi}{4}$ **b.** $\frac{n\pi}{3} + \frac{\pi}{12}$

c. $n\pi$ **d.** $n\pi \pm \frac{\pi}{4}$

27. If $\sin 3\theta = \sin\theta$, then the general value of θ is:

a. $2n\pi, (2n+1)\frac{\pi}{3}$ **b.** $n\pi, (2n+1)\frac{\pi}{4}$

c. $n\pi, (2n+1)\frac{\pi}{3}$ **d.** None of these

28. If $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$, then:

a. $\theta = \frac{(6n+1)\pi}{18}, \forall n \in I$ **b.** $\theta = \frac{(6n+1)\pi}{9}, \forall n \in I$

c. $\theta = \frac{(3n+1)\pi}{9}, \forall n \in I$ **d.** None of these

29. General value of θ satisfying the equation $\tan^2\theta + \sec 2\theta = 1$ is:

a. $m\pi, n\pi + \frac{\pi}{3}$ **b.** $m\pi, n\pi \pm \frac{\pi}{3}$

c. $m\pi, n\pi \pm \frac{\pi}{6}$ **d.** None of these

Solutions in the Case of Two Equations are given

30. The most general value of θ satisfying the equation $\tan\theta = -1$ and $\cos\theta = \frac{1}{\sqrt{2}}$ is:

a. $n\pi + \frac{7\pi}{4}$ **b.** $n\pi + (-1)^n\frac{7\pi}{4}$

c. $2n\pi + \frac{7\pi}{4}$ **d.** None of these

31. The most general value of θ which will satisfy both the equations $\sin\theta = \frac{-1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$ is:

a. $n\pi + (-1)^n\frac{\pi}{6}$ **b.** $n\pi + \frac{\pi}{6}$

c. $2n\pi \pm \frac{\pi}{6}$ **d.** None of these

Some Particular Equations

32. The solution of equation $5\sin^2 x - 7\sin x\cos x + 16\cos^2 x = 4$ is:

a. $x = n\pi + \tan^{-1} 3$ or $x = n\pi + \tan^{-1} 4$

b. $x = n\pi + \frac{\pi}{6}$ or $x = n\pi + \frac{\pi}{4}$

c. $x = n\pi$ or $x = n\pi + \frac{\pi}{4}$

d. None of these

33. If $(\cos x - \sin x)\left(2\tan x + \frac{1}{\cos x}\right) + 2 = 0$ then x =?

a. $2n\pi \pm \frac{\pi}{3}$ **b.** $n\pi \pm \frac{\pi}{3}$

c. $2n\pi \pm \frac{\pi}{6}$ **d.** None of these

34. If $\sin x + \cos x - 2\sqrt{2}\sin x\cos x = 0$ then the general solution of x is:

a. $x = 2n\pi + \frac{\pi}{4}$ **b.** $x = n\pi + (-1)^n \frac{\pi}{6} - \frac{\pi}{4}$

c. Both (a) and (b) **d.** None of these

Principal Value, Steps to solve Trigonometrical Equations and Periodic Functions

35. If $\cos\theta + \sqrt{3}\sin\theta = 2$, then $\theta =$ (only principal value):

a. $\frac{\pi}{3}$ **b.** $\frac{2\pi}{3}$ **c.** $\frac{4\pi}{3}$ **d.** $\frac{5\pi}{3}$

36. Principal value of tan $\theta = -1$is:

a. $\frac{-\pi}{4}$ **b.** $\frac{\pi}{4}$ **c.** $\frac{3\pi}{4}$ **d.** $\frac{-3\pi}{4}$

37. If the solutions for θ of $\cos p\theta + \cos q\theta = 0$, $p > 0, q > 0$ are in A.P., then the numerically smallest common difference of A.P. is:

a. $\frac{\pi}{p+q}$ **b.** $\frac{2\pi}{p+q}$

c. $\frac{\pi}{2(p+q)}$ **d.** $\frac{1}{p+q}$

38. If $\cos\theta = \frac{-1}{2}$ and $0° < \theta < 360°$, then the values of θ are:

a. 120º and 300º **b.** 60º and 120º

c. 120º and 240º **d.** 60º and 240º

39. The period of the function $y = \sin 2x$ is:

a. 2π **b.** π **c.** $\frac{\pi}{2}$ **d.** 4π

40. The function $f(x) = \sin\frac{\pi x}{2} + 2\cos\frac{\pi x}{3} - \tan\frac{\pi x}{4}$ is periodic with period:

a. 6 **b.** 3

c. 4 **d.** 12

Relation between Sides and Angles

41. In a triangle ABC, $\angle B = \frac{\pi}{3}$ and $\angle C = \frac{\pi}{4}$ and D divides BC internally in the ratio 1 : 3. Then $\frac{\sin\angle BAD}{\sin\angle CAD}$ is equal to:

a. $\frac{1}{3}$ **b.** $\frac{1}{\sqrt{3}}$ **c.** $\frac{1}{\sqrt{6}}$ **d.** $\sqrt{\frac{2}{3}}$

42. In a ΔABC, $2ac\sin\left(\frac{A-B+C}{2}\right)$ is equal to:

a. $a^2 + b^2 - c^2$ **b.** $c^2 + a^2 - b^2$

c. $b^2 - c^2 - a^2$ **d.** $c^2 - a^2 - b^2$

43. In a triangle ABC, AD is altitude from A. Given $b > c$, $\angle C = 23°$ and $AD = \frac{abc}{b^2 - c^2}$, then $\angle B$ equal to:

a. 67º **b.** 44º

c. 113º **d.** None of these

44. In a ΔABC, $\frac{\cos C + \cos A}{c+a} + \frac{\cos B}{b}$ is equal to:

a. $\frac{1}{a}$ **b.** $\frac{1}{b}$

c. $\frac{1}{c}$ **d.** $\frac{c+a}{b}$

45. AD is a median of the ΔABC, if AE and AF are medians of the triangles ABD and ADC respectively and $AD = m_1$, $AE = m_2$, $AF = m_3$, then $\frac{a^2}{8}$ is equal to:

a. $m_2^2 + m_3^2 - 2m_1^2$ **b.** $m_1^2 + m_2^2 - 2m_3^2$

c. $m_2^2 + m_3^2 - m_1^2$ **d.** None of these

Area of Triangle

46. In a triangle ABC, a, b, A are given and c_1, c_2 are two values of third side c. The sum of the areas of triangles with sides a, b, c_1 and a, b, c_2 is:

a. $\frac{1}{2}a^2\sin 2A$ **b.** $\frac{1}{2}b^2\sin 2A$

c. $b^2\sin 2A$ **d.** $a^2\sin 2A$

Half Angle Formulae

47. If in any ΔABC; $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P., then:

a. $\cot\frac{A}{2}\cot\frac{B}{2}=4$
b. $\cot\frac{A}{2}\cot\frac{C}{2}=3$
c. $\cot\frac{B}{2}\cot\frac{C}{2}=1$
d. $\cot\frac{B}{2}\tan\frac{C}{2}=0$

Ex-central Triangle

48. In a triangle ABC, let $\angle C=\frac{\pi}{2}$. If r is the in radius and R is the circum-radius of the triangle, then $2(r+R)$ is equal to:

a. $a+b$
b. $b+c$
c. $c+a$
d. $a+b+c$

Cyclic Quadrilateral

49. The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the third side is 3, the remaining fourth side is:

a. 2
b. 3
c. 4
d. 5

Regular Polygon

50. The area of the circle and the area of a regular polygon of n sides and its perimeter equal to that of the circle are in the ratio of:

a. $\tan\left(\frac{\pi}{n}\right):\frac{\pi}{n}$
b. $\cos\left(\frac{\pi}{n}\right):\frac{\pi}{n}$
c. $\sin\left(\frac{\pi}{n}\right):\frac{\pi}{n}$
d. $\cot\left(\frac{\pi}{n}\right):\frac{\pi}{n}$

NCERT EXEMPLAR PROBLEMS

More than One Answer

51. If $\tan\theta=-\frac{4}{3}$, then $\sin\theta$ is :

a. $-\frac{4}{5}$ but not $\frac{4}{5}$
b. $-\frac{4}{5}$ or $\frac{4}{5}$
c. $\frac{4}{5}$ but not $-\frac{4}{5}$
d. None of these

52. If $\alpha+\beta+\gamma=2\pi$, then :

a. $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

b. $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1$

c. $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

d. None of the above

53. Given $A=\sin^2\theta+\cos^4\theta$, then for all real values of θ :

a. $1\le A\le 2$
b. $\frac{3}{4}\le A\le 1$
c. $\frac{13}{16}\le A\le 1$
d. $\frac{3}{4}\le A\le\frac{13}{16}$

54. $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$ is equal to:

a. $\frac{1}{2}$
b. $\cos\frac{\pi}{8}$
c. $\frac{1}{8}$
d. $\frac{1+\sqrt{2}}{2\sqrt{2}}$

55. The expression $3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]$:

$-2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right]$ is equal to

a. 0
b. 0
c. 0
d. 0

56. The value of the expression $\sqrt{3}\cos ec20°-\sec 20°$ is equal to:

a. 2
b. 2sin 20°/sin 40°
c. 4
d. 4sin 20°/sin40°

57. $3(\sin x-\cos x)^4+6(\sin x+\cos x)^2+4(\sin^6 x+\cos^6 x)$ equal to:

a. 11
b. 12
c. 13
d. 14

58. Which of the following numbers is rational?

a. $\sin 15°$
b. $\cos 15°$
c. $\sin 15°\ \cos 15°$
d. $\sin 15°\ \cos 75°$

59. If $\alpha+\beta=\frac{\pi}{2}$ and $\beta+\gamma=\alpha$, then $\tan\alpha$ equals:

a. $2(\tan\beta+\tan\gamma)$
b. $\tan\beta+\tan\gamma$
c. $\tan\beta+2\tan\gamma$
d. $2\tan\beta+\tan\gamma$

60. Given both θ and ϕ are acute angles $\sin\theta=\frac{1}{2}, \cos\phi=\frac{1}{3}$, then the value of $\theta+\phi$ belongs to?

a. $\left(\frac{\pi}{3},\frac{\pi}{6}\right]$
b. $\left(\frac{\pi}{2},\frac{2\pi}{3}\right)$
c. $\left(\frac{2\pi}{3},\frac{5\pi}{6}\right]$
d. $\left(\frac{5\pi}{6},\pi\right]$

61. The number of ordered pairs (α, β), where $\alpha, \beta \in (-\pi, \pi)$ satisfying $\cos(\alpha-\beta)=1$ and $\cos(\alpha+\beta)=\frac{1}{e}$ is:

a. 0 **b.** 1
c. 2 **d.** 4

62. Let $\theta \in \left(0, \frac{\pi}{4}\right)$ and $t_1=(\tan\theta)^{\tan\theta}, t_2=(\tan\theta)^{\cos\theta}, t_3=(\cot\theta)^{\tan\theta}$ and $t_4=(\cot\theta)^{\cot\theta}$, then?

a. $t_1>t_2>t_3>t_4$ **b.** $t_4>t_3>t_1>t_2$
c. $t_3>t_1>t_2>t_4$ **d.** $t_2>t_3>t_1>t_4$

63. The expression $\frac{\tan A}{1-\cot A}+\frac{\cot A}{1-\tan A}$ can be written as:

a. sin *A* cos *A* + 1 **b.** sin *A* cosec *A* + 1
c. tan *A* + cot *A* **d.** sec *A* + cosec *A*

64. The equation $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2+x^{-2}, x \le \frac{\pi}{9}$ has:

a. no real solution
b. one real solution
c. more than one real solution
d. None of the above

65. The general solution of the trigonometric equation sin *x* + cos *x* = 1 is given by:

a. $x=2n\pi; n=0,\pm1,\pm2.....$
b. $x=2n\pi+\pi/2; n=0,\pm1,\pm2.....$
c. $x=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{4}; n=0,\pm1,\pm2,.....$
d. None of the above

Assertion and Reason

Note: Read the Assertion (A) and Reason (R) carefully to mark the correct option out of the options given below:

a. If both assertion and reason are true and the reason is the correct explanation of the assertion.
b. If both assertion and reason are true but reason is not the correct explanation of the assertion.
c. If assertion is true but reason is false.
d. If the assertion and reason both are false.
e. If assertion is false but reason is true.

66. **Assertion:** $\sin 52°+\sin 78°+\sin 50° = 4\cos 26°\cos 39°\cos 25°$

Reason: If $A+B+C=\pi$, then $\sin A+\sin B+\sin C$ $=4\cos(A/2)\cos(B/2)\cos(C/2)$

67. **Assertion:** If *A*, *B*, *C* are the angles of a triangle such the angle *A* is obtuse, then $B\ \ C<1$.

Reason: In a triangle *ABC* $\tan A=\frac{\tan B+\tan C}{1-\tan B\tan C}$

68. **Assertion:** $\frac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)}=\tan A$

Reason: $\sin(A+B)+\sin(A-B)=\sin A$ and $\cos(A+B)+\cos(A-B)=\cos A$

69. **Assertion:** If $2\sin^2((\pi/2)\cos^2 x)=1-\cos(\pi\sin 2x)$, $x\ne(2n+1)\pi/2$, *n* is a integer then $\sin 2x+\cos 2x$ is equal to 1/5.

Reason: $\sin 2x+\cos 2x=\frac{2-(\tan x-1)^2}{1+\tan^2 x}$

70. **Assertion:** The system of linear equations $x+(\sin\alpha)y+(\cos\alpha)z=0$, $x+(\cos\alpha)y+(\sin\alpha)z=0$ and $x-(\sin\alpha)y+(\cos\alpha)z=0$ has a non-trivial solution for only one value of α lying between 0 and π

Reason: $\begin{vmatrix}\sin x & \cos x & \cos x\\ \cos x & \sin x & \cos x\\ \cos x & \cos x & \sin x\end{vmatrix}=0$ has only one solution lying between 0 and $\frac{\pi}{2}$

71. **Assertion:** $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}=-\frac{1}{8}$

Reason: $\cos\theta\cos 2\theta\cos 4\theta\ldots\cos 2^{n-1}\theta=-\frac{1}{2^n}$ if $\theta=\frac{\pi}{2^n-1}$

72. **Assertion:** If *x* and *y* are real number such that $x^2+y^2=27$, then the maximum possible value of $x-y$ is $3\sqrt{6}$

Reason: $-1\le\cos\left(\theta+\frac{\pi}{4}\right)\le 1$

73. **Assertion:** If $p=7+\tan\alpha\tan\beta$, $q=5+\tan\beta\tan\gamma$ and $r=3+\tan\gamma\tan\alpha$, then the maximum value of $\sqrt{p}+\sqrt{q}+\sqrt{r}$ is $4\sqrt{3}$; $\alpha,\beta,\gamma>0$ and $\alpha+\beta+\gamma=\frac{\pi}{2}$

Reason: If $\alpha+\beta+\gamma=\frac{\pi}{2}, \alpha,\beta,\gamma>0$, then $\cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha=1$

74. **Assertion:** $2\sin^2 x+3\sin x-2>0$ and $x^2-x-2<0$, if $-1<x<2$.

Reason: $x^2-x-2<0$ if $-1<x<2$.

75. **Assertion:** $\cos^7 x + \sin^4 x = 1$ has only tow non-zero solution in the interval $-\pi < x < \pi$

Reason: $\cos^5 x + \cos^2 x - 2 = 0$ is possible only when $\cos x = 1$

Comprehension Based

Paragraph –I

If $P_n = \sin^n \theta + \cos^n \theta$ where $n \in W$ (whole number) and $\theta \in R$ (real number)

76. If $P_1 = m,$ then the value of $4(1 - P_6)$ is :

a. $3(m-1)^2$ **b.** $3(m^2-1)^2$
c. $3(m+1)^2$ **d.** $3(m^2+1)^2$

77. The value of $2P_6 - 3P_4 + 10$ is:

a. 0 **b.** 6
c. 9 **d.** 15

78. The value of $6P_{10} - 15P_8 + 10P_6 + 7$ is:

a. 8 **b.** 6
c. 4 **d.** 2

79. If $P_{n-2} - P_n = \sin^2\theta \cos^2\theta P_\lambda$, then the value of λ is:

a. $n-1$ **b.** $n-2$
c. $n-3$ **d.** $n-4$

80. The value of $\frac{P_7 - P_5}{P_5 - P_3}$ is :

a. $\frac{P_7}{P_5}$ **b.** $\frac{P_5}{P_3}$
c. $\frac{P_3}{P_1}$ **d.** $\frac{P_3}{P_5}$

Paragraph –II

The method of eliminating 'θ' from two given equations involving trigonometrical functions of 'θ'. By using given equations involving 'θ' and trigonometrical identities, we shall obtain an equation not involving 'θ'.

81. If $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ and $x\sin\theta - y\cos\theta = 0$ then (x, y) lie one:

a. a circle **b.** a parabola
c. an ellipse **d.** a hyperbola

82. If $\frac{x}{a\cos\theta} = \frac{y}{b\sin\theta}$ and $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$, then (x, y) lie on:

a. a circle **b.** a parabola
c. an ellipse **d.** a hyperbola

83. If $\tan\theta + \sin\theta = m$ and $\tan\theta - \sin\theta = n$, then $(m^2 - n^2)^2$ is:

a. $4\sqrt{mn}$ **b.** $4mn$
c. $16\sqrt{mn}$ **d.** $16mn$

84. If $\sin\theta + \cos\theta = a$ and $\sin^3\theta + \cos^3\theta = b$, then we get $\lambda a^3 + \mu b + vc = 0$ when λ, μ, v are independent of θ, then the value of $\lambda^3 + \mu^3 + v^3$ is:

a. – 6 **b.** – 18
c. – 36 **d.** – 98

85. After eliminating 'θ' from equations $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ and $x\sin\theta - y\cos\theta = \sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)}$, we get:

a. $x^2 + y^2 = a^2 + b^2$

b. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

c. $\frac{x^2}{a(a+b)} + \frac{y^2}{b(a+b)} = 1$

d. $x^2 + y^2 = (a+b)^2$

Paragraph –III

Whenever the terms on the two sides of the equation are of different nature, then equations are known as Non standard form, some of them are in the form of an ordinary equation but cannot be solved by standard procedures.

Non standard problems require high degree of logic, they also require the use of graphs, inverse properties of functions, in equalities.

86. The number of solutions of the equation $2\cos\left(\frac{x}{2}\right) = 3^x + 3^{-x}$ is:

a. 1 **b.** 2
c. 3 **d.** none of these

87. The equation $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + x^{-2}, 0 < x \le \frac{\pi}{2}$ has:

a. one real solutions
b. more than one real solutions
c. no real solution
d. none of the above

88. The number of real solutions of the equation $\sin(e^x) = 5^x + 5^{-x}$ is:

a. 0 **b.** 1
c. 2 **d.** infinitely many

89. If $0 \le x \le 2\pi$ and $2^{\operatorname{cosec}^2 x}\sqrt{\left(\frac{1}{2}y^2 - y + 1\right)} \le \sqrt{2}$, then number of ordered pairs of (x, y) is:

a. 1 **b.** 2
c. 3 **d.** infinitely many

90. The number of solutions of the equation $\sin x = x^2 + x + 1$ is:

a. 0 **b.** 1
c. 2 **d.** none of these

Match the Column

Match the conditions/expressions in Column I with statement in Column II:

91. $(\sin 3\alpha)/(\cos 2\alpha)$ is:

Column I	Column II
(A) positive	**1.** $(13\pi/48, 14\pi/48)$
(B) negative	**2.** $(14\pi/48, 18\pi/48)$
	3. $(18\pi/48, 23\pi/48)$
	4. $(0, \pi/2)$

a. A→ 3; B→ 1 **b.** A→ 2, B→ 4
c. A→ 3, B→ 4 **d.** A→ 4, B→ 1

92. Observe the following columns:

Column I	Column II
(A) If maximum and minimum values of $\frac{7 + 6\tan\theta - \tan^2\theta}{(1 + \tan^2\theta)}$ for all real values of $\theta - \frac{\pi}{2}$ are λ and μ respectively, then	**1.** $\lambda + \mu = 2$
(B) If maximum and minimum values of $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$ for all real values of θ are λ and μ respectively, then	**2.** $\lambda - \mu = 6$
(C) If maximum and minimum values of $1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$ for all real values of θ are λ and μ respectively, then	**3.** $\lambda + \mu = 6$
	4. $\lambda - \mu = 10$
	5. $\lambda - \mu = 14$

a. A→ 3,4, B→ 3,5, C→ 1,2
b. A→ 2,1, B→ 3,5, C→ 1,4
c. A→ 3,5, B→ 2,4, C→ 1,2
d. A→ 2,3, B→ 2,5, C→ 1,4

Integer

93. If $4\cos 36° + \cot\left(7\frac{1}{2}°\right)$ $\sqrt{n_1} + \sqrt{n_2} + \sqrt{n_3} + \sqrt{4_4} + \sqrt{n_5} + \sqrt{n_6}$, then the value of $\sum_{i=1}^{6} n_i^2$ must be:

94. If $a\tan\alpha + \sqrt{(a^2 - 1)}\tan\beta + \sqrt{(a^2 + 1)}\tan\gamma = 2a$, where a is constant and α, β, γ are variable angles. Then the least value of $2727(\tan^2\alpha + \tan^2\beta + \tan^2\gamma)$ must be:

95. If $A + B + C = \pi$ and $\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = \lambda = \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$ then the value of $1 + 2\lambda + 3\lambda^2 + 4\lambda^3$ must be:

96. If $\frac{1}{\cos 290°} + \frac{1}{\sqrt{3}\sin 250°} = \lambda$, then the value of $9\lambda^4 + 81\lambda^2 + 97$ must be:

97. If $\tan^2\left(\frac{\pi}{16}\right) + \tan^2\left(\frac{2\pi}{16}\right) + \tan^2\left(\frac{3\pi}{16}\right) + \ldots + \tan^2\left(\frac{7\pi}{16}\right) = \lambda$, and $x^y + y^x = \lambda$, then the value of $(x + y)^2$ must be:

98. The number of values of x between 0 and 2π that satisfies the equation $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$ must be:

99. If the equation $\sec\theta + \operatorname{cosec}\theta = c$ has two real roots between 0 and 2π, then the least integer which c^2 cannot exceed must be:

100. The trigonometric equation $\operatorname{cosec} x - \operatorname{cosec} 2x = \operatorname{cosec} 4x$ has a solution $(2n + 1)\frac{\pi}{4}$, (n integer), then n should not be an integer of type $7p + \lambda$, $(0 \le \lambda \le 6)$, (p integer). The numerical quantity λ should be:

ANSWER

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
b	d	b	c	d	d	b	d	b	b
11.	**12.**	**13.**	**14.**	**15.**	**16.**	**17.**	**18.**	**19.**	**20.**
d	d	b	d	b	b	d	c	b	b
21.	**22.**	**23.**	**24.**	**25.**	**26.**	**27.**	**28.**	**29.**	**30.**
b	a	a	a	a	b	b	c	b	c
31.	**32.**	**33.**	**34.**	**35.**	**36.**	**37.**	**38.**	**39.**	**40.**
d	a	a	c	a	a	b	c	a	d
41.	**42.**	**43.**	**44.**	**45.**	**46.**	**47.**	**48.**	**49.**	**50.**
c	b	c	b	a	b	b	a	a	a
51.	**52.**	**53.**	**54.**	**55.**	**56.**	**57.**	**58.**	**59.**	**60.**
b	a	b	c	b	c	c	c	c	b
61.	**62.**	**63.**	**64.**	**65.**	**66.**	**67.**	**68.**	**69.**	**70.**
d	b	b	a	c	a	c	c	d	b
71.	**72.**	**73.**	**74.**	**75.**	**76.**	**77.**	**78.**	**79.**	**80.**
a	a	c	d	a	b	c	a	d	c
81.	**82.**	**83.**	**84.**	**85.**	**86.**	**87.**	**88.**	**89.**	**90.**
a	c	d	b	c	a	c	a	b	a
91.	**92.**	**93.**	**94.**	**95.**	**96.**	**97.**	**98.**	**99.**	**100.**
a	a	91	3636	2257	785	1225	4	8	4950

SOLUTION

Multiple Choice Questions

1. **(b)** Given the diameter of circular wire = 14 cm. Therefore length of wire $= 14\pi\ cm$

Hence, required angle, $= \frac{\text{Arc}}{\text{Radius}} = \frac{14\pi}{12} = \frac{7\pi}{6}$ radian

$\Rightarrow \quad \frac{7\pi}{6} \times \frac{180^o}{\pi} = 210^o.$

2. **(d)** We have, π radians = 180º

$\therefore \quad 1^c = \left[\frac{180}{\pi}\right]^o;$

$\therefore \quad \left[\frac{2\pi}{15}\right]^c = \left[\frac{2\pi}{15} \times \frac{180}{\pi}\right]^o = 24^o.$

3. **(b)** We know that the tip of the minute hand makes one complete round in one hour *i.e.* 60 *minutes* since the length of the hand is 10 *cm*. the distance moved by its tip in 60 *minutes* $= 2\pi \times 10cm = 20\pi\ cm$

Hence the distance in 20 *minutes*

$= \frac{20\pi}{60} \times 20cm = \frac{20\pi}{3} cm.$

4. **(c)** Required angle, $= \frac{\text{Arc}}{\text{radius}} = \frac{1}{3}$ radian.

5. **(d)** $\tan A + \cot A = 4$

$\Rightarrow \quad \tan^2 A + \cot^2 A + 2\tan A \cot A = 16$

$\Rightarrow \quad \tan^2 A + \cot^2 A = 14$

$\Rightarrow \quad \tan^4 A + \cot^4 A + 2 = 196$

$\Rightarrow \quad \tan^4 A + \cot^4 A = 194$

6. **(d)** $\sin x + \cos x = \frac{1}{5}$

$\Rightarrow \quad \sin^2 x + \cos^2 x + 2\sin x \cos x = \frac{1}{25}$

$\Rightarrow \quad \sin 2x = -\frac{24}{25}$

$\Rightarrow \quad \cos 2x = -\frac{7}{25}$

$\Rightarrow \quad \tan 2x = \frac{24}{7}.$

7. **(b)** $\cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \left(\frac{-24}{25}\right)^2} = \frac{7}{25}$

$= \sqrt{1 - \left(\frac{-24}{25}\right)^2} = \frac{7}{25}$

$\Rightarrow \quad \tan x = \frac{\sin x}{\cos x} = \frac{-24}{7}.$

8. **(d)** From $s_\infty = \frac{a}{1-r}$

We get, $x = \frac{1}{1-\cos^2\phi} = \frac{1}{\sin^2\phi},$

$y = \frac{1}{1-\sin^2\phi} = \frac{1}{\cos^2\phi},$

$z = \frac{1}{1-\cos^2\phi\sin^2\phi} = \frac{1}{1-\frac{1}{xy}} = \frac{xy}{xy-1}$

$\Rightarrow \quad xyz - z = xy$

$\Rightarrow \quad xyz = xy + z \qquad \ldots (i)$

Also, $\frac{1}{x} + \frac{1}{y} = \cos^2\phi + \sin^2\phi = 1$

$\Rightarrow \quad x + y = xy.$

From (*i*), $xyz = x + y + z.$

9. **(b)** $\sin 75° = \sin(45° + 30°)$

$= \sin 45° \cos 30° + \sin 30° \cos 45°$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$

10. **(b)** Given $4\sin\theta = 3\cos\theta$

$\Rightarrow \quad \tan\theta = \frac{3}{4}$

The given expression is $\frac{\sec^2\theta}{4[1-\tan^2\theta]} = \frac{1+\tan^2\theta}{4(1-\tan^2\theta)}$

$= \frac{1+\frac{9}{16}}{4\left(1-\frac{9}{16}\right)} = \frac{25}{28}.$

11. **(d)** $= \frac{1-\tan 12^\circ}{1+\tan 12^\circ} + \tan 147^\circ$

$= \tan(45^\circ - 12) + \tan(180^\circ - 33^\circ) = \tan 33^\circ + (-\tan 33^\circ) = 0.$

12. **(d)** We know $|\sin\theta| \le 1$; So, each θ_1, θ_2 and θ_3 must be equal to $\pi/2$

$\therefore \quad \cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0.$

13. **(b)** $\frac{2(\sin^2 A - \sin^2 B)}{2\sin A\cos A - 2\sin B\cos B} = \frac{2\sin(A+B).\sin(A-B)}{\sin 2A - \sin 2B}$

$= \frac{2\sin(A+B)\sin(A-B)}{2\sin(A-B)\cos(A+B)} = \tan(A+B).$

14. **(d)** $\sin 47^\circ + \sin 61^\circ - (\sin 11^\circ + \sin 25^\circ)$

$= 2\sin 54^\circ.\cos 7^\circ - 2\sin 18^\circ\cos 7^\circ$

$= 2\cos 7^\circ(\sin 54^\circ - \sin 18^\circ)$

$= 2\cos 7^\circ.2\cos 36^\circ.\sin 18^\circ$

$= 4.\cos 7^\circ.\frac{\sqrt{5}+1}{4}.\frac{\sqrt{5}-1}{4} = \cos 7^\circ.$

15. **(b)** $\sin(90^\circ + 73^\circ).\cos(360^\circ - 13^\circ) + \sin 73^\circ.\sin(180^\circ - 13^\circ)$

$= \cos 73^\circ.\cos 13^\circ + \sin 73^\circ.\sin 13^\circ$

$= \cos(73^\circ - 13^\circ) = \cos 60^\circ = \frac{1}{2}.$

16. **(b)** $\cot 70^\circ + 4\cos 70^\circ$

$= \frac{\cos 70^\circ + 4\sin 70^\circ.\cos 70^\circ}{\sin 70^\circ}$

$= \frac{\cos 70^\circ + 2\sin 140^\circ}{\sin 70^\circ}$

$= \frac{\cos 70^\circ + 2\sin(180 - 40^\circ)}{\sin 70^\circ}$

$= \frac{\sin 20^\circ + \sin 40^\circ + \sin 40^\circ}{\sin 70^\circ}$

$= \frac{2\sin 30^\circ\cos 10^\circ + \sin 40^\circ}{\sin 70^\circ}$

$= \frac{\sin 80^\circ + \sin 40^\circ}{\sin 70^\circ} = \frac{2\sin 60^\circ\cos 20^\circ}{\sin 70^\circ} = \sqrt{3}$

17. **(d)** $\pi < \alpha < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}$

$\Rightarrow \quad \cos\frac{\alpha}{2} = -ve$

$\therefore \quad \cos\alpha = \frac{4}{5}$

$\Rightarrow \quad \cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{1+\frac{4}{5}}{2}} = -\sqrt{\frac{9}{10}} = \frac{-3}{\sqrt{10}}.$

18. **(c)** Since $2\cos(\alpha+\beta) = 2\cos^2(\alpha+\beta) - 1,$

$2\sin^2\beta = 1 - \cos 2\beta$

$= -\cos 2\beta + 2\cos(\alpha+\beta)[2\sin\alpha\sin\beta + \cos(\alpha+\beta)]$

$= -\cos 2\beta + 2\cos(\alpha+\beta).\cos(\alpha-\beta)$

$= -\cos 2\beta + \cos 2\alpha + \cos 2\beta = \cos 2\alpha.$

19. **(b)** $\frac{1-\tan^2 15}{1+\tan^2 15} = \frac{1-[\tan(45^\circ - 30^\circ)]^2}{1+[\tan(45^\circ - 30^\circ)]^2} = \frac{1-\left[\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}\right]^2}{1+\left[\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}\right]^2}$

$= \frac{1-\left[\frac{\sqrt{3}-1}{\sqrt{3}+1}\right]^2}{1+\left[\frac{\sqrt{3}-1}{\sqrt{3}+1}\right]^2} = \frac{[\sqrt{3}+1]^2 - [\sqrt{3}-1]^2}{[\sqrt{3}+1]^2 + [\sqrt{3}-1]^2} = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$

$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} \Rightarrow \frac{1-\tan^2 15^o}{1+\tan^2 15^o} = \cos 30^o = \frac{\sqrt{3}}{2}$

20. **(b)** Given, $\sin 2\theta + \sin 2\phi = \frac{1}{2}$ . . . (i)

and $\cos 2\theta + \cos 2\phi = \frac{3}{2}$ . . . (ii)

Squaring and adding,

$\therefore \quad (\sin^2 2\theta + \cos^2 2\theta) + (\sin^2 2\phi + \cos^2 2\phi)$

$+2[\sin 2\theta.\sin 2\phi + \cos 2\theta.\cos 2\phi] = \frac{1}{4} + \frac{9}{4}$

$\Rightarrow \quad \cos 2\theta.\cos 2\phi + \sin 2\theta.\sin 2\phi = \frac{1}{4}$

$\Rightarrow \quad \cos(2\theta - 2\phi) = \frac{1}{4}$

$\Rightarrow \quad \cos^2(\theta - \phi) = \frac{5}{8}.$

21. **(b)** We have, $\frac{x}{1}=\frac{y}{-2}=\frac{z}{-2}=\lambda$ (say)

$\therefore \quad x=\lambda,\ \ y=-2\lambda, z=-2\lambda;$

$\therefore \quad xy+yz+zx=-2\lambda^2+4\lambda^2-2\lambda^2=0$

22. **(a)** Given, $\tan\theta=a/b\Rightarrow\cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{b^2-a^2}{b^2+a^2}$

$\sin\theta=\pm\frac{a}{\sqrt{a^2+b^2}};\ \ \cos\theta=\pm\frac{b}{\sqrt{a^2+b^2}}$

$$\therefore \quad \frac{\sin\theta}{\cos^8\theta}+\frac{\cos\theta}{\sin^8\theta}=\frac{\left(\frac{a}{\sqrt{a^2+b^2}}\right)}{\left(\frac{b}{\sqrt{a^2+b^2}}\right)^8}+\frac{\left(\frac{b}{\sqrt{a^2+b^2}}\right)}{\left(\frac{a}{\sqrt{a^2+b^2}}\right)^8}$$

$$=\frac{a(a^2+b^2)^4}{b^8(a^2+b^2)^{1/2}}+\frac{b(a^2+b^2)^4}{a^8(a^2+b^2)^{1/2}}=\pm\frac{(a^2+b^2)^4}{\sqrt{a^2+b^2}}\left[\frac{a}{b^8}+\frac{b}{a^8}\right].$$

23. **(a)** $\cos^2 A+\cos^2 B-\cos^2 C$

$=\cos^2 A+(1-\sin^2 B)-\cos^2 C$

$=1+[\cos^2 A-\sin^2 B]-\cos^2 C$

$=1+\cos(A+B)\cos(A-B)-\cos^2 C$

$=1+\cos(\pi-C)\cos(A-B)-\cos^2 C$

$=1-\cos C[\cos(A-B)+\cos C]$

$=1-\cos C[\cos(A-B)+\cos\{\pi-(A+B)\}]$

$=1-\cos C[\cos(A-B)-\cos(A+B)]$

$=1-\cos C[2\sin A\sin B]=1-2\sin A\sin B\cos C.$

24. **(a)** We have $\alpha+\beta+\gamma=2\pi\Rightarrow\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}=\pi$

$\Rightarrow \quad \tan\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)=\tan\pi=0$

$\Rightarrow \quad \tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}-\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}.\tan\frac{\gamma}{2}=0$

$\Rightarrow \quad \tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}.\tan\frac{\gamma}{2}$

25. **(a)** The given relation is $\frac{\tan\alpha}{\tan\beta}=\frac{1}{\cos\theta}$

Applying componendo and dividendo rule, then

$\Rightarrow \quad \frac{\tan\alpha-\tan\beta}{\tan\alpha+\tan\beta}=\frac{1-\cos\theta}{1+\cos\theta}$

$\Rightarrow \quad \frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}=\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\ \Rightarrow\ \frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}=\tan^2\frac{\theta}{2}.$

26. **(b)** $\tan 3x=\tan\frac{\pi}{4}\Rightarrow 3x=n\pi+\frac{\pi}{4}$

$\Rightarrow \quad x=\frac{n\pi}{3}+\frac{\pi}{12}.$

27. **(b)** $\sin 3\theta=\sin\theta$ or $3\theta=m\pi+(-1)^m\theta$

For (m) even *i.e.*, $m=2n$ then $\theta=\frac{2n\pi}{2}=n\pi$

And for (m) odd, *i.e.*, $m=(2n+1)$ then $\theta=(2n+1)\frac{\pi}{4}.$

28. **(c)** $\tan\theta+\tan 2\theta+\sqrt{3}\tan\theta\tan 2\theta=\sqrt{3}$

$\Rightarrow \quad \tan\theta+\tan 2\theta=\sqrt{3}(1-\tan\theta\tan 2\theta)$

$\Rightarrow \quad \frac{\tan\theta+\tan 2\theta}{1-\tan\theta\tan 2\theta}=\sqrt{3}\Rightarrow\tan 3\theta=\tan\left(\frac{\pi}{3}\right)$

$\Rightarrow \quad 3\theta=n\pi+\frac{\pi}{3}$

$\Rightarrow \quad \theta=\frac{n\pi}{3}+\frac{\pi}{9}=(3n+1)\frac{\pi}{9}.$

29. **(b)** $\tan^2\theta+\sec 2\theta=1$

$\Rightarrow \quad \tan^2\theta+\frac{1+\tan^2\theta}{1-\tan^2\theta}=1$

$\Rightarrow \quad \tan^2\theta-\tan^4\theta+1+\tan^2\theta=1-\tan^2\theta$

$\Rightarrow \quad \tan^4\theta-3\tan^2\theta=0$

$\Rightarrow \quad \tan^2\theta(\tan^2\theta-3)=0$

$\Rightarrow \quad \tan^2\theta=0$ and $\tan^2\theta=3$

$\tan^2\theta=\tan^2 0$ and $\tan^2\theta=\tan^2\frac{\pi}{3}$

$\Rightarrow \quad \theta=m\pi$ and $\theta=n\pi\pm\frac{\pi}{3}.$

30. **(c)** $\tan\theta=-1=\tan\left(2\pi-\frac{\pi}{4}\right)$ and

$\cos\theta=\frac{1}{\sqrt{2}}=\cos\left(2\pi-\frac{\pi}{4}\right)$

Hence, general value is $2n\pi+\left(2\pi-\frac{\pi}{4}\right)=2n\pi+\frac{7\pi}{4}.$

31. **(d)** $\sin\theta=\frac{-1}{2}=\sin\left(\frac{-\pi}{6}\right)=\sin\left(\pi+\frac{\pi}{6}\right)$

$\tan\theta=\left(\frac{1}{\sqrt{3}}\right)=\tan\left(\frac{\pi}{6}\right)=\tan\left(\pi+\frac{\pi}{6}\right)$

$\Rightarrow \theta=\left(\pi+\frac{\pi}{6}\right)$

Hence, general value of θ is $2n\pi+\frac{7\pi}{6}.$

32. **(a)** To solve this kind of equation; we use the fundamental formula trigonometrical identity, $\sin^2 x+\cos^2 x=1$ writing the equation in the form, $5\sin^2 x-7\sin x\cos x+16\cos^2 x=4\ (\sin^2 x+\cos^2 x)$

$\Rightarrow\quad \sin^2 x-7\sin x\cos x+12\cos^2 x=0$

Dividing by $\cos^2 x$ on both sides we get,

$\tan^2 x-7\tan x+12=0$

Now it can be factorized as; $(\tan x-3)(\tan x-4)=0$

$\Rightarrow\quad \tan x=3,4$

i.e., $\tan x=\tan(\tan^{-1}3)$ or $\tan x=\tan(\tan^{-1}4)$

$\Rightarrow\quad x=n\pi+\tan^{-1}3$ or $x=n\pi+\tan^{-1}4$.

33. **(a)** Let $t=\tan\frac{x}{2}$, and using the formula. We get,

$$\left\{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}-\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right\}\left\{\frac{4\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}+\frac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}}\right\}+2=0$$

$$\left(\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}\right)\left(\frac{4t}{1-t^2}+\frac{1+t^2}{1-t^2}\right)+2=0$$

$$\Rightarrow\quad \frac{3t^4+6t^3+8t^2-2t-3}{(t^2+1)(1-t^2)}=0$$

Its roots are; $t_1=\frac{1}{\sqrt{3}}$ and $t_2=-\frac{1}{\sqrt{3}}$

Thus the solution of the equation reduces to that of two elementary equations, $\tan\frac{x}{2}=\frac{1}{\sqrt{3}}, \tan\frac{x}{2}=-\frac{1}{\sqrt{3}}$

$\Rightarrow\quad \frac{x}{2}=n\pi\pm\frac{\pi}{6}$

$\Rightarrow\quad x=2n\pi\pm\frac{\pi}{3}$, is required solution.

34. **(c)** Let $(\sin x+\cos x)=t$ and using the equation

$\sin x.\cos x=\frac{t^2-1}{2}$, we get $t-2\sqrt{2}\left(\frac{t^2-1}{2}\right)=0$

$\Rightarrow\quad \sqrt{2}t^2-t-\sqrt{2}=0$

The numbers $t_1=\sqrt{2}, t_2=-\frac{1}{\sqrt{2}}$ are roots of this quadratic equation.Thus the solution of the given equation reduces to the solution of two trigonometrical equation;

$\sin x+\cos x=\sqrt{2}$ or $\sin x+\cos x=-\frac{1}{\sqrt{2}}$

or $\quad \frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x=1$

or $\quad \frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x=-\frac{1}{2}$

or $\quad \sin x.\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\cos x=1$

or $\quad \sin x\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\cos x=-\frac{1}{2}$

$\Rightarrow\quad \sin\left(x+\frac{\pi}{4}\right)=1$ or $\sin\left(x+\frac{\pi}{4}\right)=-\frac{1}{2}$

$\Rightarrow\quad x+\frac{\pi}{4}=(4n+1)\frac{\pi}{2}$ or $x+\frac{\pi}{4}=n\pi+(-1)^n.\left(\frac{\pi}{6}\right)$

$\Rightarrow\quad x=2n\pi+\frac{\pi}{4}$ or $x=n\pi+(-1)^n\frac{\pi}{6}-\left(\frac{\pi}{4}\right)$.

35. **(a)** $\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta=\frac{2}{2}$

$\Rightarrow\quad \cos\theta\cos\frac{\pi}{3}+\sin\theta\sin\frac{\pi}{3}=1$

$\Rightarrow\quad \cos\left(\theta-\frac{\pi}{3}\right)=1=\cos 0^\circ$

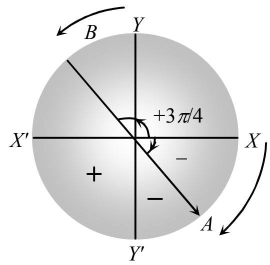
$\Rightarrow\quad \theta-\frac{\pi}{3}=0^\circ$

$\Rightarrow\quad \theta=\frac{\pi}{3}$.

36. **(a)** $\tan\theta$ is negative.

$\therefore\quad \theta$ will lie in 2[nd] or 4[th] quadrant.

For 2[nd] quadrant we will select anticlockwise and for 4[th] quadrant, we will select clockwise direction.



In the first circle two values $\frac{-\pi}{4}$ and $\frac{3\pi}{4}$ are obtained.

Among these two, $\frac{-\pi}{4}$ is numerically least angle. Hence principal value is $\frac{-\pi}{4}$.

37. **(b)** Given, $\cos p\theta=-\cos q\theta=\cos(\pi+q\theta)$

$\Rightarrow\quad p\theta=2n\pi\pm(\pi+q\theta), n\in I$

$\Rightarrow\quad \theta=\frac{(2n+1)\pi}{p-q}$

or $\quad \frac{(2n-1)\pi}{p+q},\ n\in I.$

Both the solutions form an A.P. $\theta = \frac{(2n+1)\pi}{p-q}$ gives us an A.P. with common difference $= \frac{2\pi}{p-q}$ and $\theta = \frac{(2n-1)\pi}{p+q}$ gives us an A.P. with common difference $= \frac{2\pi}{p+q}$.

Certainly, $\frac{2\pi}{p+q} < \left|\frac{2\pi}{p-q}\right|$.

38. **(c)** Given, $\cos\theta = -\frac{1}{2}$ and $0° < \theta < 360°$.

We know that $\cos 60° = \frac{1}{2}$ and

$\cos(180° - 60°) = -\cos 60° = \frac{-1}{2}$ or $\cos 120° = -\frac{1}{2}$.

Similarly $\cos(180° + 60°) = -\cos 60° = -\frac{1}{2}$

or $\cos 240° = \frac{-1}{2}$. Therefore $\theta = 120°$ and $240°$.

39. **(a)** Period of $\sin(ax+b) = \frac{2\pi}{|a|}$

$\therefore$ Period of $\sin 2x = \frac{2\pi}{|2|} = \pi$.

40. **(d)** Period of $\sin\frac{\pi x}{2} = \frac{2\pi}{\pi/2} = 4$

Period of $\cos\frac{\pi x}{3} = \frac{2\pi}{\pi/3} = 6$

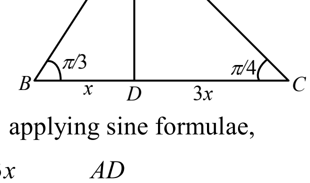
and period of $\tan\frac{\pi x}{4} = \frac{\pi}{\pi/4} = 4$

$\therefore$ Period of $f(x) =$ L.C.M. of (4, 6, 4) =12.

41. **(c)** Let $\angle BAD = \alpha, \angle CAD = \beta$

In ΔADB, applying sine formulae,

we get $\frac{x}{\sin\alpha} = \frac{AD}{\sin\left(\frac{\pi}{3}\right)}$ . . . (*i*)



In ΔADC, applying sine formulae,

we get, $\frac{3x}{\sin\beta} = \frac{AD}{\sin(\pi/4)}$ . . . (*ii*)

Dividing (*i*) by (*ii*), we get,

$\Rightarrow \frac{x}{\sin\alpha} \times \frac{\sin\beta}{3x} = \frac{AD}{\sin\left(\frac{\pi}{3}\right)} \times \frac{\sin\left(\frac{\pi}{4}\right)}{AD}$

$\Rightarrow \frac{\sin\beta}{3\sin\alpha} = \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}}{2}} = \sqrt{\frac{2}{3}} \Rightarrow \frac{\sin\beta}{\sin\alpha} = 3\sqrt{\frac{2}{3}} = \sqrt{6}$

$\therefore \frac{\sin\angle BAD}{\sin\angle CAD} = \frac{\sin\alpha}{\sin\beta} = \frac{1}{\sqrt{6}}$.

42. **(b)** $2ac\sin\frac{A-B+C}{2} = 2ac\sin\frac{\pi-2B}{2} = 2ac\cos B$

$= 2ac\frac{c^2+a^2-b^2}{2ca} = c^2+a^2-b^2$..

43. **(c)** $\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{a^2-(b^2-c^2)}{2ac}$

Now, $AD = \frac{abc}{b^2-c^2}$;

$\therefore \cos B = \frac{a^2 - \frac{abc}{AD}}{2ac}$ Also $AD = b\sin 23°$;

$\therefore \cos B = \frac{a - \frac{c}{\sin 23°}}{2c}$

By sine formulae $\Rightarrow \frac{a}{c} = \frac{\sin(B+23°)}{\sin 23°}$;

$\therefore \cos B = \frac{\left[\frac{\sin(B+23°)}{\sin 23°} - \frac{1}{\sin 23°}\right]}{2}$

$\Rightarrow \sin(23° - B) = -1 = \sin(-90°)$; therefore $23° - B = -90°$

or $B = 113°$.

44. **(b)** $\frac{\cos C + \cos A}{c+a} + \frac{\cos B}{b}$

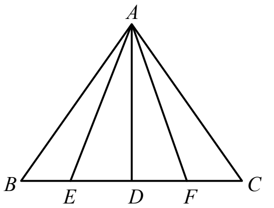
$= \frac{(b\cos C + b\cos A) + (c\cos B + a\cos B)}{b(c+a)}$

$= \frac{(b\cos C + c\cos B) + (b\cos A + a\cos B)}{b(c+a)}$

$= \frac{a+c}{b(c+a)}$ (Using projection formulae) $= \frac{1}{b}$.

45. **(a)** In ΔABC, $AD^2 = m_1^2 = \frac{c^2+b^2}{2} - \frac{a^2}{4}$

In ΔABD, $AE^2 = m_2^2 = \frac{c^2+AD^2}{2} - \frac{\left(\frac{a}{2}\right)^2}{4}$



In $\Delta ADC, AF^2 = m_3^2 = \frac{AD^2 + b^2}{2} - \frac{\left(\frac{a}{2}\right)^2}{4}$

$\therefore \quad m_2^2 + m_3^2 = AD^2 + \frac{b^2 + c^2}{2} - \frac{a^2}{8} = m_1^2 + m_1^2 + \frac{a^2}{4} - \frac{a^2}{8}$

$m_2^2 + m_3^2 = 2m_1^2 + \frac{a^2}{8} \Rightarrow \frac{a^2}{8} = m_2^2 + m_3^2 - 2m_1^2$.

46. **(b)** Let the triangles be $\Delta_1 = ABC_1$ and $\Delta_2 = ABC_2$ A, b, a are given and c has two values. Hence we apply cosine formulae $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ or $c^2 - 2bc\cos A + b^2 - a^2 = 0$.

Above is quadratic in cIf c_1, c_2 be the two values of c, then

$c_1 + c_2 = 2b\cos A,\ c_1c_2 = b^2 - a^2$

$\Delta_1 = \frac{1}{2}ab\sin C_1,\ \Delta_2 = \frac{1}{2}ab\sin C_2$

$\therefore \quad \Delta_1 + \Delta_2 = \frac{1}{2}ab(\sin C_1 + \sin C_2) = \frac{1}{2}abk(2b\cos A)$

$= b^2 ak\cos A = b^2\sin A\cos A = \frac{1}{2}b^2\sin 2A.$

47. **(b)** Take $A = B = C = 60^\circ$, then $\cot\frac{A}{2}, \cot\frac{B}{2}$ and $\cot\frac{C}{2}$ are in A.P. with common difference zero.

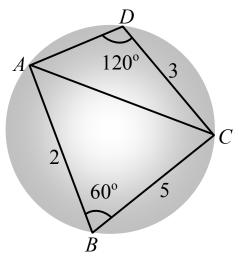
48. **(a)** $\frac{c}{\sin C} = 2R,$

$\therefore \quad c = 2R\sin 90^\circ = 2R$

Also $r = (s - c)\tan\frac{C}{2} = (s - c)$ $[\because \tan 45^\circ = 1]$

$2r = 2s - 2c = a + b - c = a + b - 2R,\ 2(r + R) = a + b.$

49. **(a)** Since $ABCD$ is cyclic quadrilateral and $\angle ABC = 60^\circ$



$\therefore \quad \angle ADC = 180^\circ - 60^\circ = 120^\circ$

Let $AB = 2,\ BC = 5$ and $CD = 3$

In $\Delta ABC,\ AB^2 + AC^2 + 2AB.BC\cos 60^\circ = AC^2$

or $4 + 25 - 2\times 2\times 5\times\frac{1}{2} = AC^2$

$\therefore \quad AC^2 = 19;$

In $\Delta ABC,\ AD^2 + CD^2 + 2AD.CD\cos 60^\circ = AC^2$

or $AD^2 + 9 + 2AD.3.\frac{1}{2} = 19$

or $AD^2 + 3AD - 10 = 0$

or $AD^2 + 5AD - 2AD - 10 = 0$

or $AD(AD + 5) - 2(AD + 5) = 0$

or $(AD - 2)(AD + 5) = 0.$

Therefore, fourth side is $AD = 2$.

50. **(a)** Let r be the radius of the circle and A_1 be its area

$\therefore \quad A_1 = \pi r^2$

Since the perimeter of the circle is the same as the perimeter of a regular polygon of n sides

$\therefore \quad 2\pi r = na$, where '$a$' is the length of one side of the regular polygon, $a = \frac{2\pi r}{n}$

Let A_2 be the area of the polygon, then

$A_2 = \frac{1}{4}na^2.\cot\frac{\pi}{n} = \frac{1}{4}n.\frac{4\pi^2 r^2}{n^2}\cot\frac{\pi}{n} = \pi r^2\cdot\frac{\pi}{n}\cdot\cot\frac{\pi}{n}$

$\therefore \quad A_1 : A_2 = \pi r^2 : \pi r^2\cdot\frac{\pi}{n}\cdot\cot\frac{\pi}{n} = 1 : \frac{\pi}{n} = \tan\frac{\pi}{n} : \frac{\pi}{n}.$

NCERT Exemplar Problems

More than One Answer

51. **(b)** Since, $\tan\theta < 0$

$\therefore$ Angle θ is either in the second or fourth quadrant.

Then, $\sin\theta > 0$ or < 0

$\therefore \quad \sin\theta$ may be $\frac{4}{5}$ or $-\frac{4}{5}$

52. **(a)** Since, $\frac{\alpha}{2} + \frac{\beta}{2} = \left(\pi - \frac{\gamma}{2}\right)$

$\therefore \quad \tan\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \tan\left(\pi - \frac{\gamma}{2}\right)$

$\Rightarrow \quad \frac{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}} = -\tan\frac{\gamma}{2}$

$\Rightarrow \quad \tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

53. **(b)** Given, $A = \sin^2\theta + (1-\sin^2\theta)^2$

$\Rightarrow \quad A = \sin^4\theta - \sin^2\theta + 1$

$\Rightarrow \quad A = \left(\sin^2\theta - \frac{1}{2}\right)^2 + \frac{3}{4}$

$\because \quad 0 \le \sin^2\theta \le 1$

$\Rightarrow \quad 0 \le \left(\sin^2\theta - \frac{1}{2}\right)^2 \le \frac{1}{4}$

$\therefore \quad \frac{3}{4} \le A \le 1$

54. **(c)** $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos 3\frac{\pi}{8}\right)\left(1+\cos 5\frac{\pi}{8}\right)\left(1+\cos 7\frac{\pi}{8}\right)$

$= \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos 3\frac{\pi}{8}\right)\left(1-\cos 3\frac{\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)$

$= \left(1-\cos^2\frac{\pi}{8}\right)\left(1-\cos^2 3\frac{\pi}{8}\right)$

$= \frac{1}{4}\left(2-1-\cos\frac{\pi}{4}\right)\left(2-1-\cos 3\frac{\pi}{4}\right)$

$= \frac{1}{4}\left(1-\cos\frac{\pi}{4}\right)\left(1-\cos 3\frac{\pi}{4}\right)$

$= \frac{1}{4}\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{2}}\right) = \frac{1}{4}\left(1-\frac{1}{2}\right) = \frac{1}{8}$

55. **(b)** $3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]-2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right]$

$= 3(\cos^4\alpha + \sin^4\alpha) - 2(\cos^6\alpha + \tan^6\alpha)$

$= 3(1-2\sin^2\alpha\cos^2\alpha) - 2(1-3\sin^2\alpha\cos^2\alpha)$

$= 3 - 6\sin^2\alpha\cos^2\alpha - 2 + 6\sin^2\alpha\cos^2\alpha = 1$

56. **(c)** $\sqrt{3}\operatorname{cosec} 20° - \sec 20°$

$= \tan 60° \operatorname{cosec} 20° - \sec 20°$

$= \frac{\sin 60°\cos 20° - \cos 60°\cdot\sin 20°}{\cos 60°\cdot\sin 20°\cdot\cos 20°}$

$= \frac{\sin(60°-20°)}{\cos 60°\cdot\sin 20°\cdot\cos 20°}$

$= \frac{\sin 40°}{\frac{1}{2}\cdot\sin 20°\cos 20°} = \frac{2\sin 20°\cos 20°}{\frac{1}{2}\sin 20°\cos 20°} = 4$

57. **(c)** $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4\{(\sin^2 x + \cos^2 x)^3$

$-3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)\}$

$= 3(1-2\sin 2x + \sin^2 2x) + 6 + 6\sin 2x + 4(1-3\sin^2 x\cos^2 x)$

$= 3(1-2\sin 2x + \sin^2 2x + 2 + 2\sin 2x) + 4\left(1-\frac{3}{4}\cdot\sin^2 2x\right)$

$= 13 + 3\sin^2 2x - 3\sin^2 2x = 13$

58. **(c)** Since, $\sin 15° = \frac{1}{2}\sqrt{2-\sqrt{3}}$

and $\cos 15° = \frac{1}{2}\sqrt{2+\sqrt{3}}$

and $\sin 15°\cos 75° = \sin 15°\cdot\sin 15° = \frac{1}{4}(2-\sqrt{3})$.

Therefore, all these values are irrational and

$\sin 15°\cos 15° = \frac{1}{2}\cdot 2\sin 15°\cos 15°$

$= \frac{1}{2}\cdot\sin 30° = \frac{1}{4}$, which is rational.

59. **(c)** Given, $\alpha + \beta = \pi/2 \Rightarrow \alpha = (\pi/2) - \beta$

$\Rightarrow \quad \tan\alpha = \tan(\pi/2 - \beta)$

$\Rightarrow \quad \tan\alpha = \cot\beta \Rightarrow \tan\alpha\tan\beta = 1$

Again, $\beta + \gamma = \alpha$ (given)

$\Rightarrow \quad \gamma = (\alpha - \beta)$

$\Rightarrow \quad \tan\gamma = \tan(\alpha - \beta)$

$\Rightarrow \quad \tan\gamma = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$\Rightarrow \quad \tan\gamma = \frac{\tan\alpha - \tan\beta}{1+1}$

$\therefore \quad 2\tan\gamma = \tan\alpha - \tan\beta$

$\Rightarrow \quad \tan\alpha = \tan\beta + 2\tan\gamma$

60. **(b)** Since, $\sin\theta = \frac{1}{2}$ and $\cos\phi = \frac{1}{3}$

$\Rightarrow \quad \theta = \frac{\pi}{6}$ and $0 < \left(\cos\phi = \frac{1}{3}\right) < \frac{1}{2}$ $\left\{as, 0 < \frac{1}{3} < \frac{1}{2}\right\}$

$\Rightarrow \quad \theta = \frac{\pi}{6}$ and $\cos^{-1}(0) > \phi > \cos^{-1}\left(\frac{1}{2}\right)$

[the sign changed as $\cos x$ is decreasing between $\left(0, \frac{\pi}{2}\right)$

$\Rightarrow \quad \theta = \frac{\pi}{6}$ and $\frac{\pi}{3} < \phi < \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} < \theta + \phi < \frac{2\pi}{3}$

$\therefore \quad \theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$.

61. **(d)** Since, $\cos(\alpha - \beta) = 1$

$\Rightarrow \quad \alpha - \beta = n\pi$,

But $\quad -2\pi < \alpha - \beta < 2\pi$

[as, $\alpha, \beta \in (-\pi, \pi)$]

$\alpha-\beta=0$ ...(i)

Given, $\cos(\alpha+\beta)=\frac{1}{e}$

$\Rightarrow$ $\cos 2\alpha=\frac{1}{e}<1,$

which is true for four values of α.

as, $-2\pi<2\alpha<2\pi$

62. **(b)** As when $\theta\in\left(0,\frac{\pi}{4}\right), \tan\theta<\cot\theta$

Since, $\tan\theta<1$ and $\cot\theta>1$

$\therefore$ $(\tan\theta)^{\cot\theta}<1$

and $(\cot\theta)^{\tan\theta}>1$

$\therefore$ $t_4>t_1$ which only holds in (b).

Therefore, (b) is the answer.

63. **(b)** Given expression is

$$\frac{\tan A}{1-\cot A}+\frac{\cot A}{1-\tan A}=\frac{\sin A}{\cos A}\times\frac{\sin A}{\sin A-\cos A}+\frac{\cos A}{\sin A}\times\frac{\cos A}{\cos A-\sin A}$$

$$=\frac{1}{\sin A-\cos A}\left\{\frac{\sin^3 A-\cos^3 A}{\cos A\sin A}\right\}$$

$$=\frac{\sin^2 A+\sin A\cos A+\cos^2 A}{\sin A\cos A}$$

$$=\frac{1+\sin A\cos A}{\sin A\cos A}=1+\sec A\operatorname{cosec} A$$

64. **(a)** Given equation is

$$2\cos^2\left(\frac{x}{2}\right)\sin^2 x=x^2+x^{-2}, x\le\frac{\pi}{9}$$

LHS $=2\cos^2\left(\frac{x}{2}\right)\sin^2 x<2$

RHS $=x^2+\frac{1}{x^2}\ge 2$ The equation has no real solution.

65. **(c)** Given, $\sin x+\cos x=1$

On dividing and multiplying by $\sqrt{2}$

$$\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x=\frac{1}{\sqrt{2}}$$

$\Rightarrow$ $\sin\left(x+\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{4}\right)$

$\Rightarrow$ $x+\frac{\pi}{4}=n\pi+(-1)^n\frac{\pi}{4}$

$\Rightarrow$ $x=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{4}, n\in I$

Assertion and Reason

66. **(a)** Reason is true form conditional identities $\Rightarrow$ Assertion, is also true.

67. **(c)** Reason is false because $A=\tan(\pi-(B+C))$

$$\tan=-\tan(B+C)=\frac{\tan B+\tan C}{\tan B\tan C-1}$$

In Assertion, If A is obtuse, $\tan A<0$

$\Rightarrow$ $\tan B\tan C<1$ and the Assertion is true.

68. **(c)** L.H.S in Assertion

$$=\frac{\sin A\cos B+\cos A\sin B+\sin A\cos B-\cos A\sin B}{\cos A\cos B-\sin A\sin B+\cos A\cos B+\sin A\sin B}$$

$$=\frac{2\sin A\cos B}{2\cos A\cos B}=\tan A$$

$\Rightarrow$ Assertion is true and Reason is false.

69. **(d)** $\sin 2x+\cos 2x=\frac{2\tan x+1-\tan^2 x}{1+\tan^2 x}=\frac{2-(\tan x-1)^2}{1+\tan^2 x}$

$\Rightarrow$ Reason is true

In Assertion, $2\sin^2\frac{\pi\cos^2\pi}{2}=2\sin^2\frac{\pi\sin 2x}{2}$

$\Rightarrow$ $\cos^2 x=\sin 2x$

$\Rightarrow$ $\cos x(\cos x-2\sin x)=0$

$\Rightarrow$ $\tan x=\frac{1}{2}$ as $\cos x\ne 0$

Form Reason $\sin 2x+\cos 2x=\frac{7}{5}$

$\Rightarrow$ Assertion is false.

70. **(b)** In Assertion, equations have a non-trivial solution if

$$\begin{vmatrix}1 & \sin\alpha & \cos\alpha\\ 1 & \cos\alpha & \sin\alpha\\ 1 & -\sin\alpha & \cos\alpha\end{vmatrix}=0$$

$\Rightarrow$ $2\sin\alpha(\cos\alpha-\sin\alpha)=0$

$\Rightarrow$ $\tan\alpha=1$ [$\therefore$ $\sin\alpha\ne 0$ as $0<\alpha<\pi$]

$\Rightarrow$ $\alpha=\frac{\pi}{4}$ is the only solution

$\Rightarrow$ Assertion is true.

In assertion, $(\sin x+2\cos x)\begin{vmatrix}1 & \cos x & \cos x\\ 1 & \sin x & \cos x\\ 1 & \cos x & \sin x\end{vmatrix}=0$

$\Rightarrow$ $(\sin x+2\cos x)(\cos x-\sin x)^2=0$

$\Rightarrow$ $\tan x=-2$ or $\tan x=1$ which gives only one values of x.

i.e., $x=\frac{\pi}{4}$ as $0<x<\frac{\pi}{2}$

$\Rightarrow$ Reason is also true but does not lead to assertion.

71. (a) $\cos\theta\cos 2\theta\ldots\cos 2^{n-1}\theta$

$$=\frac{1}{2\sin\theta}[2\sin\theta\cos\theta\cos 2\theta\ldots\cos 2^{n-1}\theta]$$

$$=\frac{1}{2^n\sin\theta}(\sin 2^n\theta)=\frac{1}{2^n\sin\theta}\sin(\pi+\theta)$$

$$=\frac{1}{2^n\sin\theta}(-\sin\theta)=\frac{1}{2^n}$$

So, Reason is true which implies Assertion is also true.

72. (a) Reason is true.

In Assertion, $x=3\sqrt{3}\cos\theta,\ y=3\sqrt{3}\sin\theta,$ then

$$x-y=3\sqrt{3}(\cos\theta-\sin\theta)=3\sqrt{3}\times\sqrt{2}\cos\left(\theta+\frac{\pi}{4}\right)\le 3\sqrt{6}$$

follows form Reason.

73. (c) If $\alpha+\beta+\gamma=\frac{\pi}{2}\tan(\alpha+\beta)=\tan\left(\frac{\pi}{2}-\gamma\right)=\frac{1}{\tan\gamma}$

$\Rightarrow$ $\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{1}{\tan\gamma}$

$\Rightarrow$ $\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha=1$

$\Rightarrow$ Reason is False.

In Assertion, $\left(\sqrt{p}+\sqrt{q}+\sqrt{r}\right)^2=p+q+r+2\left(\sqrt{pq}+\sqrt{qr}+\sqrt{rp}\right)$

$\le p+q+r+2(p+q+r)=3(p+q+r)=48$

74. (d) $x^2-x-2=(x-2)(x+1)<0$

$\Rightarrow$ $-1<x<2$

$\Rightarrow$ Reason is true

$2\sin^2 x+3\sin x-2=(2\sin x-1)(\sin x+2)>0$

$\Rightarrow$ $\sin x>\frac{1}{2}$

Which is true if $\frac{\pi}{6}<x\le\frac{\pi}{2}$

i.e., $\frac{11}{21}<x<\frac{11}{7}$, so Assertion is false.

75. (a) Reason is true as $-1\le\cos x\le 1$.

In Assertion, $\cos^7 x+\sin^4 x=1$

$\Rightarrow$ $\cos^7 x+(1-\cos^2 x)^2=1$

$\Rightarrow$ $\cos^7 x+\cos^4 x-2\cos^2 x=0$

$\Rightarrow$ $\cos^2 x(\cos^5 x+\cos^2 x-2)=0$

$\Rightarrow$ Either $\cos x=0$ or $\cos^5 x+\cos^2 x-2=0$

$\Rightarrow$ $\cos x=1$ from Reason

$\Rightarrow$ $x=-\frac{\pi}{2},\frac{\pi}{2}$ as $\cos x=1\ \Rightarrow\ x=0$

$\because$ $P_0=2, P_2=1$

and $P_n-P_{n-2}=(\sin^n\theta+\cos^n\theta)-(\sin^{n-2}\theta+\cos^{n-2}\theta)$

$=-\sin^{n-2}\theta(1-\sin^2\theta)-\cos^{n-2}\theta(1-\cos^2\theta)$

$=-\sin^{n-2}\theta\cos^2\theta-\cos^{n-2}\theta\sin^2\theta$

$=-\sin^2\theta\cos^2\theta(\sin^{n-4}\theta+\cos^{n-4}\theta)$

$=-\sin^2\theta\cos^2\theta P_{n-4}$ . . . (*i*)

$\therefore$ $P_n-P_{n-2}=-\sin^2\theta\cos^2\theta P_{n-4}$ for $n=4$,

$P_4-P_2=-\sin^2\theta\cos^2\theta P_0$

$\Rightarrow$ $P_4-1=-2\sin^2\theta\cos^2\theta$

$(\because P_2=1, P_0=2)$

$\therefore$ $P_4=1-2\sin^2\theta\cos^2\theta$ . . . (*ii*)

for $n=6$,

$P_6-P_4=-\sin^2\theta\cos^2\theta P_2$

$\Rightarrow$ $P_6=P_4-\sin^2\theta\cos^2\theta P_2$

$=1-2\sin^2\theta\cos^2\theta-\sin^2\theta\cos^2\theta$

$\therefore$ $P_6=1-3\sin^2\theta\cos^2\theta$ . . . (*iii*)

Comprehension Based

76. (b) $P_1=m$

$\Rightarrow$ $P_1^2=m^2\quad \sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta=m^2$

$\Rightarrow$ $\sin\theta\cos\theta=\frac{(m^2-1)}{2}$

Now, from Eq. (*iii*)

$P_6=1-3\sin^2\theta\cos^2\theta$

$\Rightarrow$ $(1-P_6)=3(\sin\theta\cos\theta)^2$

$=\frac{3(m^2-1)^2}{4}$

or $4(1-P_6)=3(m^2-1)^2$

77. (c) $2P_6-3P_4+10$

$=2(1-3\sin^2\theta\cos^2\theta)-3(1-2\sin^2\theta\cos^2\theta)+10$

(From Eqs. (*ii*) and (*iii*))

$=2-3+10\ =9$

78. (a) Let $\sin^2\theta\cos^2\theta=k$,

then from Eq. (*i*), $P_n-P_{n-2}=-k\,P_{n-4}$

From Eq. (*ii*), $P_4=1-2k$ and from Eq. (*iii*), $P_6=1-3k$

Put $n=10$,

Then $P_{10}-P_8=-k\,P_6=-k(1-3k)$

$\therefore$ $P_{10}-P_8=3k^2-k$ . . . (*iv*) and put $n = 8$,

then $P_8 - P_6 = -kP_4 = -k(1-2k)$

$\therefore \quad P_8 = P_6 + 2k^2 - k$

$= 1 - 3k + 2k^2 - k$

$\Rightarrow \quad P_8 = 2k^2 - 4k + 1$

From Eq. (iv), $P_{10} = 5k^2 - 5k + 1$

$\therefore \quad 6P_{10} - 15P_8 + 10P_6 + 7$

$= 6(5k^2 - 5k + 1) - 15(2k^2 - 4k + 1) + 10(1 - 3k) + 7 = 8$

79. **(d)** From Eq. (i), $P_{n-2} - P_n = \sin^2\theta\cos^2\theta\, P_{n-4}$

$\therefore \quad \lambda = n - 4$

80. **(c)** Fro Eq. (i), $\dfrac{P_n - P_{n-2}}{P_{n-4}} = -\sin^2\theta\cos^2\theta$

Put $n = 7$,

then $\dfrac{P_7 - P_5}{P_3} = -\sin^2\theta\cos^2\theta \qquad \ldots (v)$

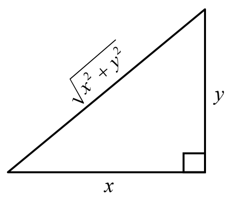
and put $n = 5$, then $\dfrac{P_5 - P_3}{P_1} = -\sin^2\theta\cos^2\theta \qquad \ldots (vi)$

From Eqs. (v) and (vi), we get $\dfrac{P_7 - P_5}{P_3} = \dfrac{P_5 - P_3}{P_1}$

$\therefore \quad \dfrac{P_7 - P_5}{P_5 - P_3} = \dfrac{P_3}{P_1}$

81. **(a)** $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta \qquad \ldots (i)$

and $x\sin\theta - y\cos\theta = 0 \qquad \ldots (ii)$



From Eq. (ii), $\tan\theta = \dfrac{y}{x}$

$\therefore \quad \sin\theta = \dfrac{y}{\sqrt{(x^2+y^2)}}$ and $\cos\theta = \dfrac{x}{\sqrt{(x^2+y^2)}}$

From Eq. (i), $x \times \dfrac{y^3}{(x^2+y^2)^{3/2}} + y \times \dfrac{x^3}{(x^2+y^2)^{3/2}}$

$= \dfrac{xy}{(x^2+y^2)}$

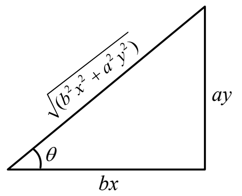
or $\dfrac{(x^2+y^2)}{(x^2+y^2)^{3/2}} = \dfrac{1}{(x^2+y^2)}$

$\Rightarrow \quad (x^2+y^2)^{1/2} = 1$

or $\quad x^2 + y^2 = 1$ which is a circle.

82. **(c)** $\dfrac{x}{a\cos\theta} = \dfrac{y}{b\sin\theta} \qquad \ldots (i)$

and $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2 \qquad \ldots (ii)$



From Eq. (i), $\tan\theta = \dfrac{ay}{bx}$

From Eq. (ii), $\dfrac{ax}{\dfrac{bx}{\sqrt{(b^2x^2+a^2y^2)}}} - \dfrac{by}{\dfrac{ay}{\sqrt{(b^2x^2+a^2y^2)}}} = (a^2 - b^2)$

$\Rightarrow \quad (a^2 - b^2)\sqrt{(b^2x^2 + a^2y^2)} = ab(a^2 - b^2)$

$\Rightarrow \quad b^2x^2 + a^2y^2 = a^2b^2$

$\therefore \quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ which is an ellipse.

83. **(d)** $m + n = 2\tan\theta, m - n\sin\theta \qquad \ldots (i)$

and $mn = \tan^2\theta - \sin^2\theta = \sin^2\theta(\sec^2\theta - 1)$

$= \sin^2\theta\tan^2\theta$

$= \left(\dfrac{m-n}{2}\right)^2\left(\dfrac{m+n}{2}\right)^2$ [from Eq. (i)]

$\therefore \quad (m^2 - n^2)^2 = 16mn$

84. **(b)** $\sin\theta + \cos\theta = a \qquad \ldots (i)$

$\sin^3\theta + \cos^3\theta = b \qquad \ldots (ii)$

From Eq. (i), $\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = a^2$

or $\quad \sin\theta\cos\theta = \dfrac{a^2 - 1}{2} \qquad \ldots (iii)$

From Eq. (ii),

$(\sin\theta + \cos\theta)^3 - 3\sin\theta\cos\theta(\sin\theta + \cos\theta) = b$

$\Rightarrow \quad a^3 - \dfrac{3(a^2-1)}{2}a = b$

[from Eqs. (i) and (iii)]

$\Rightarrow \quad 2a^3 - 3a^3 + 3a = 2b$

$\Rightarrow \quad a^3 + 2b - 3a = 0$

On comparing, we get $\lambda = 1, \mu = 2, v = -3$

$\therefore \quad \lambda + \mu + v = 0$

$\therefore \quad \lambda^3 + \mu^3 + v^3 = 3\lambda\mu v$

$= 3(1)(2)(-3) = -18$

85. **(c)** $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ . . . (i)

and $x\sin\theta-y\cos\theta=\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)}$ . . . (ii)

Squaring Eq. (i), we get

$\frac{x^2}{a^2}\cos^2\theta+\frac{y^2}{b^2}\sin^2\theta+\frac{2xy}{ab}\sin\theta\cos\theta$

$=1=\sin^2\theta+\cos^2\theta$

or $\left(\frac{x^2}{a^2}-1\right)\cos^2\theta+\left(\frac{y^2}{b^2}-1\right)\sin^2\theta+\frac{2xy}{ab}\sin\theta\cos\theta=0$ . . . (iii)

and squaring Eq. (ii),

we get $x^2\sin^2\theta+y^2\cos^2\theta-2xy\sin\theta\cos\theta$

$=a^2\sin^2\theta+b^2\cos^2\theta$

$(x^2-a^2)\sin^2\theta+(y^2-b^2)\cos^2\theta-2xy\sin\theta\cos\theta=0$

$\Rightarrow$ $\left(\frac{x^2-a^2}{ab}\right)\sin^2\theta+\left(\frac{y^2-b^2}{ab}\right)\cos^2\theta-\frac{2xy}{ab}\sin\theta\cos\theta=0$ . . . (iv)

Adding Eqs. (iii) and (iv), then

$\left(\frac{x^2-a^2}{a}\right)\left(\frac{\sin^2\theta}{b}+\frac{\cos^2\theta}{a}\right)+\left(\frac{y^2-b^2}{b}\right)\left(\frac{\sin^2\theta}{b}+\frac{\cos^2\theta}{a}\right)=0$

or $\frac{x^2-a^2}{a}+\frac{y^2-b^2}{b}=0$

or $\frac{x^2}{a}+\frac{y^2}{b}=(a+b)$

86. **(a)** $AM\geq GM$

$\therefore$ $3^x+3^{-x}\geq 2$

$\Rightarrow$ $2\cos\left(\frac{x}{2}\right)\geq 2$

or $\cos\left(\frac{x}{2}\right)\geq 1$

$\therefore$ $\cos\left(\frac{x}{2}\right)=1$ $\left(\because \cos\frac{x}{2} \text{ is never} >1\right)$

$\Rightarrow$ $\frac{x}{2}=2n\pi, n\in I$

$\therefore$ $x=4n\pi$

Hence (a) corresponding to $n=0$, because other values of n do not satisfy the equation.

87. **(c)** Let $y=2\cos^2\left(\frac{x}{2}\right)\sin^2 x=x^2+x^{-2}$

$\therefore$ $y=x^2+x^{-2}\geq 2$ $(\because AM\geq GM)$

$\Rightarrow$ $y\geq 2$ . . . (i)

and $y=2\cos^2\left(\frac{x}{2}\right)\sin^2 x$ $=(1+\cos x)\cdot\sin^2 x$

$=(\text{a number}<2)\cdot(\text{a number}\leq 1)<2$

$ie,$ $y<2$ . . . (ii)

No value of y can be obtained satisfying Eqs. (i) and (ii) simultaneously.

$\Rightarrow$ No real solution of the equation exists.

88. **(a)** $AM\geq GM$

$\therefore$ $5^x+5^{-x}\geq 2$ $(\because \sin e^x=5^x+5^{-x})$

But, the value of $\sin(e^x)$ can never be >1

Hence, the given equation has no solution.

89. **(b)** $2^{\cos ec^2 x}\cdot\sqrt{\left(\frac{1}{2}y^2-y+1\right)}\leq\sqrt{2}$

$\Rightarrow$ $2^{\cos ec^2 x}\sqrt{(y^2-2y+2)}\leq 2$

$\Rightarrow$ $2^{\cos ec^2 x}\cdot\sqrt{\{(y-1)^2+1\}}\leq 2$ . . . (i)

Since, $\cos ec^2 x\geq 1$ for all real x.

$\therefore$ $2^{\cos ec^2 x}\geq 2$ . . . (ii)

Also, $(y-1)^2+1\geq 1$

$\Rightarrow$ $\sqrt{(y-1)^2+1}\geq 1$ . . . (iii)

From Eqs. (ii) and (iii), $2^{\cos ec^2 x}\cdot\sqrt{(y-1)^2+1}\geq 2$ . . . (iv)

Now, Eqs. (i) and (iv), equality holds only when

$2^{\cos ec^2 x}=2$ and $\sqrt{\{(y-1)^2+1\}}=1$

or $\cos ec^2 x=1$

and $(y-1)^2+1=1$

$\Rightarrow$ $\sin x=\pm 1$

and $y=1$

$\Rightarrow$ $x=\pi/2, 3\pi/2$ and $y=1$

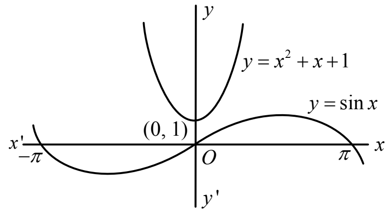
Hence, the solution of the given inequality is

$(x,y)\equiv\left(\frac{\pi}{2},1\right),\left(\frac{3\pi}{2},1\right)$

90. **(a)** Let $y=\sin x=x^2+x+1$

or $y=\sin x$

and $y=x^2+x+1$



$\because$ Graphs of both sides of the equation do not intersect, so the equation has no roots. No. of solutions is zero.

Match the Column

91. **(a)** In the interval $\left(\frac{13\pi}{48}, \frac{14\pi}{48}\right), \cos 2\alpha < 0$

and $\sin 3\alpha > 0$.

$\Rightarrow$ $\frac{\sin 3\alpha}{\cos 2\alpha}$ is negative, therefore, $B \to 1$

Again, in the interval $\left(\frac{18\pi}{48}, \frac{23\pi}{48}\right)$, both $\sin 3\alpha$

and $\cos 2\alpha$ are negative,

So, $\frac{\sin 3\alpha}{\cos 2\alpha}$ is positive, therefore

92. **(a) (A)** Let $y = \frac{7 + 6\tan\theta - \tan^2\theta}{(1 + \tan^2\theta)}$

$= 7\cos^2\theta + 6\sin\theta\cos\theta - \sin^2\theta$

$= 7\left(\frac{1 + \cos 2\theta}{2}\right) + 3\sin 2\theta - \left(\frac{1 - \cos 2\theta}{2}\right)$

$= 3\sin 2\theta + 4\cos 2\theta + 3 - \sqrt{(3^2 + 4^2)} + 3 \le 3\sin 2\theta$

$-\left(\frac{1 - \cos 2\theta}{2}\right) \le \sqrt{(3^2 + 4^2)} + 3$

$\therefore$ $-2 \le y \le 8$

$\Rightarrow$ $\lambda = 8, \mu = -2$

$\Rightarrow$ $\lambda + \mu = 6, \lambda - \mu = 10 (3, 4)$

(B) Let $y = 5\cos\theta + 3\cos(\theta + \pi/3) + 3$

$= 5\cos\theta + 3\left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right) + 3$

$= \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3$

$\therefore$ $3 - \sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{-3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos\theta \frac{3\sqrt{3}}{2}\sin\theta + 3 \le 3$

$+\sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{-3\sqrt{3}}{2}\right)^2}$

$\Rightarrow$ $3 - 7 \le y \le 3 + 7$

$\Rightarrow$ $-4 \le y \le 10$

$\therefore$ $\lambda = 10, \mu = -4$

$\Rightarrow$ $\lambda + \mu = 6, \lambda - \mu = 14 (3, 5)$

(C) Let $y = 1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$

$= 1 + \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{4} + \theta\right)\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$

$= 1 + \cos\left(\frac{\pi}{4} - \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right) = 1 + 3\cos\left(\frac{\pi}{4} - \theta\right)$

$\therefore$ $-1 \le \cos\left(\frac{\pi}{4} - \theta\right) \le 1$

$\Rightarrow$ $-3 \le 3\cos\left(\frac{\pi}{4} - \theta\right) \le 3$

$\Rightarrow$ $1 - 3 \le 1 + 3\cos\left(\frac{\pi}{4} - \theta\right) \le 1 + 3$

$\therefore$ $-2 \le y \le 4$

$\Rightarrow$ $\lambda = 4, \mu = -2$

$\therefore$ $\lambda + \mu = 2, \lambda - \mu = 6 (1, 2)$

Integer

93. **(91)** $\cot\left(7\tfrac{1}{2}°\right) = \frac{1 + \cos 15°}{\sin 15°}$

$= \frac{1 + \frac{\sqrt{3} + 1}{2\sqrt{2}}}{\frac{\sqrt{3} - 1}{2\sqrt{2}}} = 1 + \frac{2\sqrt{2} + \sqrt{3} + 1}{\sqrt{3} - 1}$

$= \frac{(2\sqrt{2} + \sqrt{3} + 1)(\sqrt{3} + 1)}{2}$

$= \frac{2\sqrt{6} + 2\sqrt{2} + 3 + \sqrt{3} + \sqrt{3} + 1}{2}$

$= \sqrt{6} + \sqrt{2} + 2 + \sqrt{3} = \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6}$

and $4\cos 36° = 4\left(\frac{\sqrt{5} + 1}{4}\right) = \sqrt{5} + 1 = \sqrt{5} + \sqrt{1}$

Hence $4\cos 36° + \cot\left(7\frac{1}{2}°\right) = \sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6}$

$\therefore$ $n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 4, n_5 = 5$

and $n_6 = 6$

$\therefore$ $\sum_{i=1}^{6} n_i^6 = n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^3 + n_6^2$

$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 91$

94. **(3636)** We have,

$(a\tan\beta - \sqrt{(a^2 - 1)}\tan\alpha)^2 + (\sqrt{(a^2 - 1)}\tan y - \sqrt{(a^2 + 1)}\tan\beta)^2$

$+ (\sqrt{(a^2 + 1)}\tan\alpha - a\tan\gamma)^2 \ge 0$

$\Rightarrow$ $(a^2 + a^2 - 1 + a^2 + 1)(\tan^2\alpha + \tan^2\beta + \tan^2\gamma)$

$-\{a\tan\alpha + \sqrt{(a^2 - 1)}\tan\beta + \sqrt{(a^2 + 1)}\tan\gamma\}^2 \ge 0$

(using Lagrange's identity)

$\Rightarrow$ $3a^2(\tan^2\alpha + \tan^2\beta + \tan^2\gamma) - (2a)^2 \ge 0$

$\therefore \quad 3(\tan^2\alpha+\tan^2\beta+\tan^2\gamma)\geq 4$

Hence, 2727 $(\tan^2\alpha+\tan^2\beta+\tan^2\gamma)\geq 3636$

$\therefore$ Least value is 3636

95. **(2257)** $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$

$$=32\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

and $\sin A+\sin B+\sin C=4=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(from conditional identities)

$$\therefore \quad \frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=8\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

On comparing, we get $\lambda=8$

Then, $1+2\lambda+3\lambda^2+4\lambda^3=1+16+3(64)+4(512)=2257$

96. **(785)** Here, $\cos 290°=\cos(270°+20°)=\sin 20°$

and $\sin 250°=\sin(270-20°)=-\cos 20°$

$\therefore$ The given expression $\frac{1}{\sin 20°}-\frac{1}{\sqrt{3}\cos 20°}=\lambda$

$$\Rightarrow \quad \frac{1}{\sin 20°}-\frac{\cos 60°}{\sin 60°\cos 20°}=\lambda$$

$$\Rightarrow \quad \frac{\sin 60°\cos 20°-\cos 60°\sin 20°}{\sin 20°\cos 20°\sin 60°}=\lambda$$

$$\Rightarrow \quad \frac{\sin(60°-20°)}{\frac{\sin 40°}{2}\times\frac{\sqrt{3}}{2}}=\lambda$$

$$\therefore \quad \lambda=\frac{4}{\sqrt{3}}$$

$$\Rightarrow \quad \lambda^2=\frac{16}{3}$$

Then $9\lambda^4+81\lambda^2+97=9\times\frac{256}{9}+81\times\frac{16}{3}+97$

$$=256-432+97=785$$

97. **(1225)** $\lambda=\left\{\tan^2\left(\frac{\pi}{16}\right)+\tan^2\left(\frac{7\pi}{16}\right)\right\}+$

$$\left\{\tan^2\left(\frac{2\pi}{16}\right)+\tan^2\left(\frac{6\pi}{16}\right)\right\}$$

$$+\left\{\tan^2\left(\frac{3\pi}{16}\right)+\tan^2\left(\frac{5\pi}{16}\right)\right\}+\tan^2\left(\frac{4\pi}{16}\right)$$

$$=\left\{\tan^2\left(\frac{\pi}{16}\right)+\cot^2\left(\frac{\pi}{2}-\frac{7\pi}{16}\right)\right\}+\left\{\tan^2\left(\frac{2\pi}{16}\right)+\cot^2\left(\frac{\pi}{2}-\frac{6\pi}{16}\right)\right\}$$

$$+\left\{\tan^2\left(\frac{3\pi}{16}\right)+\cot^2\left(\frac{\pi}{2}-\frac{5\pi}{16}\right)\right\}+1$$

$$=\left\{\tan^2\left(\frac{\pi}{16}\right)+\cot^2\left(\frac{\pi}{16}\right)\right\}+\left\{\tan^2\left(\frac{2\pi}{16}\right)+\cot^2\left(\frac{2\pi}{16}\right)\right\}$$

$$+\left\{\tan^2\left(\frac{3\pi}{16}\right)+\cot^2\left(\frac{3\pi}{16}\right)\right\}+1$$

$$=\left\{\tan\left(\frac{\pi}{16}\right)+\cot\left(\frac{\pi}{16}\right)\right\}^2+\left\{\tan\left(\frac{2\pi}{16}\right)+\cot\left(\frac{2\pi}{16}\right)\right\}^2$$

$$+\left\{\tan\left(\frac{3\pi}{11}\right)+\cot\left(\frac{3\pi}{11}\right)\right\}^2-2-2-2+1$$

$$=\frac{1}{\left\{\tan\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{16}\right)\right\}^2}+\frac{1}{\left\{\sin\left(\frac{2\pi}{16}\right)\cos\left(\frac{2\pi}{16}\right)\right\}^2}$$

$$+\frac{1}{\left\{\sin\left(\frac{3\pi}{11}\right)\cos\left(\frac{3\pi}{11}\right)\right\}^2}-5$$

$$=\frac{4}{\sin^2\frac{\pi}{8}}+\frac{4}{\sin^2\frac{\pi}{4}}+\frac{4}{\sin^2\left(\frac{3\pi}{8}\right)}-5$$

$$=4\left\{\frac{1}{\sin^2(\pi/8)}+\frac{1}{\sin^2(3\pi/8)}\right\}+4\cdot 2-5$$

$$=4\left\{\frac{1}{\sin^2(\pi/8)}+\frac{1}{\cos^2(\pi/8)}\right\}+3$$

$$=\frac{4}{\left\{\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)\right\}^3}+3=\frac{16}{\left\{\sin\left(\frac{\pi}{8}\right)\right\}^2}+3$$

$$=32+3=35$$

$\therefore \quad \lambda=35$

Then, $x^y+y^x=35$

$\Rightarrow \quad x=34,\ y=1$

or $\quad x=1,\ y=34$

$$(x+y)^2=(35)^2=1225$$

98. **(4)** $(\sin 3x+\sin x)+\sin 2x$

$$=(\cos 3x+\cos x)+\cos 2x$$

or $\quad 2\sin 2x\cos x+\sin 2x$

$$=2\cos 2x\cos x+\cos 2x$$

$$\sin 2x(2\cos x+1)-\cos 2x(2\cos x+1)=0$$

or $\quad (2\cos x+1)(\sin 2x-\cos 2x)=0$

or $\quad \cos x=-\frac{1}{2},\ \tan 2x=1$

$\therefore \quad x=\frac{2\pi}{3},\frac{4\pi}{3}$ or $\frac{\pi}{8},\frac{5\pi}{8}$

99. **(8)** Given, $\sec\theta + \operatorname{cosec}\theta = c$

$\Rightarrow \sqrt{(1+\tan^2\theta)} + \sqrt{(1+\cot^2\theta)} = c$

$\Rightarrow \sqrt{(1+\lambda^2)} + \sqrt{\left(1+\frac{1}{\lambda^2}\right)} = c$, where $\tan\theta = \lambda$

$\Rightarrow \sqrt{(1+\lambda^2)}\left(\frac{1+\lambda}{\lambda}\right) = c$

$\Rightarrow (1+\lambda^2)^2 + 2\lambda(1+\lambda^2) + \lambda^2 = (c^2+1)\lambda^2$

$\Rightarrow (\lambda^2+\lambda+1)^2 = (c^2+1)\lambda^2$

$\Rightarrow \lambda^2+\lambda+1 \pm \lambda\sqrt{(c^2+1)} = 0$

$\Rightarrow \lambda^2+\lambda+1+\lambda\sqrt{(c^2+1)} = 0$ . . . (*i*)

or $\lambda^2+\lambda+1-\lambda\sqrt{(c^2+1)} = 0$ . . . (*ii*)

Discriminant of equation (*i*) is $D_1 = \{1+\sqrt{(c^2+1)}\}^2 - 4$

Discriminant of equation (*ii*) is $D_2 = \{1-\sqrt{(c^2+1)}\}^2 - 4$

Now, $D_1 = \{3+\sqrt{(c^2+1)}\}\{\sqrt{(c^2+1)}-1\} > 0$

$\therefore$ The equation always has two real roots. It will have only two roots provided $D_2 < 0$

$\Rightarrow 1+c^2+1-2\sqrt{(c^2+1)}-4 < 0$

$\Rightarrow c^2-2 < 2\sqrt{(c^2+1)}$

It is always true, if $c^2 \le 2$. If $c^2 > 2$,

then on squaring we get $c^4 - 4c^2 + 4 < 4c^2 + 4$

$\Rightarrow c^2(c^2-8) < 0$

$\Rightarrow 0 < c^2 < 8$

$\Rightarrow c^2 = 8$

100. **(4950)** On putting $\cos 2x = t$,

We get $2t^2 - 1 + 6 = 7t$

$\Rightarrow t = 1, \frac{5}{2}$

$t = \frac{5}{2}$ (impossible)

$\therefore t = 1$

$\Rightarrow \cos 2x = 1$

$\Rightarrow 2x = 2n\pi$

$\Rightarrow x = n\pi,\ n \in I$

The roots over $[0, 314]$

are $\pi, 2\pi, 3\pi, \ldots 99\pi$

$\because 100\pi > 314$

$\Rightarrow$ sum of roots $= \pi + 2\pi + 3\pi + \ldots + 99\pi = 4950\pi$

$\Rightarrow \lambda = 4950$

* * *