Inverse Trigonometric Functions

Inverse Trigonometric Functions

- Trigonometric functions are not one-one and onto in their usual natural domain. Hence, they are not invertible.

  They can be made one-one and onto (i.e., invertible) by restricting their domain. In this case, the

- Range of the inverse of trigonometric function is the proper subset of the domain of that trigonometric function.

- The branch of the inverse trigonometric function with the restricted range is called the Principal Value Branch.

- There are 6 inverse trigonometric functions. They can be described as

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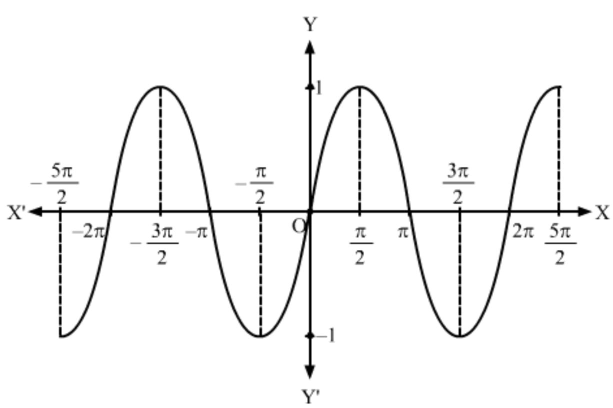
<tr><td rowspan="4">

$y = \sin x$</td><td>Trigonometric function</td><td>$y = \sin x$</td></tr>
<tr><td>Inverse trigonometric function</td><td>$y = \sin^{-1} x$</td></tr>
<tr><td>Domain</td><td>$[-1, 1]$</td></tr>
<tr><td>Principal Value Branch (Range)</td><td>$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$</td></tr>
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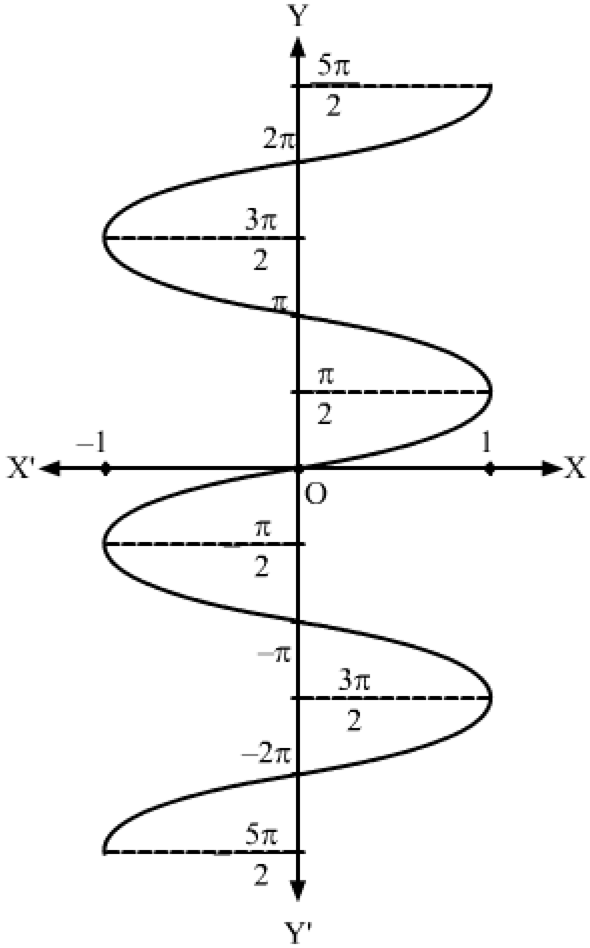
<td>

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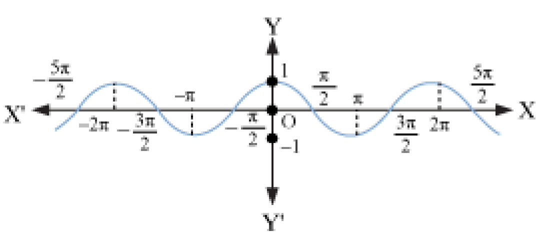
<td rowspan="4">

$y = \cos x$</td>
<td>Trigonometric function</td>
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<td>Domain</td>
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<td>Principal Value Branch (Range)</td>
<td>[0, π]</td>
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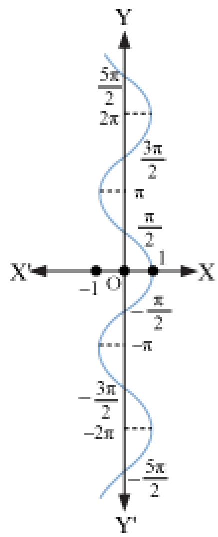
<td>

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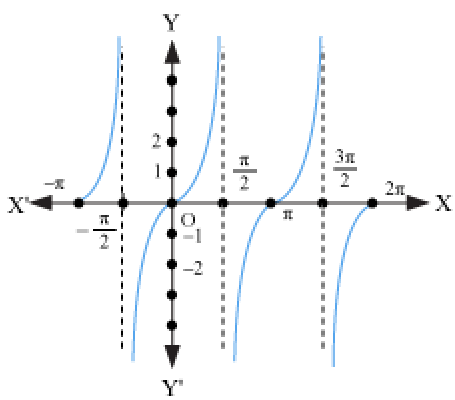
<td rowspan="4">

$y = \tan x$</td>
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<td>Domain</td>
<td>R</td>
</tr>
<tr>
<td>Principal Value Branch (Range)</td>
<td>$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$</td>
</tr>
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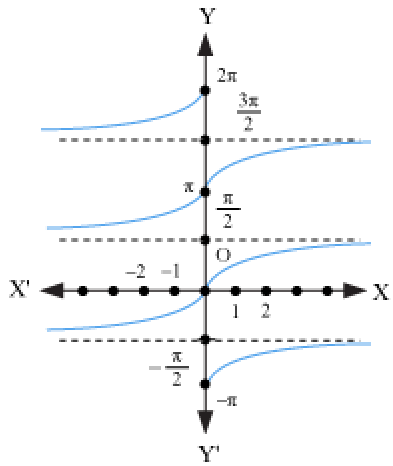
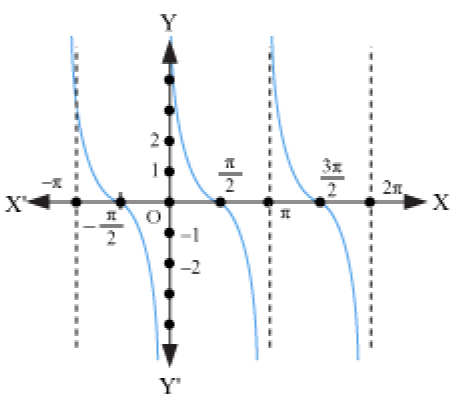
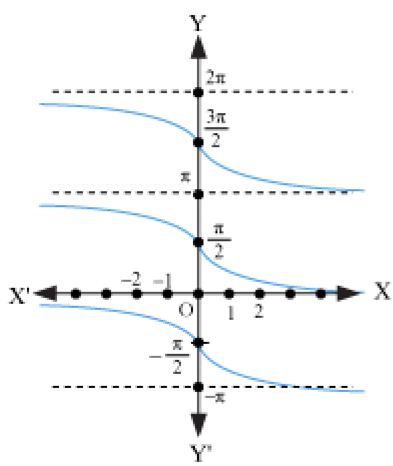
 $y = \tan^{-1}x$		
 $y = \cot x$  $y = \cot^{-1}x$	Trigonometric function	$y = \cot x$
	Inverse trigonometric function	$y = \cot^{-1}x$
	Domain	**R**
	Principal Value Branch (Range)	$(0, \pi)$

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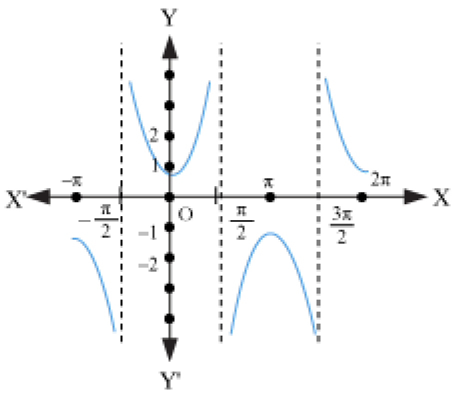
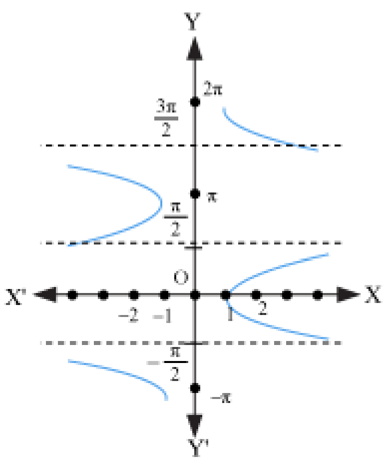
<tr><td rowspan="4">

y = secx


y = sec⁻¹x</td><td>Trigonometric function</td><td>$y = \sec x$</td></tr>
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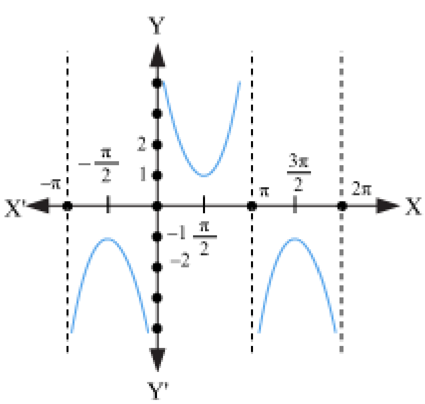
<tr><td rowspan="4">

y = cosecx</td><td>Trigonometric function</td><td>$y = \text{cosec}\, x$</td></tr>
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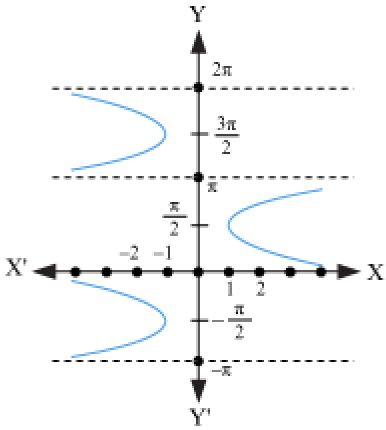 $y = \text{cosec}^{-1}x$		

Solved Examples

Example 1

Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

Solution:

$$\text{Let } \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$$

Accordingly, $\cos y = \frac{\sqrt{3}}{2}$

We know that the range of principal value branch of $\cos^{-1}$ is $[0, \pi]$.

$$\cos y = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right), \text{ where } \frac{\pi}{6} \in [0, \pi]$$

Thus, the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$.

Example 2

Find the value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)+3\cos^{-1}\left(\frac{1}{2}\right)+\tan^{-1}(1)$.

Solution:

$$\text{Let } \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=y$$

$$\text{Accordingly, } \sin y=\frac{\sqrt{3}}{2}.$$

The range of the principal value branch of $\sin^{-1}$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

$$\sin y=\frac{\sqrt{3}}{2}=\sin\left(\frac{\pi}{3}\right), \text{ where } \frac{\pi}{3}\in\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$$

$$\therefore \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3} \qquad \ldots(1)$$

$$\text{Let } \cos^{-1}\left(\frac{1}{2}\right)=u. \text{ Accordingly, } \cos u=\frac{1}{2}$$

The range of the principal value branch of $\cos^{-1}$ is $[0, \pi]$.

$$\cos u=\frac{1}{2}=\cos\left(\frac{\pi}{3}\right), \text{ where } \frac{\pi}{3}\in[0,\ \pi]$$

$$\therefore \cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3} \qquad \ldots(2)$$

$$\text{Let } \tan^{-1}(1)=v. \text{ Accordingly, } \tan v=1.$$

The range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$.

$$\tan v=1=\tan\left(\frac{\pi}{4}\right), \text{ where } \frac{\pi}{4}\in\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$$

$$\therefore \tan^{-1}(1)=\frac{\pi}{4} \qquad \ldots(3)$$

From (1), (2) and (3), we obtain

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)+3\cos^{-1}\left(\frac{1}{2}\right)+\tan^{-1}(1)$$
$$=\frac{\pi}{3}+3.\frac{\pi}{3}+\frac{\pi}{4}$$
$$=\frac{4\pi+12\pi+3\pi}{12}$$
$$=\frac{19\pi}{12}$$

Properties of Inverse Trigonometric Functions

- **Properties of $\sin^{-1}x$**
- $y = \sin^{-1} x \Rightarrow x = \sin y$
- $x = \sin y \Rightarrow y = \sin^{-1} x$
- $\sin(\sin^{-1} x) = x$
- $\sin^{-1}(\sin x) = x$
- $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x$
- $\sin^{-1}(-x) = -\sin^{-1}x$
- **Properties of $\cos^{-1}x$.**
- $y = \cos^{-1} x \Rightarrow x = \cos y$
- $x = \cos y \Rightarrow y = \cos^{-1} x$
- $\cos(\cos^{-1} x) = x$
- $\cos^{-1}(\cos x) = x$
- $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$
- $\cos^{-1}(-x) = \pi - \cos^{-1}x$
- **Properties of $\tan^{-1}x$**

- $x = \tan y \Rightarrow y = \tan^{-1} x$
- $\tan^{-1} (\tan x) = x$
- $\tan (\tan^{-1} x) = x$
- $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x$
- $\tan^{-1} (-x) = -\tan^{-1} x$
- **Properties of $\cot^{-1} x$.**
- $\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1} x$
- $\cot^{-1} (-x) = \pi - \cot^{-1} x$
- **Properties of $\operatorname{cosec}^{-1} x$.**
- $\operatorname{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1} x$
- $\operatorname{cosec}^{-1} (-x) = -\operatorname{cosec}^{-1} x$
- **Properties of $\sec^{-1} x$.**
- $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1} x$
- $\sec^{-1} (-x) = \pi - \sec^{-1} x$
- $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

$$\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

- To understand the proof of the formula $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$,

- $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), xy<1$

  $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), xy>-1$

- To understand the proof of the formula of $\tan^{-1}x+\tan^{-1}y$,

- $$2\tan^{-1}x=\begin{cases}\sin^{-1}\left(\frac{2x}{1+x^2}\right), |x|\le 1\\ \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), x\ge 1\\ \tan^{-1}\left(\frac{2x}{1-x^2}\right), -1<x<1\end{cases}$$

- To understand the proof of the formula $2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

- $2\sin^{-1}x=\sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}$

  $2\cos^{-1}x=\sin^{-1}\left(2x\sqrt{1-x^2}\right), \frac{1}{\sqrt{2}}\le x\le 1$

Solved Examples

Example 1

Solve the equation $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$.

Solution:

$$\left(\tan^{-1} x\right)^2 + \left(\cot^{-1} x\right)^2 = \frac{5\pi^2}{8}$$

$$\Rightarrow \left(\tan^{-1} x + \cot^{-1} x\right)^2 - 2\tan^{-1} x \cot^{-1} x = \frac{5\pi^2}{8} \qquad \left[a^2 + b^2 = (a+b)^2 - 2ab\right]$$

$$\Rightarrow \left(\frac{\pi}{2}\right)^2 - 2\tan^{-1} x.\cot^{-1} x = \frac{5\pi^2}{8} \qquad \left[\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right]$$

$$\Rightarrow 2\tan^{-1} x \cot^{-1} x = \frac{\pi^2}{4} - \frac{5\pi^2}{8}$$

$$\Rightarrow 2\tan^{-1} x \cot^{-1} x = -\frac{3\pi^2}{8}$$

$$\Rightarrow 2\tan^{-1} x \left(\frac{\pi}{2} - \tan^{-1} x\right) = -\frac{3\pi^2}{8}$$

$$\Rightarrow \pi \tan^{-1} x - 2\left(\tan^{-1} x\right)^2 = -\frac{3\pi^2}{8}$$

Let $\tan^{-1} x = y$

$$\Rightarrow 8\pi y - 16y^2 = -3\pi^2$$

$$\Rightarrow 16y^2 - 8\pi y - 3\pi^2 = 0$$

$$\Rightarrow y = \frac{8\pi \pm \sqrt{64\pi^2 + 192\pi^2}}{32} = \frac{8\pi \pm 16\pi}{32}$$

$$\Rightarrow y = \frac{\pi \pm 2\pi}{4}$$

$$\Rightarrow y = -\frac{\pi}{4} \text{ and } \frac{3\pi}{4}$$

$$\Rightarrow \tan^{-1} x = -\frac{\pi}{4} \text{ and } \tan^{-1} x = \frac{3\pi}{4}$$

The principal value branch of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \tan^{-1} x = -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\Rightarrow x = \tan\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow x = -1$$

Example 2

If $\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x$, $x > 0$, then prove that $x=\pm\frac{1}{\sqrt{3}}$.

Solution:

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x$$

$$\Rightarrow 2\tan^{-1}\left(\frac{1-x}{1+x}\right)=\tan^{-1}x$$

$$\Rightarrow \tan^{-1}\left\{\frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2}\right\}=\tan^{-1}x$$

$$\Rightarrow \frac{2\left(1-x^2\right)}{4x}=x$$

$$\Rightarrow 1-x^2=2x^2$$

$$\Rightarrow x^2=\frac{1}{3}$$

$$\Rightarrow x=\pm\frac{1}{\sqrt{3}}$$