

QUADRATIC EQUATION

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➤ BASIC CONCEPT

- ◆ Every algebraic polynomial of second degree is called a quadratic polynomial.

For Example :

(i) $3x^2 + 5x + 7$

(ii) $8x^2 - 6x$

(iii) $5x^2 - 7$

(iv) $\sqrt{2}x^2 + 6x - \sqrt{3}$

- ◆ The general form of quadratic polynomial is $ax^2 + bx + c$; where a, b, c are real numbers, $a \neq 0$ and x is variable.

For a particular quadratic polynomial the values of a, b, c are constant and for this reason a, b and c are also called real constants. For example, in quadratic polynomial $3x^2 - 5x + 8$; $3, -5$ and 8 are constant where as x is variable.

➤ VALUE OF A QUADRATIC POLYNOMIAL

The value of a quadratic polynomial $ax^2 + bx + c$

(i) at $x = \alpha$ is $a(\alpha)^2 + b(\alpha) + c = a\alpha^2 + b\alpha + c$

(ii) at $x = \beta$ is $a\beta^2 + b\beta + c$

(iii) at $x = 5$ is $a(5)^2 + b(5) + c = 25a^2 + 5a + c$

In the same way :

(i) Value of $5x^2 - 3x + 4$ at $x = 2$ is
$$= 5(2)^2 - 3(2) + 4$$
$$= 20 - 6 + 4 = 18$$

(ii) Value of $x^2 - 8x - 15$ at $x = -1$ is
$$= (-1)^2 - 8(-1) - 15$$
$$= 1 + 8 - 15 = -6$$

(iii) Value of $7x^2 - 4$ at $x = \frac{2}{3}$ is $7\left(\frac{2}{3}\right)^2 - 4$
$$= 7 \times \frac{4}{9} - 4$$
$$= \frac{28 - 36}{9} = \frac{8}{9}$$

➤ ZEROS OF A QUADRATIC POLYNOMIAL

The value of the polynomial $x^2 - 7x + 10$ at :

- (i) $x = 1$ is $(1)^2 - 7 \times 1 + 10 = 1 - 7 + 10 = 4$
- (ii) $x = 2$ is $(2)^2 - 7 \times 2 + 10 = 4 - 14 + 10 = 0$
- (iii) $x = 3$ is $(3)^2 - 7 \times 3 + 10 = 9 - 21 + 10 = -2$
- (iv) $x = 5$ is $(5)^2 - 7 \times 5 + 10 = 25 - 35 + 10 = 0$

It is observed here that for $x = 2$ and $x = 5$; the value of polynomial $x^2 - 7x + 10$ is zero. These two values of x are called zeros of the polynomial.

Thus, if for $x = \alpha$, where α is a real number, the value of given quadratic polynomial is zero; the real number α is called zero of the quadratic polynomial.

❖ EXAMPLES ❖

Ex.1 Show that :

- (i) $x = 3$ is a zero of quadratic polynomial $x^2 - 2x - 3$.
- (ii) $x = -2$ is a zero of quadratic polynomial $3x^2 + 7x + 2$.
- (iii) $x = 4$ is not a zero of quadratic polynomial $2x^2 - 7x - 5$.

Sol.(i) The value of $x^2 - 2x - 3$ at $x = 3$ is

$$(3)^2 - 2 \times 3 - 3 = 9 - 6 - 3 = 0$$

$\Rightarrow x = 3$ is a zero of quadratic polynomial $x^2 - 2x - 3$.

(ii) The value of $3x^2 + 7x + 2$ at $x = -2$ is

$$3(-2)^2 + 7(-2) + 2 = 12 - 14 + 2 = 0$$

$\Rightarrow x = -2$ is a zero of quadratic polynomial $3x^2 + 7x + 2$

(iii) The value of $2x^2 - 7x - 5$ at $x = 4$ is

$$2(4)^2 - 7(4) - 5 = 32 - 28 - 5 = -1 \neq 0$$

$\Rightarrow x = 4$ is not a zero of quadratic polynomial $2x^2 - 7x - 5$.

Ex.2 Find the value of m , if $x = 2$ is a zero of quadratic polynomial $3x^2 - mx + 4$.

Sol. Since, $x = 2$ is a zero of $3x^2 - mx + 4$

$$\Rightarrow 3(2)^2 - m \times 2 + 4 = 0$$

$$\Rightarrow 12 - 2m + 4 = 0, \text{ i.e., } m = 8.$$

➤ QUADRATIC EQUATION & IT'S ROOTS

Since, $ax^2 + bx + c$, $a \neq 0$ is a quadratic polynomial, $ax^2 + bx + c = 0$, $a \neq 0$ is called a quadratic equation.

- (i) $-x^2 - 7x + 2 = 0$ is a quadratic equation, as $-x^2 - 7x + 2$ is a quadratic polynomial.
- (ii) $5x^2 - 7x = 0$ is a quadratic equation.
- (iii) $5x^2 + 2 = 0$ is a quadratic equation, but
- (iv) $-7x + 2 = 0$ is not a quadratic equation.

❖ EXAMPLES ❖

Ex.3 Which of the following are quadratic equations, give reason :

(i) $x^2 - 8x + 6 = 0$

(ii) $3x^2 - 4 = 0$

(iii) $2x + \frac{5}{x} = x^2$

(iv) $x^2 + \frac{2}{x^2} = 3$

Sol. (i) Since, $x^2 - 8x + 6$ is a quadratic polynomial

$$\Rightarrow x^2 - 8x + 6 = 0 \text{ is a quadratic equation.}$$

(ii) $3x^2 - 4 = 0$ is a quadratic equation.

(iii) $2x + \frac{5}{x} = x^3$

$$\Rightarrow 2x^2 + 5 = x^3$$

$\Rightarrow x^3 - 2x^2 - 5 = 0$; which is cubic and not a quadratic equation.

(iv) $x^2 + \frac{2}{x^2} = 3$

$$\Rightarrow x^4 + 2 = 2x^2$$

$\Rightarrow x^4 - 2x^2 + 2 = 0$; which is biquadratic and not a quadratic equation.

Ex.4 In each of the following, determine whether the given values are solutions (roots) of the equation or not :

(i) $3x^2 - 2x - 1 = 0$; $x = 1$

(ii) $x^2 + 6x + 5 = 0$; $x = -1$, $x = -5$

(iii) $x^2 + \sqrt{2}x - 4 = 0$; $x = \sqrt{2}$, $x = -2\sqrt{2}$

Sol. (i) Θ Value of $3x^2 - 2x - 1$ at $x = 1$ is

$$3(1)^2 - 2(1) - 1 = 3 - 2 - 1 = 0 = \text{RHS}$$

$\therefore x = 1$ is a solution of the given equation.

(ii) For $x = -1$, L.H.S. $= (-1)^2 + 6(-1) + 5$

$$= 1 - 6 + 5 = 0 = \text{R.H.S.}$$

$\Rightarrow x = -1$ is a solution of the given equation

For $x = -5$, L.H.S. $= (-5)^2 + 6(-5) + 5$

$$= 25 - 30 + 5 = 0 = \text{R.H.S.}$$

$\Rightarrow x = -5$ is a solution of the given equation.

(iii) For $x = \sqrt{2}$, L.H.S. $= x^2 + \sqrt{2}x - 4$

$$= (\sqrt{2})^2 + \sqrt{2}(\sqrt{2}) - 4$$

$$= 2 + 2 - 4 = 0$$

$= \text{R.H.S.}$

$\therefore x = \sqrt{2}$ is a solution of the given equation

For $x = -2\sqrt{2}$,

$$\text{L.H.S.} = (-2\sqrt{2})^2 + \sqrt{2} \times -2\sqrt{2} - 4$$

$$= 4 \times 2 - 2 \times 2 - 4 = 0 \text{ R.H.S.}$$

$\therefore x = -2\sqrt{2}$ is a solution of the given equation.

➤ SOLVING A QUADRATIC EQUATION BY FACTORISATION

Since, $3x^2 - 5x + 2$ is a quadratic polynomial;
 $3x^2 - 5x + 2 = 0$ is a quadratic equation.

Also,

$$3x^2 - 5x + 2 = 3x^2 - 3x - 2x + 2 \text{ [Factorising]}$$

$$= 3x(x - 1) - 2(x - 1)$$

$$= (x - 1)(3x - 2)$$

In the same way :

$$3x^2 - 5x + 2 = 0 \Rightarrow 3x^2 - 3x - 2x + 2 = 0$$

[Factorising L.H.S.]

$$\Rightarrow (x - 1)(3x - 2) = 0$$

$$\text{i.e., } x - 1 = 0 \quad \text{or} \quad 3x - 2 = 0$$

$$\Rightarrow x = 1 \quad \text{or} \quad x = \frac{2}{3};$$

which is the solution of given quadratic equation.

In order to solve the given Quadratic Equation:

1. Clear the fractions and brackets, if given.
2. By transferring each term to the left hand side; express the given equation as ; $ax^2 + bx + c = 0$ or $a + bx + cx^2 = 0$
3. Factorise left hand side of the equation obtained (**the right hand side being zero**).
4. By putting each factor equal to zero; solve it.

❖ EXAMPLES ❖

Ex.5 Solve :

$$(i) \quad x^2 + 3x - 18 = 0 \quad (ii) \quad (x - 4)(5x + 2) = 0$$

$$(iii) \quad 2x^2 + ax - a^2 = 0; \text{ where 'a' is a real number.}$$

Sol. (i) $x^2 + 3x - 18 = 0$

$$\Rightarrow x^2 + 6x - 3x - 18 = 0$$

$$\Rightarrow x(x + 6) - 3(x + 6) = 0$$

$$\text{i.e., } (x + 6)(x - 3) = 0 \Rightarrow x + 6 = 0$$

$$\text{or } x - 3 = 0$$

$$\Rightarrow x = -6 \quad \text{or} \quad x = 3$$

$\therefore$ Roots of the given equation are : -6 and 3

$$(ii) \quad (x - 4)(5x + 2) = 0 \Rightarrow x - 4 = 0$$

$$\text{or } 5x + 2 = 0$$

$$\Rightarrow x = 4 \quad \text{or} \quad x = -\frac{2}{5}$$

$$(iii) \quad 2x^2 + ax - a^2 = 0$$

$$\Rightarrow 2x^2 + 2ax - ax - a^2 = 0$$

$$\Rightarrow 2x(x + a) - a(x + a) = 0$$

$$\text{i.e., } (x + a)(2x - a) = 0$$

$$\Rightarrow x + a = 0 \quad \text{or} \quad 2x - a = 0$$

$$\Rightarrow x = -a \quad \text{or} \quad x = \frac{a}{2}$$

Ex.6 Solve the following quadratic equations :

$$(i) \quad x^2 + 5x = 0 \quad (ii) \quad x^2 = 3x$$

$$(iii) \quad x^2 = 4$$

Sol. (i) $x^2 + 5x = 0 \Rightarrow x(x + 5) = 0$

$$\Rightarrow x = 0 \quad \text{or} \quad x + 5 = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = -5$$

$$(ii) x^2 = 3x$$

$$\Rightarrow x^2 - 3x = 0$$

$$\Rightarrow x(x - 3) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 3$$

$$(iii) x^2 = 4$$

$$\Rightarrow x = \pm 2$$

Ex.7 Solve the following quadratic equations :

$$(i) 7x^2 = 8 - 10x$$

$$(ii) 3(x^2 - 4) = 5x$$

$$(iii) x(x + 1) + (x + 2)(x + 3) = 42$$

Sol.

$$(i) 7x^2 = 8 - 10x$$

$$\Rightarrow 7x^2 + 10x - 8 = 0$$

$$\Rightarrow 7x^2 + 14x - 4x - 8 = 0$$

$$\Rightarrow 7x(x + 2) - 4(x + 2) = 0$$

$$\Rightarrow (x + 2)(7x - 4) = 0$$

$$\Rightarrow x + 2 = 0 \text{ or } 7x - 4 = 0$$

$$\Rightarrow x = -2 \text{ or } x = \frac{4}{7}$$

$$(ii) 3(x^2 - 4) = 5x$$

$$\Rightarrow 3x^2 - 5x - 12 = 0$$

$$\Rightarrow 3x^2 - 9x + 4x - 12 = 0$$

$$\Rightarrow 3x(x - 3) + 4(x - 3) = 0$$

$$\Rightarrow (x - 3)(3x + 4) = 0$$

$$\Rightarrow x - 3 = 0 \text{ or } 3x + 4 = 0$$

$$\Rightarrow x = 3 \text{ or } x = -\frac{4}{3}$$

$$(iii) x(x + 1) + (x + 2)(x + 3) = 42$$

$$\Rightarrow x^2 + x + x^2 + 3x + 2x + 6 - 42 = 0$$

$$\Rightarrow 2x^2 + 6x - 36 = 0$$

$$\Rightarrow x^2 + 3x - 18 = 0$$

$$\Rightarrow x^2 + 6x - 3x - 18 = 0$$

$$\Rightarrow x(x + 6) - 3(x + 6) = 0$$

$$\Rightarrow (x + 6)(x - 3) = 0$$

$$\Rightarrow x = -6 \text{ or } x = 3$$

$$\text{Ex.8} \quad \text{Solve for } x : 12abx^2 - (9a^2 - 8b^2)x - 6ab = 0$$

Sol. Given equation is :

$$12abx^2 - 9a^2x + 8b^2x - 6ab = 0$$

$$\Rightarrow 3ax(4bx - 3a) + 2b(4bx - 3a) = 0$$

$$\Rightarrow (4bx - 3a)(3ax + 2b) = 0$$

$$\Rightarrow 4bx - 3a = 0 \text{ or } 3ax + 2b = 0$$

$$\Rightarrow x = \frac{3a}{4b} \text{ or } x = -\frac{2b}{3a}$$

➤ SOLVING A QUADRATIC EQUATION BY COMPLETING THE SQUARE

Every quadratic equation can be converted in the form :

$$(x + a)^2 - b^2 = 0 \text{ or } (x - a)^2 - b^2 = 0.$$

Steps :

1. Bring, if required, all the term of the quadratic equation to the left hand side.
2. Express the terms containing x as $x^2 + 2xy$ or $x^2 - 2xy$.
3. Add and subtract y^2 to get $x^2 + 2xy + y^2 - y^2$ or $x^2 - 2xy + y^2 - y^2$; which gives $(x + y)^2 - y^2$ or $(x - y)^2 - y^2$.

Thus,

$$(i) x^2 + 8x = 0 \Rightarrow x^2 + 2x \times 4 = 0$$

$$\Rightarrow x^2 + 2x \times 4 + 4^2 - 4^2 = 0$$

$$\Rightarrow (x + 4)^2 - 16 = 0$$

$$(ii) x^2 - 8x = 0 \Rightarrow x^2 - 2 \times x \times 4 = 0$$

$$\Rightarrow x^2 - 2 \times x \times 4 + 4^2 - 4^2 = 0$$

$$\Rightarrow (x - 4)^2 - 16 = 0$$

❖ EXAMPLES ❖

Ex.9 Find the roots of the following quadratic equations (if they exist) by the method of completing the square.

$$(i) 2x^2 - 7x + 3 = 0 \quad (ii) 4x^2 + 4\sqrt{3}x + 3 = 0$$

$$(iii) 2x^2 + x + 4 = 0$$

Sol. (i) $2x^2 - 7x + 3 = 0 \Rightarrow x^2 - \frac{7}{2}x + \frac{3}{2} = 0$

[Dividing each term by 2]

$$\Rightarrow x^2 - 2 \times x \times \frac{7}{4} + \frac{3}{2} = 0$$

$$\Rightarrow x^2 - 2 \times x \times \frac{7}{4} + \left(\frac{7}{4}\right)^2 - \left(\frac{7}{4}\right)^2 + \frac{3}{2} = 0$$

$$\Rightarrow \left(x - \frac{7}{4}\right)^2 - \frac{49}{16} + \frac{3}{2} = 0$$

$$\Rightarrow \left(x - \frac{7}{4}\right)^2 - \left(\frac{49-24}{16}\right) = 0$$

$$\Rightarrow \left(x - \frac{7}{4}\right)^2 - \frac{25}{16} = 0$$

$$\text{i.e., } \left(x - \frac{7}{4}\right)^2 = \frac{25}{16} \Rightarrow x - \frac{7}{4} = \pm \frac{5}{4}$$

$$\text{i.e., } x - \frac{7}{4} = \frac{5}{4} \quad \text{or} \quad x - \frac{7}{4} = -\frac{5}{4}$$

$$\Rightarrow x = \frac{7}{4} + \frac{5}{4} \quad \text{or} \quad x = \frac{7}{4} - \frac{5}{4}$$

$$\Rightarrow x = 3 \quad \text{or} \quad x = \frac{1}{2}$$

$$(ii) 4x^2 + 4\sqrt{3}x + 3 = 0$$

$$\Rightarrow x^2 + \sqrt{3}x + \frac{3}{4} = 0$$

$$\text{i.e., } x^2 + 2 \times x \times \frac{\sqrt{3}}{2} + \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{3}{4} = 0$$

$$\Rightarrow \left(x + \frac{\sqrt{3}}{2}\right)^2 - \frac{3}{4} + \frac{3}{4} = 0$$

$$\text{i.e., } \left(x + \frac{\sqrt{3}}{2}\right)^2 = 0$$

$$\Rightarrow x + \frac{\sqrt{3}}{2} = 0 \quad \text{and} \quad x = -\frac{\sqrt{3}}{2}$$

$$\therefore \text{ Roots are : } -\frac{\sqrt{3}}{2} \text{ and } -\frac{\sqrt{3}}{2}$$

$$(iii) 2x^2 + x + 4 = 0 \Rightarrow x^2 + \frac{x}{2} + 2 = 0$$

$$\text{i.e., } x^2 + 2 \times x \times \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + 2 = 0$$

$$\Rightarrow \left(x + \frac{1}{4}\right)^2 - \frac{1}{16} + 2 = 0$$

$$\Rightarrow \left(x + \frac{1}{4}\right)^2 + \frac{31}{16} = 0$$

$$\left[-\frac{1}{16} + 2 = \frac{-1+32}{16} = \frac{31}{16}\right]$$

$$\text{i.e., } \left(x + \frac{1}{4}\right)^2 = -\frac{31}{16}$$

This is not possible as the square of a real number can not be negative.

➤ SOLVING A QUADRATIC EQUATION BY USING QUADRATIC FORMULA

Hindu Method (Sri Dharacharya Method) :

By completing the perfect square as

$$ax^2 + bx + c = 0 \Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Adding and subtracting $\left(\frac{b}{2a}\right)^2$

$$\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] = 0$$

$$\text{Which gives, } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence the Quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$) has two roots, given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Note : Every quadratic equation has at most two and only two real roots.

❖ EXAMPLES ❖

Ex.10 Solve the following quadratic equations by using quadratic formula :

$$(i) x^2 - 7x + 12 = 0$$

$$(ii) 3x^2 - x - 10 = 0$$

Sol. (i) Comparing the given equation $x^2 - 7x + 12 = 0$ with standard quadratic equation $ax^2 + bx + c = 0$; we get : $a = 1$, $b = -7$ and $c = 12$

$$\therefore b^2 - 4ac = (-7)^2 - 4 \times 1 \times 12$$

$$= 49 - 48 = 1$$

$$\text{and } \sqrt{b^2 - 4ac} = \sqrt{1} = 1$$

$$\Theta \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{7 \pm 1}{2 \times 1} = \frac{7+1}{2} \text{ or } \frac{7-1}{2} = 4 \text{ or } 3$$

(ii) Comparing the given equation

$$3x^2 - x - 10 = 0 \text{ with equation}$$

$$ax^2 + bx + c = 0; \text{ we get : } a = 3, b = -1 \text{ and } c = -10$$

$$\therefore b^2 - 4ac = (-1)^2 - 4 \times 3 \times -10 = 1 + 120 = 121$$

$$\text{and } \sqrt{b^2 - 4ac} = \sqrt{121} = 11$$

$$\text{Hence, } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{1 \pm 11}{2} = \frac{1+11}{2} \text{ or } \frac{1-11}{2} = 6 \text{ or } -5$$

Ex.11 For a quadratic equation $ax^2 + bx + c = 0$,

$$\text{where } a \neq 0, \text{ prove that : } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sol. $ax^2 + bx + c = 0$

$$\Rightarrow 4a^2x^2 + 4abx + 4ac = 0$$

[Multiplying by '4a']

$$\Rightarrow (2ax)^2 + 2 \times 2ax \times b + b^2 - b^2 + 4ac = 0$$

$$\Rightarrow (2ax + b)^2 - b^2 + 4ac = 0$$

$$\Rightarrow (2ax + b)^2 = b^2 - 4ac$$

$$\Rightarrow 2ax + b = \pm \sqrt{b^2 - 4ac}$$

$$\Rightarrow 2ax = -b \pm \sqrt{b^2 - 4ac}$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Ex.12 Solve, by using quadratic formula, each of the following equations :

(i) $2x^2 + 5\sqrt{3}x + 6 = 0$

(ii) $3x^2 + 2\sqrt{5}x - 5 = 0$

Sol. (i) Comparing $2x^2 + 5\sqrt{3}x + 6 = 0$ with

$$ax^2 + bx + c = 0, \text{ we get :}$$

$$a = 2, b = 5\sqrt{3} \text{ and } c = 6$$

$$b^2 - 4ac = (5\sqrt{3})^2 - 4 \times 2 \times 6$$

$$= 25 \times 3 - 48 = 27$$

$$\sqrt{b^2 - 4ac} = \sqrt{27} = \sqrt{3 \times 3 \times 3} = 3\sqrt{3}$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5\sqrt{3} \pm 3\sqrt{3}}{2 \times 2}$$

$$= \frac{-5\sqrt{3} + 3\sqrt{3}}{4} \text{ or } \frac{-5\sqrt{3} - 3\sqrt{3}}{4}$$

$$= \frac{-2\sqrt{3}}{4} \text{ or } \frac{-8\sqrt{3}}{4} = -\frac{\sqrt{3}}{2} \text{ or } -2\sqrt{3}$$

(ii) Comparing $3x^2 + 2\sqrt{5}x - 5 = 0$ with $ax^2 + bx + c = 0$, we get :

$$a = 3, b = 2\sqrt{5} \text{ and } c = -5$$

$$b^2 - 4ac = (2\sqrt{5})^2 - 4 \times 3 \times -5 = 4 \times 5 + 60 = 80$$

$$\sqrt{b^2 - 4ac} = \sqrt{80} = \sqrt{16 \times 5} = 4\sqrt{5}$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2\sqrt{5} \pm 4\sqrt{5}}{2 \times 3}$$

$$= \frac{-2\sqrt{5} + 4\sqrt{5}}{6} \text{ or } \frac{-2\sqrt{5} - 4\sqrt{5}}{6}$$

$$= \frac{2\sqrt{5}}{6} \text{ or } \frac{-6\sqrt{5}}{6} = \frac{\sqrt{5}}{3} \text{ or } -\sqrt{5}$$

Ex.13 Using the quadratic formula, solve the equation:

$$a^2b^2x^2 - (4b^2 - 3a^4)x - 12a^2b^2 = 0$$

Sol. Comparing given equation with

$$Ax^2 + Bx + C = 0, \text{ we get :}$$

$$A = a^2b^2, B = -(4b^2 - 3a^4) \text{ and } C = -12a^2b^2$$

$$\therefore B^2 - 4AC = (4b^2 - 3a^4)^2 - 4 \times a^2b^2 \times (-12a^2b^2)$$

$$= 16b^8 + 9a^8 - 24a^4b^4 + 48a^4b^4$$

$$= 16b^8 + 9a^8 + 24a^4b^4 = (4b^4 + 3a^4)^2$$

$$\sqrt{B^2 - 4AC} = 4b^4 + 3a^4$$

$$\therefore x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$\begin{aligned}
&= \frac{(4b^4 - 3a^4) \pm (4b^2 + 3a^4)}{2 \times a^2 b^2} \\
&= \frac{4b^4 - 3a^4 + 4b^4 + 3a^4}{2a^2 b^2} \\
&\text{or } \frac{4b^4 - 3a^4 - 4b^4 - 3a^4}{2a^2 b^2} \\
&= \frac{8b^4}{2a^2 b^2} \text{ or } \frac{-6a^4}{2a^2 b^2} = \frac{4b^2}{a^2} \text{ or } \frac{-3a^2}{b^2}
\end{aligned}$$

► NATURE OR CHARACTER OF THE ROOTS OF A QUADRATIC EQUATION

The nature of the roots depends on the value of $b^2 - 4ac$. $b^2 - 4ac$ is called the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$ and is generally, denoted by D .

$$\therefore D = b^2 - 4ac$$

◆ If $D > 0$, i.e., $b^2 - 4ac > 0$, i.e., $b^2 - 4ac$ is positive; **the roots are real and unequal**. Also,

(i) If $b^2 - 4ac$ is a perfect square, the roots are rational and unequal.

(ii) If $b^2 - 4ac$ is positive but not perfect square, the roots are irrational and unequal.

◆ If $D = 0$, i.e., $b^2 - 4ac = 0$; **the roots are real and equal**.

◆ If $D < 0$, i.e., $b^2 - 4ac < 0$; i.e., $b^2 - 4ac$ is negative; the roots are not real, i.e., **the roots are imaginary**.

❖ EXAMPLES ❖

Ex.14 Without solving, examine the nature of roots of the equations :

(i) $2x^2 + 2x + 3 = 0$

(ii) $2x^2 - 7x + 3 = 0$

(iii) $x^2 - 5x - 2 = 0$

(iv) $4x^2 - 4x + 1 = 0$

Sol. (i) Comparing $2x^2 + 2x + 3 = 0$

with $ax^2 + bx + c = 0$; we get : $a = 2$, $b = 2$ and $c = 3$

$$D = b^2 - 4ac = (2)^2 - 4 \times 2 \times 3 = 4 - 24$$

$$= -20; \text{ which is negative.}$$

$\therefore$ The roots of the given equation are imaginary.

(ii) Comparing $2x^2 - 7x + 3 = 0$

with $ax^2 + bx + c = 0$;

we get : $a = 2$, $b = -7$ and $c = 3$

$$D = b^2 - 4ac = (-7)^2 - 4 \times 2 \times 3$$

$$= 49 - 24 = 25, \text{ which is perfect square.}$$

$\therefore$ The roots of the given equation are rational and unequal.

(iii) Comparing $x^2 - 5x - 2 = 0$

with $ax^2 + bx + c = 0$;

we get : $a = 1$, $b = -5$ and $c = -2$

$$D = b^2 - 4ac = (-5)^2 - 4 \times 1 \times -2$$

$$= 25 + 8 = 33 ; \text{ which is positive but not a perfect square.}$$

$\therefore$ The roots of the given equation are irrational and unequal.

(iv) Comparing $4x^2 - 4x + 1 = 0$

with $ax^2 + bx + c = 0$;

we get : $a = 4$, $b = -4$, and $c = 1$

$$D = b^2 - 4ac = (-4)^2 - 4 \times 4 \times 1$$

$$= 16 - 16 = 0$$

$\therefore$ Roots are real and equal.

Ex.15 For what value of m , are the roots of the equation $(3m + 1)x^2 + (11 + m)x + 9 = 0$ equal?

Sol. Comparing the given equation

with $ax^2 + bx + c = 0$;

we get : $a = 3m + 1$, $b = 11 + m$ and $c = 9$

$\therefore$ Discriminant, $D = b^2 - 4ac$

$$= (11 + m)^2 - 4(3m + 1) \times 9$$

$$= 121 + 22m + m^2 - 108m - 36$$

$$= m^2 - 86m + 85$$

$$= m^2 - 85m - m + 85$$

$$= m(m - 85) - 1(m - 85)$$

$$= (m - 85)(m - 1)$$

Since the roots are equal, $D = 0$

$$\Rightarrow (m - 85)(m - 1) = 0$$

$$\Rightarrow m - 85 = 0 \text{ or } m - 1 = 0$$

$$\Rightarrow m = 85 \text{ or } m = 1$$

SUM AND PRODUCT OF THE ROOTS

Let α and β be the two roots of the quadratic equation $ax^2 + bx + c = 0$.

$$\text{Since, } ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{then, let : } \alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$\text{and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$\therefore$ The sum of the roots $= \alpha + \beta$

$$\begin{aligned} &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2b}{2a} = -\frac{b}{a} \end{aligned}$$

And, the product of the roots $= \alpha \cdot \beta$

$$\begin{aligned} &= \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a} \right) \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a} \right) \\ &= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2} = \frac{b^2 - (b^2 - 4ac)}{4a^2} \\ &= \frac{b^2 - b^2 + 4ac}{4a^2} \\ &= \frac{4ac}{4a^2} = \frac{c}{a} \end{aligned}$$

$\therefore$ If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$; then :

(i) The sum of the roots

$$\begin{aligned} &= \alpha + \beta = -\frac{b}{a} \\ &= -\frac{\text{coefficient of } x}{\text{coefficient of } x^2} \end{aligned}$$

(ii) The product of the roots

$$= \alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

TO CONSTRUCT A QUADRATIC EQUATION WHOSE ROOTS ARE GIVEN

$$x^2 - (\text{sum of roots})x + \text{product of roots} = 0$$

To get the quadratic equation with given roots :

(i) Find the sum of the roots.

(ii) Find the product of the roots.

(iii) Substitute the values of steps (i) and (ii) in

$$x^2 - (\text{sum of the roots})x + (\text{product of roots}) = 0$$

and get the required quadratic equation.

❖ EXAMPLES ❖

Ex.16 For each quadratic equation given below, find the sum of the roots and the product of the roots :

$$(i) \quad x^2 + 3x - 6 = 0 \quad (ii) \quad 2x^2 + 5\sqrt{3}x + 6 = 0$$

$$(iii) \quad 3x^2 + 2\sqrt{5}x - 5 = 0$$

Sol.(i) Comparing $x^2 + 3x - 6 = 0$

$$\text{with } ax^2 + bx + c = 0,$$

we get :

$$a = 1, b = 3 \text{ and } c = -6$$

$$\therefore \text{ The sum of the roots} = -\frac{b}{a} = -\frac{3}{1}$$

$$\text{And, the product of the roots} = \frac{c}{a} = \frac{-6}{1} = -6$$

(ii) Comparing $2x^2 + 5\sqrt{3}x + 6 = 0$ with $ax^2 + bx + c = 0$; we get :

$$a = 2, b = 5\sqrt{3} \text{ and } c = 6$$

$$\therefore \text{ The sum of the roots} = -\frac{b}{a} = -\frac{5\sqrt{3}}{2}$$

$$\text{And, the product of the roots} = \frac{c}{a} = \frac{6}{2} = 3$$

(iii) Comparing $3x^2 + 2\sqrt{5}x - 5 = 0$

$$\text{with } ax^2 + bx + c = 0; \text{ we get :}$$

$$a = 3, b = 2\sqrt{5} \text{ and } c = -5$$

$$\therefore \text{ The sum of the roots} = -\frac{b}{a} = -\frac{2\sqrt{5}}{3}$$

$$\text{and, the product of the roots} = \frac{c}{a} = \frac{-5}{3}$$

Ex.17 Construct the quadratic equation whose roots are given below -

(i) $3, -3$

(ii) $3 + \sqrt{3}, 3 - \sqrt{3}$

(iii) $\frac{2+\sqrt{5}}{2}, \frac{2-\sqrt{5}}{2}$

Sol. (i) Since, the sum of the roots

$$= (3) + (-3) = 3 - 3 = 0$$

and, the product of the roots

$$= (3)(-3) = -9$$

$\therefore$ The required quadratic equation is :

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

$$\Rightarrow x^2 - (0)x + (-9) = 0, \text{ i.e., } x^2 - 9 = 0$$

(ii) Since, the sum of the roots

$$= 3 + \sqrt{3} + 3 - \sqrt{3} = 6$$

and, the product of the roots

$$= (3 + \sqrt{3})(3 - \sqrt{3}) = 9 - 3 = 6$$

$\therefore$ The required quadratic equation is :

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

$$\Rightarrow x^2 - 6x + 6 = 0$$

(iii) Since, the sum of the roots

$$= \frac{2+\sqrt{5}}{2} + \frac{2-\sqrt{5}}{2} = \frac{2+\sqrt{5}+2-\sqrt{5}}{2} = \frac{4}{2} = 2$$

and, the product of the roots

$$= \left(\frac{2+\sqrt{5}}{2} \right) \left(\frac{2-\sqrt{5}}{2} \right) = \frac{4-5}{4} = -\frac{1}{4}$$

$\therefore$ The required quadratic equation is :

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

$$\Rightarrow x^2 - 2x + \left(-\frac{1}{4} \right) = 0$$

$$\Rightarrow x^2 - 2x - \frac{1}{4} = 0,$$

$$\text{i.e., } 4x^2 - 8x - 1 = 0$$

Ex.18 If a and c are such that the quadratic equation $ax^2 - 5x + 3 = 0$ has 10 as the sum of the roots and also as the product of the roots, find a and c .

Sol. For $ax^2 - 5x + c = 0$

$$\text{the sum of roots} = - \frac{\text{coefficient of } x}{\text{coefficient of } x^2}$$

$$= - \frac{-5}{a} = \frac{5}{a}$$

and the product of roots

$$= \frac{\text{constant term}}{\text{coefficient of } x^2} = \frac{c}{a}$$

Given : The sum of the roots = 10

$$\Rightarrow \frac{5}{a} = 10, \text{ i.e., } 10a = 5 \Rightarrow a = \frac{5}{10} = \frac{1}{2}$$

The product of roots = 10

$$\Rightarrow \frac{c}{a} = 10 \Rightarrow c = 10a = 10 \times \frac{1}{2} = 5$$

$$\Rightarrow a = \frac{1}{2} \text{ and } c = 5$$

Ex.19 If one of the roots of the quadratic equation $2x^2 + px + 4 = 0$ is 2, find the value of p . also find the value of the other roots.

Sol. As, 2 is one of the roots, $x = 2$ will satisfy the equation $2x^2 + px + 4 = 0$

$$\Rightarrow 2(2)^2 + p(2) + 4 = 0$$

$$\Rightarrow 8 + 2p + 4 = 0$$

$$\text{i.e., } 2p = -12 \text{ and } p = -6$$

Substituting $p = -6$ in the equation

$$2x^2 + px + 4 = 0; \text{ we get : } 2x^2 - 6x + 4 = 0$$

$$\Rightarrow x^2 - 3x + 2 = 0$$

[Dividing each term by 2]

$$\Rightarrow x^2 - 2x - x + 2 = 0$$

$$\Rightarrow x(x - 2) - (x - 1) = 0$$

$$\Rightarrow x - 2 = 0 \quad \text{or} \quad x - 1 = 0$$

$$\Rightarrow x = 2 \quad \text{or} \quad x = 1$$

$\therefore$ The other (second) root is 1.

Ex.20 In the following, find the value (s) of p so that the given equation has equal roots.

(i) $3x^2 - 5x + p = 0$

(ii) $2px^2 - 8x + p = 0$

Sol. (i) Comparing $3x^2 - 5x + p = 0$

with $ax^2 + bx + c = 0$,

we get : $a = 3$, $b = -5$ and $c = p$

Since, the roots are equal ; the discriminant

$$b^2 - 4ac = 0$$

$$\text{i.e., } (-5)^2 - 4 \times 3 \times p = 0$$

$$\Rightarrow 25 - 12p = 0 \text{ and } p = \frac{25}{12} = 2\frac{1}{12}$$

(ii) Comparing $2px^2 - 8x + p = 0$

with $ax^2 + bx + c = 0$;

we get : $a = 2p$, $b = -8$ and $c = p$

$$b^2 - 4ac = 0 \text{ [Given, that the roots are equal]}$$

$$\Rightarrow (-8)^2 - 4 \times 2p \times p = 0$$

$$\Rightarrow 64 - 8p^2 = 0$$

$$\Rightarrow -8p^2 = -64, p^2 = 8 \text{ and } p = \pm\sqrt{8}$$

$$\text{i.e., } p = \pm 2\sqrt{2}$$

Ex.21 If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, ($a \neq 0$) then find the values of :

$$(i) \alpha^2 + \beta^2$$

$$(ii) \alpha^3 + \beta^3$$

$$(iii) \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$$

Sol. We know, sum of roots $(\alpha + \beta) = -\frac{b}{a}$

And, product of roots $(\alpha \beta) = \frac{c}{a}$; therefore :

$$(i) (\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$$

$$\Rightarrow \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= \left(-\frac{b}{a}\right)^2 - 2\frac{c}{a} = \frac{b^2}{a^2} - 2\frac{c}{a} = \frac{b^2 - 2ac}{a^2}$$

$$(ii) \alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta)$$

$$= \left(-\frac{b}{a}\right) \left(\frac{b^2 - 2ac}{a^2} - \frac{c}{a}\right)$$

$$= \left(-\frac{b}{a}\right) \left(\frac{b^2 - 2ac - ac}{a^2}\right) = -\frac{b(b^2 - 3ac)}{a^3}$$

$$(iii) \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{\frac{b^2 - 2ac}{a^2}}{\frac{c}{a}} = \frac{b^2 - 2ac}{a^2} \times \frac{a}{c} = \frac{b^2 - 2ac}{ca}$$

Relation between Roots and Coefficients :

If roots of quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$) are α and β then :

$$(i) (\alpha - \beta) = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$$

$$= \pm \frac{\sqrt{b^2 - 4ac}}{a} = \pm \frac{\sqrt{D}}{a}$$

$$(ii) \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{b^2 - 2ac}{a^2}$$

$$(iii) \alpha^2 - \beta^2 = (\alpha + \beta) \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$$

$$= -\frac{b\sqrt{b^2 - 4ac}}{a^2} = \pm \frac{\sqrt{D}}{a}$$

$$(iv) \alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta).$$

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$= -\frac{b(b^2 - 3ac)}{a^3}$$

$$(v) \alpha^3 - \beta^3 = (\alpha - \beta)(\alpha^2 + \beta^2 + \alpha\beta).$$

$$\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$$

$$= \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} \{(\alpha + \beta)^2 - \alpha\beta\}$$

$$= \frac{(b^2 - ac)\sqrt{b^2 - 4ac}}{a^3}$$

$$(vi) \alpha^4 + \beta^4 = \{(\alpha + \beta)^2 - 2\alpha\beta\}^2 - 2\alpha^2\beta^2$$

$$= \left(\frac{b^2 - 2ac}{a^2}\right)^2 - 2\frac{c^2}{a^2}$$

$$(vii) \alpha^4 - \beta^4 = (\alpha^2 - \beta^2)(\alpha^2 + \beta^2)$$

$$= \frac{-b(b^2 - 2ac)\sqrt{b^2 - 4ac}}{a^4}$$

$$(viii) \alpha^2 + \alpha\beta + \beta^2 = (\alpha + \beta)^2 - \alpha\beta$$

$$(ix) \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$(x) \alpha^2\beta + \beta^2\alpha = \alpha\beta(\alpha + \beta)$$

$$(xi) \left(\frac{\alpha}{\beta}\right)^2 + \left(\frac{\beta}{\alpha}\right)^2$$

$$= \frac{\alpha^4 + \beta^4}{\alpha^2\beta^2} = \frac{(\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2}{\alpha^2\beta^2}$$

➤ EQUATIONS REDUCIBLE TO QUADRATIC EQUATIONS

Type 1 : Equations of the form $ax^4 + bx^2 + c = 0$,

Method : Substitute $x^2 = y$ and solve.

❖ EXAMPLES ❖

Ex.22 Solve the following equations :

$$(i) x^4 - 26x^2 + 25 = 0 \quad (ii) z^4 - 10z^2 + 9 = 0$$

Sol. (i) Substituting $x^2 = y$:

$$x^4 - 26x^2 + 25 = 0$$

$$\Rightarrow y^2 - 26y + 25 = 0$$

$$\text{i.e., } y^2 - 25y - y + 25 = 0$$

$$\Rightarrow y(y - 25) - 1(y - 25) = 0$$

$$\text{i.e., } (y - 25)(y - 1) = 0$$

$$\Rightarrow y - 25 = 0 \text{ or } y - 1 = 0$$

$$\text{i.e., } y = 25 \text{ or } y = 1$$

$$y = 25 \Rightarrow x^2 = 25 \mid y = 1 \Rightarrow x^2 = 1$$

$$\Rightarrow x = \pm 5 \mid \Rightarrow x = \pm 1$$

$\therefore$ Roots of the given equation are : $\pm 5, \pm 1$

(ii) Substituting $z^2 = x$

$$z^4 - 10z^2 + 9 = 0 \Rightarrow x^2 - 10x + 9 = 0$$

$$\text{i.e., } x^2 - 9x - x + 9 = 0$$

$$\Rightarrow x(x - 9) - 1(x - 9) = 0$$

$$\text{i.e., } (x - 9)(x - 1) = 0$$

$$\Rightarrow x - 9 = 0 \text{ or } x - 1 = 0$$

$$x = 9 \Rightarrow z^2 = 9 \mid x = 1 \Rightarrow z^2 = 1$$

$$\Rightarrow z = \pm 3 \mid \Rightarrow z = \pm 1$$

$\therefore$ Solution of the given equation is : $\pm 3, \pm 1$.

Type 2 : Equation of the form : $px + \frac{q}{x} = r$

Method : (i) Multiply each term by x .

(ii) Solve the quadratic equation obtained to get the non-zero value(s) of x .

Ex.23 Solve :

$$(i) x + \frac{5}{x} = 6 \quad (ii) 3y + \frac{5}{16y} = 2$$

Sol. (i) $x + \frac{5}{x} = 6$

$$\Rightarrow x^2 + 5 = 6x \quad [\text{Multiplying each term by } x]$$

$$\Rightarrow x^2 - 6x + 5 = 0 \Rightarrow x^2 - 5x - x + 5 = 0$$

$$\text{i.e., } x(x - 5) - 1(x - 5) = 0$$

$$\Rightarrow (x - 5)(x - 1) = 0 \text{ i.e., } x - 5 = 0$$

$$\text{or } x - 1 = 0 \Rightarrow x = 5 \text{ or } x = 1.$$

$\therefore$ Required solution is 5, 1

(ii) $3y + \frac{5}{16y} = 2$

$$\Rightarrow 3y \times 16y + 5 = 2 \times 16y$$

$$\Rightarrow 48y^2 - 32y + 5 = 0$$

$$\Rightarrow 48y^2 - 12y - 20y + 5 = 0$$

$$\text{i.e., } 12y(4y - 1) - 5(4y - 1) = 0$$

$$\Rightarrow (4y - 1)(12y - 5) = 0$$

$$\text{i.e., } 4y - 1 = 0 \text{ or } 12y - 5 = 0$$

$$\Rightarrow 4y = 1 \text{ or } 12y = 5 \text{ i.e., } y = \frac{1}{4} \text{ or } y = \frac{5}{12}$$

$\therefore$ Required solutions is : $\frac{1}{4}, \frac{5}{12}$

Type 3 :

Equations involving one radical :

$$\sqrt{a - x^2} = bx + c$$

Method :

1. Square both the sides to get :

$$a - x^2 = (bx + c)^2$$

2. Now simplify it to get a quadratic equation.

3. Solve the quadratic equation obtained.

Ex.24 Solve :

(i) $\sqrt{x} + 2x = 1$ (ii) $\sqrt{3x^2 - 2} + 1 = 2x$

(iii) $\sqrt{2x^2 + 9} + x = 13$

Sol. (i) $\sqrt{x} + 2x = 1 \Rightarrow \sqrt{x} = 1 - 2x$

i.e., $x = (1 - 2x)^2$

$\Rightarrow x = 1 + 4x^2 - 4x$

i.e., $1 + 4x^2 - 4x - x = 0$

$\Rightarrow 4x^2 - 5x + 1 = 0$ i.e., $4x^2 - 4x - x + 1 = 0$

$\Rightarrow 4x(x - 1) - 1(x - 1) = 0$

i.e., $(x - 1)(4x - 1) = 0$

$\Rightarrow x - 1 = 0$

or $4x - 1 = 0$

i.e., $x = 1$ or $x = \frac{1}{4}$

► PROBLEMS ON QUADRATIC EQUATIONS

For solving problems based on quadratic equations, the following steps must be adopted :

1. Read the given statement of the problem carefully to find the required unknown quantity.
2. Take the unknown quantity as 'x' and according to the given statement, form an equation in terms of 'x'.
3. Simplify and solve the equation to get the value/values of 'x'.

❖ EXAMPLES ❖

Ex.25 Find two consecutive natural numbers, whose product is equal to 20.

Sol. Let the required two consecutive natural numbers be x and x + 1.

Given : $x(x + 1) = 20 \Rightarrow x^2 + x - 20 = 0$

$\Rightarrow (x + 5)(x - 4) = 0 \Rightarrow x = -5$, or $x = 4$

Since, x must be a natural number,

$\therefore x = 4$

And required numbers are x and x + 1 i.e., and 5.

Ex.26 The sum of the squares of two consecutive whole numbers is 61. Find the numbers.

Sol. Let the required consecutive whole numbers be x and x + 1.

$\therefore x^2 + (x + 1)^2 = 61$

$\Rightarrow x^2 + x^2 + 2x + 1 - 61 = 0 \Rightarrow 2x^2 + 2x - 60 = 0$

$\Rightarrow x^2 + x - 30 = 0$ [Dividing each term by 2]

$\Rightarrow (x + 6)(x - 5) = 0$ [On factorising]

$\Rightarrow x = -6$, or $x = 5$

Θ x is a whole number, $\therefore x = 5$

And, required numbers are x and x + 1 = 5 and 5 + 1 i.e., 5 and 6

Ex.27 The sum of two natural numbers is 8. If the sum of their reciprocals is $\frac{8}{15}$, find the two numbers.

Sol. Let the numbers be x and 8 - x.

$\therefore \frac{1}{x} + \frac{1}{8-x} = \frac{8}{15} \Rightarrow \frac{8-x+x}{x(8-x)} = \frac{8}{15}$

$\Rightarrow \frac{8}{8x-x^2} = \frac{8}{15}$ i.e., $120 = 64x - 8x^2$

$\Rightarrow 8x^2 - 64x + 120 = 0$

$\Rightarrow x^2 - 8x + 15 = 0$ [Dividing by 8]

$\Rightarrow (x - 5)(x - 3) = 0$ [On factorizing]

$\Rightarrow x = 5$, or $x = 3$

When $x = 5$, the number are x and $8 - x = 5$ and 3, and when $x = 3$, the numbers are x and $8 - x = 3$ and 5.

$\therefore$ Required numbers are 5 and 3.

Ex.28 Divide 16 into two parts such that twice the square of the larger part exceeds the square of the smaller part by 164.

Sol. Let larger part be x, therefore the smaller part = 16 - x

Given : $2x^2 - (16 - x)^2 = 164$

$\Rightarrow 2x^2 - (256 + x^2 - 32x) - 164 = 0$

i.e., $2x^2 - 256 - x^2 + 32x - 164 = 0$

$\Rightarrow x^2 + 32x - 420 = 0$

On factorizing, it gives : $(x + 42)(x - 10) = 0$

i.e., $x = -42$ or $x = 10$

$\therefore x = 10$

Hence the larger part = 10 and the smaller part = $16 - x = 16 - 10 = 6$

Ex.29 Two positive numbers are in the ratio 2 : 5. If difference between the squares of these numbers is 189 ; find the numbers.

Sol. Let numbers be $2x$ and $5x$

$$\therefore (5x)^2 - (2x)^2 = 189$$

$$\Rightarrow 25x^2 - 4x^2 = 189 \text{ and } 21x^2 = 189$$

$$\text{i.e., } x^2 = \frac{189}{21} = 9 \Rightarrow x = \pm 3$$

Since, the required numbers are positive,

$$\therefore x = 3$$

And, required numbers = $2x$ and $5x = 2 \times 3$ and $5 \times 3 = 6$ and 15

Ex.30 A two digit number is such that the product of the digits is 35. When 18 is added to this number the digits interchange their places. Determine the number.

Sol. Let ten's digit of the numbers = x and its unit digit = y .

$\therefore$ The two digit number is $10x + y$.

$$\text{Given : } x \cdot y = 35 \text{ and } 10x + y + 18 = 10y + x$$

$$\Rightarrow y = \frac{35}{x} \text{ and } 9x + 18 = 9y$$

$$\text{i.e., } x + 2 = y$$

On substituting $y = \frac{35}{x}$ in $x + 2 = y$; we get:

$$x + 2 = \frac{35}{x}$$

$$\Rightarrow x^2 + 2x = 35$$

$$\text{and } x^2 + 2x - 35 = 0$$

$$\text{On factorising, we get : } (x + 7)(x - 5) = 0$$

$$\text{i.e., } x = -7 \text{ or } x = 5$$

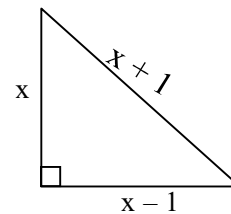
Since, x is digit, therefore $x = 5$ and

$$y = \frac{35}{x} = 7$$

$$\therefore \text{ The required two digit number} = 10x + y \\ = 10 \times 5 + 7 = 57$$

Ex.31 The sides (in cm) of a right triangle are $x - 1$, x and $x + 1$. Find the sides of triangle.

Sol. It is clear that the largest side $x + 1$ is hypotenuse of the right triangle.



According to Pythagoras Theorem, we have :

$$x^2 + (x - 1)^2 = (x + 1)^2$$

$$\Rightarrow x^2 + x^2 - 2x + 1 = x^2 + 2x + 1$$

$$\text{This gives } x^2 - 4x = 0$$

$$\Rightarrow x(x - 4) = 0 \text{ i.e., } x = 0 \text{ or } x = 4$$

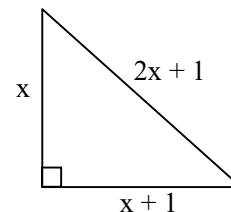
Since, with $x = 0$ the triangle is not possible; hence $x = 4$.

$$\therefore \text{ Sides, are } x - 1, x \text{ and } x + 1 = 4 - 1$$

$$\text{i.e., } 3 \text{ cm, } 4 \text{ cm and } 5 \text{ cm}$$

Ex.32 The hypotenuse of a right triangle is 1 m less than twice the shortest side. If the third side is 1 m more than the shortest side, find the sides of the triangle.

Sol. Let the shortest side be x m.



$$\therefore \text{ Hypotenuse} = (2x - 1) \text{ m and the third side} \\ = (x + 1) \text{ m}$$

Applying Pythagoras theorem, we get ;

$$(2x - 1)^2 = x^2 + (x + 1)^2$$

$$\Rightarrow 4x^2 - 4x + 1 = x^2 + x^2 + 2x + 1$$

$$\text{i.e., } 2x^2 - 6x = 0$$

$$\Rightarrow x^2 - 3x = 0$$

$$\text{i.e., } x(x - 3) = 0 \Rightarrow x = 0 \text{ or } x = 3$$

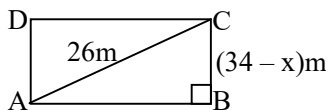
Since, $x = 0$ makes the triangle impossible. therefore, $x = 3$

And, sides of the triangle are $= x, 2x - 1$ and $x + 1 = 3, 2 \times 3 - 1$ and $3 + 1 = 3\text{m, } 5\text{m}$

and 4m

Ex.33 If the perimeter of a rectangular plot is 68 m and its diagonal is 26 m. Find its area.

Sol. Let the length of plot = x m



$$\ominus \quad 2(\text{length} + \text{breadth}) = \text{perimeter}$$

$$\Rightarrow 2(x + \text{breadth}) = 68$$

$$\Rightarrow x + \text{breadth} = \frac{68}{2} \text{ and breadth} = (34 - x) \text{ m}$$

Given its diagonal = 26 m and we know each angle of the rectangle = 90° .

$$\therefore x^2 + (34 - x)^2 = 26^2$$

[Applying Pythagoras Theorem]

$$\Rightarrow x^2 + 1156 - 68x + x^2 - 676 = 0$$

$$\Rightarrow 2x^2 - 68x + 480 = 0$$

$$\Rightarrow x^2 - 34x + 240 = 0$$

$$\text{i.e., } x^2 - 34x + 240 = 0$$

On factorising, we get : $(x - 24)(x - 10) = 0$

$$\text{i.e., } x = 24 \text{ or } x = 10$$

$$x = 24$$

$$\Rightarrow \text{length} = 24 \text{ m and breadth}$$

$$= (34 - 24) \text{ m} = 10 \text{ m}$$

$$\text{and, } x = 10$$

$$\Rightarrow \text{length} = 10 \text{ m and breadth}$$

$$= (34 - 10) \text{ m} = 24 \text{ m}$$

$\therefore$ Dimensions of the given rectangular plot are 24 m and 10 m.

Hence, its area = length $\times$ breadth

$$= 24 \text{ m} \times 10 \text{ m} = 240 \text{ m}^2$$

Ex.34 A train travels a distance of 300 km at a uniform speed. If the speed of the train is increased by 5 km an hour, the journey would have taken two hours less. Find the original speed of the train.

Sol. Let the original speed of the train be x km/hr.

In 1st case, Distance = 300 km and speed = x km/hr.

$$\Rightarrow \text{Time taken} = \frac{\text{distance}}{\text{speed}} = \frac{300}{x} \text{ hrs.}$$

In 2nd case, Distance = 300 km and

speed = $(x + 5)$ km/hr.

$$\therefore \text{Time taken} = \frac{\text{distance}}{\text{speed}} = \frac{300}{x + 5} \text{ hrs.}$$

$$\text{Given : } \frac{300}{x} - \frac{300}{x + 5} = 2$$

$$\Rightarrow \frac{300(x + 5) - 300x}{x(x + 5)} = 2$$

$$\text{i.e., } \frac{300x + 1500 - 300x}{x^2 + 5x} = 2$$

$$\Rightarrow 2(x^2 + 5x) = 1500$$

$$\Rightarrow x^2 + 5x - 750 = 0$$

On factorising, we get : $(x + 30)(x - 25) = 0$

$$\text{i.e., } x = -30 \text{ or } x = 25$$

Neglecting $x = -30$; we get $x = 25$

$$\text{i.e., } x = 25 \text{ km per hour}$$

Ex.35 A motor boat, whose speed is 15 km/hr in still water, goes 30 km downstream and comes back in a total of 4 hours 30 minutes. Determine the speed of the stream.

Sol. Let the speed of the stream = x km/hr

$$\Rightarrow \text{The speed of the boat downstream} = (15 + x) \text{ km/hr.}$$

$$\text{and, the speed of the boat upstream} = (15 - x) \text{ km/hr}$$

Now, time taken to go 30 km downstream

$$= \frac{30}{15 + x} \text{ hrs.}$$

and, time take to come back 30 km upstream

$$= \frac{30}{15 - x} \text{ hrs.}$$

Given : the time taken for both the journeys

$$= 4 \text{ hours } 30 \text{ min.} = 4 \frac{1}{2} \text{ hrs} = \frac{9}{2} \text{ hrs}$$

$$\therefore \frac{30}{15 + x} + \frac{30}{15 - x} = \frac{9}{2}$$

$$\Rightarrow \frac{30(15 - x) + 30(15 + x)}{(15 + x)(15 - x)} = \frac{9}{2}$$

$$\text{i.e., } \frac{450 - 30x + 450 + 30x}{225 - x^2} = \frac{9}{2}$$

$$\Rightarrow 2 \times 900 = 9(225 - x^2)$$

On dividing both the sides by 9, we get :

$$2 \times 100 = 225 - x^2$$

$$\text{i.e., } x^2 = 225 - 200 \Rightarrow x^2 = 25 \text{ and } x = \pm 5$$

Rejecting the negative value of x ,

$$\text{we get : } x = 5$$

$$\text{i.e., the speed of the steam} = 5 \text{ km/hr}$$

Ex.36 The hotel bill for a number of people for overnight stay in Rs. 4, 800. If there were 4 people more, the bill each person had to pay would have reduced by Rs. 200. Find the number of people staying overnight.

Sol. Let the number of people staying overnight be x .

$$\therefore \text{ For } x \text{ people, the hotel bill} = \text{Rs } 4,800$$

$$\Rightarrow \text{ For 1 person, the hotel bill} = \text{Rs } \frac{4,800}{x}$$

When 4 people were more :

$$\text{For } (x + 4) \text{ people, the hotel bill} = \text{Rs } 4,800$$

$$\Rightarrow \text{ For 1 person, the hotel bill} = \text{Rs } \frac{4,800}{x + 4}$$

It is given that now the bill paid by each person is reduced by Rs 200.

$$\therefore \frac{4,800}{x} - \frac{4,800}{x + 4} = 200$$

$$\Rightarrow \frac{4,800(x + 4) - 4,800x}{x(x + 4)} = 200$$

$$\text{i.e., } 200(x^2 + 4x) = 4800x + 19200 - 4800x$$

$$\Rightarrow x^2 + 4x = \frac{19200}{200} = 96$$

$$\text{i.e., } x^2 + 4x - 96 = 0$$

$$\text{On factorising, we get : } (x + 12)(x - 8) = 0$$

$$\text{i.e., } x = -12 \quad \text{or} \quad x = 8$$

⊖ No. of people can not be negative

$$\Rightarrow \text{ No. of people staying overnight} = 8$$

Ex.37 In an auditorium, the number of rows was equal to the number of seats in each row. If the number of rows is doubled and the number of seats in each row is reduced by 5, then the total number of seats is increased by 375. How many rows were there ?

Sol. Let the number of rows be x

$$\Rightarrow \text{ No. of seats in each row} = x$$

$$\therefore \text{ The total number of seats in the auditorium}$$

$$= x \times x = x^2$$

Now, the new no. of rows = $2x$ and the new no. of seats in each row = $x - 5$

$$\therefore \text{ The new no. of total seats in the auditorium}$$

$$= 2x(x - 5).$$

$$\text{Given : } 2x(x - 5) - x^2 = 375$$

$$\Rightarrow 2x^2 - 10x - x^2 = 375 \text{ and } x^2 - 10x - 375 = 0$$

$$\text{On factorising, we get : } (x - 25)(x + 15) = 0$$

$$\text{i.e., } x = 25 \text{ or } x = -15$$

$$\text{Neglecting } x = -15, \text{ we get : no. of rows} = 25$$

Ex.38 Two years ago, a man's age was three times the square of his son's age. In three years time, his age will be four times his son's age. Find their present ages.

Sol. Let present age of son = x years

Two years ago : The age of son was $(x - 2)$ years and so the age of the man was $3(x - 2)^2$

$$\therefore \text{ Man's present age} = 3(x - 2)^2 + 2$$

$$= 3(x^2 - 4x + 4) + 2 = 3x^2 - 12x + 14$$

In 3 years time : The age of son will be $(x + 3)$ years and the age of man will be

$$(3x^2 - 12x + 14) + 3 = 3x^2 - 12x + 17 \text{ years}$$

$$\text{Given : } 3x^2 - 12x + 17 = 4(x + 3)$$

$$\Rightarrow 3x^2 - 12x + 17 = 4x + 12 \text{ i.e., } 3x^2 - 16x + 5 = 0$$

$$\text{On factorising, we get : } (x - 5)(3x - 1) = 0$$

$$\text{i.e., } x = 5 \text{ or } x = \frac{1}{3}$$

$$\text{Since, } x = \frac{1}{3} \text{ is not possible ; } x = 5$$

$$\therefore \text{ The present age of man} = 3x^2 - 12x + 14$$

$$= 3 \times 5^2 - 12 \times 5 + 14 = 29 \text{ years.}$$

And, the present age of son = $x = 5$ years

Ex.39 Find the roots of the equation $x^2 - 2x - 8 = 0$.

Sol. Quadratic Equation $x^2 - 2x - 8 = 0$

$$\text{After factorization } (x - 4)(x + 2) = 0$$

$$\Rightarrow x = 4, -2$$

Ex.40 Find the roots of the equation $x^2 - 4x + 1 = 0$.

Sol. Here $a = 1, b = 4, c = 1$

Using Hindu Method

$$x = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

Ex.41 Find the nature of the roots of the quadratic equation $7x^2 - 9x + 2 = 0$.

Sol. $b^2 - 4ac = 81 - 56 = 25 > 0$ and a perfect square so roots are rational and different.

Ex.42 Find the nature of the roots of the quadratic equation $2x^2 - 7x + 4 = 0$.

Sol. $b^2 - 4ac = 49 - 32 = 17 > 0$ (not a perfect square) Its roots are irrational and different.

Ex.43 Find the nature of the roots of the quadratic equation $x^2 - 2(a+b)x + 2(a^2 + b^2) = 0$.

Sol. $A = 1, B = -2(a+b), C = 2(a^2 + b^2)$
 $B^2 - 4AC = 1[2(a+b)]^2 - 4(1)(2a^2 + 2b^2)$
 $= 4a^2 + 4b^2 + 8ab - 8a^2 - 8b^2$
 $= -4a^2 - 4b^2 + 8ab$
 $= -4(a-b)^2 < 0$

So roots are imaginary and different.

Ex.44 Find the nature of roots of the equation

$$x^2 - 2\sqrt{2}x + 1 = 0.$$

Sol. The discriminant of the equation

$(-2\sqrt{2})^2 - 4(1)(1) = 8 - 4 = 4 > 0$ and a perfect square so roots are real and different but we can't say that roots are rational because coefficients are not rational therefore.

$$\alpha, \beta = \frac{2\sqrt{2} \pm \sqrt{(2\sqrt{2})^2 - 4}}{2}$$

$$= \frac{2\sqrt{2} \pm 2}{2} = \sqrt{2} \pm 1$$

this is irrational.

$\therefore$ the roots are real and different.

Ex.45 Find the nature of the roots of the equation

$$(b+c)x^2 - (a+b+c)x + a = 0, (a, b, c \in \mathbb{Q}) ?$$

Sol. The discriminant of the equation is

$$(a+b+c)^2 - 4(b+c)(a)$$

$$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - 4(b+c)a$$

$$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - 4ab - 4ac$$

$$= a^2 + b^2 + c^2 - 2ab + 2bc - 2ca$$

$$(a-b-c)^2 > 0$$

So roots are rational and different.

Ex.46 If the roots of the equation $x^2 + 2x + P = 0$ are real then find the value of P.

Sol. Here $a = 1, b = 2, c = P$

$$\therefore \text{discriminant} = (2)^2 - 4(1)(P) = 0$$

(Since roots are real)

$$\Rightarrow 4 - 4P \geq 0 \Rightarrow 4 \geq 4P$$

$$\Rightarrow P \leq 1$$

Ex.47 If the product of the roots of the quadratic equation $mx^2 - 2x + (2m-1) = 0$ is 3 then find the value of m is -

Sol. Product of the roots $c/a = 3 = \frac{2m-1}{m}$

$$\therefore 3m - 2m = -1 \Rightarrow m = -1$$

Ex.48 If α and β are roots of the equation

$$x^2 - 5x + 6 = 0 \text{ then find the value of } \alpha^3 + \beta^3.$$

Sol. Here $\alpha + \beta = 5, \alpha\beta = 6$

$$\text{Now } \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$= (5)^3 - 3 \cdot 6 \cdot 5 = 125 - 90 = 35$$

Ex.49 If the equation $(k-2)x^2 - (k-4)x - 2 = 0$ has difference of roots as 3 then find the value of k.

Sol. $(\alpha - \beta) = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

$$\text{Now } \alpha + \beta = \frac{(k-4)}{(k-2)}, \alpha\beta = \frac{-2}{k-2}$$

$$\therefore (\alpha - \beta) = \sqrt{\left(\frac{k-4}{k-2}\right)^2 + \frac{8}{(k-2)}}$$

$$= \frac{\sqrt{k^2 + 16 - 8k + 8(k-2)}}{(k-2)}$$

$$\Rightarrow 3 = \frac{\sqrt{k^2 + 16 - 8k + 8k - 16}}{(k-2)}$$

$$\Rightarrow 3k - 6 = \pm k$$

$$\therefore k = 3, 3/2$$

Ex.50 If α, β are roots of the equation $ax^2 + bx + c = 0$ then find the value of

$$\frac{1}{(a\alpha + b)^2} + \frac{1}{(a\beta + b)^2}.$$

Sol. Since α, β are the root of the $ax^2 + bx + c$ then $a\alpha^2 + b\alpha + c = 0$

$$\Rightarrow \alpha(a\alpha + b) + c = 0$$

$$\Rightarrow (a\alpha + b) = -c / \alpha \quad \dots(1)$$

Similarly

$$(a\beta + b) = -c / \beta \quad \dots(2)$$

$$\therefore \frac{1}{(a\alpha + b)^2} + \frac{1}{(a\beta + b)^2} = \frac{1}{(-c/\alpha)^2} + \frac{1}{(-c/\beta)^2}$$

$$\Rightarrow \frac{\alpha^2}{c^2} + \frac{\beta^2}{c^2} = \frac{\alpha^2 + \beta^2}{c^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{c^2}$$

$$= \frac{b^2/a^2 - 2c/a}{c^2} = \frac{b^2 - 2ac}{a^2c^2}$$

Ex.51 Find the equation whose roots are 3 and 4.

Sol. The quadratic equation is given by

$$x^2 - (\text{sum of the roots})x + (\text{product of roots}) = 0$$

$\therefore$ The required equation

$$= x^2 - (3 + 4)x + 3 \cdot 4 = 0$$

$$= x^2 - 7x + 12 = 0$$

Ex.52 Find the quadratic equation with rational coefficients whose one root is $2 + \sqrt{3}$ -

Sol. The required equation is

$$x^2 - \{(2 + \sqrt{3}) + (2 - \sqrt{3})\}x + (2 + \sqrt{3})(2 - \sqrt{3}) = 0$$

$$\text{or } x^2 - 4x + 1 = 0$$

Ex.53 If α, β are the root of a quadratic equation $x^2 - 3x + 5 = 0$ then find the equation whose roots are $(\alpha^2 - 3\alpha + 7)$ and $(\beta^2 - 3\beta + 7)$.

Sol. Since α, β are the roots of equation

$$x^2 - 3x + 5 = 0$$

$$\text{So } \alpha^2 - 3\alpha + 5 = 0 \text{ \& } \beta^2 - 3\beta + 5 = 0$$

$$\therefore \alpha^2 - 3\alpha = -5 \text{ \& } \beta^2 - 3\beta = -5$$

$$\text{putting in } (\alpha^2 - 3\alpha + 7) \text{ \& } (\beta^2 - 3\beta + 7) \quad \dots(1)$$

$$-5 + 7, -5 + 7$$

2 and 2 are the roots

the required equation is

$$x^2 - 4x + 4 = 0.$$

Ex.54 If α, β are roots of the equation $x^2 - 5x + 6 = 0$ then find the equation whose roots are $\alpha + 3$ and $\beta + 3$ is -

Sol. Let $\alpha + 3 = x$

$$\therefore \alpha = x - 3 \text{ (Replace } x \text{ by } x - 3)$$

So the required equation is

$$(x - 3)^2 - 5(x - 3) + 6 = 0 \quad \dots(1)$$

$$\Rightarrow x^2 - 6x + 9 - 5x + 15 + 6 = 0$$

$$\Rightarrow x^2 - 11x + 30 = 0 \quad \dots(2)$$

Ex.55 If α, β are roots of the equation $2x^2 + x - 1 = 0$ then find the equation whose roots are $1/\alpha, 1/\beta$ will be -

Sol. From the given equation

$$\alpha + \beta = -1/2, \alpha\beta = -1/2$$

The required equation is -

$$x^2 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)x + \frac{1}{\alpha\beta} = 0$$

$$\Rightarrow x^2 - \left(\frac{\alpha + \beta}{\alpha\beta}\right)x + \frac{1}{\alpha\beta} = 0$$

$$\Rightarrow x^2 - \left(\frac{-\frac{1}{2}}{-\frac{1}{2}}\right)x + \frac{1}{-\frac{1}{2}} = 0 \Rightarrow x^2 - x - 2 = 0$$

Short cut : Replace x by $1/x$

$$\Rightarrow 2(1/x)^2 + 1/x - 1 = 0 \Rightarrow x^2 - x - 2 = 0$$

❖ SOME MORE EXAMPLES ❖

Ex.56 If r and s are positive, then find the nature of roots of the equation $x^2 - rx - s = 0$

Sol. Here Discriminant

$$= r^2 + 4s > 0 \quad (\because r, s > 0)$$

$\Rightarrow$ roots are real

Again $a = 1 > 0$ and $c = -s < 0$

$\Rightarrow$ roots are of opposite signs.

Ex.57 Find the nature of both roots of the equation

$$(x-b)(x-c) + (x-c)(x-a) + (x-a)(x-b) = 0.$$

Sol. The given equation can be written in the following form :

$$3x^2 - 2(a+b+c)x + (ab+bc+ca) = 0$$

Here discriminant

$$= 4(a+b+c)^2 - 12(ab+bc+ca)$$

$$= 4[(a^2+b^2+c^2) - (ab+bc+ca)] > 0$$

$$[a^2+b^2+c^2 > ab+bc+ca]$$

$\therefore$ Both roots are real.

Ex.58 If one root of the equation $4x^2 + 2x - 1 = 0$ is α , then find the other root.

Sol. Let α and β are roots of the given equation,

$$\text{then } \alpha + \beta = -\frac{1}{2} \Rightarrow \beta = -\frac{1}{2} - \alpha$$

$$\text{Now } 4\alpha^2 + 2\alpha - 1 = 0$$

$$4\alpha^2 = 1 - 2\alpha \quad \dots(1)$$

$$\text{Now } 4\alpha^3 = \alpha - 2\alpha^2$$

$$= \alpha - \frac{1}{2}(1 - 2\alpha) \quad [\text{from (1)}]$$

$$\therefore 4\alpha^3 - 3\alpha = -2\alpha - \frac{1}{2}(1 - 2\alpha)$$

$$= -\frac{1}{2} - \alpha = \beta$$

Ex.59 Find the quadratic equation, whose one root is

$$\frac{1}{2+\sqrt{5}}.$$

Sol. Given root $= \frac{1}{2+\sqrt{5}} = \sqrt{5} - 2$

So the other root $= -\sqrt{5} - 2$. Then sum of the roots $= -4$, product of the roots $= -1$

Hence the equation is $x^2 + 4x - 1 = 0$

EXERCISE # 1

A. Very Short Answer Type Question

Q.1 Which of the following are quadratic polynomials

- (i) $5x^2 - 8x + 12$ (ii) $3 + 4x - 7x^2$
 (iii) $8x^2 - 15$ (iv) $8x - 15$
 (v) $8x^3 - 3x$ (vi) $x^2 - \sqrt{5}x + 2\sqrt{3}$
 (vii) $\sqrt{3}x^2 - 10x - 5\sqrt{3}$
 (viii) $\sqrt{7} - \sqrt{5}x - \sqrt{3}x^3$
 (ix) $\sqrt{15}x^2 - \sqrt{5}x + 7$

Q.2 Find the value of each given polynomial at the given value of its variable :

- (i) $5x^2 - 7x + 2$ at $x = 3$
 (ii) $x^2 + 15x - 4$ at $x = -1$
 (iii) $2y^2 - y + 2$ at $y = -2$
 (iv) $3y + 8 - 2y^2$ at $y = -3$
 (v) $\sqrt{2}x^2 + 3x + 1$ at $x = \sqrt{2}$
 (vi) $x^3 - 3x^2 + 5x + 2$ at $x = -4$
 (vii) $5\sqrt{2}x^3 + 2x^2 - \sqrt{2}x + 1$ at $x = 2\sqrt{2}$

Q.3 Find the value of constant 'm' if :

- (i) $x = -2$ is a zero of quadratic polynomial $4x^2 - 3mx + 5$.
 (ii) $y = -5$ is a zero of quadratic polynomial $7 + 4(m+2)y - y^2$

Q.4 Which of the following are quadratic equations:

- (i) $x^2 - 9x + 5 = 0$ (ii) $x^2 - \frac{3}{x} = 2$

Q.5 Which of the following are quadratic equations:

- (i) $x - \frac{3}{x} = 2x^2$ (ii) $15x^2 + 27x - 33 = 0$

Q.6 Which of the following are quadratic equations:

- (i) $\sqrt{3}x^2 + 8x = 3\sqrt{2}$
 (ii) $\frac{7}{8}x^2 - \frac{3}{5}x + \frac{5}{7} = 0$

Q.7 Determine whether $x = -\frac{2}{\sqrt{3}}$ and $x = -3\sqrt{3}$

are solutions of given equation or not :

$$\sqrt{3}x^2 + 11x + 6\sqrt{3} = 0$$

Q.8 Determine if $x = 5$ is a root of equation given below or not :

$$\sqrt{2x^2 + 4x - 5} - \sqrt{x^2 - 4x + 4} = \sqrt{1 - 12x + 3x^2}$$

Q.9 In each case given below; find the value of 'm' for which the given value of the variable is a solution of the equation :

- (i) $(2m + 1)x^2 + 2x - 3 = 0$; $x = 2$
 (ii) $3x^2 + 2mx - 3 = 0$; $2x - 1 = 0$
 (iii) $x^2 + 2ax - m = 0$; $x + a = 0$

Q.10 Determine whether $x = \frac{1}{2}$ and $x = \frac{3}{2}$ are solutions of the equation $2x^2 - 5x + 3 = 0$ or not.

Solve each of the following quadratic equations (Q.11 to Q. 27)

Q.11 $x^2 + 5x + 6 = 0$

Q.12 $x^2 - 8x - 33 = 0$

Q.13 $x^2 + 4x - 32 = 0$

Q.14 $x^2 + 5x - 6 = 0$

Q.15 $x^2 - 5x - 6 = 0$

Q.16 $x^2 - 5x + 6 = 0$

Q.17 $5x^2 - 2ax - 3a^2 = 0$

Q.18 $x^2 + 8x = 0$

Q.19 $3x^2 + 2ax - a^2 = 0$

Q.20 $4x^2 - 25x - 21 = 0$

Q.21 $10x^2 - 7x - 12 = 0$

Q.22 $8x^2 - 2x - 3 = 0$

Q.23 $3x^2 - 7x - 6 = 0$

Q.24 $x(4x - 7) = 0$

Q.25 $x(x + 1) + (x + 2)(x + 3) = 26$

Q.26 $x(x - 1) + (x - 2)(x - 3) = 42$

Q.27 $4x^2 = 25$

Q.28 Without solving, examine the nature of roots of the equations :

(i) $3x^2 + 2x - 1 = 0$ (ii) $4x^2 + 3x - 1 = 0$

(iii) $6x^2 - 5x - 6 = 0$ (iv) $x^2 - 6x + 9 = 0$

(v) $2x^2 - 5x + 5 = 0$ (vi) $3x^2 + 7x + 3 = 0$

(vii) $4x^2 - 4x + 1 = 0$ (viii) $5x^2 - 8x + 2 = 0$

(ix) $x^2 + px - q^2 = 0$

Q.29 Find the discriminant of the following quadratic equations :

(i) $x^2 - 3x + 1 = 0$ (ii) $4x^2 + 3x - 2 = 0$

(iii) $x^2 - x + 1 = 0$ (iv) $9x^2 - px + 2 = 0$

(v) $ax^2 - 3x - 5 = 0$ (vi) $4x^2 - 5x + c = 0$

(vii) $\sqrt{2}x^2 + 5\sqrt{3}x - 2\sqrt{2} = 0$

(viii) $3\sqrt{5}x^2 - 8x + 2\sqrt{5} = 0$

Q.30 Find the sum and the product of the roots of the following equations :

(i) $2x^2 - 7x + 4 = 0$

(ii) $3x^2 + 4\sqrt{2}x + 9 = 0$

(iii) $2x^2 + 5\sqrt{3}x - 3 = 0$

(iv) $x^2 - 2\sqrt{5}x - 15 = 0$

(v) $5x^2 - 10x + 3\sqrt{5} = 0$

B. Short Answer Type Question

Find the roots (if they exist) of the following quadratic equations by the method of completing the square : (Q.31 to Q.37)

Q.31 $x^2 - 2\sqrt{5}x + 1 = 0$

Q.32 $4x^2 + x - 5 = 0$

Q.33 $9x^2 + x + 15 = 0$

Q.34 $x^2 - 5x + 7 = 0$

Q.35 $x^2 + 4x - 9 = 0$

Q.36 $2x^2 - 5x + 3 = 0$

Q.37 $5x^2 - 6x - 2 = 0$

Solve each of the following equations by using quadratic formula (Q.38 to Q.45)

Q.38 $x^2 - 2\sqrt{2}x - 6 = 0$

Q.39 $\sqrt{6}x^2 - 4x - 2\sqrt{6} = 0$

Q.40 $\sqrt{3}x^2 + 11x + 6\sqrt{3} = 0$

Q.41 $16x^2 - 1 = 0$

Q.42 $5x^2 - x - 4 = 0$

Q.43 $4x^2 - 7x + 3 = 0$

Q.44 $x^2 = 3x$

Q.45 $3x^2 - 5x = 0$

Q.46 In the following, determine the set of values of p for which the quadratic equation has real roots :

(i) $px^2 + 4x + 1 = 0$ (ii) $2x^2 + 3x + p = 0$

(iii) $2x^2 + px + 3 = 0$ (iv) $3x^2 - 2px - 5 = 0$

(v) $2px^2 - 6x - 3 = 0$

Q.47 In the following, determine whether the given quadratic equations have real roots and if so find the roots :

(i) $x^2 + 6x + 6 = 0$

(ii) $x^2 - 3x + 4 = 0$

(iii) $4x^2 + x - 3 = 0$

(iv) $9x^2 + 30x + 25 = 0$

(v) $4x^2 - 12x + 9 = 0$

(vi) $3x^2 - 3x + 1 = 0$

(vii) $3x^2 - 3x - 1 = 0$

(viii) $4x^2 + 5\sqrt{3}x + 3 = 0$

(ix) $5x^2 - 2\sqrt{5}x - 3 = 0$

Q.48 Find the quadratic equation whose roots are :

- (i) 5 and -5 (ii) 8 and 3
- (iii) -8 and -3 (iv) -8 and 3
- (v) $\sqrt{3}$ and $5\sqrt{3}$ (vi) $2\sqrt{2}$ and $-3\sqrt{2}$
- (vii) $-3\sqrt{5}$ and $-4\sqrt{5}$
- (viii) $1 + \sqrt{2}$ and $1 - \sqrt{2}$
- (ix) $4 - \sqrt{5}$ and $4 + \sqrt{5}$
- (x) $7 + \sqrt{7}$ and $7 - \sqrt{7}$
- (xi) $\frac{3 + \sqrt{2}}{3}$ and $\frac{3 - \sqrt{2}}{3}$
- (xii) $\frac{4 - \sqrt{5}}{2}$ and $\frac{4 + \sqrt{5}}{2}$

C. Long Answer Type Question

Q.49 Find the value of 'm' so that the roots of the equation :

$(4 - m)x^2 + (2m + 4)x + (8m + 1) = 0$ may be equal.

Q.50 For the quadratic equation $ax^2 + 7x + c = 0$; the sum of roots is -1 and the product of roots is 1 ; find the values of 'a' and 'c'.

Q.51 For the quadratic equation $ax^2 - 3x - b = 0$; the sum of roots is 6 and the product of roots is -8 ; find the values of 'a' and 'b'.

Q.52 Find the value of p ; if one root of quadratic equation $3x^2 - px - 6 = 0$ is 3. Also, find the second (other) roots of the equation.

Q.53 If α and β are the roots of the equation $2x^2 + 5x - 4 = 0$; find the value of :

- (i) $\alpha^2 + \beta^2$ (ii) $\alpha^2 + \beta^2 - 3\alpha - 3\beta$
- (iii) $\alpha^2 + \beta^2 - 4\alpha\beta$ (iv) $\alpha^3 + \beta^3$
- (v) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

Q.54 If α and β are the roots of the equation, $x^2 - 6x + 1 = 0$; find the value of :

- (i) $\alpha^2 + \beta^2$ (ii) $\alpha^4 + \beta^4$
- (iii) $\alpha^3 + \beta^3$ (iv) $\alpha^2 + \beta^2 - 2\alpha\beta$

Q.55 For each equation, given below, find the value (s) of p so that the equation has equal roots :

- (i) $2x^2 - 7x + p = 0$
- (ii) $6x^2 + 12x - p = 0$
- (iii) $px^2 + 4x + p = 0$
- (iv) $2px^2 - 20x + (13p - 1) = 0$
- (v) $3px^2 + 18x + p = 0$

ANSWER KEY

A. VERY SHORT ANSWER TYPE :

1. (i), (ii), (iii), (vi), (vii), (ix) 2. (i) 26 (ii) -18 (iii) 12 (iv) -19 (v) $5\sqrt{2} + 1$ (vi) -130 (vii) 173
3. (i) $-\frac{7}{2}$ (ii) $-\frac{29}{10}$ 4. (i) 5. (ii) 6. (i), (ii) 7. yes
8. no 9. (i) $-\frac{5}{8}$ (ii) $\frac{9}{4}$ (iii) $-a^2$ 10. no [only $x = 3/2$ is a solution of the given equation]
11. -3, -2 12. 11, -3 13. -8, 4 14. -6, 1 15. 6, -1 16. 3, 2 17. a, $-\frac{3a}{5}$
18. 0, -8 19. $\frac{a}{3}, -a$ 20. $7, -\frac{3}{4}$ 21. $\frac{3}{2}, -\frac{4}{5}$ 22. $\frac{3}{4}, -\frac{1}{2}$ 23. $3, -\frac{2}{3}$ 24. $0, \frac{7}{4}$
25. 2, -5 26. 6, -3 27. $\frac{5}{2}, -\frac{5}{2}$
28. (i) Rational (real) and unequal. (ii) Rational (real) and unequal. (iii) Rational and unequal. (iv) Real and equal.
 (v) Imaginary (vi) Irrational and unequal. (vii) Real and equal
 (viii) Irrational and unequal. (ix) Irrational and unequal.
29. (i) 5 (ii) 41 (iii) -3 (iv) $p^2 - 72$ (v) $9 + 20a$ (vi) $25 - 16c$ (vii) 91 (viii) -56
30. (i) $\frac{7}{2}, 2$ (ii) $\frac{-4\sqrt{2}}{3}, 3$ (iii) $\frac{-5\sqrt{3}}{2}, -\frac{3}{2}$ (iv) $2\sqrt{5}, -15$ (v) $2, \frac{3\sqrt{5}}{5}$

B. SHORT ANSWER TYPE :

31. $\sqrt{5} + 2, \sqrt{5} - 2$ 32. $1, -\frac{5}{4}$ 33. no real root. 34. $\frac{5+\sqrt{7}}{2}, \frac{5-\sqrt{7}}{2}$
35. 1, 5 36. $1, \frac{3}{2}$ 37. $\frac{3+\sqrt{19}}{5}, \frac{3-\sqrt{19}}{5}$ 38. $3\sqrt{2}, -\sqrt{2}$
39. $\sqrt{6}, -\frac{2}{\sqrt{6}}$ 40. $-3\sqrt{3}, -\frac{2}{\sqrt{3}}$ 41. $\frac{1}{4}, -\frac{1}{4}$ 42. $1, -\frac{4}{5}$
43. $1, \frac{3}{4}$ 44. 0, 3 45. $0, \frac{5}{3}$
46. (i) $p \leq 4$ (ii) $p \leq \frac{9}{8}$ (iii) $p^2 \geq 24$ (iv) $p^2 + 15 \geq 0$ (v) $p \geq -\frac{3}{2}$
47. (i) yes, $-3 \pm \sqrt{3}$ (ii) no (iii) yes, $-1, \frac{3}{4}$ (iv) yes, $-\frac{5}{3}, -\frac{5}{3}$ (v) yes, $\frac{3}{2}, \frac{3}{2}$
- (vi) no (vii) yes, $\frac{3 \pm \sqrt{21}}{6}$ (viii) yes, $-\sqrt{3}, -\frac{\sqrt{3}}{4}$ (ix) yes, $\frac{3\sqrt{5}}{5}$

48. (i) $x^2 - 25 = 0$

(ii) $x^2 - 11x + 24 = 0$

(iii) $x^2 + 11x + 24 = 0$

(iv) $x^2 + 5x - 24 = 0$

(v) $x^2 - 6\sqrt{3}x + 15 = 0$

(vi) $x^2 + \sqrt{2}x - 12 = 0$

(vii) $x^2 + 7\sqrt{5}x + 60 = 0$

(viii) $x^2 - 2x - 1 = 0$

(ix) $x^2 - 8x - 11 = 0$

(x) $x^2 - 14x + 42 = 0$

(xi) $9x^2 - 18x + 7 = 0$

(xii) $4x^2 - 16x + 11 = 0$

C. LONG ANSWER TYPE :

49. 0, 3

50. $a = 7, c = 7$

51. $a = \frac{1}{2}, b = 4$

52. $p = 7, -\frac{2}{3}$

53. (i) $\frac{41}{4}$ (ii) $\frac{71}{4}$ (iii) $\frac{73}{4}$ (iv) $-\frac{245}{8}$ (v) $-\frac{41}{8}$

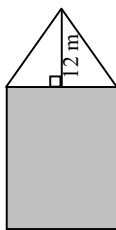
54. (i) 34 (ii) 1154 (iii) 198 (iv) 32

55. (i) $6\frac{1}{8}$ (ii) -6 (iii) ± 2 (iv) $2, -\frac{25}{13}$ (v) $\pm 3\sqrt{3}$

EXERCISE # 2

- Q.1** Find the value(s) of k for which each of the following quadratic equation has equal roots :
- (i) $2x^2 + kx + 3 = 0$
 - (ii) $(k + 4)x^2 + (k + 1)x + 1 = 0$
 - (iii) $(k - 4)x^2 + 2(k - 4)x + 4 = 0$
 - (iv) $kx(x - 2) + 6 = 0$.
- Q.2** Find the value(s) of k for which the following quadratic equation has real roots
- (i) $px^2 + 4x + 1 = 0$
 - (ii) $4x^2 + 8x - p = 0$
 - (iii) $2x^2 + px + 8 = 0$
- Q.3** Find the value(s) of k for which the following simultaneous linear equation have a unique solution :
- $$(k - 1)x - 3y = 4, 3x - (4k + 1)y = 5$$
- Q.4** (i) Find two consecutive natural numbers such that the sum of their squares is 365.
- (ii) Find two consecutive integers such that the sum of their squares is 365.
- (iii) If the product of two consecutive positive integers is 306, find the integers.
- Q.5** (i) If the product of two positive consecutive odd integers is 255, find the integers.
- (ii) If the product of two consecutive odd integers is 255, find the integers.
- Q.6** The sum of the squares of two consecutive odd natural numbers is 290, find the numbers.
- Q.7** The sum of the squares of two natural numbers is 116. If the square of the larger is 25 times the smaller, find the numbers.
- Q.8** The difference of squares of two numbers is 180. The square of the smaller number is 8 times the large number. Find the two numbers.
- Q.9** (i) The sum of the square of three consecutive natural numbers is 110, find the numbers.
- (ii) Three consecutive natural numbers are such that the square of the middle number exceeds the different of the square of the other two by 60, find the numbers.
- Q.10** Divide 14 into two parts such that the sum of their reciprocals is $\frac{7}{20}$.
- Q.11** The difference of two natural number is 4 and the difference of their reciprocals is $\frac{1}{8}$, find the numbers.
- Q.12** The sum of the numerator and denominator of a certain fraction is 11. If 1 is added to both numerator and denominator, the fraction increases by $\frac{3}{56}$. Find the fraction.
- Q.13** The denominator of a fraction is one more than twice the numerator. If the sum of the fraction and its reciprocal is $2\frac{16}{21}$, find the fraction.
- Q.14** A two digit number contains the bigger at ten's place. The product of the digit is 27 and the difference between the two digits is 6. find the number.
- Q.15** A two digit number is such that the product of its digits is 24. When 18 is subtracted from this number, the digits interchange their places. Find the number.
- Q.16** Mukesh and Suresh together have 45 marbles. Both of them lost 5 marbles each, and the product of the number of marbles they now have is 124. How many marbles they had to start with.
- Q.17** A cottage industry produces a certain number of toys in a day. The cost of production of each toy (in rupees) was found to be 55 minus the number of toys produces in a day. On a particular day, the total cost reduction was Rs. 750. Find the number of toys produced on that day.
- Q.18** The area of rectangular plot is 528 m^2 . The length of the plot (in meters) is one more than twice its breadth. find the length and breadth of the plot.

- Q.19** A rectangle has an area of 24 cm^2 . If the perimeter is 20 cm , find its length.
- Q.20** A rectangular garden 10 m by 16 m is to be surrounded by a concrete walk of uniform width. If the area of the walk is 120 m^2 , find the width of the walk.
- Q.21** Harish made a rectangular garden with its length 5 meters more than its breadth. Next year, he increased the length by 3 metres and decreased the width by 2 meters. If the area of the garden is 119 sq. m , was this garden larger or smaller ?
- Q.22** The sum of the areas of two squares is 468 m^2 . If the difference of their perimeters is 24 m , find the sides of the two squares.
- Q.23** The altitude of a right triangle is 7 cm less than its base. If the hypotenuse is 13 cm , find the other two sides.
- Q.24** A wire, 112 cm long is bent to form a right angled triangle. If the hypotenuse is 50 cm long, find the area of the triangle.
- Q.25** A rectangular park is to be designed whose length is 3m more than its breadth. Its area is 4 square metres more than the area of a park that has already been made in the shape of an isosceles triangle with its base as the breadth of the rectangular park and of altitude 12 m (Shown in the adjoining figure). Find the dimensions of the rectangular park.



- Q.26** A school bus transported an excursion party to a picnic spot 150 km away. While returning it was raining and the bus had to reduce its speed by 5 km / hr , and it took one hour longer to make the return trip. Find the time taken in return trip.

- Q.27** An aeroplane flying with a wind of 30 km / hr takes 40 minutes less of fly 3600 km , than what it would have taken to fly the same wind. Find the plane's speed in still air.
- Q.28** A motor boat whose speed is 18 km/hr in still water takes 1 hour more to go 24 km upstream than to return to the same spot. Find the speed of the stream.
- Q.29** When the price of an article is reduced by Rs 2 , 5 more articles could be bought for Rs 120 . Find the original price of each article.
- Q.30** A trader bought a number of articles for Rs 900 , five were damaged and he sold each of the rest at Rs 2 more than what he paid for it thus getting a profit of Rs 80 on the whole transaction. Find the number of articles he bought.
- Q.31** Rahul sold an article for Rs 56 which cost him Rs x . He finds that he has gained $x\%$ on his outlay. Find x .
- Q.32** (i) Rohan's mother is 26 years older than him. The product of their ages (in years) 3 years from now will be 360 . What is Rohan's present age ?
(ii) Forty years hence Mr. Pratap's age will be square of what it was 32 years ago. Find his present age.
(iii) The sum of the ages of father and his son is 45 years. Five years ago, the product of their ages (in years) was 124 . Find the present ages.
- Q.33** The age of a man is twice the square of the age of his son. Eight years hence, the age of the man will be 4 years more than three times the age of his son. Find their present ages.
- Q.34** B takes 16 days less than A to do a piece of work. If both working together can do it in 15 days, in how many days will be alone complete the work ?

ANSWER KEY

1. (i) $2\sqrt{6}, -2\sqrt{6}$ (ii) 5, -3 (iii) 8 (iv) 6 2. (i) $p \leq 4$ (ii) $p \geq -4$ (iii) $p \geq 8$ or $p \leq -8$
3. All real nos except 2 and $-\frac{5}{4}$ 4. (i) 13, 14 (ii) 13, 14 or -13, -14 (iii) 17, 18
5. (i) 15, 17 (ii) 15, 17 or -15, -17 6. 11, 13 7. 4, 10 8. 18, 12 or 18, -12
9. (i) 5, 6, 7 (ii) 9, 10, 11 10. 4, 10 11. 8, 4 12. $\frac{4}{7}$
13. $\frac{3}{7}$ 14. 93 15. 64 16. 9, 36
17. 25 or 30 18. 33 m, 16 m 19. 6 cm
20. 2 m. 21. smaller by 7 sq. m. 22. 18 m, 12 m
23. 12 cm, 5 cm 24. 336 cm^2 25. $7 \text{ m} \times 4 \text{ m}$
26. 6 hours 27. 570 km/hr 28. 6 km/ hr 29. Rs. 8
30. 75 31. 40 32. (i) 7 years (ii) 41 years (iii) 36 years, 9 years
33. 32 years, 4 years 34. 24 days