Chapter 5

Quadratic Expressions and Equations

REMEMBER

Before beginning this chapter, you should be able to:

- Understand simple quadratic equations
- Know natural numbers, integers and fractions
- Aware of basic algebra and simple indices

KEY IDEAS

After completing this chapter, you should be able to:

- Study about quadratic expression, its zeroes and quadratic equations
- Find solutions/roots of a quadratic equation
- Understand the nature and sign of the roots of a quadratic equation
- Learn reciprocal equation and maximum or minimum value of a quadratic equation

INTRODUCTION

In previous topic, we have learnt linear expression, linear equation and linear inequation. In this topic, we shall learn quadratic expression and quadratic equation.

QUADRATIC EXPRESSION

A polynomial of second degree in one variable is termed as a quadratic polynomial or quadratic expression. The general form of a quadratic expression in x is $ax^2 + bx + c$ where a, b, c are real numbers and $a \neq 0$.

Example: $2x^2 + 3x$, $x^2 - 2$, $3x^2 + 11x - 108$ are some quadratic expressions.

> **Note** The expressions $x^2 + \frac{1}{x^2} - 3$, $x^2 + \sqrt[3]{x} + 3$, $2x^2 - \frac{1}{x} + 4$ are not quadratic expressions.

Zeroes of a Quadratic Expression

If a quadratic expression, $ax^2 + bx + c$ becomes zero for $x = \alpha$, where α is a real number, then α is called a zero of the expression $ax^2 + bx + c$. A quadratic expression can have at the most two zeroes.

QUADRATIC EQUATION

The equation of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$ is known as a quadratic equation, or an equation of the second degree.

Example: $2x^2 + 3x + 5 = 0$, $x^2 - 5 = 0$ and $3x^2 - 4x + \sqrt{5} = 0$ are some quadratic equations.

Solutions or Roots of a Quadratic Equation

The values of x for which the equation $ax^2 + bx + c = 0$ is satisfied are called the roots of the quadratic equation. A quadratic equation cannot have more than two roots.

EXAMPLE 5.1

Verify whether $x = 2$, is a solution of $2x^2 + x - 10 = 0$.

SOLUTION

On substituting $x = 2$ in $2x^2 + x - 10$, we get

$$2(2)^2 + 2 - 10 = 10 - 10 = 0.$$

$\therefore$ 2 is a solution (or) root of $2x^2 + x - 10 = 0$.

Finding the Solutions or Roots of a Quadratic Equation

There are two ways of finding the roots of a quadratic equation.

1. Factorization method
2. Application of formula

Factorization Method to Obtain the Roots of a Quadratic Equation

The steps involved in obtaining the roots of $ax^2 + bx + c = 0$ are as follows:

1. Resolve $ax^2 + bx + c$ into factors and express $ax^2 + bx + c$ as a product of its factors.
2. For this product to be zero, one of the factors should be zero. The zeroes of the factors give the roots of the equation, $ax^2 + bx + c = 0$.

EXAMPLE 5.2

(a) Solve $x^2 - 15x + 26 = 0$.

(b) Solve $x + \frac{1}{x} = \frac{5}{2}$.

SOLUTION

(a) First, let us resolve $x^2 - 15x + 26$ into factors.

$$\Rightarrow x^2 - 15x + 26$$
$$= x^2 - 13x - 2x + 26$$
$$= x(x - 13) - 2(x - 13)$$
$$= (x - 13)(x - 2).$$

The given equation, $x^2 - 15x + 26 = 0$ is reduced to $(x - 13)(x - 2) = 0$

$$\Rightarrow x - 13 = 0 \text{ (or) } x - 2 = 0$$
$$\Rightarrow x = 13 \text{ (or) } x = 2.$$

$\therefore$ $x = 2, 13$ are the roots of the given equation.

(b) $\frac{x^2 + 1}{x} = \frac{5}{2}$

$$\Rightarrow 2x^2 + 2 = 5x$$
$$\Rightarrow 2x^2 - 5x + 2 = 0$$
$$\Rightarrow 2x^2 - 4x - x + 2 = 0$$
$$\Rightarrow 2x(x - 2) - 1(x - 2) = 0$$
$$\Rightarrow (2x - 1)(x - 2) = 0$$
$$\Rightarrow 2x - 1 = 0 \text{ or } x - 2 = 0$$
$$\Rightarrow x = \frac{1}{2} \text{ or } x = 2.$$

$\therefore$ $x = \frac{1}{2}$, 2 are the roots of the given equation.

Finding the Solutions/Roots of a Quadratic Equation by the Application of Formula

Factorization of $ax^2 + bx + c$ might not be always possible using the above method. So here, we derive a formula to find the roots of the equation, $ax^2 + bx + c = 0$, where $a \neq 0$, $b, c \in R$.

$$ax^2 + bx + c = 0$$
$$\Rightarrow ax^2 + bx = -c$$
$$\Rightarrow a\left(x^2 + \frac{b}{a}x\right) = -c$$
$$\Rightarrow x^2 + \frac{b}{a}x = \frac{-c}{a}$$
$$\Rightarrow x^2 + 2\cdot\frac{b}{2a}\cdot x = \frac{-c}{a}$$

Adding $\left(\frac{b}{2a}\right)^2$ on both sides to make LHS a perfect square, we get,

$$\Rightarrow (x)^2 + 2\left(\frac{b}{2a}\right)(x) + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$
$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$
$$\Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$
$$\Rightarrow x = \frac{-b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{(2a)^2}}$$
$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\therefore$ The roots of the equation $ax^2 + bx + c = 0$ are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Relation Between the Roots and the Coefficients of a Quadratic Equation

Let us assume that α, β are the roots of $ax^2 + bx + c = 0$.

Then, $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Now, the sum of the roots, $\alpha + \beta = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a} + \dfrac{-b - \sqrt{b^2 - 4ac}}{2a} = \dfrac{-2b}{2a} = \dfrac{-b}{a}$.

$\therefore$ The sum of the roots, $\alpha + \beta = \dfrac{-b}{a} = \dfrac{-(\text{coefficient of } x)}{(\text{coefficient of } x^2)}$.

The product of the roots, $\alpha\beta = \left(\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$

$$= \frac{(-b)^2 - \left(\sqrt{b^2 - 4ac}\right)^2}{4a^2}$$

$$= \frac{b^2 - b^2 + 4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

$\therefore$ The product of the roots, $\alpha\beta = \frac{c}{a} = \frac{\text{(constant term)}}{\text{(coefficient of } x^2)}$.

Nature of the Roots of a Quadratic Equation

We know that the roots of the equation $ax^2 + bx + c = 0$ are

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We call, $\Delta = b^2 - 4ac$, the discriminant of $ax^2 + bx + c = 0$.

The value of the discriminant determines the nature of the roots of the equation.

In this context, we study the following cases. We shall limit ourselves to cases where the coefficients a, b and c are real.

Case 1: If $b^2 - 4ac = 0$, i.e., $\Delta = 0$, then

$$\alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{-b}{2a}.$$

So, $\alpha = \beta = \frac{-b}{2a}$.

Hence, when $\Delta = b^2 - 4ac = 0$, the two roots of the equation are real and equal.

Case 2: If $b^2 - 4ac > 0$, then the roots are real and distinct.

We shall consider, the nature of the roots when a, b and c are rational numbers.

1. If $b^2 - 4ac > 0$, i.e., $\Delta > 0$ and a perfect square, then the roots are rational and distinct.
2. If $b^2 - 4ac > 0$, i.e., $\Delta > 0$ and not a perfect square, then the roots are irrational and distinct.

In this case one root is a surd conjugate of the other. If one root is of the form $a + \sqrt{b}$, then the other root will be in the form of $a - \sqrt{b}$ and *vice-versa*.

Case 3: If $b^2 - 4ac < 0$, i.e., $\Delta < 0$, then the roots of the equation are imaginary.

Signs of the Roots of a Quadratic Equation

When the signs of the sum of the roots and the product of the roots of a quadratic equation are known, the signs of the roots of the quadratic equation can be determined. Let us consider the following four cases.

Case 1: When the sum and the product of the roots of a quadratic equation are both positive, then each root is positive.

For example, if the sum of the roots is 5 and the product of the roots is 6, then the roots are 2 and 3.

Case 2: When the sum of the roots is positive and the product of the roots is negative, the root having the greater magnitude is positive and the other root is negative.

For example, the sum of the roots is 6 and the product of the roots is −16, the roots are +8 and −2. Because $8 \times (-2) = -16$ and $+8 - 2 = 6$. The root with the greater magnitude, i.e., 8, has a positive sign and the root with the lesser magnitude, i.e., 2 has the negative sign.

Case 3: When the sum of the roots is negative and the product of the roots is positive, then both the roots are negative.

For example, the sum of the roots is −10 and the product of the roots is 24, the roots are −6 and −4.

$$-6 - 4 = -10 \text{ and } (-6)(-4) = 24.$$

In this case, both the roots, i.e., −6 and −4 bear the negative sign.

Case 4: When the sum of the roots is negative and the product of the roots is negative, the root having the greater magnitude has the negative sign and the other root has the positive sign.

For example, the sum of the roots is −4 and the product of the roots is −21, then the roots are −7 and +3.

$$(-7) \times 3 = -21 \text{ and } -7 + 3 = -4.$$

The root with the greater magnitude, i.e., 7, has the negative sign and the root with the lesser magnitude, i.e., 3, has the positive sign.

The above information can be tabulated as follows:

Sign of the Sum of the Roots	Sign of the Product of the Roots	Sign of the Roots
+ve	+ve	Both roots are positive
+ve	−ve	One root is positive, the other is negative. Numerically larger root is positive.
−ve	−ve	One root is positive and the other is negative. Numerically larger root is negative.
−ve	+ve	Both roots are negative.

Constructing the Quadratic Equation when its Roots are Given

Let us say that α and β are the roots of a quadratic equation.

The quadratic equation can be written as

$$(x - \alpha)(x - \beta) = 0.$$

$$\Rightarrow x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

That is, x^2 − (sum of the roots) x + (product of the roots) = 0

$$\Rightarrow x^2 - \left(\frac{-b}{a}\right)x + \frac{c}{a} = 0.$$

EXAMPLE 5.3

Solve the equation $x^2 - 11x + 30 = 0$ by using the formula.

SOLUTION

Given equation is $x^2 - 11x + 30 = 0$.

The roots of $ax^2 + bx + c = 0$, are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Here, $a = 1$, $b = -11$ and $c = 30$.

That is,

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(1)(30)}}{2 \times 1} = \frac{11 \pm \sqrt{121 - 120}}{2}$$

$$x = \frac{11 \pm 1}{2}$$

$$x = \frac{11+1}{2} \text{ and } \frac{11-1}{2} \Rightarrow 6 \text{ and } 5.$$

∴ The roots are 5 and 6.

EXAMPLE 5.4

Find the nature of the roots of the equations given below:

(a) $x^2 - 13x + 11 = 0$

(b) $18x^2 - 14x + 17 = 0$

(c) $9x^2 - 36x + 36 = 0$

(d) $3x^2 - 5x - 8 = 0$

SOLUTION

(a) Given $x^2 - 13x + 11 = 0$.

Comparing it with $ax^2 + bx + c = 0$, we have $a = 1$, $b = -13$ and $c = 11$.

Now, $b^2 - 4ac = (-13)^2 - 4(1)(11)$

$= 169 - 44 = 125 > 0.$

$\Rightarrow b^2 - 4ac > 0$ and is not a perfect square.

$\therefore$ The roots are distinct and irrational.

(b) Given $18x^2 - 14x + 17 = 0$.

Comparing it with $ax^2 + bx + c = 0$, $a = 18$, $b = -14$ and $c = 17$.

Now, $b^2 - 4ac = (-14)^2 - 4(18)(17)$

$= 196 - 1224$

$= -1028 < 0$

$\Rightarrow b^2 - 4ac < 0$.

$\therefore$ The roots are imaginary.

(c) Given $9x^2 - 36x + 36 = 0$.

$\Rightarrow 9(x^2 - 4x + 4) = 0$

$\Rightarrow x^2 - 4x + 4 = 0$

Comparing the above equation, with $ax^2 + bx + c = 0$, $a = 1$, $b = -4$ and $c = 4$.

Now, $b^2 - 4ac = (-4)^2 - 4(1)(4)$

$= 16 - 16 = 0$

$\Rightarrow b^2 - 4ac = 0$

$\therefore$ The roots are real and equal.

(d) Given $3x^2 - 5x - 8 = 0$.

Comparing the above equation with $ax^2 + bx + c = 0$, we get, $a = 3$, $b = -5$ and $c = -8$.

Now, $b^2 - 4ac = (-5)^2 - 4(3)(-8)$

$= 25 + 96 = 121 > 0$

$\Rightarrow b^2 - 4ac > 0$ and is a perfect square.

$\therefore$ The roots are rational and distinct.

EXAMPLE 5.5

If α, β are the roots of the equation $x^2 - lx + m = 0$, then find the value of $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$ in terms of l and m.

SOLUTION

Given α, β are the roots of $x^2 - lx + m = 0$.

$$\Rightarrow \text{Sum of the roots } = \alpha + \beta = \frac{-(-l)}{1} = l \qquad (1)$$

$$\Rightarrow \text{Product of the roots} = \alpha\beta = \frac{m}{1} = m \qquad (2)$$

Now,

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}$$

$$=\frac{(\alpha+\beta)^2-2(\alpha\beta)}{(\alpha\beta)^2}$$

Substituting the values of $\alpha+\beta$ and $\alpha\beta$ in the above equation, we get,

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{l^2-2m}{m^2}$$

$\therefore$ The value of $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{l^2-2m}{m^2}$.

EXAMPLE 5.6

Write a quadratic equation whose roots are $\frac{5}{2}$ and $\frac{8}{3}$.

SOLUTION

Sum of the roots $=\frac{5}{2}+\frac{8}{3}=\frac{31}{6}$.

Product of the roots $=\frac{5}{2}\left(\frac{8}{3}\right)=\frac{20}{3}$.

The required quadratic equation is, x^2 − (sum of the roots)x + (product of the roots) = 0.

$$\Rightarrow x^2-\left(\frac{31}{6}\right)x+\frac{20}{3}=0$$

$$\Rightarrow 6x^2-31x+40=0$$

$\therefore$ A quadratic equation with roots $\frac{5}{2}$ and $\frac{8}{3}$ is $6x^2 - 31x + 40 = 0$.

EXAMPLE 5.7

If one root of the equation, $x^2 - 11x + (p - 3) = 0$ is 3, then find the value of p and also its other root.

SOLUTION

Given that 3 is one of the roots of the equation $x^2 - 11x + p - 3 = 0$.

$\Rightarrow x = 3$ satisfies the given equation.

$$\Rightarrow (3)^2 - 11(3) + p - 3 = 0$$

$$\Rightarrow p = 33 + 3 - 9$$

$$\Rightarrow p = 27$$

$\therefore$ The value of p is 27.

Since the sum of the roots of the equation is 11 and one of the roots is 3, the other root of the equation is 8.

Equations which can be Reduced to Quadratic Form

EXAMPLE 5.8

Solve $(x^2 - 2x)^2 - 23(x^2 - 2x) + 120 = 0$.

SOLUTION

Let us assume that $x^2 - 2x = y$

$\Rightarrow$ The given equation reduced to a quadratic equation in y

That is, $y^2 - 23y + 120 = 0$

$$\Rightarrow y^2 - 15y - 8y + 120 = 0$$
$$\Rightarrow y(y - 15) - 8(y - 15) = 0$$
$$\Rightarrow (y - 8)(y - 15) = 0$$
$$\Rightarrow y - 8 = 0 \text{ (or) } y - 15 = 0$$
$$\Rightarrow y = 8 \text{ (or) } y = 15$$

But $x^2 - 2x = y$

When $y = 8$, $x^2 - 2x = 8$

$$\Rightarrow x^2 - 2x - 8 = 0$$
$$\Rightarrow x^2 - 4x + 2x - 8 = 0$$
$$\Rightarrow x(x - 4) + 2(x - 4) = 0$$
$$\Rightarrow (x + 2)(x - 4) = 0$$
$$\Rightarrow x + 2 = 0 \text{ (or) } x - 4 = 0$$
$$\Rightarrow x = -2 \text{ (or) } x = 4$$

When $y = 15$, $x^2 - 2x = 15$

$$\Rightarrow x^2 - 2x - 15 = 0$$
$$\Rightarrow x^2 - 5x + 3x - 15 = 0$$
$$\Rightarrow x(x - 5) + 3(x - 5) = 0$$
$$\Rightarrow (x - 5)(x + 3) = 0$$
$$\Rightarrow x - 5 = 0 \text{ (or) } x + 3 = 0$$
$$\Rightarrow x = 5 \text{ (or) } x = -3$$

$\therefore$ $x = -2, -3, 4$ and 5 are the required solutions of the given equation.

EXAMPLE 5.9

Solve $\sqrt{x+5} + \sqrt{5-x} = 4$.

SOLUTION

Squaring the terms on both the sides, we get

$$(\sqrt{x+5}+\sqrt{5-x})^2 = 4^2$$

$$\Rightarrow x+5+5-x+2\sqrt{(x+5)(5-x)} = 16$$

$$\Rightarrow 10+2\sqrt{25-x^2} = 16$$

$$\Rightarrow \sqrt{25-x^2} = 3$$

Squaring the terms on both the sides again, we get

$$25 - x^2 = 3^2$$

$$\Rightarrow x^2 = 25 - 9$$

$$\Rightarrow x^2 = 16 \Rightarrow x = \pm 4.$$

$\therefore$ −4 and 4 are the required solutions of the given equation.

Reciprocal Equation

Any equation of the form $ax^4 + bx^3 + cx^2 + bx + a = 0$, in which the coefficients of terms equidistant from first and last are equal in magnitude, is called a reciprocal equation. This is one of the case of a reciprocal equation. This can be reduced to quadratic form by dividing by x^2 on both sides and with a proper substitution.

EXAMPLE 5.10

Solve $3x^4 - 8x^3 - 6x^2 + 8x + 3 = 0$.

SOLUTION

The above equation is a reciprocal equation. Dividing the equation by x^2, we get

$$\frac{3x^4 - 8x^3 - 6x^2 + 8x + 3}{x^2} = 0$$

$$\Rightarrow 3\left(x^2+\frac{1}{x^2}\right)-8\left(x-\frac{1}{x}\right)-6=0 \qquad (1)$$

Now put $x - \dfrac{1}{x} = y$

$$\therefore\ y^2 = \left(x-\frac{1}{x}\right)^2$$

$$\Rightarrow y^2 = x^2+\frac{1}{x^2}-2$$

$$\Rightarrow x^2+\frac{1}{x^2} = y^2+2$$

Substituting $x - \frac{1}{x}$ and $x^2 + \frac{1}{x^2}$ in terms of y in the Eq. (1) we get,

$$3(y^2 + 2) - 8(y) - 6 = 0.$$
$$\Rightarrow 3y^2 + 6 - 8y - 6 = 0$$
$$\Rightarrow 3y^2 - 8y = 0$$
$$\Rightarrow y(3y - 8) = 0$$

When $y = 0$, $x - \frac{1}{x} = 0$

$$\Rightarrow x^2 = 1$$
$$\Rightarrow x = \pm 1$$

When $y = \frac{8}{3}$, $x - \frac{1}{x} = \frac{8}{3}$

$$\Rightarrow 3x^2 - 3 = 8x$$
$$\Rightarrow 3x^2 - 8x - 3 = 0$$
$$\Rightarrow 3x^2 - 9x + x - 3 = 0$$
$$\Rightarrow 3x(x - 3) + 1(x - 3) = 0$$
$$\Rightarrow (3x + 1)(x - 3) = 0$$
$$\Rightarrow x = -\frac{1}{3} \text{ (or) } x = 3$$

$\therefore$ $x = \pm 1$, $\frac{-1}{3}$ and 3 are the required solutions of the given equation.

Constructing a New Quadratic Equation by Changing the Roots of a Given Quadratic Equation

If we are given a quadratic equation, we can build a new quadratic equation by changing the roots of the given equation as directed.

For example, consider the quadratic equation $ax^2 + bx + c = 0$, whose roots are α and β.

The new equations can be constructed in the following manner:

1. A quadratic equation whose roots are the reciprocals of the roots of the equation $ax^2 + bx + c = 0$, can be formed by substituting $\frac{1}{x}$ for x. The new equation is $a\left(\frac{1}{x}\right)^2 + b\left(\frac{1}{x}\right) + c = 0$, i.e., $cx^2 + bx + a = 0$.
2. A quadratic equation, whose roots are k more than the roots of the equation $ax^2 + bx + c = 0$, is obtained by substituting $(x - k)$ for x in the given equation.
3. A quadratic equation whose roots are k less than the roots of the equation $ax^2 + bx + c = 0$ can be obtained by substituting $(x + k)$ for x in the given equation.

4. A quadratic equation whose roots are k times the roots of the equation $ax^2 + bx + c = 0$ can be obtained by substituting $\frac{x}{k}$ for x in the given equation.

5. A quadratic equation whose roots are $\frac{1}{k}$ times the roots of the equation $ax^2 + bx + c = 0$ can be obtained by substituting kx for x in the given equation.

EXAMPLE 5.11

The roots of $x^2 - (a + 1)x + b^2 = 0$ are equal. Then choose the correct value of a, b from the following option:

(a) 5, 2 **(b)** 3, 4 **(c)** 5, −3 **(d)** 5, 4

SOLUTION

The roots of $x^2 - (a + 1)x + b^2 = 0$ are equal

$\Rightarrow (a + 1)^2 - 4b^2 = 0$

$\Rightarrow a + 1 = \pm 2b$

From the options

$a = 5$, $b = -3$ satisfies the above relation.

Maximum or Minimum Value of a Quadratic Expression

The quadratic expression $ax^2 + bx + c$ takes different values, as x takes different values.

As x varies from $-\infty$ to $+\infty$ (i.e., when x is real), the quadratic expression $ax^2 + bx + c$

1. has the minimum value, when $a > 0$.

2. has the maximum value, when $a < 0$.

The minimum or the maximum value of the quadratic expression $ax^2 + bx + c$ occurs at $x = \frac{-b}{2a}$ and is equal to $\frac{4ac - b^2}{4a}$.

When x has an imaginary value, $ax^2 + bx + c$ may have a real value or an imaginary value. For some imaginary values of x, $ax^2 + bx + c$ will be real and it may have minimum or maximum value. But such cases will be dealt in the higher stages of learning.

EXAMPLE 5.12

Find the value of x, to get the maximum value of $-3x^2 + 6x + 5$.

(a) 1 **(b)** $\frac{5}{3}$ **(c)** $\frac{-5}{6}$ **(d)** 6

SOLUTION

Maximum value of a quadratic expression occurs at $x = \frac{-b}{2a}$.

$\Rightarrow$ For $-3x^2 + 6x + 5$, maximum value occurs at $x = \frac{-6}{2(-3)}$, i.e., 1.

EXAMPLE 5.13

Choose the minimum value of $\frac{2x^2+12x-3}{1+18x-3x^2}$ from the following options:

(a) $\frac{-15}{29}$ **(b)** $\frac{15}{28}$ **(c)** $\frac{-15}{28}$ **(d)** None of these

SOLUTION

For the minimum value of $\frac{2x^2-12x+3}{1+18x-3x^2}$, $2x^2 - 12x + 3$ is minimum and $1 + 18x - 3x^2$ is maximum.

The minimum value of $2x^2 - 12x + 3$ occurs at $x=\frac{-b}{2a}=\frac{-(-12)}{2\times 2}=3$.

The maximum value of $1 + 18x - 3x^2$ occurs at $x=\frac{-b}{2a}=\frac{-18}{2\times -3}=3$.

Minimum value of given expression is $\frac{2(3)^2-12(3)+3}{1+18(3)-3(3)^2}=\frac{18-36+3}{55-27}=\frac{-15}{28}$.

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. For the expression $ax^2 + 7x + 2$ to be quadratic, the possible values of a are _____.
2. The equation, $ax^2 + bx + c = 0$ can be expressed as $a\alpha^2 + b\alpha + c = 0$, only when '$\alpha$' is _____ of the equation.
3. If -3 and 4 are the roots of the equation $(x + k)(x - 4) = 0$, then the value of k is _____.
4. The polynomial, $\sqrt{3}x^2 + 2x + 1$ is a _____ expression.
5. For the equation $2x^2 - 3x + 5 = 0$, sum of the roots is _____.
6. The quadratic equation having roots $-a$, $-b$ is _____.
7. A quadratic equation whose roots are 2 more than the roots of the quadratic equation $2x^2 + 3x + 5 = 0$ can be obtained by substituting _____ for x. [$(x - 2)/(x + 2)$]
8. For the expression $7x^2 + bx + 4$ to be quadratic, the possible values of b are _____.
9. If $(x - 2)(x + 3) = 0$, then the values of x are _____.
10. The roots of the equation $x^2 + ax + b = 0$ are _____.
11. $x = 2$ is a root of the equation $x^2 - 5x + 6 = 0$. Is the given statement true?
12. If the equation $3x^2 - 2x - 3 = 0$ has roots α and β, then $\alpha \cdot \beta =$ _____.
13. If the discriminant of the equation $ax^2 + bx + c = 0$ is greater than zero, then the roots are _____.
14. If the roots of a quadratic equation $ax^2 + bx + c$ are complex, then $b^2 <$ _____.
15. The roots of a quadratic equation $ax^2 + bx + c = 0$ are 1 and $\frac{c}{a}$, then $a + b + c =$ _____.
16. If the roots of a quadratic equation are equal, then the discriminant of the equation is _____.
17. For what values of b, the roots of $x^2 + bx + 9 = 0$ are equal?
18. If the sum of the roots of a quadratic equation is positive and product of the roots is negative, the numerically greater root has _____ sign. [positive/negative]
19. If $x = 1$ is a solution of the quadratic equation $ax^2 - bx + c = 0$, then b is equal to _____.

Short Answer Type Questions

20. Factorize the following quadratic expressions:
    (a) $x^2 + 5x + 6$
    (b) $x^2 - 5x - 36$
    (c) $2x^2 + 5x - 18$
21. Determine the nature of the roots of the following equations:
    (a) $x^2 + 2x + 4 = 0$
    (b) $3x^2 - 10x + 3 = 0$
    (c) $x^2 - 24x + 144 = 0$

Solve the following quadratic equations:

22. If $f(x) = x^2 - 5x - 36$ and $g(x) = x^2 + 9x + 20$, then for what values of x is $2f(x) = 3g(x)$?
23. Solve: $16x^4 - 28x^2 - 8 = 0$
24. For what value of m does the equation, $mx^2 + (3x - 1)m + 2x + 5 = 0$ have equal roots but of opposite sign?
25. Find the value of m for which the quadratic equation, $3x^2 - 10x + (m - 3) = 0$ has roots which are reciprocal to each other.
26. If a, b are the roots of the equation $x^2 - px + q = 0$, then find the equation which has $\frac{a}{b}$ and $\frac{b}{a}$ as its roots.
27. If one root of the equation $x^2 - mx + n = 0$ is twice the other root, then show that $2m^2 = 9n$.
28. The square of one-sixth of the number of students in a class are studying in the library and the

remaining eight students are playing in the ground. What is the total number of students of the class?

29. If 2α and 3β are the roots of the equation $x^2 + ax + b = 0$, then find the equation whose roots are a, b.

30. If α, β are the roots of the quadratic equation $lx^2 + mx + n = 0$, then evaluate the following expressions.

(a) $\alpha^2 + \beta^2$

(b) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

(c) $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$

31. If the price of sugar is reduced by ₹1 per kg, 5 kilograms more can be purchased for ₹1200. What was the original price of sugar per kilogram?

32. The zeroes of the quadratic polynomial $x^2 - 24x + 143$ are

33. Find the quadratic equation in x whose roots are $\frac{-7}{2}$ and $\frac{8}{3}$.

34. If k_1 and k_2 are the roots of $x^2 - 5x - 24 = 0$, then find the quadratic equation whose roots are $-k_1$ and $-k_2$.

35. The product of the roots of the equations $\frac{1}{x+1} + \frac{1}{x-2} = \frac{1}{x+2}$ is ______.

36. The roots of the equation $2x^2 + 3x + c = 0$ (where $c < 0$) could be ______.

37. The roots of the equation $30x^2 - 7\sqrt{3x} + 1 = 0$ are

38. If α and β are the roots of the quadratic equation $x^2 + 3x - 4 = 0$, then $\alpha^{-1} + \beta^{-1} =$ ______.

39. The roots of the equation $\frac{x-3}{x-2} + \frac{2x}{x+3} = 1$, where $x \neq 2, -3$, are

40. If the roots of the quadratic equation $4x^2 - 16x + p = 0$ are real and unequal, then find the value/s of p.

41. If one root of the quadratic equation $ax^2 + bx + c = 0$ is $15 + 2\sqrt{56}$ and a, b and c are rational, then find the quadratic equation.

42. If the roots of the equation $ax^2 + bx + 4c = 0$ are in the ratio of 3 : 4, then find the relation between a, b and c.

43. For which value of p among the following, does the quadratic equation $3x^2 + px + 1 = 0$ have real roots?

44. If the product of the roots of $ax^2 + bx + 2 = 0$ is equal to the product of the roots of $px^2 + qx - 1 = 0$, then $a + 2p =$ ______.

45. Find the roots of quadratic equation $ax^2 + (a - b + c)\, x - b + c = 0$.

46. If $(2x - 9)$ is a factor of $2x^2 + px - 9$, then $p =$ ______.

Essay Type Questions

Solve the given equations:

47. $(x + 3)\,(x + 4)\,(x + 6)\,(x + 7) = 1120$.

48. $(x^2 + 3x)^2 - 16\,(x^2 + 3x) - 36 = 0$.

49. $\sqrt{x-3} + \sqrt{3x+4} = 5$.

50. $3x^4 - 10x^3 - 3x^2 + 10x + 3 = 0$.

CONCEPT APPLICATION

Level 1

Direction for questions 1 to 20: Select the correct alternative from the given choices.

1. The solution of the equation $x^2 + x + 1 = 1$ is are

(a) $x = 0$

(b) $x = -1$

(c) Both (a) and (b)

(d) Cannot be determined

2. The discriminant of the equation $x^2 - 7x + 2 = 0$ is

(a) 47 (b) 40

(c) 41 (d) −41

3. Find the maximum value of the quadratic expression $-3x^2 + 7x + 4$.

PRACTICE QUESTIONS

(a) $8\frac{1}{6}$ (b) $8\frac{1}{12}$

(c) $8\frac{1}{4}$ (d) 12

4. If α and β are the roots of the equation $x^2 + 3x - 2 = 0$, then $\alpha^2\beta + \alpha\beta^2 = ?$

(a) −6 (b) −3

(c) 6 (d) 3

5. If one of the roots of an equation, $x^2 - 2x + c = 0$ is thrice the other, then $c = ?$

(a) $\frac{1}{2}$ (b) $\frac{4}{3}$

(c) $-\frac{1}{2}$ (d) $\frac{3}{4}$

6. The number of real roots of the quadratic equation $3x^2 + 4 = 0$ is

(a) 0 (b) 1

(c) 2 (d) 4

7. If α and β are the roots of the equation $x^2 + px + q = 0$ then $(\alpha - \beta)^2 =$ ______.

(a) $q^2 - 4p$ (b) $4q^2 - p$

(c) $p^2 - 4q$ (d) $p^2 + 4q$

8. Which of the following equations does not have real roots?

(a) $x^2 + 4x + 4 = 0$ (b) $x^2 + 9x + 16 = 0$

(c) $x^2 + x + 1 = 0$ (d) $x^2 + 3x + 1 = 0$

9. The sum of the roots of the equation, $ax^2 + bx + c = 0$ where a, b and c are rational and whose one of the roots is $4 - \sqrt{5}$, is

(a) 8 (b) $-2\sqrt{5}$

(c) $2\sqrt{5}$ (d) 11

10. For the quadratic equation $x^2 + 3x - 4 = 0$ which of the following is a solution?

(A) $x = -4$ (B) $x = 3$

(C) $x = 1$

(a) A and B (b) B and C

(c) A and C (d) Only A

11. Find the quadratic equation whose roots are reciprocals of the roots of the equation $7x^2 - 2x + 9 = 0$.

(a) $9x^2 - 2x + 7 = 0$

(b) $9x^2 - 2x - 7 = 0$

(c) $9x^2 + 2x - 7 = 0$

(d) $9x^2 + 2x + 7 = 0$

12. The number of real roots of the quadratic equation $(x - 4)^2 + (x - 5)^2 + (x - 6)^2 = 0$ is

(a) 1 (b) 2

(c) 3 (d) None of these

13. The number of distinct real solutions of $|x|^2 - 5|x| + 6 = 0$ is

(a) 4 (b) 3

(c) 2 (d) 1

14. The number of real solutions of $|x|^2 - 5|x| + 6 = 0$ is

(a) 1 (b) 2

(c) 3 (d) 4

15. In writing a quadratic equation of the form $x^2 + px + q = 0$, a student makes a mistake in writing the coefficient of x and gets the roots as 8 and 12. Another student makes a mistake in writing the constant term and gets the roots as 7 and 3. Find the correct quadratic equation.

(a) $x^2 - 10x + 96 = 0$

(b) $x^2 - 20x + 21 = 0$

(c) $x^2 - 21x + 20 = 0$

(d) $x^2 - 96x + 10 = 0$

16. The roots of the equation $6x^2 - 8\sqrt{2x} + 4 = 0$ are

(a) $\frac{1}{3}, \sqrt{2}$ (b) $\frac{\sqrt{2}}{3}, 1$

(c) $\frac{\sqrt{2}}{3}, \sqrt{2}$ (d) $\frac{3}{\sqrt{2}}, \sqrt{2}$

17. Which of the following equations has roots as a, b and c?

(a) $x^3 + x^2(a + b + c) + x(ab + bc + ca) + abc = 0$

(b) $x^3 + x^2(a + b + c) + x(ab + bc + ca) - abc = 0$

(c) $x^3 - x^2(a + b + c) + x(ab + bc + ca) - abc = 0$

(d) $x^3 - x^2(a + b + c) - x(ab + bc + ca) - abc = 0$

18. If the roots of the equation $2ax^2 + (3b - 9)x + 1 = 0$ are −2 and 3, then the values of a and b respectively are

(a) $\frac{1}{12}, \frac{5}{18}$ (b) $\frac{-1}{12}, \frac{-53}{18}$

(c) $\frac{-1}{12}, \frac{-5}{8}$ (d) $\frac{-1}{12}, \frac{55}{18}$

19. The roots of the equation $x^2 + 5x + 1 = 0$ are

(a) $\frac{5+\sqrt{21}}{2}, \frac{5-\sqrt{21}}{2}$

(b) $\frac{-5-\sqrt{21}}{2}, \frac{5+\sqrt{21}}{2}$

(c) $\frac{-5+\sqrt{21}}{2}, \frac{-5-\sqrt{21}}{2}$

(d) $\frac{-5+\sqrt{29}}{2}, \frac{-5-\sqrt{29}}{2}$

20. If α and β are the roots of the equation $3x^2 - 2x - 8 = 0$, then $\alpha^2 - \alpha\beta + \beta^2 =$ ______.

(a) $\frac{76}{9}$ (b) $\frac{25}{3}$

(c) $\frac{16}{3}$ (d) $\frac{32}{3}$

21. If $2x^2 + (2p - 13)\, x + 2 = 0$ is exactly divisible by $x - 3$, then the value of p is

(a) $\frac{-16}{6}$ (b) $\frac{19}{6}$

(c) $\frac{16}{6}$ (d) $\frac{-19}{6}$

22. If $x^2 - ax - 6 = 0$ and $x^2 + ax - 2 = 0$ have one common root, then a can be ______.

(a) -1 (b) 2

(c) -3 (d) 0

23. The root of the equation $\frac{2}{x-1} + \frac{1}{x+2} + \frac{3x(x+1)}{(x-1)(x+2)} = 0$ among the following is ______.

(a) 2 (b) 3

(c) -1 (d) 0

24. If α and β are the roots of the equation, $2x^2 - 5x + 2 = 0$, then $(\alpha - 1)^{\beta-1} =$ ______. where $(\alpha > \beta)$.

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$

(c) $\frac{1}{\sqrt{2}}$ (d) 1

25. The age of a father is 25 years more than his son's age. The product of their ages is 84 in years. What will be son's age in years, after 10 years?

(a) 3 (b) 28

(c) 13 (d) 18

26. If the roots of the equation $ax^2 - bx + 5c = 0$ are in the ratio of 4 : 5, then

(a) $ab = 18c^2$

(b) $81b^2 = 4ac$

(c) $bc = a^2$

(d) $4b^2 = 81ac$

27. The speed of Uday is 5 km/h more than that of Subash. Subash reaches his home from office 2 hours earlier than Uday. If Subash and Uday stay 12 km and 48 km from their respective offices, find the speed of Uday.

(a) 10 km/h

(b) 4 km/h

(c) 9 km/h

(d) 8 km/h

28. If the roots of the quadratic equation $c(a - b)x^2 + a(b - c)x + b(c - a) = 0$ are equal, then

(a) $2b^{-1} = a^{-1} + c^{-1}$

(b) $2c^{-1} = a^{-1} + b^{-1}$

(c) $2a^{-1} = b^{-1} + c^{-1}$

(d) None of these

29. If the roots of the quadratic equation $x^2 - 3x - 304 = 0$ are α and β, then the quadratic equation with roots 3α and 3β is

(a) $x^2 + 9x - 2736 = 0$

(b) $x^2 - 9x - 2736 = 0$

(c) $x^2 - 9x + 2736 = 0$

(d) $x^2 + 9x + 2736 = 0$

30. If $x^2 + \alpha_1 x + \beta_1 = 0$ and $x^2 + \alpha_2 x + \beta_2 = 0$ have a common root $(x - k)$, then find k.

(a) $k = \frac{\alpha_2 - \alpha_1}{\beta_2 - \beta_1}$ (b) $k = \frac{\alpha_2 - \alpha_1}{\beta_2 - \beta_1}$

(c) $k = \frac{\alpha_2 - \alpha_1}{\beta_2 - \beta_1}$ (d) $k = \frac{\alpha_2 - \alpha_1}{\beta_2 - \beta_1}$

PRACTICE QUESTIONS

Level 2

31. If one of the roots of $x^2 + (1 + k)x + 2k = 0$ is twice the other, then $\frac{a^2+b^2}{ab}$ _______.

(a) 2 (b) 1

(c) 4 (d) 7

32. If α and β are the roots of $2x^2 - x - 2 = 0$, then $(\alpha^{-3} + \beta^{-3} + 2\alpha^{-1}\beta^{-1})$ is equal to

(a) $-\frac{17}{8}$ (b) $\frac{23}{6}$

(c) $\frac{37}{9}$ (d) $-\frac{29}{8}$

33. In a right angled triangle, one of the perpendicular sides is 4 cm greater than the other and 4 cm lesser than the hypotenuse. Find the area of triangle in cm^2.

(a) 72 (b) 48

(c) 36 (d) 96

34. In a fraction, the denominator is 1 less than the numerator. The sum of the fraction and its reciprocal is $2\frac{1}{56}$. Find the fraction.

(a) $\frac{3}{2}$ (b) $\frac{13}{12}$

(c) $\frac{18}{17}$ (d) $\frac{8}{7}$

35. The length of the rectangular surface of a table is 10 m more than its breadth. If the area of the surface is 96 m^2, its perimeter is (in m) _______.

(a) 64 (b) 44

(c) 52 (d) 48

36. If α and β are the roots of the equation $x^2 + 9x + 18 = 0$, then the quadratic equation having the roots $\alpha + \beta$ and $\alpha - \beta$ is _______, where $(\alpha > \beta)$.

(a) $x^2 + 6x - 27 = 0$ (b) $x^2 - 9x + 27 = 0$

(c) $x^2 - 9x + 7 = 0$ (d) $x^2 + 6x + 27 = 0$

37. Find the minimum value of the quadratic expression $4x^2 - 3x + 4$.

(a) $\frac{-55}{16}$ (b) $\frac{55}{16}$

(c) $\frac{16}{15}$ (d) $\frac{161}{22}$

38. If $(x + 2)$ is a common factor of the expressions $x^2 + ax - 6$, $x^2 + bx + 2$ and $kx^2 - ax - (a + b)$, then $k =$ _______.

(a) 2 (b) 3

(c) 1 (d) −2

39. The roots of a pure quadratic equation exists only if _______.

(a) $a > 0, c < 0$ (b) $c > 0, a < 0$

(c) $a > 0, c \le 0$ (d) Both (a) and (b)

40. The roots of the equation $\frac{x-1}{x-2}+\frac{x-3}{x-4}=2\frac{1}{3}$, (where $x \neq 2, 4$) are

(a) $6+\sqrt{10}, 6-\sqrt{10}$

(b) $6+2\sqrt{10}, 6-\sqrt{10}$

(c) $6+6\sqrt{10}, 6-6\sqrt{10}$

(d) $2+2\sqrt{10}, 6-2\sqrt{10}$

41. If $x + 3$ is the common factor of the expressions $ax^2 + bx + 1$ and $px^2 + qx - 3$, then $-\frac{(9a+3p)}{3b+q}=$ _______.

(a) −2 (b) 2

(c) 3 (d) −1

42. If the sum of the roots of an equation $x^2 + px + 1 = 0$ $(p > 0)$ is twice the difference between them, then $p =$ _______.

(a) $-\frac{1}{4}$ (b) $\frac{3}{4}$

(d) $\frac{4}{\sqrt{3}}$ (d) $\frac{\sqrt{3}}{2}$

43. The equation $x+\frac{5}{3-x}=3+\frac{5}{3-x}$ has

(a) no real root.

(b) one real root.

(c) two equal roots.

(d) infinite roots.

44. The roots of the equation $\frac{1}{2x-3}-\frac{1}{2x+5}=8$ are

PRACTICE QUESTIONS

(a) $2-\frac{1}{\sqrt{2}}, 2-\frac{1}{\sqrt{2}}$

(b) $\frac{-1+\sqrt{17}}{2}, \frac{-1-1\sqrt{17}}{2}$

(c) $2+\frac{1}{\sqrt{2}}, 2-\frac{1}{\sqrt{2}}$

(d) $\frac{1+\sqrt{17}}{2}, \frac{1-\sqrt{17}}{2}$

45. If the quadratic expression $x^2 + (a - 4)x + (a + 4)$ is a perfect square, then $a =$ _______.

(a) 0 and −4 (b) 0 and 6
(c) 0 and 12 (d) 6 and 12

46. The minimum value of $2x^2 - 3x + 2$ is _______.

(a) $\frac{7}{8}$ (b) $\frac{4}{7}$ (c) 4 (d) −3

47. The roots of $x^2 - 2x - 1 = 0$ are _______.

(a) $\sqrt{2}+1, \sqrt{2}-1$

(b) $1, \sqrt{2}$

(c) $1+\sqrt{2}, 1-\sqrt{2}$

(d) 2, 1

48. If $2x^2 + 4x - k = 0$ is same as $(x - 5)\left(x+\frac{k}{10}\right) = 0$, then find the value of k.

(a) 100 (b) 90
(c) 70 (d) 35

49. If the numerically smaller root of $x^2 + mx = 2$ is 3 more than the other one, find the value of m.

(a) −1 (b) 1
(c) −2 (d) 2

Level 3

50. Two persons A and B solved a quadratic equation of the form $x^2 + bx + c = 0$. A made a mistake in noting down the coefficient of x and obtained the roots as 18 and 2, where as B obtained the roots as −9 and −3 by misreading the constant term. The correct roots of the equation are

(a) −6, −3 (b) −6, 6
(c) −6, −5 (d) −6, −6

51. If α and β are the roots of $x^2 - x + 2 = 0$, then find the value of $(\alpha^{-6} + \beta^{-6} + 2\alpha^{-3}\beta^{-3})\alpha^6\beta^6$.

(a) 16 (b) 25
(c) 30 (d) 36

52. If $b_1, b_2, b_3, \ldots, b_n$ are positive, then the least value of $(b_1 + b_2 + b_3 + \cdots + b_n)\left(\frac{1}{b_1}+\frac{1}{b_2}+\cdots+\frac{1}{b_n}\right)$ is

(a) $b_1b_2 \cdots b_n$ (b) $n^2 + 1$
(c) $n(n + 1)$ (d) n^2

53. The equation $\sqrt{x+1}-\sqrt{4x-1}=\sqrt{x-1}$ has

(a) no solution. (b) one solution.
(c) two solutions. (d) more than two solutions.

54. Out of the group of employees, twice the square root of the number of the employees are on a trip to attend a conference held by the company, half the number are in the office and the remaining six employees are on leave. What is the number of employees in the group?

(a) 49 (b) 64
(c) 36 (d) 100

55. Find the quadratic equation whose roots are 2 times the roots of $x^2 - 12x - 13 = 0$.

(a) $x^2 - 24x - 52 = 0$

(b) $x^2 - 24x - 26 = 0$

(c) $x^2 - 14x - 15 = 0$

(d) None of these

56. If one of the roots of $ax^2 + bx + c = 0$ is thrice that of the other root, then b can be

(a) $\frac{4ac}{3}$ (b) $\frac{16ac}{9}$

(c) $4\sqrt{\frac{ac}{3}}$ (d) $\sqrt{\frac{4ac}{3}}$

57. If α, β are the roots of $px^2 + qx + r = 0$, then $\alpha^3 + \beta^3 =$ _______.

(a) $\frac{3qpr-q^3}{p^3}$ (b) $\frac{3pqr-3q}{p^3}$

(c) $\frac{pqr-3q}{p^3}$ (d) $\frac{3pqr-q}{p^3}$

PRACTICE QUESTIONS

58. If α and β are the roots of $x^2 - (a+1)x + \frac{1}{2}(a^2 + a + 1) = 0$ then $\alpha^2 + \beta^2 =$ ______.

(a) a (b) a^2

(c) $2a$ (d) 1

59. The number of roots of the equation $2|x|^2 - 7|x| + 6 = 0$

(a) 4 (b) 3

(c) 2 (d) 1

60. In a quadratic equation $ax^2 - bx + c = 0$, a, b, c are distinct primes and the product of the sum of the roots and product of the roots is $\frac{91}{9}$. Find the difference between the sum of the roots and the product of the roots.

(a) 2 (b) 3

(c) 4 (d) Cannot be determined

61. Maximum value of $\frac{2+12x-3x^2}{2x^2-8x+9}$ is ______.

(a) 14 (b) 17

(c) 11 (d) Cannot be determined

62. If $x^2 + ax + b$ and $x^2 + bx + c$ have a common factor $(x - k)$, then $k =$ ________.

(a) $\frac{a-b}{b-c}$ (b) $\frac{b-c}{c-a}$

(c) $\frac{c-b}{b-a}$ (d) $\frac{c-b}{a-b}$

63. If $9x - 3y + z = 0$, then the value of $\frac{y}{2x} + \sqrt{\frac{y^2-4xz}{4x^2}}$ (where x, y, z are constants).

(a) 9 (b) 2

(c) 3 (d) 6

64. If the roots of $3x^2 - 12x + k = 0$ are complex, then find the range of k.

(a) $k < 22$ (b) $k < -10$

(c) $k > 11$ (d) $k > 12$

65. If α, β are the roots of $ax^2 + bx + c = 0$, then find the quadratic equation whose roots are $\alpha + \beta$, $\alpha\beta$.

(a) $ax^2 + (ab - ac)x - c = 0$

(b) $ax^2 + (b - c)x - bc = 0$

(c) $a^2x^2 + (b - c)x - ac = 0$

(d) $a^2x^2 + (ab - ac)x - bc = 0$

66. Ramu swims a distance of 3 km each upstream and downstream. The total time taken is one hour. If the speed of the stream is 4 km/h, then find the speed of Ramu in still water.

(a) 12 km/h (b) 9 km/h

(c) 8 km/h (d) 6 km/h

67. In solving a quadratic equation $x^2 + px + q = 0$ a student made a mistake in copying the coefficient of x and obtained the roots as 4, −3 but one of the actual roots is 2 what is the difference between the actual and wrong values of the coefficients of x?

(a) 5 (b) 4

(c) 7 (d) +6

68. The roots of $ax^2 - bx + 2c = 0$ are in the ratio of 2 : 3, then ______.

(a) $a^2 = bc$ (b) $3b^2 = 25ac$

(c) $2b^2 = 75c$ (d) $5b^2 = ac$

69. If the roots of $9x^2 - 2x + 7 = 0$ are 2 more than the roots of $ax^2 + bx + c = 0$, then $4a - 2b + c$ can be

(a) −2 (b) 7

(c) 9 (d) 10

70. If the roots of $ax^2 + bx + c = 0$ are 2 more than the roots of $px^2 + qx + r = 0$, then the value of c in terms of p, q and r is

(a) $p + q + r$ (b) $4p - 2q + r$

(c) $3p - q + 2r$ (d) $2p + q - r$

71. If the roots of $2x^2 + 7x + 5 = 0$ are the reciprocal roots of $ax^2 + bx + c = 0$, then $a - c =$ ______.

(a) 3 (b) −3

(c) −2 (d) −5

72. If the roots of the equation $ax^2 + bx + c = 0$ is $\frac{1}{k}$ times the roots of $px^2 + qx + r = 0$, then which of the following is true?

(a) $a = pk$ (b) $\frac{a}{b} = \frac{p}{q}$

(c) $aq = pbk$ (d) $ab = pqk^2$

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. non-zero real numbers
2. root
3. 3
4. quadratic
5. $\frac{3}{2}$
6. $x^2 + x(a + b) + ab = 0$
7. $(x - 2)$
8. real numbers
9. 2, −3
10. $\frac{-a \pm \sqrt{a^2 - 4b}}{2}$
11. Yes
12. −1
13. real and distinct
14. $4ac$
15. Zero
16. zero
17. ±6
18. positive
19. $a + c$

Short Answer Type Questions

20. (a) $(x + 2)(x + 3)$
    (b) $(x - 9)(x + 4)$
    (c) $(x - 2)(2x + 9)$
21. (a) Complex conjugates
    (b) Real and distinct
    (c) Real and equal
22. −4 and −33
23. $\pm\sqrt{2}$
24. $\frac{-2}{3}$
25. 6
26. $qx^2 - (p^2 - 2q)x + q = 0$
27. 8 km/h
28. 12 or 24
29. $x^2 - (6\alpha\beta - 2\alpha - 3\beta)x - (2\alpha + 3\beta)(6\alpha\beta) = 0$
30. (a) $\frac{m^2 - 2ln}{l^2}$ (b) $\frac{m^2 - 2ln}{ln}$ (c) $\frac{3lmn - m^3}{n^3}$
31. ₹16
32. 11, 13
33. $6x^2 + 5x - 59 = -3$
34. $x^2 + 5x - 24 = 0$
35. zero
36. rational or irrational, but unequal
37. $\frac{1}{2\sqrt{3}}, \frac{1}{5\sqrt{3}}$
38. $\frac{3}{4}$
39. $3, \frac{-1}{2}$
40. $p < 16$
41. $x^2 - 30x + 1 = 0$
42. $3b^2 = 49ac$
43. 4
44. 0
45. $-1, \frac{b - c}{a}$
46. −7

Essay Type Questions

47. $x = 1, x = -11$
48. −1, −2, 3 and −6
49. 4
50. $\frac{1 \pm \sqrt{37}}{6}, \frac{3 \pm \sqrt{13}}{2}$

ANSWER KEYS

CONCEPT APPLICATION

Level 1

1. (c)	**2.** (c)	**3.** (b)	**4.** (c)	**5.** (d)	**6.** (a)	**7.** (c)	**8.** (c)	**9.** (a)	**10.** (c)
11. (a)	**12.** (d)	**13.** (c)	**14.** (d)	**15.** (a)	**16.** (c)	**17.** (c)	**18.** (d)	**19.** (c)	**20.** (a)
21. (b)	**22.** (a)	**23.** (c)	**24.** (d)	**25.** (c)	**26.** (d)	**27.** (d)	**28.** (c)	**29.** (b)	**30.** (a)

Level 2

31. (d)	**32.** (d)	**33.** (d)	**34.** (d)	**35.** (b)	**36.** (a)	**37.** (b)	**38.** (c)	**39.** (d)	**40.** (a)
41. (d)	**42.** (c)	**43.** (a)	**44.** (b)	**45.** (c)	**46.** (a)	**47.** (c)	**48.** (c)	**49.** (b)	

Level 3

50. (d)	**51.** (b)	**52.** (d)	**53.** (a)	**54.** (c)	**55.** (a)	**56.** (c)	**57.** (a)	**58.** (a)	**59.** (a)
60. (a)	**61.** (a)	**62.** (d)	**63.** (c)	**64.** (d)	**65.** (d)	**66.** (c)	**67.** (a)	**68.** (b)	**69.** (b)
70. (b)	**71.** (a)	**72.** (c)							

CONCEPT APPLICATION

Level 1

1. Simplify and factorize.
2. Use the formula to find the discriminant.
3. Use the formula to find the maximum value.
4. Find the sum and product of the roots. Let $\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha + \beta)$.
5. Let the roots be α and 3α.
6. Solve for x.
7. $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$.
8. Find the value of the discriminant for each of the equations.
9. If one root is $4-\sqrt{5}$, then the other root is $4 + \sqrt{5}$, because the coefficients of the x^n terms are rational.
10. Solve for x.
11. The quadratic equation with reciprocals of the roots of the equation $f(x) = 0$ is $f\left(\frac{1}{x}\right) = 0$.
12. (i) Use the concept of perfect square of a number.
    (ii) If $a^2 + b^2 + c^2 = 0$ is true only when $a = b = c = 0$.
13. (i) Solve the equation to find the number of real roots.
    (ii) Replace $|x|$ by y and solve for y.
    (iii) Now, $x = \pm y$.
14. (i) Use the concept $|x|$ and find the roots.
    (ii) Replace $|x|$ by y and solve for y.
    (iii) Now, $x = \pm y$.
15. (i) Use the concept of sum and product of the roots of a quadratic equation.
    (ii) The product of the roots obtained by the first student is product of the roots of the required quadratic equation.
    (iii) The sum of the roots obtained by the second student is sum of the roots of the required quadratic equation.
16. Take 2 as common and then factorize.
17. Let $(x - a)(x - b)(x - c) = 0$ and expand.
18. Substitute $x = -2$ and $x = 3$ to get to equation in a and b and then solve for a and b.
19. Apply formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
20. Find $\alpha\beta$ and $(\alpha + \beta)$ and use, $\alpha^2 - \alpha\beta + \beta^2 = (\alpha + \beta)^2 - 3\alpha\beta$.
21. Substitute $x = 3$ in the given equation and simplify.
22. Find x in terms of a and substitute $x = a$ in either of the equations.
23. Simplify and solve for x.
24. Factorize LHS of the given equation to find α and β.
25. (i) Form a quadratic equation by assuming the age of son as r years. The roots of quadratic equation are 1 and $\frac{c}{a}$ only if sum of all the coefficient $s = 0$.
    (ii) Assume the ages of father and son as x and $(25 - x)$ years.
    (iii) Write the relation in terms of x according to the data and then solve the equation.
26. (i) Use the concept of sum and product of the roots of a quadratic equation.
    (ii) Take the roots as 4α, 5α.
    (iii) Using the sum of the roots and product of the roots eliminate α.
27. (i) Frame the quadratic equation from the given data.
    (ii) Assume the speed of Subhash as x and Subhash reaches his home in t hours and Uday reaches his home in $(t + 2)$ hours.
    (iii) Now, use time $= \frac{\text{distance}}{\text{speed}}$, then solve the equation for x.
28. (i) If sum of all coefficients is zero, then 1, $\frac{c}{a}$ are the roots of $ax^2 + bx + c = 0$.

(ii) If the sum of the coefficients is 0, then 1 and $\frac{c}{a}$ are the roots of the equation.

(iii) Use product of the roots concept.

29. (i) The quadratic equation with thrice the roots of $f(x) = 0$ as roots is $f\left(\frac{1}{x}\right) = 0$.

(ii) The quadratic equation whose roots are m times of the roots of the equation $f(x) = 0$ is $\frac{-1}{12}, \frac{55}{18} = 0$.

30. (i) Use the concept of common root of given equations.

(ii) Put $x = k$ in the given equations and solve for k.

Level 2

31. (i) Use the concept of sum and product of the roots of a quadratic equation.

(ii) Assume the roots as α and 2α.

(iii) Find the sum of the roots and product of the roots.

(iv) From the above equation eliminate 'α'.

(v) Then obtain the value of $\frac{k^2+1}{k}$.

32. (i) Simplify the required expression and find $\alpha + \beta$ and $\alpha\beta$.

(ii) Find the sum of the roots and product of the roots.

(iii) Use relation, $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$.

33. (i) Use Pythagorean theorem to find the sides of the triangle.

(ii) Assume the sides as x, $x - 4$ and hypotenuse as $x + 4$.

(iii) Find the value of x using the relation $(\text{hypotenuse})^2$ = sum of the squares of the other two sides.

(iv) The area of triangle = $\frac{1}{2} \times$ base $\times$ height.

34. (i) Form the quadratic equation and solve for x.

(ii) Assume the fraction as $\frac{x}{x-1}$.

(iii) $\frac{x}{x-1} + \frac{x-1}{x} = \frac{21}{16}$.

(iv) Solve the above equation.

35. (i) Use the formula to find the area of the rectangle.

(ii) Assume the length and breadth as lm and $(l - 10)$ m.

(iii) Find the value of 'l', using $l(l - 10) = 96$.

(iv) Calculate the perimeter of the rectangle using $2(l + l - 10)$.

36. Find $\alpha + \beta$, $\alpha\beta$ and using these values find $\alpha - \beta$.

37. Use the formula to find the minimum value.

38. Substitute $x = -2$ in the first two expressions, equated to zero.

39. (i) A pure quadratic equation is $ax^2 + c = 0$.

(ii) Pure quadratic equation is $ax^2 + c = 0$.

40. (i) Simplify the equation 1.

(ii) Take the LCM of the equation.

(iii) Convert it into quadratic equation.

(iv) Solve the equation for x.

41. (i) If $x + k$ is the common root, then $x = -k$ satisfies both the equations.

(ii) If $x + a$ is factor of $f(x)$, then $f(-a) = 0$.

(iii) Write p in terms of q and b in terms of a.

(iv) Now substitute these values in the given expression and simplify.

42. Find the sum and product of the roots and form the equation as per the condition given in the problem.

43. (i) Simplify the equation.

(ii) A rational function $\frac{f(x)}{g(x)}$ is defined only when $g(x) > 0$.

44. (i) Simplify the equation.

(ii) Take the LCM.

(iii) Convert it into quadratic equation.

(iv) Solve the equation by using formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

45. (i) If the equation is a perfect square, then it has equal roots.

(ii) Quadratic equation is a perfect square, if $b^2 - 4ac = 0$.

(iii) Substitute the value of b and c in the above equation and obtained the value of a.

46. The minimum value of $ax^2 + bx + c$ is $\frac{4ac - b^2}{4a}$. $(a > 0)$

The minimum value of $2x^2 - 3x + 2 = \frac{4 \times 2 \times 2 - (-3)^2}{4 \times 2} = \frac{7}{8}$.

47. Given $x^2 - 2x - 1 = 0$

$$x = \frac{2 \pm \sqrt{4 - 4 \times 1 \times (-1)}}{2 \times 1}$$

$$x = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

$$x = 1 + \sqrt{2} \text{ or } 1 - \sqrt{2}.$$

48. $2x^2 + 4x - k = 0$ (1)

$\therefore \Rightarrow (x - 5)$ is a factor of Eq. (1).

$\Rightarrow x - 5 = 0 \Rightarrow x = 5$

$\therefore 2(5)^2 + 4(5) - k = 0$

$50 + 20 - k = 0$

$\therefore k = 70$.

49. The difference of the roots of $ax^2 + bx + c = 0$ is $\frac{\sqrt{b^2 - 4ac}}{a}$ for $a > 0$.

$\therefore$ For $x^2 + mx - 2 = 0$. It is $\sqrt{m^2 + 8} = 3$
$\Rightarrow m = \pm 1$.

That is, the equation could be $x^2 + x - 2 = 0$ or $x^2 - x - 2 = 0$.

That is, $(x + 2)(x - 1) = 0$ or $(x - 2)(x + 1) = 0$

The roots are -2, 1 or -1, 2.

As the numerically smaller root is greater, the roots are -2, 1 and $m = 1$.

Level 3

50. (i) Use the concept of sum of the roots and product of the roots of a quadratic equation.

(ii) The product of the roots obtained by A and sum of the roots obtained by B is equal to the product and sum of the roots of the required equation respectively.

51. (i) Simplify the expression and find $(\alpha + \beta)$ and $\alpha\beta$.

(ii) First find the sum of the roots and product of the roots.

(iii) $\alpha^{-6} + \beta^{-6} + \frac{2}{\alpha^3\beta^3} = \frac{(\alpha^3 + \beta^3)}{\alpha^6\beta^6}$

52. (i) Take $b_1 = b_2 = b_3 \ldots b_n = k$ and find the value.

(ii) AM $(a_1, a_2, \ldots, a_n) \geq$ HM $(a_1, a_2, \ldots, a_n)$.

(iii) $\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_2}}$.

53. (i) Simplify the equation.

(ii) Square the given expression twice and then solve for x.

54. (i) Form the quadratic equation and solve for x.

(ii) Assume number of employees in the group as x. Then write the quadratic equation in x according to the data and solve it.

55. $f(x) = x^2 - 12x - 13 = 0$

If the roots of $g(x)$ are 2 times the roots of $f(x)$, then

$$g(x) = f\left(\frac{x}{2}\right) = 0.$$

$$\Rightarrow \left(\frac{x}{2}\right)^2 - 12\left(\frac{x}{2}\right) - 13 = 0$$

$$\Rightarrow \frac{x^2}{4} - \frac{12x}{2} - 13 = 0$$

$$\Rightarrow x^2 - 24x - 52 = 0.$$

56. Let the roots be k and $3k$.

Sum of the roots $= k + 3k$

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$\Rightarrow 4k = \frac{-b}{a} \Rightarrow k = \frac{-b}{4a}$.

Product of the roots = $k \times 3k$

$\Rightarrow 3k^2 = \frac{c}{a}$

$\Rightarrow 3\left(\frac{-b}{4a}\right)^2 = \frac{c}{a}$

$\Rightarrow \frac{3b^2}{16a^2} = \frac{c}{a}$

$\Rightarrow 3b^2 = 16ac$

$\Rightarrow b = \pm 4\sqrt{\frac{ac}{3}}$.

57. $\alpha + \beta = \frac{-q}{p}$

$\alpha \cdot \beta = \frac{r}{p}$

$\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$= \left(\frac{-q}{p}\right)^3 - 3\left(\frac{r}{p}\right)\left(\frac{-q}{p}\right)$

$= \frac{-q^3 + 3pqr}{p^3}$

$\therefore \alpha^3 + \beta^3 = \frac{3pqr - q^3}{p^3}$.

58. $x^2 - (a+1)x + \frac{1}{2}(a^2 + a + 1) = 0$

$\alpha + \beta = a + 1$

$\alpha\beta = \frac{1}{2}(a^2 + a + 1)$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (a+1)^2 - 2\left[\frac{1}{2}(a^2 + a + 1)\right]$

$= a^2 + 2a + 1 - a^2 - a - 1 = a$.

59. $2|x|^2 - 7|x| + 6 = 0$

Let $|x| = y$

$2y^2 - 7y + 6 = 0$

$2y^2 - 3y - 4y + 6 = 0$

$y(2y - 3) - 2(2y - 3) = 0$

$(2y - 3)(y - 2) = 0$

$2y - 3 = 0$ or $y - 2 = 0$

$y = \frac{3}{2}$ or $y = 2$

$|x| = \frac{3}{2}$ or $|x| = 2$

$x = \pm\frac{3}{2}$ or $x = \pm 2$.

$\therefore$ x has 4 real solutions.

60. Product of the sum of the roots and product of the roots is $\frac{91}{9}$, i.e.,

$\left(\frac{b}{a} \times \frac{c}{a}\right) = \frac{91}{9}$

$\frac{bc}{a^2} = \frac{91}{9}$

$\frac{bc}{a \times a} = \frac{13 \times 7}{3 \times 3}$

$\Rightarrow \frac{b}{a} = \frac{13}{3}, \frac{c}{a} = \frac{7}{3}$ or $\frac{b}{a} = \frac{7}{3}, \frac{c}{a} = \frac{13}{3}$

The required difference is $\frac{13}{3} - \frac{7}{3} = 2$.

61. For the maximum value of $\frac{2 + 12x - 3x^2}{2x^2 - 8x + 9}$, $2 + 12x - 3x^2$ is maximum and $2x^2 - 8x + 9$ is minimum.

The maximum value of $2 + 12x - 3x^2$ and minimum value of $2x^2 - 8x + 9$ occurs at $x = \frac{-b}{2a}$, i.e., 2.

When $x = 2$,

$\frac{2 + 12x - 3x^2}{2x^2 - 8x + 9} = \frac{2 + 24 - 12}{8 - 16 + 9} = 14$.

62. $x^2 + ax + b$ and $x^2 + bx + c$ have a common factor $(x - k)$

$\Rightarrow k^2 + ak + b = 0$ and $k^2 + bk + c = 0$

$\Rightarrow k^2 + ak + b = k^2 + bk + c$

$ak + b = bk + c$

$k = \frac{c - b}{a - b}$.

63. $9x - 3y + z = 0$ consider $xa^2 - ya + z = 0$, a quadratic equation in a. Where x, y, z are constants.

Let $a = 3 \Rightarrow (3)^2 x - 3y + z = 0$ is a quadratic equation in 3.

$$3 = \frac{-(-y) + \sqrt{y^2 - 4 \cdot x \cdot z}}{2.x}$$

$$\Rightarrow 3 = \frac{y}{2x} + \sqrt{\frac{y^2 - 4xz}{4x^2}}.$$

64. Given the roots of the given equation are complex

$\Rightarrow b^2 - 4ac < 0$

$\Rightarrow (-12)^2 - 4(3)\ k < 0$

$144 - 12\ k < 0$

$-12k < -144$

$12\ k > 144$

$k > 12.$

65. α, β are the roots of $ax^2 + bx + c = 0$

$$\Rightarrow \alpha + \beta = \frac{-b}{a}$$

$$\alpha \cdot \beta = \frac{c}{a}.$$

Quadratic equation whose roots are $\alpha + \beta$, and $\alpha\beta$

is $x^2 - \left(\frac{-b}{c} + \frac{c}{a}\right)x + \frac{-b}{a} \times \frac{c}{a} = 0.$

$$x^2 + \left(\frac{b-c}{a}\right)x - \frac{bc}{a^2} = 0$$

$a^2 x^2 + (ab - ac)x - bc = 0.$

66. Let the speed of Ramu = x km/h

Total time taken is = 1 hour

That is,

$$\frac{3}{x-4} + \frac{3}{x+4} = 1$$

$$\frac{3x + 12 + 3x - 12}{x^2 - 16} = 1$$

$6x = x^2 - 16$

$x^2 - 6x - 16 = 0$

$x^2 - 8x + 2x - 16 = 0$

$x(x - 8) + 2(x - 8) = 0$

$(x - 8)\ (x + 2) = 0$

$x = 8$ km/h ($\because$ speed cannot be -2 km/h).

67. Quadratic equation with 4, -3 as roots is $x^2 - 1x - 12 = 0$, quadratic equation whose product of the roots is -12. (1)

As one of the actual roots is 2, the other root is -6. (from (1))

The quadratic equation is $x^2 - (-6 + 2)\ x - 12 = 0$

$x^2 + 4x - 12 = 0.$

$\therefore$ The difference between the coefficients of x = $4 - (-1) = 5$.

68. Let α, β be the roots of $ax^2 - bx + 2c = 0$

Given $= \frac{\alpha}{\beta} = \frac{2}{3}$

$$\Rightarrow \alpha = \frac{2\beta}{3}$$

Product of the roots

$$\alpha \cdot \beta = \frac{2c}{a}$$

$$\frac{2\beta}{3} \times \beta = \frac{2c}{a}$$

$$\beta^2 = \frac{3c}{a} \qquad (1)$$

Sum of the roots = $\alpha + \beta$

$$\Rightarrow \frac{2\beta}{3} + \beta = \frac{b}{a}$$

$$\Rightarrow \frac{5\beta}{3} = \frac{b}{a}$$

$$\beta = \frac{3b}{5a}$$

$$\beta^2 = \frac{9b^2}{25a^2} \qquad (2)$$

From Eqs. (1) and (2), $\frac{3c}{a} = \frac{9b^2}{25a^2}$ $3b^2 = 25ac.$

69. $f(x) = ax^2 + bx + c = 0$

$f(x - 2) = 9x^2 - 2x + 7 = 0$

$\because\ a(x - 2)^2 + b(x - 2) + c = 9(x)^2 - 2x + 7$

$a(x^2 - 4x + 4) + bx - 2b + c = 9x^2 - 2x + 7$

$ax^2 - (4a - b)x + 4a - 2b + c = 9x^2 - 2x + 7$

$\Rightarrow 4a - 2b + c = 7.$

70. Let $f(x) \equiv px^2 + qx + r = 0$

Given the quadratic equation whose roots are 2 more than the roots of $f(x)$ as $ax^2 + bx + c = 0$.

$\Rightarrow f(x-2) = ax^2 + bx + c$

$\Rightarrow p(x-2)^2 + q(x-2) + r = ax^2 + bx + c$

$\Rightarrow px^2 - 4px + 4p + qx - 2q + r = ax^2 + bx + c$

$\Rightarrow px^2 + (q-4p)x + (4p - 2q + r) = ax^2 + bx + c$

$\Rightarrow c = 4p - 2q + r.$

71. If the roots of $2x^2 + 7x + 5 = 0$ (1) are the reciprocal roots of $ax^2 + bx + c = 0$, then $ax^2 + bx + c = 0$ is obtained by substituting $\frac{1}{x}$ in Eq. (1).

That is, $2\left(\frac{1}{x}\right)^2 + 7\left(\frac{1}{x}\right) + 5 = 0$

$\Rightarrow 2 + 7x + 5x^2 = 0 \Rightarrow 5x^2 + 7x + 2 = 0$

$a = 5,\ b = 7,\ c = 2$

$a - c = 5 - 2 = 3.$

72. $ax^2 + bx + c$

$= p(kx)^2 + q(kx) + r = 0$

$\Rightarrow a = pk^2,\ b = qk,\ c = r$

$\frac{a}{b} = \frac{pk^2}{qk}$

$\Rightarrow aq = pbk.$

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