DPP - Daily Practice Problems

Date : | **Start Time :** | **End Time :**

MATHEMATICS CM20

SYLLABUS : Continuity and Differentiability

Max. Marks : 120 **Marking Scheme :** (+4) for correct & (–1) for incorrect answer **Time : 60 min.**

INSTRUCTIONS : This Daily Practice Problem Sheet contains 30 MCQ's. For each question only one option is correct. Darken the correct circle/ bubble in the Response Grid provided on each page.

1. Let $f : R \to R$ be a function such that

$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}$, $f(0) = 0$ and $f'(0) = 3$. Then

(a) f(x) is a quadratic function
(b) f(x) is continuous but not differentiable
(c) f(x) is differentiable in R
(d) f(x) is bounded in R

2. If the function $f(x) = \begin{cases} (\cos x)^{1/x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at x = 0, then the value of k is

(a) 1 (b) –1
(c) 0 (d) e

3. If $x^2 + y^2 = a^2$ and $k = \frac{1}{a}$, then k is equal to

(a) $\frac{|y''|}{\sqrt{1+y'}}$ (b) $\frac{|y''|}{\sqrt{(1+y'^2)^3}}$

(c) $\frac{2y''}{\sqrt{1+y'^2}}$ (d) $\frac{y''}{2\sqrt{(1+y'^2)^3}}$

4. Let a function $f : R \to R$ satisfy the equation f (x + y) = f(x) + f(y) for all x, y, If the function f(x) is continuous at x = 0, then

(a) f(x) = 0 for all x
(b) f(x) is continuous for all positive real x
(c) f(x) is continuous for all x
(d) None of these

RESPONSE GRID	1. ⓐⓑⓒⓓ	2. ⓐⓑⓒⓓ	3. ⓐⓑⓒⓓ	4. ⓐⓑⓒⓓ

5. Differential coefficient of $\tan^{-1}\frac{2x}{1-x^2}$ with respect to $\sin^{-1}\frac{2x}{1+x^2}$ will be

(a) 1 (b) – 1

(c) – 1/2 (d) x

6. The values of a, b and c which make the function

$$f(x)=\begin{cases}\frac{\sin(a+1)x+\sin x}{x}, & x<0\\ c & ,x=0\\ \frac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}} & ,x>0\end{cases}$$

continuous at x = 0 are

(a) $a=\frac{-3}{2}, c=\frac{1}{2}, b=0$ (b) $a=\frac{3}{2}, c=\frac{1}{2}, b\neq 0$

(c) $a=\frac{-3}{2}, c=\frac{1}{2}, b\neq 0$ (d) None of these

7. Let f(x), g(x) be two continuously differentiable functions satisfying the relationships $f'(x) = g(x)$ and $f''(x) = -f(x)$. Let $h(x) = [f(x)]^2 + [g(x)]^2$. If h(0) = 5, then h (10)=

(a) 10 (b) 5

(c) 15 (d) None of these

8. The function $f(x) = [x]^2 - [x^2]$ (where [y] is the greatest integer function less than or equal to y), is discontinuous at :

(a) all integers

(b) all integers except 0 and 1

(c) all integers except 0

(d) all integers except 1

9. If $f(x)=\lim_{n\to\infty}\frac{\tan \pi x^2+(x+1)^n \sin x}{x^2+(x+1)^n}$, then

(a) f is continuous at x = 0

(b) f is differentiable at x = 0

(c) f is continuous but not differentiable at x = 0

(d) None of these

10. The value of p for which the function

$$f(x)=\begin{cases}\frac{(4^x-1)^3}{\sin\frac{x}{p}\log\left[1+\frac{x^2}{3}\right]}, & x\neq 0\\ 12(\log 4)^3, & x=0\end{cases}$$

may be continuous at $x = 0$, is

(a) 1 (b) 2

(c) 3 (d) None of these

11. In the mean value theorem $\frac{f(b)-f(a)}{b-a}=f'(c)$, if a = 0, b = 1/2 and f(x) = x (x – 1) (x – 2), the value of c is –

(a) $1-\frac{\sqrt{15}}{6}$ (b) $1+\sqrt{15}$

(c) $1-\frac{\sqrt{21}}{6}$ (d) $1+\sqrt{21}$

12. If $y=(1+1/x)^x$, then $\frac{2\sqrt{y_2(2)+1/8}}{(\log 3/2-1/3)}$ is equal to –

(a) 3 (b) 4

(c) 1 (d) 2

13. If $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}(\log a)^n$, then at x = 0, f(x)

(a) has no limit

(b) is discontinuous

(c) is continuous but not differentiable

(d) is differentiable

14. Let $f(x)=g(x).\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}$, where g is a continuous function then $\lim_{x\to 0} f(x)$ does not exist if

(a) g(x) is any constant function

(b) g(x) = x

(c) $g(x) = x^2$

(d) g(x) = x h (x), where h(x) is a polynomial

RESPONSE GRID					
	5. ⓐⓑⓒⓓ	6. ⓐⓑⓒⓓ	7. ⓐⓑⓒⓓ	8. ⓐⓑⓒⓓ	9. ⓐⓑⓒⓓ
	10. ⓐⓑⓒⓓ	11. ⓐⓑⓒⓓ	12. ⓐⓑⓒⓓ	13. ⓐⓑⓒⓓ	14. ⓐⓑⓒⓓ

15. If $f(x) = \cot^{-1}\left(\frac{x^x - x^{-x}}{2}\right)$, then $f'(1)$ is equal to

(a) -1 (b) 1
(c) $\log 2$ (d) $-\log 2$

16. Let $f: R \to R$ be a function defined by $f(x) = \max\{x, x^3\}$. The set of all points where $f(x)$ is NOT differentiable is

(a) $\{-1, 1\}$ (b) $\{-1, 0\}$
(c) $\{0, 1\}$ (d) $\{-1, 0, 1\}$

17. The function $f(x) = (\sin 2x)^{\tan^2 2x}$ is not defined at $x = \frac{\pi}{4}$. The value of $f\left(\frac{\pi}{4}\right)$ so that f is continuous at $x = \frac{\pi}{4}$ is

(a) $\sqrt{e}$ (b) $\frac{1}{\sqrt{e}}$
(c) 2 (d) None of these

18. If g is the inverse function of f and $f'(x) = \sin x$, then $g'(x)$ is

(a) $\text{cosec}\{g(x)\}$ (b) $\sin\{g(x)\}$
(c) $-\frac{1}{\sin\{g(x)\}}$ (d) $\cos\{g(x)\}$

19. Let $f(x) = \begin{cases} \frac{1-\sin^3 x}{3\cos^2 x}, & x < \frac{\pi}{2} \\ p, & x = \frac{\pi}{2} \\ \frac{q(1-\sin x)}{(\pi - 2x)^2}, & x > \frac{\pi}{2} \end{cases}$

If f(x) is continuous at $x = \frac{\pi}{2}$, (p, q) =

(a) (1,4) (b) $\left(\frac{1}{2}, 2\right)$
(c) $\left(\frac{1}{2}, 4\right)$ (d) None of these

20. Which of the following functions is differentiable at $x = 0$?

(a) $\cos(|x|) + |x|$ (b) $\cos(|x|) - |x|$
(c) $\sin(|x|) + |x|$ (d) $\sin(|x|) - |x|$

21. Let $f(x) = \frac{(e^x - 1)^2}{\sin\left(\frac{x}{a}\right)\log\left(1 + \frac{x}{4}\right)}$ for $x \neq 0$, and $f(0) = 12$. If f is continuous at x = 0, then the value of a is equal to

(a) 1 (b) -1
(c) 2 (d) 3

22. If the equation $a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots\ldots\ldots + a_1 x = 0$ $a_1 \neq 0, n \geq 2$, has a positive root $x = \alpha$, then the equation $n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \ldots\ldots\ldots + a_1 = 0$ has a positive root, which is

(a) greater than α
(b) smaller than α
(c) greater than or equal to α
(d) equal to α

23. If $f(x) = \begin{cases} \frac{\sin[x]}{[x]+1}, & \text{for } x > 0 \\ \frac{\cos\frac{\pi}{2}[x]}{[x]}, & \text{for } x < 0 \\ k, & \text{at } x = 0 \end{cases}$; where [x] denotes the greatest integer less than or equal to x, then in order that f be continuous at x = 0, the value of k is

(a) equal to 0 (b) equal to 1
(c) equal to -1 (d) indeterminate

24. If $f(x) = \frac{\tan[x]\pi}{[1 + |\log(\sin^2 x + 1)|]}$, where [.] denotes the greatest integer function and |.| stands for the modulus of the function, then f(x) is

(a) discontinuous $\forall x \in \mathbf{I}$
(b) continuous $\forall$ x
(c) non differentiable $\forall x \in \mathbf{I}$
(d) a periodic function with fundamental period 1.

RESPONSE GRID					
	15. ⓐⓑⓒⓓ	16. ⓐⓑⓒⓓ	17. ⓐⓑⓒⓓ	18. ⓐⓑⓒⓓ	19. ⓐⓑⓒⓓ
	20. ⓐⓑⓒⓓ	21. ⓐⓑⓒⓓ	22. ⓐⓑⓒⓓ	23. ⓐⓑⓒⓓ	24. ⓐⓑⓒⓓ

25. If $\sqrt{1-x^{2n}}+\sqrt{1-y^{2n}}=a(x^n-y^n)$, then $\sqrt{\frac{1-x^{2n}}{1-y^{2n}}}\frac{dy}{dx}$ is equal to

(a) 1 (b) x/y

(c) $\frac{x^{n-1}}{y^{n-1}}$ (d) None of these

26. The equation $e^{x-8}+2x-17=0$ has

(a) two real roots (b) one real root

(c) eight real roots (d) four real roots

27. If $f''(x)=-f(x)$ and $g(x)=f'(x)$ and

$F(x)=\left(f\left(\frac{x}{2}\right)\right)^2+\left(g\left(\frac{x}{2}\right)\right)^2$ and given that

F (5) =5, then F (10) is equal to –

(a) 5 (b) 10

(c) 0 (d) 15

28. Choose the correct statements –

(a) If $f'(a^+)$ and $f'(a^-)$ exist finitely at a point, then f is continuous at x = a.

(b) The function $f(x)=3\tan 5x-7$ is differentiable at all points in its domain.

(c) The existence of $\lim_{x\to c}(f(x)+g(x))$ does not imply of existence of $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$.

(d) All of these

29. **Statement-1 :** If g (x) is a differentiable function $g(1)\neq 0$, $g(-1)\neq 0$ and Rolles theorem is not applicable to $f(x)=\frac{x^2-1}{g(x)}$ in [–1,1], then g (x) has atleast one root in (–1, 1).

Statement-2 : If f (a) = f (b), then Rolles theorem is applicable for $x\in(a, b)$.

(a) Statement- 1 is True, Statement-2 is True, Statement-2 is a correct explanation for Statement -1

(b) Statement -1 is True, Statement -2 is True ; Statement-2 is NOT a correct explanation for Statement - 1

(c) Statement -1 is False, Statement -2 is True

(d) Statement - 1 is True, Statement- 2 is False

30. **Statement-1 :** $f(x)=|x|\sin x$, is differentiable at x = 0.

Statement-2 : If f (x) is not differentiable and g (x) is differentiable at x = a, then f(x) . g (x) can still be differentiable at x = a.

(a) Statement-1 is true, Statement-2 is true, Statement-2 is a correct explanation for Statement -1

(b) Statement -1 is True, Statement -2 is True ; Statement-2 is NOT a correct explanation for Statement - 1

(c) Statement -1 is False, Statement -2 is True

(d) Statement - 1 is True, Statement- 2 is False

RESPONSE GRID	25. ⓐⓑⓒⓓ	26. ⓐⓑⓒⓓ	27. ⓐⓑⓒⓓ	28. ⓐⓑⓒⓓ	29. ⓐⓑⓒⓓ
	30. ⓐⓑⓒⓓ				

DAILY PRACTICE PROBLEM DPP CHAPTERWISE 20 - MATHEMATICS			
Total Questions	30	Total Marks	120
Attempted		Correct	
Incorrect		Net Score	
Cut-off Score	41	Qualifying Score	60
Success Gap = Net Score – Qualifying Score			
Net Score = (Correct × 4) – (Incorrect × 1)			

1. (c) We have

$f\left(\frac{x+y}{3}\right)=\frac{f(x)+f(y)}{3}$, f(0) =0 and $f'(0)=3$

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{f\left(\frac{3x+3h}{3}\right)-f(x)}{h}$$

$$=\lim_{h\to0}\frac{\frac{f(3x)+f(3h)}{3}-\frac{f(3x)+f(0)}{3}}{h}=\lim_{h\to0}\frac{f(3h)-f(0)}{3h}=3$$

$\therefore$ $f(x)=3x+c$, $\because f(0)=0\Rightarrow c=0$

$\therefore$ $f(x)=3x$

2. (a) $\lim_{x\to0}(\cos x)^{1/x}=k\Rightarrow\lim_{x\to0}\frac{1}{x}\log(\cos x)=\log k$

$$\Rightarrow\lim_{x\to0}\frac{1}{x}\lim_{x\to0}\log\cos x=\log k$$

$$\Rightarrow\lim_{x\to0}\frac{1}{x}\times0=\log_e k\Rightarrow k=1.$$

3. (b) $x^2+y^2=a^2\Rightarrow2x+2yy'=0\Rightarrow y'=-x/y$

$\Rightarrow yy'+x=0$

$$\Rightarrow yy''+y'^2+1=0\Rightarrow y=-\left(\frac{1+y'^2}{y''}\right)\quad....(i)$$

$$\therefore k=\frac{1}{a}=\left|\frac{1}{\sqrt{x^2+y^2}}\right|=\left|\frac{1}{y\sqrt{1+\frac{x^2}{y^2}}}\right|=\left|\frac{1}{y\sqrt{1+y_1^2}}\right|\left[\because y'=-\frac{x}{y}\right]$$

$$=\left|\frac{-y''}{(1+y'^2)\sqrt{1+y'^2}}\right|=\frac{|y''|}{(1+y'^2)^{3/2}}$$

4. (c) Since f(x) is continuous at x = 0 $\therefore\lim_{x\to0}f(x)=f(0)$

Take any point x = a, then at x = a

$\lim_{x\to a}f(x)=\lim_{h\to0}f(a+h)$

$=\lim_{h\to0}[f(a)+f(h)]$ $[\because f(x+y)=f(x)+f(y)]$

$=f(a)+\lim_{h\to0}f(h)=f(a)+f(0)=f(a+0)=f(a)$

$\therefore$ f(x) is continuous at x = a. Since x = a is any arbitrary point, therefore f(x) is continuous for all x.

5. (a) Let $u=\tan^{-1}\frac{2x}{1-x^2}$ ...... (i)

and $v=\sin^{-1}\frac{2x}{1+x^2}$ ...... (ii)

In equation (i) put, $x=\tan\theta$

$$\therefore u=\tan^{-1}\left[\frac{2\tan\theta}{1-\tan^2\theta}\right]=\tan^{-1}(\tan2\theta)$$

$$\Rightarrow u=2\theta\Rightarrow\frac{du}{d\theta}=2\text{ ...... (a)}$$

In equation (ii), put $x=\tan\theta$

$$\therefore v=\sin^{-1}\left[\frac{2\tan\theta}{1+\tan^2\theta}\right]=\sin^{-1}(\sin2\theta)$$

$$\Rightarrow v=2\theta\Rightarrow\frac{dv}{d\theta}=2\text{ .... (b)}$$

From equations (a) and (b),

$$\frac{du}{dv}=\frac{du}{d\theta}\times\frac{d\theta}{dv}=2\times\frac{1}{2}=1$$

$\therefore$ Required differential coefficient will be 1.

6. (c) In the definition of the function, $b\neq0$, then f(x) will be undefined in $x>0$.

$\because$ f(x) is continuous at x = 0, $\therefore$ LHL = RHL = f(0)

$$\Rightarrow\lim_{\substack{x\to0\\x<0}}\frac{\sin(a+1)x+\sin x}{x}=\lim_{\substack{x\to0\\x<0}}\frac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}}=c$$

$$\Rightarrow\lim_{x\to0}\left(\frac{\sin(a+1)x}{x}+\frac{\sin x}{x}\right)=\lim_{x\to0}\frac{\sqrt{1+bx}-1}{bx}=c$$

$$\Rightarrow(a+1)+1=\lim_{x\to0}\frac{(1+bx)-1}{bx(\sqrt{1+bx}+1)}=c$$

$$\Rightarrow a+2=\lim_{x\to0}\frac{1}{\sqrt{1+bx}+1}=c$$

$$\Rightarrow a+2=\frac{1}{2}=c\quad\therefore a=-\frac{3}{2},\ c=\frac{1}{2},\ b\neq0$$

7. (b) Since $f'(x)=g(x)$, $f''(x)=g'(x)$

Put $f''(x)=-f(x)$. Hence $g'(x)=-f(x)$

we have $h'(x)=2f(x)f'(x)+2g(x)g'(x)$

$=2[f(x)g(x)+g(x)[-f(x)]]=2[f(x)g(x)-f(x)g(x)]=0$

$\therefore$ h(x) = C, a constant

$\therefore$ h(0) = C i.e. C = 5

h(x) = 5 for all x. Hence h (10) = 5.

8. (d) $f(x)=[x]^2-[x^2]=(-1)^2-(0)^2=0,-1<x<0\Rightarrow0<x^2<1$

$=0-0=0,\ 0\le x<1$ and $=1-1=0,\ 1\le x<\sqrt{3}$

and $=1-3=-2,\ \sqrt{3}\le x<\sqrt{4}$

$\therefore$ From above it is clear that the function is discontinuous at $\sqrt{n}\ \forall\ n\in\mathbf{I}$ except at x = 1.

- -

9. **(d)** $f(0)=\dfrac{0+1\times 0}{0+1}=0$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\lim_{n\to\infty}\frac{\tan \pi x^2+(x+1)^n \sin x}{x^2+(x+1)^n}$$

$$=\lim_{x\to 0^-}\frac{\tan \pi x^2}{x^2}\quad (\text{If } x\to 0^-,\ x+1<1)$$

$=\pi$ $\qquad\therefore$ LHL $\neq f(0)$

$\therefore$ f(x) is not continuous at x = 0 hence not differentiable also.

10. **(d)** For $f(x)$ to be continuous at $x = 0$, we should have

$$\lim_{\infty\to 0} f(x)=f(0)=12(\log 4)^3$$

$$\lim_{\infty\to 0} f(x)=\lim_{x\to 0}\left(\frac{4^x-1}{x}\right)^3\times\frac{\left(\frac{x}{p}\right)}{\left(\sin\frac{x}{p}\right)}\cdot\frac{px^2}{\log\left(1+\frac{1}{3}x^2\right)}$$

$$=(\log 4)^3\cdot 1\cdot p\cdot\lim_{x\to 0}\left(\frac{x^2}{\frac{1}{3}x^2-\frac{1}{18}x^4+\dots}\right)$$

$=3p(\log 4)^3\cdot$ Hence p = 4.

11. **(c)** $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

$$\Rightarrow 3c^2-6c+2=\frac{3/8-0}{1/2-0}=\frac{3}{4}$$

$$\Rightarrow c=1\pm\frac{\sqrt{21}}{6}\Rightarrow c=1+\frac{\sqrt{21}}{6}\notin\left(0,\frac{1}{2}\right)\Rightarrow c=1-\frac{\sqrt{21}}{6}$$

12. **(a)** Let $y=\left(1+\dfrac{1}{x}\right)^x$

Taking logarithm of both sides, we get

$$\log y=x\left[\log\left(1+\frac{1}{x}\right)\right]$$

$$\Rightarrow\ \frac{1}{y}y_1(x)=\frac{x^2}{x+1}\left(-\frac{1}{x^2}\right)+\log\left(1+\frac{1}{x}\right)$$

$$=-\frac{1}{x+1}+\log\left(1+\frac{1}{x}\right)\qquad\text{......... (1)}$$

Since, $y(2)=(1+1/2)^2=9/4$

so, $y_1(2)=(9/4)\left(-\dfrac{1}{3}+\log\dfrac{3}{2}\right)$

Again differentiate eq (1) w.r.t (x), we get

$$\frac{y(x)y_2(x)-[y_1(x)]^2}{(y(x))^2}=\frac{1}{(1+x)^2}-\frac{1}{x(x+1)}$$

By putting $x = 2$, we get

$$\frac{y(2)y_2(2)-(y_1(2))^2}{(y(2))^2}=\frac{-1}{18}$$

Now, put value of $y(2)$ and $y_1(2)$

$$\Rightarrow y_2(2)=\left(\frac{9}{4}\right)\left(-\frac{1}{3}+\log\frac{3}{2}\right)^2-\frac{1}{8}$$

$$4\left(y_2(2)+\frac{1}{8}\right)=9\left(\log\frac{3}{2}-\frac{1}{3}\right)^2$$

$\Rightarrow$ Required expression = 3

13. **(d)** We have, $f(x)=\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}(\log a)^n=\sum_{n=0}^{\infty}\frac{(x\log a)^n}{n!}$

$$=e^{x\log a}=e^{\log a^x}=a^x$$

$$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{a^{-h}-1}{-h}=\log_e a$$

$$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{a^h-1}{h}=\log_e a$$

Since $Lf'(0)=Rf'(0)$, $\quad\therefore$ f(x) is differentiable at x = 0

Since every differentiable function is continuous, therefore, f(x) is continuous at x = 0.

14. **(a)** $\displaystyle\lim_{x\to 0^+}\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}=\lim_{x\to 0^+}\frac{1-e^{-2/x}}{1+e^{-2/x}}=1$

and $\displaystyle\lim_{x\to 0^-}\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}=\lim_{x\to 0^-}\frac{e^{2/x}-1}{e^{2/x}+1}=-1.$

Hence $\displaystyle\lim_{x\to 0} f(x)$ exists if $\displaystyle\lim_{x\to 0} g(x)=0$.

If $g(x)=a\neq 0$ (constant) then

$\displaystyle\lim_{x\to 0+} f(x)=a$ and $\displaystyle\lim_{x\to 0-} f(x)=-a.$

Thus $\displaystyle\lim_{x\to 0} f(x)$ doesn't exist in this case.

$\therefore\ \displaystyle\lim_{x\to 0} f(x)$ exists in case of (b), (c) and (d) each.

15. **(a)** Let $f(x)=\cot^{-1}\left(\dfrac{x^x-x^{-x}}{2}\right)$

Take out x^{-x} common

$$f(x)=\cot^{-1}\left(\frac{x^{2x}-1}{2x^x}\right)$$

Put $x^x=\tan\theta$

$$\therefore\ f(x)=\cot^{-1}\left\{\frac{\tan^2\theta-1}{2\tan\theta}\right\}=\cot^{-1}(-\cot 2\theta)$$

$$=\pi-\cot^{-1}(\cot 2\theta)\quad\left[\because \tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}\right]$$

$$\Rightarrow\ f(x)=\pi-2\theta=\pi-2\tan^{-1}(x^x)$$

Differentiate w.r.t. x, we get

$$f'(x) = -\frac{2}{1+x^{2x}}.x^x(1+\log x)$$

$\therefore$ At $x=1$

$$f'(1) = \frac{-2}{1+1}(1+0) = -1.$$

16. (d) $f(x)$ = max. $\{x, x^3\}$

$$= \begin{cases} x\ ; & x<-1 \\ x^3; & -1 \le x \le 0 \\ x\ ; & 0 \le x \le 1 \\ x^3; & x \ge 1 \end{cases}$$

$$\therefore \quad f'(x) = \begin{cases} 1\ ; & x<-1 \\ 3x^2; & -1 \le x \le 0 \\ 1\ ; & 0 \le x \le 1 \\ 3x^2; & x \ge 1 \end{cases}$$

Clearly f is not differentiable at – 1, 0 and 1.

17. (b) f is continuous at $x = \pi/4$, if $\lim_{x\to\pi/4} f(x) = f(\pi/4)$.

Now, $L = \lim_{x\to\pi/4} (\sin 2x)^{\tan^2 2x}$

$\Rightarrow \log L = \lim_{x\to\pi/4} \tan^2 2x \log\sin 2x$

$= \lim_{x\to\pi/4} \frac{\log\sin 2x}{\cot^2 2x}\left(\frac{\infty}{\infty}\right)$

$= \lim_{x\to\pi/4} \frac{2\cot 2x}{-2\cot 2x \operatorname{cosec}^2 2x.2} = -\frac{1}{2}$

or $L = e^{-1/2}$ $\qquad \therefore f(\pi/4) = e^{-1/2} = 1/\sqrt{e}$

18. (a) Given $f^{-1}(x) = g(x)$

$\Rightarrow x = f[g(x)]$

Diff. both side w.r.t (x)

$\Rightarrow 1 = f'[g(x)].g'(x) \Rightarrow g'(x) = \frac{1}{f'(g(x))}$

Given, $f'(x) = \sin x$

$\therefore f'(g(x)) = \sin[g(x)]$

$\Rightarrow \frac{1}{f'(g(x))} = \operatorname{cosec}[g(x)]$

Hence, $g'(x) = \operatorname{cosec}[g(x)]$

19. (c) $f[(\pi/2)^-] = \lim_{h\to 0} \frac{1-\sin^3[(\pi/2)-h]}{3\cos^2[(\pi/2)-h]}$

$= \lim_{h\to 0} \frac{1-\cos^3 h}{3\sin^2 h} = \frac{1}{2}$

$f[(\pi/2)^+] = \lim_{h\to 0} \frac{q[1-\sin\{(\pi/2)+h\}]}{[\pi-2\{(\pi/2)+h\}]^2} = \lim_{h\to 0} \frac{q(1-\cos h)}{4h^2} = \frac{q}{8}$

$\therefore p = \frac{1}{2} = \frac{q}{8} \Rightarrow p = \frac{1}{2}, q = 4.$

20. (d) $|x|$ is non-differentiable function at $x = 0$ as L.H.D = –1 and R.H. D = 1

$$\because |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$$

But $\cos|h|$ is differentiable

$\therefore$ Any combination of two such functions will be non-differentiable. Hence option (a) and (b) are ruled out.

Now, consider $\sin|x| + |x|$

$L' = \lim_{h\to 0} \frac{\sin|-h| + |-h|}{-h}$

$= \lim_{h\to 0} \frac{\sin h}{-h} - 1 = -1-1 = -2$

$R' = \lim_{h\to 0} \frac{\sin|h| + |h|}{h}$

$= \lim_{h\to 0} \frac{\sin h}{h} + 1 = 1+1 = 2$

Consider $\sin|x| - |x|$

$L' = \lim_{h\to 0} \frac{\sin|-h| - |-h|}{-h}$

$= \lim_{h\to 0} \frac{\sin h}{-h} + 1 = 0$

$R' = \lim_{h\to 0} \frac{\sin|h| - |h|}{h}$

$= \lim_{h\to 0} \frac{\sin h}{h} - 1 = 0$

Hence, $\sin|x| - |x|$ is differentiable at $x = 0$.

21. (d) $\underset{x\to 0}{\text{Lt}} \frac{(e^x-1)^2}{\sin\left(\frac{x}{a}\right)\log\left(1+\frac{x}{4}\right)}$

$$= \underset{x\to 0}{\text{Lt}} \frac{\frac{(e^x-1)^2}{x}.x^2}{\frac{x}{a}.\frac{\sin\left(\frac{x}{a}\right)}{\left(\frac{x}{a}\right)}.\frac{\log\left(1+\frac{x}{4}\right)}{\frac{x}{4}}.\frac{x}{4}}$$

$\Rightarrow 4a = 12 \Rightarrow a = 3$

22. (b) Let $f(x)=a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x = 0$

The other given equation,

$$na_n x^{n-1} + (n-1)\, a_{n-1}x^{n-2} + ... + a_1 = 0 = f'(x)$$

Given $a_1 \neq 0 \Rightarrow f(0)=0$

Again f (x) has root α, $\Rightarrow f(\alpha)=0$

$\therefore\ f(0)=f(\alpha)$

$\therefore$ By Rolle's theorem,

$f'(x)=0$ has root between $(0, \alpha)$

Hence $f'(x)$ has a positive root smaller than α.

23. (a) If f is continuous at x = 0, then

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\Rightarrow f(0) = \lim_{x\to 0^-} f(x)$$

$$k = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{\cos\frac{\pi}{2}[0-h]}{[0-h]}$$

$$k = \lim_{h\to 0} \frac{\cos\frac{\pi}{2}[-h]}{[-h]} = \lim_{h\to 0} \frac{\cos\frac{\pi}{2}[-h-1]}{[-h-1]}$$

$$k = \lim_{h\to 0} \frac{\cos\left(-\frac{\pi}{2}\right)}{-1}; k = 0$$

24. (b) The denominator of the given function is always defined
Also, tan $[x]\pi = \tan n\pi = 0$ [[x] = integer, say n]

$\therefore f(x)=0\ \forall x$

$\therefore$ f(x) is continuous and differentiable for all x.

25. (c) Put $x^n = \cos\alpha$, $y^n = \cos\beta$

$$\Rightarrow a = \frac{\sin\alpha + \sin\beta}{\cos\alpha - \cos\beta} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$= -\cot\left(\frac{\alpha-\beta}{2}\right)$$

$\Rightarrow 2\cot^{-1}(-a) = \alpha - \beta$

$\Rightarrow \cos^{-1}(x^n) - \cos^{-1}(y^n) = 2\cot^{-1}(-a)$

$$\Rightarrow \frac{y^{n-1}}{\sqrt{1-y^{2n}}}\frac{dy}{dx} = \frac{x^{n-1}}{\sqrt{1-x^{2n}}} \Rightarrow \sqrt{\frac{1-x^{2n}}{1-y^{2n}}}\frac{dy}{dx} = \frac{x^{n-1}}{y^{n-1}}$$

26. (b) Clearly x = 8 satisfies the given equation. Assume that $f(x) = e^{x-8} + 2x - 17 = 0$ has a real root α other than x = 8. We may suppose that $\alpha > 8$ (the case for $\alpha < 8$ is exactly similar). Applying Rolle's theorem on $[8, \alpha]$, we get $\beta \in (8, \alpha)$, such that $f'(\beta) = 0$.

But $f'(\beta) = e^{\beta-8} + 2$, so that $e^{\beta-8} = -2$ which is not possible, Hence there is no real root other than 8.

27. (a) $F'(x) = \left[f\left(\frac{x}{2}\right).f'\left(\frac{x}{2}\right) + g\left(\frac{x}{2}\right)g'\left(\frac{x}{2}\right)\right]$

Here, $g(x) = f'(x)$

and $g'(x) = f''(x) = -f(x)$

so $F'(x) = f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right) - f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right) = 0$

$\Rightarrow$ F (x) is constant function

so F (10) = 5

28. (d) (a) $\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$ exist finitely

$\therefore \lim_{h\to 0^+} f(a+h) - f(a)$

$$= \lim_{h\to 0^+}\left(\frac{f(a+h)-f(a)}{h}\right)h = 0$$

$\Rightarrow \lim_{h\to 0^+} f(a+h) = f(a)$

Similarly, $\lim_{h\to 0^-} f(a+h) = f(a)$

$\therefore$ f is continuous at x = a

(b) Function is not differentiable at $5x = (2n+1)\frac{\pi}{2}$ only, which are not in domain

(c) Let $f(x) = \frac{1}{x^2}$ and $g(x) = -\frac{1}{x^2}$,

$\lim_{x\to 0} f(x)+g(x)$ exists whatever $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ does not exist.

29. (d) **Statement 1 :** As $f(-1) = f(1)$ and Rolles theorem is not applicable, then it implies f (x) is either discontinuous or f ' (x) does not exist at atleast one point in (–1, 1)
$\Rightarrow$ g (x) = 0 at atleast one value of x in (–1, 1).
Statement 2 is false. Consider the example in statement-1.

30. (a) $f(x) = |x| \sin x$

$$\text{L.H.D.} = \lim_{h\to 0}\frac{|0-h|\sin(0-h) - 0}{h}$$

$$= \lim_{h\to 0}\frac{-h\sin h}{h} = 0$$

$$\text{R.H.D.} = \lim_{h\to 0}\frac{|0+h|\sin(0+h) - 0}{h}$$

f(x) is differentiable at x = 0