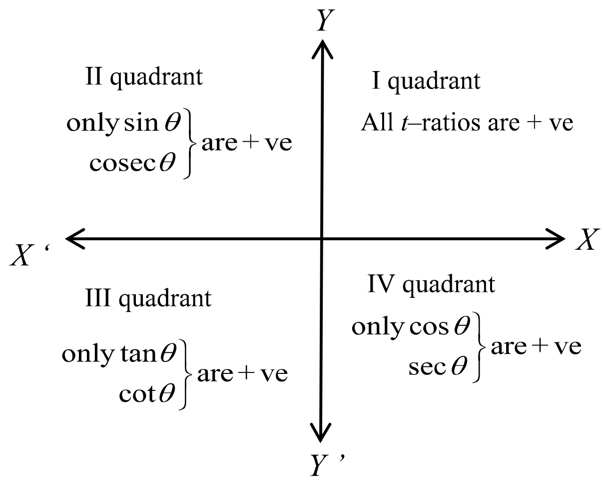
Trigonometric Ratios, Identities & Equations

Chapter 12

BASIC FORMULAE

1. $\sin^2 A + \cos^2 A = 1$ or $\cos^2 A = 1 - \sin^2 A$ or $\sin^2 A = 1 - \cos^2 A$.
2. $1 + \tan^2 A = \sec^2 A$ or $\sec^2 A - \tan^2 A = 1$.
3. $1 + \cot^2 A = \text{cosec}^2 A$, or $\text{cosec}^2 A - \cot^2 A = 1$
4. $\sin A \operatorname{cosec} A = \tan A \cot A = \cos A \sec A = 1$.

A system of rectangular coordinate axes divides a plane into four quadrants. An angle θ lies in one and only one of these quadrants. The values of the trigonometric ratios in the various quadrants are shown in Fig.



Formulae for the trigonometric ratios of sum and differences of two angles

1. $\sin(A+B) = \sin A \cos B + \cos A \sin B$
2. $\sin(A-B) = \sin A \cos B - \cos A \sin B$
3. $\cos(A+B) = \cos A \cos B - \sin A \sin B$
4. $\cos(A-B) = \cos A \cos B + \sin A \sin B$
5. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
6. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$
7. $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$
8. $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$
9. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
10. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
11. $\tan A \pm \tan B = \dfrac{\sin A}{\cos A} \pm \dfrac{\sin B}{\cos B} = \dfrac{\sin A \cos B \pm \cos A \sin B}{\cos A \cos B} = \dfrac{\sin(A \pm B)}{\cos A . \cos B}$,
12. $\cot A \pm \cot B = \dfrac{\sin(B \pm A)}{\sin A \sin B}$
13. $\tan A + \cot B = \dfrac{\cos(B - A)}{\cos A \sin B}$

14. $\tan A-\cot B=\dfrac{-\cos(B+A)}{\cos A\sin B}$

Formulae for the trigonometric ratios of sum and differences of three angle

1. $\sin(A+B+C)=\sin A\cos B\cos C+\cos A\sin B\cos C+\cos A\cos B\sin C-\sin A\sin B\sin C$

   or $\sin(A+B+C)=\cos A\cos B\cos C\,(\tan A+\tan B+\tan C-\tan A.\tan B.\tan C)$

2. $\cos(A+B+C)=\cos A\cos B\cos C-\sin A\sin B\cos C-\sin A\cos B\sin C-\cos A\sin B\sin C$

   $\cos(A+B+C)=\cos A\cos B\cos C(1-\tan A\tan B-\tan B\tan C-\tan C\tan A)$

3. $\tan(A+B+C)=\dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$

4. $\cot(A+B+C)=\dfrac{\cot A\cot B\cot C-\cot A-\cot B-\cot C}{\cot A\cot B+\cot B\cot C+\cot C.\cot A-1}=\dfrac{\cot A+\cot B+\cot C-\cot A\cot B\cot C}{1-\cot A\cot B-\cot B\cot C-\cot C\cot A}$

In general

5. $\sin(A_1+A_2+...+A_n)=\cos A_1\cos A_2...\cos A_n(S_1-S_3+S_5-S_7+...)$

6. $\cos(A_1+A_2+....+A_n)=\cos A_1\cos A_2....\cos A_n(1-S_2+S_4-S_6....)$

7. $\tan(A_1+A_2+...A_n)=\dfrac{S_1-S_3+S_5-S_7+...}{1-S_2+S_4-S_6+...}$

   where, $S_1=\tan A_1+\tan A_2+...+\tan A_n$ = The sum of the tangents of the separate angles.

   $S_2=\tan A_1\tan A_2+\tan A_1\tan_3+...=$ The sum of the tangents taken two at time.

   $S_3=\tan A_1\tan A_2\tan A_3+\tan A_2\cdot\tan A_3\tan A_4+...$ = Sum of tangents three at a time, and so on.

   If $A_1=A_2=...=A_n=A,\ then\ \ S_1=n\tan A$

   $S_2={}^nC_2\tan^2A, S_3={}^nC_3\tan^3A,...$

8. $\sin nA=\cos^nA\left({}^nC_1\tan A-{}^nC_3\tan^3A+{}^nC_5\tan^5A-...\right)$

9. $\cos nA=\cos^nA\left(1-{}^nC_2\tan^2A+{}^nC_4\tan^4A-...\right)$

10. $\tan nA=\dfrac{{}^nC_1\tan A-{}^nC_3\tan^3A+{}^nC_5\tan^5A-...}{1-{}^nC_2\tan^2A+{}^nC_4\tan^4A-{}^nC_6\tan^6A+...}$

11. $\sin(\alpha)+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+...+\sin(\alpha+(n-1)\beta)=\dfrac{\sin\left\{\alpha+(n-1)\left(\beta/2\right)\right\}.\sin\left(n\beta/2\right)}{\sin\left(\beta/2\right)}$

12. $\cos(\alpha)+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+...+\cos(\alpha+(n-1)\beta)=\dfrac{\cos\left\{\alpha+(n-1)\left(\beta/2\right)\right\}.\sin\left(n\beta/2\right)}{\sin\left(\beta/2\right)}$

Formulae to transform the product into sum or difference

1. $2\sin A\cos B=\sin(A+B)+\sin(A-B)$
2. $2\cos A\sin B=\sin(A+B)-\sin(A-B)$
3. $2\cos A\cos B=\cos(A+B)+\cos(A-B)$
4. $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

Let $A+B=C$ *and* $A-B=D$ Then $A=\frac{C+D}{2}$ and $B=\frac{C-D}{2}$

Formulae to transform the sum or difference into product

1. $\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$
2. $\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$
3. $\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$
4. $\cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

Trigonometric ratio of multiple of an angle

1. $\sin 2A=2\sin A\cos A=\frac{2\tan A}{1+\tan^2 A}$
2. $\cos 2A=2\cos^2 A-1=1-2\sin^2 A=\cos^2 A-\sin^2 A=\frac{1-\tan^2 A}{1+\tan^2 A}$.
3. $\tan 2A=\frac{2\tan A}{1-\tan^2 A}$
4. $\sin 3A=3\sin A-4\sin^3 A=4\sin\left(60^o-A\right).\sin A.\sin\left(60^o+A\right)$
5. $\cos 3A=4\cos^3 A-3\cos A=4\cos\left(60^o-A\right).\cos A.\cos\left(60^o+A\right)$
6. $\tan 3A=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}=\tan\left(60^o-A\right).\tan A.\tan\left(60^o+A\right)$
7. $\sin 4\theta=4\sin\theta.\cos^3\theta-4\cos\theta\sin^3\theta$
8. $\cos 4\theta=8\cos^4\theta-8\cos^2\theta+1$
9. $\tan 4\theta=\frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$
10. $\sin 5A=16\sin^5 A-20\sin^3 A+5\sin A$
11. $\cos 5A=16\cos^5 A-20\cos^3 A+5\cos A$

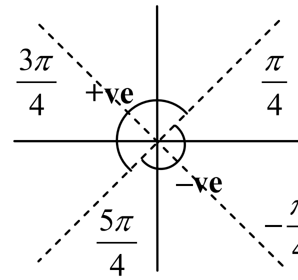
Trigonometrical values

1. $\sin 22\frac{1}{2}°=\cos 67\frac{1}{2}°=\frac{1}{2}\sqrt{2-\sqrt{2}}$
2. $\sin 67\frac{1}{2}°=\cos 22\frac{1}{2}°=\frac{1}{2}\sqrt{2+\sqrt{2}}$
3. $\sin 15°=\cos 75°=\frac{\sqrt{3}-1}{2\sqrt{2}}$
4. $\cos 15°=\sin 75°=\frac{\sqrt{3}+1}{2\sqrt{2}}$
5. $\tan 15°=\cot 75°=\frac{\sqrt{3}-1}{\sqrt{3}+1}=2-\sqrt{3}$
6. $\tan 75°=\cot 15°=\frac{\sqrt{3}+1}{\sqrt{3}-1}=2+\sqrt{3}$
7. $\tan\left(22\frac{1}{2}\right)^\circ=\cot\left(67\frac{1}{2}\right)^\circ=\sqrt{2}-1$
8. $\tan\left(67\frac{1}{2}\right)^\circ=\cot\left(22\frac{1}{2}\right)^\circ=\sqrt{2}+1$
9. $\sin 18°=\frac{\sqrt{5}-1}{4}=\cos 72°$
10. $\sin 36°=\frac{\sqrt{10-2\sqrt{5}}}{4}=\cos 54°$
11. $\sin 54°=\frac{\sqrt{5}+1}{4}=\cos 36°$
12. $\sin 72°=\frac{\sqrt{10+2\sqrt{5}}}{4}=\cos 18°$

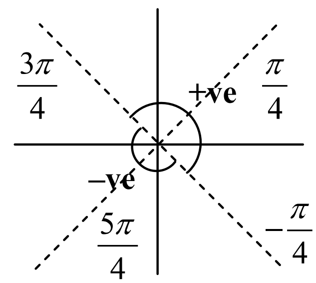
CONDITIONAL TRIGONOMETRICAL IDENTITIES

If $A+B+C=180^o$, then

1. $\sin 2A+\sin 2B+\sin 2C = 4\sin A\sin B\sin C$
2. $\sin 2A+\sin 2B-\sin 2C = 4\cos A\cos B\sin C$
3. $\cos 2A+\cos 2B+\cos 2C = -1-4\cos A\cos B\cos C$
4. $\cos 2A+\cos 2B-\cos 2C = 1-4\sin A\sin B\cos C$
5. $\sin A+\sin B+\sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
6. $\sin A+\sin B-\sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$
7. $\cos A+\cos B+\cos C = 1+4\sin\frac{A}{2}\sin\frac{B}{2}.\sin\frac{C}{2}$
8. $\cos A+\cos B-\cos C = -1+4\cos\frac{A}{2}.\cos\frac{B}{2}.\sin\frac{C}{2}$
9. (i) $\sin^2 A+\sin^2 B-\sin^2 C = 2\sin A\sin B\cos C$
   (ii) $\cos^2 A+\cos^2 B-\cos^2 C = 1-2\sin A\sin B\cos C$
   (iii) $\cos^2 A+\cos^2 B+\cos^2 C = 1-2\cos A\cos B\cos C$
   (iv) $\sin^2 A+\sin^2 B+\sin^2 C = 2+2\cos A\cos B\cos C$
10. (i) $\tan A+\tan B+\tan C = \tan A\tan B\tan C$
    (ii) $\cot B\cot C+\cot C\cot A+\cot A.\cos B = 1$
    (iii) $\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{A}{2}.\tan\frac{B}{2} = 1$
    (iv) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$



$\sin\theta-\cos\theta$ is +ve or –ve in interval shown above



$\sin\theta+\cos\theta$ is +ve or –ve in interval shown above

TIPS AND TRICKS

1. $\sin n\pi = 0$
2. $\cos n\pi = (-1)^n$
3. $\sin(n\pi\pm\theta) = \pm(-1)^n\sin\theta$
4. $\cos(n\pi\pm\theta)=(-1)^n\cos\theta$
5. $\tan(n\pi\pm\theta)=\pm\tan\theta$
6. $\sin\left(\frac{n\pi}{2}+\theta\right) = \begin{cases}(-1)^{\frac{n-1}{2}}\cos\theta, & \text{if } n \text{ is odd}\\ (-1)^{n/2}\sin\theta, & \text{if } n \text{ is even}\end{cases}$
7. $\cos\left(\frac{n\pi}{2}+\theta\right) = \begin{cases}(-1)^{\frac{n+1}{2}}\sin\theta, & \text{if } n \text{ is odd}\\ (-1)^{n/2}\cos\theta, & \text{if } n \text{ is even}\end{cases}$
8. $\tan\frac{A}{2} = \frac{-1\pm\sqrt{1+\tan^2 A}}{\tan A}$

9. $\prod_{r=0}^{n-1} \cos 2^r A = \frac{\sin 2^n A}{2^n \sin A}$

10. $\tan\alpha = \cot\alpha - 2\cot 2\alpha$, $\operatorname{cosec} 2\alpha = \cot\alpha - \cot 2\alpha$

11. $\sec\alpha.\sec(\alpha+\beta) = \operatorname{cosec}\beta\{\tan(\alpha+\beta) - \tan\alpha\}$

12. $\tan\alpha.\tan(\alpha+\beta) = \cot\beta\{\tan(\alpha+\beta) - \tan\alpha\} - 1$

13. $\operatorname{cosec}\alpha\operatorname{cosec}(\alpha+\beta) = \operatorname{cosec}\beta\{\cot\alpha - \cot(\alpha+\beta)\}$

14. $\tan\alpha.\sec 2\alpha = \tan 2\alpha - \tan\alpha$

15. $\tan^2\alpha.\tan 2\alpha = \tan 2\alpha - 2\tan\alpha$

16. $\tan A + \tan B = \frac{\sin(A+B)}{\cos A \cos B}$

17. $\tan A - \tan B = \frac{\sin(A-B)}{\cos A \cos B}$

18. $\cot A + \cot B = \frac{\sin(A+B)}{\sin A \sin B}$

19. $\cot A - \cot B = \frac{\sin(B-A)}{\sin A \sin B}$

20. $1 + \tan A \tan B = \frac{\cos(A-B)}{\cos A \cos B}$

21. $1 - \tan A \tan B = \frac{\cos(A+B)}{\cos A \cos B}$

22. $1 + \cot A \cot B = \frac{\cos(A-B)}{\sin A \sin B}$

23. $1 - \cot A \cot B = \frac{-\cos(A+B)}{\sin A \sin B}$

24. If $\alpha + \beta + \gamma = 0$ then

(a) $\sin\alpha + \sin\beta + \sin\gamma = -4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$, (b) $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

(c) $\cos\alpha + \cos\beta + \cos\gamma = -1 + 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

24. $\tan\theta + \cot\theta = 2\operatorname{cosec} 2\theta$ 25. $\cot\theta - \tan\theta = 2\cot 2\theta$

26. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$.

27. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$.

28. $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$

29. $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma) = 4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}$

30. $\tan\alpha + \tan\beta + \tan\gamma - \tan(\alpha+\beta+\gamma) = \tan\alpha\tan\beta\tan\gamma\left[1 - \frac{\cot\alpha + \cot\beta + \cot\gamma}{\cot(\alpha+\beta+\gamma)}\right]$

31. For any α, β and γ we have the following identities.

(a) $\sin\alpha\sin(\beta-\gamma)+\sin\beta\sin(\gamma-\alpha)+\sin\gamma\sin(\alpha-\beta) = 0$

(b) $\cos\alpha\sin(\beta-\gamma)+\cos\beta\sin(\gamma-\alpha)+\cos\gamma\sin(\alpha-\beta) = 0$.

32. (a) $(\cos x+\sin x)(\cos y+\sin y)=\cos(x-y)+\sin(x+y)$

(b) $(\cos x-\sin x)(\cos y-\sin y)=\cos(x-y)-\sin(x+y)$

TRIGONOMETRICAL EQUATIONS WITH THEIR GENERAL SOLUTION.

Trigonometrical equation	General solution
$\sin\theta=\sin\alpha$	$\theta=n\pi+(-1)^n\alpha$ where $n\in I$
$\cos\theta=\cos\alpha$	$\theta=2n\pi\pm\alpha$ where $n\in I$
$\tan\theta=\tan\alpha$	$\theta=n\pi+\alpha$ where $n\in I$
$\sin^2\theta=\sin^2\alpha$ $\tan^2\theta=\tan^2\alpha$ $\cos^2\theta=\cos^2\alpha$	$\theta=n\pi\pm\alpha$ where $n\in I$

But it is better to remember the following results instead of using above formulae in the following cases.

Trigonometrical equation	General solution
$\sin\theta=0$	$\theta=n\pi$ where $n\in I$
$\cos\theta=0$	$\theta=n\pi+\pi/2$ where $n\in I$
$\sin\theta=1$	$\theta=2n\pi+\pi/2$ where $n\in I$
$\sin\theta=-1$	$\theta=2n\pi-\pi/2$ where $n\in I$
$\cos\theta=1$	$\theta=2n\pi$ where $n\in I$
$\cos\theta=-1$	$\theta=(2n+1)\pi$ where $n\in I$
$\sin\theta=\pm1$	$\theta=(2n+1)\pi/2$ where $n\in I$
$\cos\theta=\pm1$	$\theta=n\pi$ where $n\in I$

GENERAL SOLUTION OF THE FORM $a\cos\theta+b\sin\theta=c$

In $a\cos\theta+b\sin\theta=c$, put $a=r\cos\alpha$ and $b=r\sin\alpha$ where $r=\sqrt{a^2+b^2}$ and $|c|\le\sqrt{a^2+b^2}$.

Then, $r(\cos\alpha\cos\theta+\sin\alpha\sin\theta)=c \Rightarrow \cos(\theta-\alpha)=\dfrac{c}{\sqrt{a^2+b^2}}=\cos\beta$, (say)

$\Rightarrow \theta-\alpha=2n\pi\pm\beta \Rightarrow \theta=2n\pi\pm\beta+\alpha$, where $\tan\alpha=\dfrac{b}{a}$ is the general solution.